## INNER MODELS FOR LARGE CARDINALS

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One new idea leads to another, that to a third, and so on through a course of time until someone, with whom no one of these is original, combines all together, and produces what is justly called a new invention.<sup>1</sup>

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## 1 INTRODUCTION

- <sup>6</sup> This chapter will discuss the development of inner models for large cardinals during the last part of the last century, beginning at the end of Kanamori's chapter [Kanamori, 2010b], which is to say at about 1965. There are two major themes
- $_{9}$  in this development: The first of these is the expansion of Gödel's class L of constructible sets to obtain L-like models of ZFC which are capable of containing any larger cardinals existing in the universe, and the second is the development of
- <sup>12</sup> core model theory, which provides the possibility of proving that the universe is well approximated by such a model.
- The main part of the chapter is accordingly divided into two themes. The first <sup>15</sup> is in Section 2, which describes the process of incorporating large cardinal properties into the constructibility paradigm, and the discovery of strong uniqueness properties of the resulting models, allowing the class of large cardinals to be seen <sup>18</sup> as a well-ordered extension of the ordinals.

The second theme is begun in Section 3, which describes the evolution of combinatorial principles in L, leading through the development of fine structure to

<sup>21</sup> the discovery of the Covering Lemma. The generalization of this development into a core model theory for larger cardinals is described in section 4. The two themes merge in section 5, which describes the extension of the models to the level

- of Woodin cardinals, where the models involved rely on fine structure and other techniques of core model theory, even when the maximality implied by the core model is not assumed.
- Section 1 provides an introduction, which discusses the basic concepts of constructibility, on the one hand, and of large cardinals, on the other, as they were understood prior to the start of the main narrative. Both of these topics are cov ered more fully in other chapters of this volume: constructibility in [Kanamori,
- 2010b] (also see [Kanamori, 2007]) and large cardinals in [Kanamori, 2010a].

<sup>&</sup>lt;sup>1</sup>Thomas Jefferson, quoted in Bedini, "Godfather of American Invention" 82

## 1.1 Constructibility

According to Kanamori [Kanamori, 2010b], "An *inner model* is a transitive class containing all the ordinals such that, with membership and quantification restricted to it, the class satisfies each axiom of ZF." If we restrict consideration to definable classes, we have three principal examples. The first example is the

- <sup>6</sup> universe V which, although not generally thought of as an inner model, does satisfy the definition; and it also provides, via the cumulative hierarchy defined by von Neumann in 1929, the beginnings of the structural analysis which will be seen
- <sup>9</sup> in other inner models to be discussed.

The second example of an inner model is the class of *hereditarily ordinal definable* sets (HOD) which was introduced independently by several people to give an easier proof of the consistency of the Axiom of Choice. The model HOD has no intrinsic structure: the cumulative hierarchy in V is needed to to see that the class HOD is even definable, and moreover the formula defining it is not absolute, so that the model HOD<sup>(HOD)</sup>, that is, the model HOD as defined inside HOD, may well be a proper subclass of HOD. Its primary use has been to produce, with the help of forcing, models which do not satisfy the Axiom of Choice. However, a precursor to the core model and covering lemma can be seen in Vopěnka's theorem [Vopěnka and Hájek, 1972] stating that any set of ordinals is in a generic extension of HOD. The theme of HOD as a core model has recently been investigated; see, for example, [Steel, 1995].

The third example of an inner model, and the prototype of the models with which this chapter is concerned, is Gödel's class L of constructible sets, introduced in [Gödel, 1938; Gödel, 1939; Gödel, 1940]. The model L is defined using a hierarchy like that of the first example, but the hierarchy incorporates definability by modifying the successor step to  $L_{\alpha+1} = def(L_{\alpha})$  where def(X) is the set of

- <sup>27</sup> subsets of x which are first-order definable over the model  $(X, \in)$ , using parameters from X. In this case the definability is restricted to the emerging class being defined, and hence the definition of the class L is independent of the universe in
- which it is being defined. Thus the class L is minimal in the sense that it is contained in any model of ZF which contains all of the ordinals. A simple induction shows that it satisfies the Axiom of Choice and that  $|L_{\alpha}| = |\alpha|$  for all infinite or-
- dinals  $\alpha$ . To prove the Generalized Continuum Hypothesis (GCH) it then suffices to show that  $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$  for every cardinal  $\kappa$ . This is a consequence of what has become known as the principle of *condensation*:
- <sup>36</sup> LEMMA 1 Condensation. Suppose that  $\alpha$  is an ordinal and  $X \prec L_{\alpha}$ . Then  $X \cong L_{\bar{\alpha}}$  for some ordinal  $\bar{\alpha} \leq \alpha$ .

This principle is one of the major themes in the developments described in this <sup>39</sup> chapter.

## Relative Constructibility

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Relative constructibility was introduced independently by Hajnal [Hajnal, 1956; Hajnal, 1961] and Levy [Levy, 1957; Levy, 1960], who gave definitions which were

- equivalent in their original applications, but are quite different in general. Hajnal modified the definition of L to define L(A), where A is an arbitrary transitive set, by setting  $L_0(A) = A$ . Thus L(A) is the smallest model of ZF which contains
- $A \cup \{A\}$ . The model L(A) need not satisfy the Axiom of Choice, and indeed is most commonly used in applications where the Axiom of Choice does not hold.
- <sup>9</sup> Levy definition of L[A], on the other hand, retained from L the definition of  $L_0[A] = \emptyset$ , and instead modified the successor step, defining  $L_{\alpha+1}[A] = \det(L_{\alpha}[A], A)$ , the set of subsets definable in  $L_{\alpha}[A]$  with parameters from  $L_{\alpha}[A]$  and the predicate
- <sup>12</sup> A. Thus L[A] is the smallest model M of ZF which has  $A \cap M$  as a member, and L[A] always satisfies the Axiom of Choice; however A is not, in general, a subset of L[A]. The model L[A] can be viewed as the smallest model which contains the
- <sup>15</sup> structure of the set A, and since this chapter is concerned with models including large cardinal structure, essentially all models discussed will be of this type. In both cases the definability of the model requires the use of A as a parameter.
- <sup>18</sup> For this reason, the condensation principle, Lemma 1, is severely weakened. It is, in general, valid for L(A) only in the case that  $A \subseteq X$ . For L[A] there is the general statement
- <sup>21</sup> LEMMA 2. Suppose  $\pi: M \cong X \prec L_{\alpha}[A]$  is the inverse of the the transitive collapse of X. Then  $M = L_{\bar{\alpha}}[\bar{A}]$  for some ordinal  $\bar{\alpha} \leq \alpha$ , where  $\bar{A} = \pi^{-1}[A \cap X]$ . In particular, if  $\bar{A} = A \cap L_{\bar{a}}[\bar{A}]$  then  $X \cong L_{\bar{\alpha}}[A]$ .
- Levy originally applied this condensation principle in cases in which the set A is contained in a transitive subset of X, so that  $A = \overline{A}$ . Using this he could prove, for example, that if  $A \subseteq \kappa$  then  $2^{\lambda} = \lambda^+$  in L[A] for all  $\lambda \geq \kappa$ . The successful application of models of the form L[A] to large cardinal theory has come from
- choosing the set A to encode the large cardinal structure, and then using that structure to see that  $\bar{A} = A \cap L_{\bar{\alpha}}[\bar{A}]$  holds even in cases that  $\pi$  is not the identity on A.

In the model L[A] the set A is being regarded as providing structure. If U is a ultrafilter on a cardinal  $\kappa$ , for example, then the model  $L(U \cup \{U\})$  would include all subsets of  $\kappa$ , whether or not they have any relation to measurability. The model L[U], on the other hand, contains only those subsets of  $\kappa$  which are required by the possibility of using U as a predicate. As will be seen, these are exactly the sets which are required by the existence of any measure on  $\kappa$ .

## 1.2 Large Cardinals

Of the large cardinals properties known at the start of this history, two will dominate it: measurability and supercompactness.

Measurable cardinals were introduced by Ulam [Ulam, 1930], but we will use the later formulation in terms of elementary embeddings and normal ultrafilters:

A cardinal  $\kappa$  is *measurable* if there is an elementary embedding  $i: V \to M$  into a well-founded model M with *critical point*  $\kappa$ , that is, such that  $i(\kappa) > \kappa$ , but  $i(\nu) = \nu$  for all  $\nu < \kappa$ . The normal ultrafilter, or *measure*, associated with this embedding is  $U = \{ x \subseteq \kappa : \kappa \in j(x) \}$ . The measure U in turn leads to an embedding via the ultrapower construction,  $i^U: V \to \text{Ult}(V, U)$ .

- The concept of a measurable cardinal will serve as both a goal and a benchmark.
   The first part of the story will largely concern the effect which the presence of a measurable cardinal has on the constructible universe L, and the understanding
   of the minimal model L[U] containing a measurable cardinal. The rest of the
- <sup>9</sup> of the minimal model L[U] containing a measurable cardinal. The rest of the story will concern the generalization of this model and associated techniques to accommodate larger cardinals.
- <sup>12</sup> A number of large cardinals weaker than a measurable cardinal were known at the start of this history: these include inaccessible, Mahlo, and weakly compact cardinals. The most important for our purposes are Ramsey cardinals, which <sup>15</sup> satisfy the partition relation  $\kappa \to (\kappa)_2^{\leq \omega}$  and their generalization the  $\alpha$ -Erdős cardinals, which satisfy the partition relation  $\kappa \to (\alpha)_2^{\leq \omega}$ .

Supercompact cardinals were introduced, at the start of the period of this history, by Reinhardt and Solovay although they were only published later in [Solovay et al., 1978]: a cardinal  $\kappa$  is  $\lambda$ -supercompact if there is an elementary embedding  $i: V \to M$  with critical point  $\kappa$  such that  ${}^{\lambda}M \subseteq M$ , and is supercompact if it

- <sup>21</sup> is  $\lambda$ -supercompact for all  $\lambda$ . In contrast to measurable cardinals, it enters this history only as a goal: as this is being written, there is still no fully developed *L*-like model for even a  $\kappa^+$ -supercompact cardinal.
- <sup>24</sup> Given this fact, it is natural to ask about cardinal properties intermediate between measurable cardinals; at the start of this history, however there were, with one possible exception, no such cardinals. The possible exception is a *strongly*
- <sup>27</sup> compact cardinal. This notion was defined by Tarski [Tarski, 1962] using infinitary logic but characterized by Reinhardt and Solovay as being supercompactness without normality: that is, with the condition  ${}^{\lambda}M \subseteq M$  replaced by the covering
- <sup>30</sup> property  $\forall x \in [M]^{\lambda} \exists y \in M \ (x \subseteq y \land |y| < j(\kappa))$ . It is conjectured that having a strongly compact cardinal is equiconsistent with having a supercompact cardinal, and proving this has been described as the holy grail problem of inner model

A number of intermediate large cardinal properties have since been discovered, and will be described in due course. Most of these are modifications either of mea-<sup>36</sup> surability or of supercompactness, but one exception is worth mentioning here: *Woodin cardinals*, discovered in 1984, have come to have an independent importance comparable to that of measurable and supercompact cardinals.

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<sup>33</sup> theory.

## 2 DEVELOPMENT OF INNER MODELS WITHOUT FINE STRUCTURE

## 2.1 Large cardinals in L

- <sup>3</sup> The study of *L*-like models for large cardinals naturally starts with *L* itself. Gödel [Gödel, 1938] observed that any inaccessible or Mahlo cardinal in *V* has the same property in *L*, simply because these properties are downward absolute to any inner
- $_{6}$  model. After the discovery of weakly compact cardinals it was quickly realized that they too are downward absolute to L, by a more complicated argument. The same was true of several related stronger properties.
- <sup>9</sup> The study of inner models for larger cardinals can be said to have started with Dana Scott's proof in [Scott, 1961] that there cannot be any measurable cardinals in L, by observing that otherwise there would be an elementary embedding
- i<sup>U</sup>: L → M = Ult(L, U) in M such that i(κ) > κ, where κ is the least measurable cardinal in L. This is impossible because M, as an inner model of the minimal model L, must be equal to L; but M satisfies the sentence asserting that i(κ) is
  the least measurable cardinal.
- The response to Scott's observation took two directions. The first direction, which concentrated on the model L in order to further characterizing the consequences of the existence in V of a measurable cardinal, led to Silver's discovery of  $0^{\#}$ . The other direction developed the model L[U] as the minimal model, analogous to L, which contains a measurable cardinal. These two approaches use many of the same techniques, and they eventually merged in Dodd and Jensen's construction of the core model.

## 2.2 What a measurable cardinal says about L

- <sup>24</sup> Scott's result was followed up in the model L by Gaifman [Gaifman, 1974] and Rowbottom [Rowbottom, 1971]. Gaifman, working in 1963, realized that the ultrapower embedding used by Scott could be iterated, and in the next year he <sup>27</sup> realized that this iteration could be carried out using a countable elementary
- substructure of an appropriate  $V_{\lambda}$ . This showed, for example, that every cardinal of V is inaccessible in L. Rowbottom obtained similar results using partition
- <sup>30</sup> properties. He reproved an earlier result of Erdős and Hajnal that any measurable cardinal is a Ramsey cardinal, If U is a measure on  $\kappa$  and  $f: [\kappa]^{<\omega} \to \lambda$  for  $\lambda < \kappa$  then there is a set  $A \in U$  such that f is constant on  $[A]^n$  for each  $n \in \omega$ .
- <sup>33</sup> He then used this to show that  $\kappa$  is what came to be known as a *Rowbottom* cardinal, which is a cardinal  $\kappa$  with the following property: Suppose that  $(\kappa, A, ...)$ is any structure with universe  $\kappa$ , that  $\rho$  is a cardinal less than  $\kappa$ , and  $A \subseteq \kappa$  with
- <sup>36</sup>  $\rho < |A| < \kappa$ . Then there is a set  $X \subseteq \kappa$  with  $\rho \subseteq X$  and  $|X \cap A| = \rho$  such that  $(X, A \cap X, \dots) \prec (\kappa, A, \dots)$ . He then used condensation to show that this implies that every successor cardinal of V is inaccessible in L.

This analysis of the structure of L in the presence of a measurable cardinal took its final form with Silver's discovery of  $0^{\#}$ . He observed that starting from a Ramsey cardinal, or even an  $\omega_1$ -Erdős cardinal, one could obtain an uncountable

set of indiscernibles for L, that is, a set of ordinals A such that any two finite increasing sequences of members of A of the same length satisfy the same formulas in L. Using the condensation property of L, together with model theoretic methods of Ehrenfeucht and Mostowski, he then showed that any such set could be extended to a proper class I of indiscernibles with the property that every set  $x \in L$  is definable in L using only parameters from I. Furthermore, he noted that if the class I of indiscernibles is chosen so that its  $\omega$ th member is as small as possible, then the class I would be closed and unbounded, and hence uniquely determined. Finally he defined the set  $0^{\#} \subseteq \omega$  to be

$$0^{\#} = \{ \ulcorner \phi(x_0, \dots, x_{n-1}) \urcorner : L \models \phi(c_0, \dots, c_{n-1}) \}$$
(1)

and showed that  $0^{\#}$  can be used to construct, and can be constructed from, the class I of indiscernibles. This class has since become known as the class of *Silver* indiscernibles.

Solovay independently discovered the set  $0^{\#}$ , and observed that the singleton  $\{0^{\#}\}$  is  $\Pi_2^1$ -definable over the reals, and thus  $0^{\#}$  is a  $\Delta_3^1$  subset of  $\omega$  which is not constructible.

The existence of  $0^{\#}$  has three important consequences. (i) The set  $\{0^{\#}\}$  is definable over the real numbers, by a  $\Pi_2^1$  formula which describes the process of obtaining I from  $0^{\#}$ . (ii) It takes the results of Rowbottom and Gaifman to a

natural conclusion: if  $\alpha$  is any member of I, and in particular any uncountable cardinal of V, then  $\alpha$  is, in L, inaccessible and weakly compact—indeed  $\alpha$  has revery large cardinal property which can hold in L. (iii) It implies that there is

a nontrivial elementary embedding  $L \to L$ . Indeed every increasing embedding from I into I can be extended to an elementary embedding from L into L, and

<sup>15</sup> every elementary embedding  $i: L \to L$  is determined by its restriction to the class I. This analysis was rounded out by Kunen, who proved that the existence  $0^{\#}$  follows from the existence of any nontrivial embedding  $i: L \to L$ .

- <sup>18</sup> Other work of Silver reinforced the view that the existence of  $0^{\#}$  is the weakest large cardinal property not compatible with L. He noted that although the existence of  $0^{\#}$ , and hence  $V \neq L$ , follows from that of an  $\omega_1$ -Erdős cardinal, any <sup>21</sup> cardinal which is  $\alpha$ -Erdős for any ordinal — or even all ordinals — less than  $\omega_1$
- has the same property in L [Silver, 1970].

It is evident that the same analysis which gave  $0^{\#}$  can be carried out for any set  $a \subseteq \omega$ , by defining  $a^{\#}$  to be the theory of a closed unbounded class of indiscernibles for the structure  $(L[a], \in, a)$ . In particular the sharp construction can be iterated: setting  $0^{\alpha \#} = \langle 0^{\nu \#} : \nu < \alpha \rangle^{\#}$  for any countable ordinal  $\alpha$ . This construction gives

- <sup>27</sup> the beginning of a hierarchy of inner models,  $L[\langle 0^{\alpha \#} : \alpha < \theta \rangle]$ . The case when  $\theta$  is equal to  $\omega_1$  in the resulting model gives a minimal model for the statement that  $a^{\#}$  exists for all reals a, which is the correct hypothesis for many results giving
- $_{30}$  consequences for the continuum of the existence of a measurable cardinal. For example, results of Martin and Solovay [Martin and Solovay, 1969], including the  $\Sigma_3^1$ -absoluteness result, follow from this hypothesis.

2.3 L[U]

The other direction, defining a model analogous to L with a measurable cardinal, <sup>3</sup> was initiated by Solovay. If U is a measure on a cardinal  $\kappa$  then it is easy to see that  $U \cap L[U] \in L[U]$ , and that  $U \cap L[U]$  is a measure on  $\kappa$ . In addition, the same arguments as were used for L can be used to show that L[U] satisfies the axioms <sup>6</sup> of ZFC.

Condensation, however, is more problematic. It was straightforward to show that it holds for any set  $X \prec L[U]$  such that  $\kappa \subseteq X$  and it follows that  $L[U] \models$  $2^{\lambda} = \lambda^{+}$  for all  $\lambda \geq \kappa$ . Solovay also showed that  $\kappa$  is the only measurable cardinal in L[U].

Condensation does not hold for arbitrary sets  $X \prec L_{\alpha}[U]$ ; however in [Silver, 12 1971] Silver extended Gödel's proof that  $2^{\lambda} = \lambda^+$  in L to cardinals  $\lambda < \kappa$  in L[U]by using Rowbottom's theorem [Rowbottom, 1971] to find, for each  $x \subseteq \lambda$  in L[U], a set  $X \prec L_a[U]$  with  $X \cap \kappa \in U$  and  $\lambda \cup \{x\} \subseteq X$  but  $|\mathcal{P}(\lambda) \cap X| = \lambda$ . Then condensation does hold for such X: if  $\pi : L_{\bar{\alpha}}[\bar{U}] \cong X$  is the transitive collapse then  $\{\lambda \in \kappa : \pi(\lambda) = \lambda\} \in U$  and hence  $\bar{U} = \pi^{-1}[U \cap X] = U \cap L_{\bar{\alpha}}[U]$ . This is precisely

what is needed to show that  $L[U] \models 2^{\lambda} = \lambda^+$ .

The final step towards establishing L[U] as a close analog of the model L was taken by Kunen in his thesis [Kunen, 1970], which adapted from Gaifman the use of iterated ultrapowers. The thesis began by refining Gaifman's theory of iterated

<sup>21</sup> ultrapowers. He defined an *M*-ultrafilter on a transitive model *M* of set theory to be an ultrafilter *U*, not necessarily a member of *M*, on  $\mathcal{P}(\kappa) \cap M$  such that  $X \cap U \in M$  whenever  $X \in M$  and  $|X|^M = \kappa$ , and he showed that this property <sup>24</sup> sufficed for the construction of an iterated ultrapower  $\mathrm{Ult}_{\alpha}(M, U)$ .

He then turned to the model L[U], showing that if the models L[U] and L[U']satisfy that U and U', respectively, are measures on the same cardinal  $\kappa$ , then  $_{27}$  L[U] = L[U'] and  $U \cap L[U] = U' \cap L[U]$ .

It follows that  $U \cap L[U]$  is the only measure in L[U], and that L[U] is a definable subset of any class model of ZFC + " $\kappa$  is a measurable cardinal" o GCH fails at a measurable cardinal  $\kappa$ , or that every  $\kappa$ -complete filter can be extended to an ultrafilter.

Kunen's hypothesis, that there were fewer than measurably many measurable cardinals, was not by any means a rigid limit on the effectiveness of his techniques; however longer sequences of measures did entail additional complications, and a new framework was needed to deal with these complications in a general way. This

- framework came out of consideration of a much less important question: how many different measures could a single cardinal carry? Kunen showed that the model L[U] had only one measure, and Kunen and Paris had presented in [Kunen and
- <sup>39</sup> Paris, 1970/1971] a model in which there are  $2^{2^{\kappa}}$  measures, the maximum possible. However no model was known with any intermediate number of measures, and they ended their paper with the following question: "Can the number of normal
- <sup>42</sup> ultrafilters on a measurable cardinal be intermediate between 1 and  $2^{2^{\kappa}}$ ? Can this number be 2?"

Kunen gave an inductive proof in [Kunen, 1970] that every measurable cardinal has a measure concentrating on smaller nonmeasurable cardinals: Suppose that

- <sup>3</sup> for each measurable cardinal  $\lambda < \kappa$  we have such a measure  $U_{\lambda}$  on  $\lambda$ , and let U be a measure on  $\kappa$  which does concentrate on smaller measurable cardinals. Then there is a second measure on  $\kappa$ , concentrating on nonmeasurable cardinals, can be
- defined by setting  $U_{\kappa} = \{x \subseteq \kappa : \{\lambda < \kappa : x \cap \lambda \in U_{\lambda}\} \in U\}$ , or equivalently  $U_{\kappa} = i(\langle U_{\lambda} : \lambda < \kappa \rangle)_{\kappa}$ . Seeing this Mitchell, then finishing up his graduate work at Berkeley, asked whether there could be a model in which these are the
- only two measures on  $\kappa$ ? It should be noted that the model  $L[U, U_{\kappa}]$  would not do: this would be equal to L[U], since the measures U and  $U_{\kappa}$  would agree on every set constructed using them as predicates. Two years later, driving back to Berkeley for the summer after a digression into category theory, he realized how to
- <sup>12</sup> Berkeley for the summer after a digression into category theory, he realized how to construct such a model. In [Mitchell, 1974] he defined an order on the ultrafilters on  $\kappa$ , by saying that  $U \triangleleft U'$  if  $U \in \text{Ult}(V, U')$ . Thus, for example,  $U_{\kappa} \triangleleft U$  in
- <sup>15</sup> Kunen's example discussed above. By an argument like that which Scott had used to prove that there are no measurable cardinals in L, Mitchell showed that the partial order  $\triangleleft$  is well-founded, and this fact allowed a definition of the order of a
- <sup>18</sup> measure,  $o(U) = \sup\{o(U') + 1 : U' \triangleleft U\}$ , and of a cardinal,  $o(\kappa) = \sup\{o(U) + 1 : U \text{ is a measure on } \kappa\}$ . He then defined a *coherent sequence of ultrafilters* to be a function  $\mathcal{U}$ , with domain of the form  $\{(\kappa, \beta) : \beta < o^{\mathcal{U}}(\kappa)\}$  for some function  $o^{\mathcal{U}}$ ,
- which witnesses that  $o(\mathcal{U}(\kappa,\beta)) = \beta$  in the sense that  $i^{\mathcal{U}(\kappa,\beta)}(\mathcal{U})(\kappa,\alpha) = \mathcal{U}(\kappa,\alpha)$ for all  $\alpha < \beta < o^{\mathcal{U}}(\kappa)$ , and he showed that if  $\mathcal{U}$  is a coherent sequence then many of the basic properties of L[U] can be extended to  $L[\mathcal{U}]$ . In particular,  $L[\mathcal{U}]$  is a model
- of ZFC + GCH, and the only normal measures in  $L[\mathcal{U}]$  are the sets  $\mathcal{U}(\kappa, \beta) \cap L[\mathcal{U}]$ with  $\beta < o^{\mathcal{U}}(\kappa)$ . Thus, provided suitable coherent sequences could be found, this construction provided models with any desired number  $\delta$  of normal measures on a cardinal  $\kappa$ , with  $0 \le \delta \le \kappa^{++}$ . Since the models satisfied GCH, this was the
- $2^{\prime}$  a calculat  $\lambda$ , with  $0 \leq 0 \leq \lambda^{\prime}$ . Since the models satisfied GCH, the maximum possible number of measures.

For any finite *n* it was easy, if given a cardinal  $\kappa$  with  $o(\kappa) \geq n$ , to find a coherent sequence  $\mathcal{U}$  with  $o^{\mathcal{U}}(\kappa) = n$ . This gave models which exactly *n* measures, for any finite *n*. Finding coherent sequences with  $o^{\mathcal{U}}(\kappa)$  infinite was more difficult. Mitchell used what he later termed a  $\mu$ -measurable cardinal to obtain sequences

- with  $o(\kappa) = \lambda$  for any  $\lambda \leq \kappa^{++}$ . It is a commentary on the changes which our picture of large cardinals has changed that the smallest cardinal property generally known at the time which implied the existence of a  $\mu$ -measurable cardinal —
- <sup>36</sup> which is barely stronger than  $\exists \kappa o(\kappa) = \kappa^{++}$  was supercompactness. Later it was determined that  $\exists \kappa o(\kappa) = \kappa^{++}$  in V was sufficient: first by using core model theory but later, in [Mitchell, 1983], with more elementary methods: If
- $o_{\alpha}(\kappa) = \delta$  then there is an inner model  $L[\mathcal{U}]$  in which either  $o^{\mathcal{U}}(\kappa) = \delta$  or else  $o^{\mathcal{U}}(\kappa) = \kappa^{++L[\mathcal{U}]} < \delta$ . This was achieved by using a sequence  $\mathcal{U}$  which might not be fully coherent in V: instead the measures on the sequence  $\mathcal{U}$  were only required
- <sup>42</sup> to have the correct order. In  $L[\mathcal{U}]$  the sequence becomes fully coherent, because no set is constructed in  $L[\mathcal{U}]$  which would witness a failure of coherence. This concept was independently discovered by A. Dodd and has come to be used as an

important tool, referred to as *Doddages*, in core model theory.

### Iterating the Least Difference

- <sup>3</sup> The essential tool in the analysis of  $L[\mathcal{U}]$  was a generalization of Kunen's use of iterated ultrapowers which has come to be known as the process of *iterating the least difference*. This process is central to the development of inner model theory,
- and it seems appropriate to describe it a bit more fully so as to have a framework for discussing how its use has influenced the development of inner models. To this end we will give a brief sketch of the proof of a simple application: every measure
  in L[U] is a member of the sequence U.
- Supposing the contrary, we may assume that  $\mathcal{U}$  is the shortest sequence for which is the assertion to be proved is false, and  $\kappa$  is the smallest cardinal for <sup>12</sup> which there is a measure W in  $L[\mathcal{U}]$  which is not on the sequence  $\mathcal{U}$ .
  - The proof uses a comparison of  $L[\mathcal{U}]$  with the ultrapower  $\text{Ult}(L[\mathcal{U}], W)$  of that model by the supposed extraneous ultrafilter W. The comparison is made by an iterated ultrapower of the two models,

$$i_{0,\theta}: \qquad L[\mathcal{U}] = M_0 \xrightarrow{i_{01}} M_1 \xrightarrow{i_{12}} \cdots M_{\theta} \qquad (2)$$

$$\downarrow^{i^W}_{i^W}$$

$$j_{0,\theta}: \qquad \text{Ult}(L[\mathcal{U}], W) = N_0 \xrightarrow{j_{01}} M_1 \xrightarrow{j_{12}} \cdots N_{\theta}$$

The term "iterating the least difference" describes the way the iterated ultrapowers  $i^0$  and  $i^1$  are defined: at stage  $\nu$  we have  $M_{\nu} = L[i_{0\nu}(\mathcal{U})]$  and  $N_{\nu} = L[j_{0\nu} \circ i^W(\mathcal{U})]$ .

<sup>15</sup> The next models  $M_{\nu+1}$  and  $N_{\nu+1}$  in the sequence are defined by simply identifying the first ultrafilter occurring in one of the sequences  $i_{0\nu}(\mathcal{U})$  and  $j_{0\nu}i^U(\mathcal{U})$  which does not occur in the other, and taking the ultrapower of that model by that <sup>18</sup> ultrafilter. A crucial fact, the proof of which we skip over, is that this process

always terminates, with one of the models  $M_{\theta}$  and  $N_{\theta}$  being an initial segment of the other.

Now we can conclude the proof of the statement: we want to show that the ultrafilter W, supposedly not on the sequence  $\mathcal{U}$ , is in fact equal to the first ultrafilter used in the iterated ultrapower  $i_{0\theta}$ . It can be shown that this ultrafilter is on  $\kappa$ ; write it as  $U = \mathcal{U}(\kappa, \beta)$ , so  $i_{01} = i^U$ . Now suppose to the contrary that  $U \neq W$ , and let x be the first subset of  $\kappa$  on which W and U disagree. The argument is concluded by showing that

$$x \in U \iff \kappa \in i^U(x) = i_{01}(x) \iff \kappa \in i_{0\phi}(x) \tag{3}$$

$$x \in W \iff \kappa \in i^W(x) \iff \kappa \in j_{0\phi} \circ i^W(x) \tag{4}$$

and noting that the whole construction, and in particular the set x, is definable in  $L[\mathcal{U}]$ , so that  $i_{0\theta}(x) = j_{0,\theta} \circ i^W(x)$ . Hence W = U.

This concludes the proof of the assertion that there are no measures in  $L[\mathcal{U}]$ other than those on the sequence  $\mathcal{U}$ . We now isolate, for future reference, three factors which are necessary to the success of this argument:

**Iterability.** All iterated ultrapowers of  $L[\mathcal{U}]$ , and in particular the final models  $M_{\theta}$  and  $N_{\theta}$  in the comparison, are well-founded.

**Equal comparison.** The final models of the iteration are equal: neither of  $M_{\theta}$  or  $N_{\theta}$  is a proper initial segment of the other.

**No moving generators.** None of the critical points of ultrafilters used in the iterated ultrapowers of diagram (2) are moved by later ultrapowers in the same iteration.

Two of these terms, *iterability* and *moving generators*, are standard. For the models currently under consideration, iterability can be insured by using countably complete ultrafilters: either, as in this case, countably complete in  $L[\mathcal{U}]$  or, in the case of a proof where the iteration is not definable in  $L[\mathcal{U}]$ , countably complete

- <sup>15</sup> in V. Later, for larger cardinals, other methods became necessary. At the time of writing, iterability is still the principal limiting factor for the provable existence of inner models.
- In the proof of  $x \in U \iff x \in W$  sketched above, the lack of moving generators was used only with respect to the critical point  $\kappa$  of the embeddings  $i^W$  and  $i_{0,1} = i^U$ . It is also needed, with respect to all critical points, in the omitted
- <sup>21</sup> proof that the comparison terminates. It was not recognized as a difficulty at this point in the history, since it is ensured by the coherence of the sequence  $\mathcal{U}$  of measures. Later, in the 1980s when models for cardinals beyond a strong cardinal
- <sup>24</sup> are developed (as we shall describe), it will be seen to have (in a more general form) a critical influence: it will require a major change in the structure of the iterated ultrapowers used for this method of comparison by iterating the least difference.
- <sup>27</sup> (The term "moving generators" comes from the extenders used in such models.) The definability technique used in the proof above to ensure equal comparison was implicitly used by Kunen, but was isolated by Mitchell in[Mitchell, 1974] and
- 30 given the name  $\phi$ -minimality. An important theme in the development of core models has been the introduction of new techniques to ensure equal comparison.

## 3 FINE STRUCTURE AND THE COVERING LEMMA IN ${\cal L}$

- <sup>33</sup> Most of Jensen's work was originally distributed as handwritten, mimeographed notes made shortly after the results were obtained. These notes, though mostly undated, provide a valuable source for viewing his evolving views.
- The first suggestion of fine structure came in his Habilitationsschrift [Jensen, 1967], completed in 1967. The introduction states that the main problem is to determine, given a transitive set M, for which ordinals  $\alpha$  is the structure  $L_{\alpha}(M)$

closed under given logical operations, and states that some of this is related to interests of his advisor, G. Hasenjaeger. The first theorem stated asserts that

- <sup>3</sup>  $\mathcal{P}(M) \cap L_{\alpha+1}(M) \subseteq L_{\alpha}(M)$  if and only if there is no  $L_{\alpha}(M)$ -definable map from M onto  $L_{\alpha}$ , which recognizably presages (in contrapositive form) the basic theorems of fine structure. An appendix specifically gives the  $\Sigma_1$  case: every subset of
- <sup>6</sup> M which is  $\Sigma_1$ -definable over  $L_{\alpha}(M)$  is a member of  $L_{\alpha}(M)$  if and only if there is no map from M onto  $L_{\alpha}(M)$  which is  $\Sigma_1$ -definable over  $L_{\alpha}(M)$ . Jensen has stated that the name "fine structure" came from Gandy's comments on the Habil-
- 9 itationsschrift. Jensen has regretted the name, as it also names an unfashionable area of algebraic geometry.
- In [Jensen, c] Jensen proved that there is a Suslin tree in L. The proof was based on Tennenbaum's [Tennenbaum, 1968] forcing construction of a Suslin tree: Jensen observed that the tree T given by Tennenbaum's construction, the nodes  $T_{\alpha}$  of T at level  $\alpha < \omega_1$  were generic branches through the first  $\alpha$  levels of T, and
- <sup>15</sup> noted that it would be enough if they were generic only over the countable model  $L_{\beta}$ , where  $\beta \geq \alpha$  is least such that  $\mathcal{P}(\omega) \cap L_{\beta+1} \not\subseteq L_{\beta}$ . Shortly after that, Solovay used a similar method to prove that there is a Kurepa tree in L, and several people
- <sup>18</sup> then independently discovered the combinatorial principle  $\Diamond$ , and proved that  $\Diamond$  is true in *L* and allows the construction of a Suslin tree. In [Jensen, 1969], written at the Rockefeller University in 1969, Jensen discussed results of Kunen and himself
- <sup>21</sup> concerning  $\Diamond$  as well as the stronger principle  $\Diamond^+$  which is needed to prove that there is a Kürepa tree.

In the notes [Jensen, 1970], expanded from lectures at Kiev, he presented a number of results relating to  $\diamond$  and trees, including generalizations of  $\diamond$  (which is  $\diamond_{\omega_1}$ ) and Suslin trees to larger cardinals. At the end he considered the question of a Suslin tree on an inaccessible cardinal  $\kappa$  in L. Now if  $\kappa$  is weakly compact then there is no Aronszajn tree on  $\kappa$ , and hence no Suslin tree. His argument needed another constraint. The construction of a  $\omega_1$ -Suslin tree T proceeded by recursion on the level  $\alpha$  of T, and for limit  $\alpha$  the definition of the  $\alpha$ th level of the tree had two parts: first showing that tree of height  $\alpha$  constructed so far had branches, and then selecting countably many of these branches to continue the tree. To define a

- $\kappa$ -Suslin tree for larger  $\kappa$ , these two parts had conflicting requirements: there must be a stationary set  $E \subseteq \kappa$  on which fewer than  $\kappa$  many nodes would be selected
- to continue the tree, but to ensure that there are branches at each level  $\alpha$  there must be a closed unbounded subset  $C_{\alpha} \subseteq \alpha$  whose limit points are disjoint from
- <sup>36</sup> E in order that a branch can be defined by recursion along  $C_{\alpha}$ . This was not a problem in defining a  $\kappa^+$ -Suslin tree for a regular cardinal  $\kappa$ , as one could take E to be the sets of cofinality  $\kappa$  and let the sets  $C_a$  be of order type at most  $\kappa$ . Thus
- <sup>39</sup> the limit points of  $C_{\alpha}$  all have cofinality less than  $\kappa$ . In [Jensen, 1970] Jensen described a new combinatorial principle which he called  $K_{\kappa}^*$ , and shows that  $K_{\kappa}^*$ +  $\Diamond_{\kappa}$  implies that there is a  $\kappa$ -Suslin tree. The principle  $K_{\kappa}^*$  asserts that there
- <sup>42</sup> are sets E and  $C_{\nu}$  as needed, and Jensen states as a theorem that  $K_{\kappa}^{*}$  holds in L for every cardinal  $\kappa$  which is inaccessible, but not weakly compact. He omits the proof: "Der Beweis hiervon ist leider sehr lang and wird deshalb nicht gebracht".

The principle  $K_{\kappa}^*$  is essentially a square principle, but omits the vital coherence condition:  $C_{\beta} = C_{\alpha} \cap \beta$  whenever  $\beta$  is a limit point of  $C_{\alpha}$ . This coherence condition

- <sup>3</sup> appears at the end of [Jensen, b], in an addendum in which Jensen corrects an error in section I of the note pointed out by Solovay. The correction, which Jensen characterizes as "restoring the 'unsimplified' proof", uses a principle which he
- <sup>6</sup> calls +, and which we can recognize as essentially  $\Box_{\omega_1}$ . In this note Jensen uses a preliminary forcing extension to make the principle + true.
- In [Jensen, 1972], Jensen finally gave a complete exposition of the fine structure of L, ending up with the definition of  $\Box_{\kappa}$  and of global square, and completing the proof that in L there is a Suslin tree on every regular cardinal  $\kappa > \omega$  which is not weakly compact. In this result the case in which  $\kappa$  was the successor of a singular
- cardinal  $\lambda$  was new, relying on the principle  $\Box_{\lambda}$ . This principle was also critical in a final section written by Silver which proved that L satisfies the transfer property  $(\kappa, \delta) \rightarrow (\lambda^+, \lambda)$  for any cardinals  $\lambda$  and  $\kappa > \delta$ . This notation means that any
- theory with a model of type  $(\kappa, \delta)$  also has a model of type  $(\lambda^+, \lambda)$ , where a model  $(M, U, \cdots)$  is said to have type  $(\kappa, \lambda)$  if M has size  $\kappa$  and the unary predicate U has size  $\lambda$ .
- <sup>18</sup> Details of the fine structure construction are far out of the scope of this paper, but it will be useful to have a simplified (and in some aspects inaccurate) description. We begin with an innovation which, though inessential and technical,
- <sup>21</sup> was important to the later study of L and of fine structure models in general: the introduction of the  $J_{\alpha}$  hierarchy in place of the usual  $L_{\alpha}$  hierarchy. The  $J_{\alpha}$ hierarchy was based on the set of rudimentary functions, defined by Gandy in
- <sup>24</sup> [Gandy, 1974]. The two hierarchies are essentially equivalent (whence the characterization of the innovation as 'inessential'), however the new sequence allowed a much cleaner analysis, largely because every finite sequence of members of  $J_{\alpha}$
- <sup>27</sup> is a member of  $J_{\alpha}$ . The fact that the parameters, which were finite sequences of ordinals, could not be treated as single objects in  $L_{\alpha}$  had greatly complicated the fine structure analysis of the  $L_{\alpha}$  hierarchy. The  $J_{\alpha}$  hierarchy has been widely used
- since, and is universally used in any inner model theory involving fine structure. We will use the  $J_{\alpha}$  hierarchy henceforth; however a reader who prefers to read  $J_{\alpha}$ as  $L_{\alpha}$  will lose little or nothing in accuracy or comprehension.
- Jensen took from Kripke and Platek the term *projectum*. He defined the projectum of  $J_{\alpha}$  to be the least ordinal  $\rho \leq \alpha$  such that there is a subset of  $\rho$  which is in  $J_{\alpha+1} \setminus J_{\alpha}$ , and for  $n \in \omega$  he defined the  $\Sigma_n$  projectum of  $J_{\alpha}$  to be the least ordinal
- $\rho_n^{\alpha}$  such that such a subset is  $\Sigma_n$ -definable over  $J_{\alpha}$  with parameters from  $J_{\alpha}$ . He answered the question left open in [Jensen, 1967] by showing that there is, for each ordinal  $\alpha$  and each  $n \in \omega$ , a  $\Sigma_n$ -definable map from  $\rho_n^{\alpha}$  onto  $J_{\alpha}$ ; however what
- <sup>39</sup> is more important is the construction by which the map is obtained. He defined by recursion on  $n \in \omega$  the  $\Sigma_n$ -core  $\mathbb{C}_n(J_\alpha)$  as a structure  $(J_{\rho_n^\alpha}, A_n^\alpha)$  in which  $A_n^\alpha$ coded the  $\Sigma_n$  theory of  $J_\alpha$ , with parameters from  $\rho_n^\alpha$ . In the recursion step, he
- <sup>42</sup> used the method from his Habilitationsschrift to define a  $\Sigma_1^{\mathbb{C}_n(J_\alpha)}$ -definable map from the  $\Sigma_1$ -projectum of  $\mathbb{C}_n(J_\alpha)$ , which is the  $\Sigma_{n+1}$  projectum  $\rho_{n+1}^{\alpha}$  of  $J_\alpha$ . He then defined  $\mathbb{C}_{n+1}(J_\alpha) = (J_{\rho_{n+1}^{\alpha}}, A_{n+1}^{\alpha})$  where  $A_{n+1}^{\alpha}$  encoded in a natural way the

 $\Sigma_1$  theory of  $\mathbb{C}_n(J_\alpha)$  and hence, by extension, the  $\Sigma_{n+1}$  theory of  $J_\alpha$ .

The construction was entirely canonical. In particular, given  $\mathbb{C}_n(J_\alpha)$  one could recover  $J_\alpha$ . In addition—and this is a critical point—it is preserved by embeddings: he showed that any substructure  $X \prec_{\Sigma_1} \mathbb{C}_n(J_\alpha)$  of the  $\Sigma_n$ -code of  $J_\alpha$  is isomorphic

- to the  $\Sigma_n$  code of some  $J_{\bar{\alpha}}$ , and furthermore the collapse map  $\pi : \mathbb{C}_n(J_{\bar{\alpha}}) \cong X \prec_{\Sigma_1}$   $\mathbb{C}_n(J_{\alpha})$  can be extended to a  $\Sigma_{n+1}$  elementary map from  $J_{\bar{\alpha}} \to J_{\alpha}$ . We will refer to this property as *downward extension of embeddings*, but it should be noted that it can be regarded as an enhancement of the Condensation Lemma 1: it says that
- <sup>9</sup> condensation applies at each of the infinitely many levels of the Levy hierarchy between  $J_{\alpha}$  and  $J_{\alpha+1}$ .

The importance of this may be seen in the proof of the principle  $\Box_{\kappa}$ , which states that there is a sequence of closed unbounded subsets of  $C_{\alpha}$ , defined for limit ordinals  $\alpha < \kappa^+$ , such that the order type of  $C_{\alpha}$  is at most  $\kappa$ , and if  $\beta$  is a limit point of  $C_{\alpha}$  then  $C_{\beta} = C_{\alpha} \cap \beta$ . Jensen's construction has a number of cases, but the interesting case is that in which  $\kappa < \alpha < \kappa^+$  and  $\alpha$  has uncountable

- cofinality. Jensen considers the first ordinal  $\gamma$  such that  $\alpha$  is singular in  $J_{\alpha+1}$ . Then the least witness  $\sigma: \xi \to \alpha$  to this singularity is  $\Sigma_{n+1}$ -definable over  $J_{\gamma}$ for some  $n \in \omega$ . By careful use of the  $\Sigma_n$  code of  $J_{\gamma}$ , Jensen finds a tower of
- $\Sigma_{n+1}$  elementary substructures  $X_{\nu} \prec_{\Sigma_{n+1}} J_{\gamma}$  with the following property: For limit  $\nu < \xi$  set  $\alpha_{\nu} = \sup \sigma "\nu$  and let  $\pi_{\nu} \colon X_{\nu} \cong J_{\gamma_{\nu}}$  be the collapse map. Then <sup>21</sup>  $\pi_{\nu}$  preserves the definition of the map  $\sigma$ , so that  $\sigma \upharpoonright \nu$  is the least witness to the
- singularity of  $\alpha_{\nu}$ , just as  $\sigma$  was for  $\alpha$ . In particular, if  $\sigma$  is used to define  $C_{\alpha}$  and  $\beta$  is any limit member of  $C_{\alpha}$ , then  $\beta = \alpha_{\nu}$  for some  $\nu < \xi$ , so  $C_{\beta}$  is defined using  $\sigma \upharpoonright \nu$  in the same way as the definition of  $C_{\alpha}$  used  $\sigma$ . Thus  $C_{\beta} = C_{\alpha} \cap \beta$ .

The property  $\Box_{\kappa}$  has given rise to an extensive literature. In addition to its consequences, a number of weaker principles have been considered, starting with

- <sup>27</sup> weak square,  $\Box_{\kappa}^{*}$ , which was defined by Jensen by allowing  $\kappa$  many club subsets of each  $\alpha < \kappa^{+}$  instead of only one. In particular, some of the scale and club guessing principles which Shelah has introduced in connection with his pcf theory can be <sup>30</sup> regarded as weak versions of  $\Box_{\kappa}$ . Little of this theory is relevant, however, to this
- regarded as weak versions of □<sub>κ</sub>. Little of this theory is relevant, however, to this history: since square holds in the inner models we are discussing, the study of weaker properties is moot. The reader may see chapter [Kojman, 2010], covering
   singular cardinal combinatorics, for more about these properties.

Following the completion of [Jensen, 1972], Jensen went on to prove that the gap-1 transfer property is true in L. The gap-1 transfer property states that if  $\alpha > \beta^+$  are cardinals then any theory which has a model of type  $(\alpha, \beta)$  also has models of type  $(\kappa^{++}, \kappa)$  for any cardinal  $\kappa$ . For this he defined a new combinatorial structure, which he called a gap 1 morass, and used the fine structure to show that such a morasses exists in L on every cardinal. The gap 1  $\omega_2$ -morass, for

- example, expressed  $J_{\omega_2}$  as an intricate direct limit of countable structures. This structure permitted a structure of size  $\omega_2$  to be defined as a limit of recursively defined countable models, in somewhat the same way as Chang's original proof
- that  $(\alpha^+, \alpha) \to (\omega_1, \omega)$  constructed a model of size  $\omega_1$  as a limit of countable models.

From here, Jensen went on to define gap n morasses for each  $n \in \omega$ , and he used these to prove transfer theorems with larger gaps: If V = L and  $n \in \omega$  then

- $\lambda = (\lambda^{+n}, \lambda) \to (\kappa^{+n}, \kappa)$  for any infinite cardinals  $\lambda$  and  $\kappa$ . Part of the motivation here was in the hope that this would lead to a solution to the leading open problem of set theory of the time: the Singular Cardinal Hypothesis (SCH). The hope was
- <sup>6</sup> this construction would lead to a forcing notion over L which would give a model in which  $2^{\omega_n} = \omega_{n+1}$  for all  $n < \omega$ , but  $2^{\omega_\omega} > \omega_{\omega+1}$ . The actual outcome was quite different.
- <sup>9</sup> Before continuing, it is worth noting that while the combinatorial principles described above had their origin in the constructible sets L, they have largely moved away from constructivity and inner model theory in general indeed a
- $^{12}$  primary part of the motivation for their definition was to abstract away from L the combinatorial properties used in applications. This is not true of the next topic: the covering lemma remains tightly tied to the inner model program and
- <sup>15</sup> has remained at its center.

## 3.1 The Singular Cardinal Hypothesis and the Covering Lemma

The origins of the Singular Cardinal Hypothesis are described in chapter [Kanamori,

- <sup>18</sup> 2010b]. Shortly after Cohen's proof of the independence of the Continuum Hypothesis, Easton [Easton, 1970] completely settled the general question of the size of the power sets of regular cardinals; however this left the size of the power set of
- <sup>21</sup> a singular cardinal  $\lambda$  completely open. The Singular Cardinal Hypothesis, SCH, is the assertion that  $\lambda^{cf(\lambda)} = \max(\lambda^+, 2^{cf(\lambda)})$  for every singular cardinal  $\lambda$ , which for the case when  $\lambda$  is a strong limit cardinal can be simplified to  $2^{\lambda} = \lambda^+$ . Easton's
- <sup>24</sup> models satisfies SCH, but Silver showed that a model in which SCH failed could be constructed by starting with a  $\kappa^{++}$  supercompact cardinal; but it was commonly believed that no such hypothesis was necessary: there should be an Easton-like <sup>27</sup> theorem for singular cardinals with no large cardinal hypothesis.

While trying to prove such a theorem for singular cardinals, Silver discovered a new restriction [Silver, 1975] to the size of the power set of a singular cardinal:

- <sup>30</sup> Suppose that  $\lambda$  is a singular cardinal of uncountable cofinality, and  $2^{\gamma} = \gamma^+$  for stationarily many  $\gamma < \lambda$ . Then  $2^{\lambda} = \lambda^+$ . In particular, if  $\lambda$  is the first singular cardinal at which SCH fails, then  $\lambda$  has cofinality  $\omega$ .
- Silver's proof started by forcing to obtain a generic ultrafilter U on  $\mathcal{P}^V(\mathrm{cf}(\lambda))$  extending the closed unbounded filter, and then considered the embedding  $i^U : V \to M = \mathrm{Ult}(V, U)$ . The model M is not well-founded; however if we set  $\overline{\lambda} =$
- <sup>36</sup> sup{ $i(\alpha) : \alpha < \lambda$ } then the set of ordinals below  $\overline{\lambda}$  in M is  $\lambda$ -like,<sup>2</sup> and that of  $(\overline{\lambda}^+)^M$  is  $\lambda^+$ -like. Also  $M \models 2^{\overline{\lambda}} = \overline{\lambda}^+$  by the Los theorem, and from these facts he deduced that  $2^{\lambda} = \lambda^+$  in V.
- <sup>39</sup> Only a few weeks later, Jensen distributed a handwritten note entitled *Marginalia to a Theorem of Silver* [Jensen, 1974b], which gave the first statement of the

<sup>&</sup>lt;sup>2</sup>A linear ordering  $(R, \leq)$  is said to be  $\lambda$ -like, where  $\lambda$  is a cardinal, if R has size  $\lambda$  but each proper initial segment has size less than  $\lambda$ .

covering lemma. In spite of the name of the note, the proof of the covering lemma is not closely related to the proof of Silver's theorem: the only thing they have in

- <sup>3</sup> common is that both involve embeddings on a singular cardinal. A close reading of the main theorem of [Jensen, 1974b] gives a hint as to how thinking about Silver's result led Jensen to the covering lemma.
- <sup>6</sup> The main theorem of [Jensen, 1974b] is in three parts, all of which concern a singular cardinal  $\beta$  of uncountable cofinality. The first part, Theorem I(i), is essentially a straightforward application of Silver's method, though Jensen (in-
- <sup>9</sup> dependently of Baumgartner and Prikry) eliminated the use of forcing from the proof: Assume that  $A \subseteq \beta$  is such that  $^{3}L_{\beta}[A] = H_{\beta}$  and  $2^{\lambda} = \lambda^{+}$  on a stationary subset of  $\beta$ , and that  $cf(\beta)^{L} = cf(\beta)$  and  $(\beta^{+})^{L} = \beta^{+}$ . Then  $\mathcal{P}(\beta) \subseteq L[A]$ . The second part, Theorem I(ii) applies Theorem I(i) in the special case  $A = \emptyset$ . The
- hypothesis that  $2^{\lambda} = \lambda^{+}$  for all  $\lambda < \beta$  follows automatically from the hypothesis that  $H_{\beta} = L_{\beta}$ , but there remain the awkward extra assumptions  $cf(\beta)^{L} = cf(\beta)$ and  $(\beta^{+})^{L} = \beta^{+}$ . Jensen used his newly developed fine structure to show that
- these assumptions also follow automatically. Consider, for example, the statement that  $cf(\beta)^L = cf(\beta)$ . Jensen used the singularity of  $\beta$  to obtain an elementary
- <sup>18</sup> substructure X of  $L_{\beta}$  of size  $cf(\beta) < \beta$  such that  $X \cap \beta$  is cofinal in  $\beta$ . By the condensation property, there is an isomorphism  $\pi: L_{\bar{\beta}} \cong X \prec L_{\beta}$  for some  $\bar{\beta} < \beta$ . Since  $H_{\beta} = L_{\beta}$ , the preimage  $\bar{\sigma}$  of  $\sigma$  is in L. In particular,  $\bar{\beta}$  is singular in L. Let <sup>21</sup>  $\bar{\alpha} \geq \bar{\beta}$  be least such that  $\bar{\beta}$  is singular in  $L_{\bar{\alpha}+1}$ .

To complete the proof, Jensen introduced two innovations. First, he generalized the ultrapower construction to extend the embedding  $\pi$  to an embedding

 $\tilde{\pi}: L_{\bar{\alpha}} \to L_{\alpha}$  for some  $\alpha \geq \beta$ . This generalization later gave rise to notion of an *extender*, which has become central to the theory of inner models for large cardinals. Second, an *upward extension of embeddings* principle, a counterpart to the downward extension of embedding principle used for the  $\Box$  principles, enabled this extension to be further extended to preserve the definition over  $L_{\bar{\alpha}}$  of a witness to the singularity of  $\bar{\beta}$ . It followed that the same definition, applied to  $L_{\alpha}$ , gave a

witness to the singularity of  $\beta$ : thus  $\beta$  was singular in L.

In Theorem I(iii) Jensen let this argument stand on its own, and replaced the hypothesis that  $L_{\beta} = H_{\beta}$  with the much weaker assumption that  $0^{\#}$  does not exist. The result is recognizably the covering lemma (for uncountable cofinality): if  $\beta$  is a singular cardinal with  $cf(\beta) > \omega$ , and if  $cf^{L}(\beta) = \beta$  or  $(\beta^{+})^{L} < \beta^{+}$ , then  $0^{\#}$  exists.

The argument again uses the embedding  $\pi$  as an extender: If  $\bar{\sigma}$  is not singular in L then  $\pi \upharpoonright L^{\bar{M}}$  can be extended to a nontrivial map  $\tilde{\pi} \colon L \to L$ , which by Kunen's theorem implies that  $0^{\#}$  exists.

<sup>39</sup> The restriction in [Jensen, 1974b] to cardinals  $\beta$  of uncountable cofinality came from the need to show that the target of the extended embedding  $\tilde{\pi}$ , which was obtained by the use of  $\pi$  as an extender, was well-founded. The main theorem <sup>42</sup> of the second note, [Jensen, 1974c], removes this restriction: Assume  $\neg 0^{\#}$ . Then

<sup>&</sup>lt;sup>3</sup>Here  $H_{\beta}$  is the set of all sets which are heriditarily of size less than  $\beta$ , that is, sets whose transitive closure has size less than  $\beta$ .

 $cf_L(\lambda) \ge \omega_2 \implies cf(\lambda) = cf_L(\lambda)$ . The proof replaces the previous use of the assumption that  $cf(\lambda) > \omega$  with the facts that  $\lambda^+$  and  $\omega_1$  each have uncountable

- cofinality. The use of  $\omega_1$  in the proof dictates the restriction  $\operatorname{cf}_L(\lambda) \geq \omega_2$ . As he remarks, it would be "reasonable but wrong to suppose the theorem to hold for  $\operatorname{cf}(\lambda) \geq \omega_1$ " because of the counterexample in [Namba, 1971].
- The third and final note in this series, [Jensen, 1974a], gives the final form to the statement of the covering lemma: "Theorem: Assume  $\neg 0^{\#}$ . Let X be an uncountable set of ordinals. Then there is a set  $Y \in L$  such that |X| = |Y| and  $X \subset Y$ ."

As we have seen, the fine structure was a basic component of Jensen's proof of the covering lemma; however some have considered its use to be a flaw in the theory. Silver has given a proof which avoids fine structure by using what he called

- <sup>12</sup> theory. Silver has given a proof which avoids fine structure by using what he called "machines"; these machines were perhaps simpler than the fine structure but were, at least in this author's opinion, less intuitive. In any case they have not been
- <sup>15</sup> generalized to the larger theory described in the rest of this chapter. Another proof, due to Magidor [Magidor, 1990], replaces the fine structure with the naïve Skolem function for  $\Sigma_n$  sets: using, for a  $\Sigma_n$  relation R over  $L_{\alpha}$ , the function
- <sup>18</sup>  $h(x) = \mu y \ R(x, y)$ . This function is not  $\Sigma_n$ -definable over  $L_\alpha$ , but surprisingly the covering lemma can be proved by replacing the use of the  $\Sigma_n$  code  $\mathbb{C}_n(J_\alpha)$ with the closure of  $J_\alpha$  under the function h. Magidor has had some success in
- <sup>21</sup> generalizing this method to core models with sequences of measurable cardinals, but unfortunately it does not appear to be suitable even for the full theory of such sequences, let alone the larger core models we describe in section 5.3.
- The main result of Magidor's paper [Magidor, 1990] cited above gave an interesting alternate conclusion from the proof of the covering lemma: If  $0^{\#}$  does not exist then every primitively closed set of ordinals is a countable union of sets
- <sup>27</sup> in L. Another interesting development is the strong covering theorem, an observation contained in an unpublished note of Carlson: If  $0^{\#}$  does not exists, then for any pair of uncountable cardinals  $\delta < \kappa$  there is a set  $C \subseteq [\kappa]^{\delta}$  in L which
- <sup>30</sup> is unbounded and closed under increasing unions of sequences whose length has uncountable cofinality.

## 4 THE CORE MODEL

## <sup>33</sup> 4.1 Up to One Measurable Cardinal

It now seemed that, with the new machinery available, the truth of any question could be decided, and attention turned to larger models. An obvious choice was

- $_{26}$  L[U], and in fact Solovay verified that  $\Diamond_{\kappa}$  and  $\Box_{\kappa}$  held there for all  $\kappa$ . However, while the model L[U] is, in many respects, a suitable generalization of L, it has number of defects centering on the fact that its construction requires a known
- <sup>39</sup> ultrafilter U. In some applications this problem was overcome, notably in the use by Solovay and Kunen of models L[F], where F was the cardinal filter or the closed unbounded filter, under hypotheses which implied that F became an

ultrafilter in L[F]. The usefulness of such strategies was limited, however. There is, for example, no evident filter associated with the failure of Singular Cardinal Hypothesis. Furthermore, the model L[U] gives little insight into large cardinal properties lying between L and L[U]: it does not for example, provide a minimal model for a Ramsey cardinal.

3

Solovay had already investigated fine structure in the model L[U], verifying that  $\diamond$  and  $\Box$  properties held there. Jensen and Anthony Dodd approached the fine structure differently. Instead of assuming that there is a model L[U] with a measurable cardinal, they used approximations to such a model, which they called *mice*. According to Jensen, the term "mouse" was chosen because he felt that the concept was important enough to give it a name which was not used anywhere else in mathematics. A mouse is a structure  $M = J_{\alpha}[U]$  which has three properties: (i) it satisfies the statement that U is a measure on some  $\kappa$ , (ii) all of its iterated ultrapowers are well-founded, and (iii) it admits a fine structure analysis, with projectum  $\rho^M$  smaller than the critical point of the ultrapower. The paradigmatic example of a mouse comes from  $0^{\#}$ . If U is a measure on  $\kappa$ , then  $J_{\kappa+1}[U]$  is equal to  $J_{\kappa+1}$ , but  $0^{\#}$  is  $\Sigma_1$ -definable in the structure  $(J_{\kappa+1}[U], \in, U)$ : the Gödel number  $\lceil \phi(c_0, \ldots, c_n) \rceil$  is in  $0^{\#}$  if and only if

$$\exists X \in U^n \cap J_{\kappa+1}[U] \ \forall (\nu_0, \dots, \nu_{n-1}) \in X \ J_{\kappa+1}[U] \models \phi(\nu_0, \dots, \nu_{n-1}).$$

6 This fact is rather awkward for the fine structure of L[U], taken by itself: what is "fine" about a subset of ω which does not appear until after a measurable cardinal? Taking the transitive collapse of the Σ<sub>1</sub>-Skolem hull of J<sub>κ+1</sub>[U], however, gives a
9 countable structure M = J<sub>k+1</sub>[Ū], over which 0<sup>#</sup> is similarly definable. Thus M and 0<sup>#</sup> are equiconstructable.

Another way to view the equivalence between M and  $0^{\#}$  is to take an iterated <sup>12</sup> ultrapower of M of length  $\Omega$ , the class of ordinals. In the resulting class model, the measurable cardinal is  $\Omega$  and the initial segment below  $\Omega$  is isomorphic to L, with the critical points of the embedding being the Silver indiscernibles. This can <sup>15</sup> be compared with Gaifman [Gaifman, 1974], in which he started with a countable

substructure of an appropriate  $V_{\tau}$  containing a measurable cardinal, and obtained a countable model X whose iterations generated a closed unbounded class of inaccessibles of L containing the cardinals of V. The 0<sup>#</sup> mouse M is the ultimate refinement of Gaifman's model X, and is the weakest Dodd-Jensen mouse.

It is important to realize that the measure U in a mouse  $M = J_{\alpha}[U]$  is not a measure in V; indeed the fine structure analysis ensures that  $J_{\alpha+1} \models |\alpha| = \rho^M < \operatorname{crit}(U)$ , so that U is not a measure in any model larger than M. Thus if  $M = J_{\alpha}[U]$  and  $J_{\alpha'}[U']$  are two mice then neither is an initial part of the other, as always happens at two levels  $J_{\alpha}$  and  $J_{\alpha'}$  of the L hierarchy. However the mice M and M' can be compared by use of iterated ultrapowers, as Kunen did with L[U] and L[U'], because the fact that a mouse M has a fine structure, and that its projectum  $\rho$  is smaller than its critical point, ensures that the iterated ultrapowers will preserve the definition of the new subset of its projectum  $\rho$ . Thus the fine structure of a mouse can be used to ensure the criterion of equal comparison from

section 2.3, similarly to the way  $\phi$ -minimality was used at that point.

- The use of comparison by iterated ultrapowers gave a well-ordering of the mice, defined by setting  $M \prec M'$  if M is a member of an iterated ultrapower of M'. This opened the way for Dodd and Jensen to define the core model as  $L[\mathcal{M}]$ , where
- $\mathcal{M}$  is the class of all mice. Although they called this model K, we will refer to it as  $K^{\mathrm{DJ}}$  to distinguish it from later generalizations. Using properties of mice, they showed that  $K^{\mathrm{DJ}}$  had most of the basic properties of L: It is a model of ZFC, it has a condensation property and hence satisfies GCH, and it satisfies  $\Diamond_{\kappa}$  and  $\Box_{\kappa}$  for
- all cardinals  $\kappa$ . One important difference was in the complexity of its definition: while a model  $M = J_{\alpha}$  need only be well-founded to be an initial segment of L, a model  $M = J_{\alpha}[U]$  must have all of its iterated ultrapowers well-founded. This meant that for  $x \subseteq \omega$ , the formula " $x \in K^{\mathrm{DJ}}$ " is  $\Sigma_3^1$  while the sentence " $x \in L$ " is
- meant that for  $x \subseteq \omega$ , the formula " $x \in K^{D_3}$ " is  $\Sigma_3^1$  while the sentence " $x \in L$ " is  $\Sigma_2^1$ , and thus the canonical well-ordering of the reals of K is  $\Delta_3^1$  as in L[U], instead of  $\Delta_2^1$  as in L.
- <sup>15</sup> Dodd and Jensen showed that  $K^{\text{DJ}}$  bridged the gap between L and L[U] in the sense that we would have  $K^{\text{DJ}} = L$  in the event  $0^{\#}$  did not exist, while if there were a model L[U] with a measure U on  $\kappa$ , then  $K^{\text{DJ}}$  was equal to the initial ordinal
- length segment of  $\text{Ult}_{ON}(L[U], U)$  or, equivalently, to  $\bigcap_{\alpha \in ON} \text{Ult}_{\alpha}(L[U], U)$ . In particular,  $V_{\kappa}^{K^{\text{DJ}}} = V_{\kappa}^{L[U]}$ .

Finally, they proved a pair of covering lemmas. The covering lemma for  $K^{\text{DJ}}$ , which assumed that there is no model with a measurable cardinal, had the same

- statement as that for L: If there is no model with a measurable cardinal, then for any uncountable set x of ordinals there is a set  $y \supseteq x$  in  $K^{\text{DJ}}$  of the same cardinality. The covering lemma for the case that a model L[U] with a measurable
- cardinal existed assumed that  $0^{\dagger}$ , the sharp of U, does not exist, but the conclusion bifurcated. As Prikry had shown in his thesis [Prikry, 1970], there is a forcing
- <sup>27</sup> notion which, given a measurable cardinal  $\kappa$ , adds an  $\omega$  sequence sequence C of ordinals, cofinal in  $\kappa$ , without collapsing any cardinals. However the necessary modification to the covering lemma was minimal. With the term "covering prop-
- <sup>30</sup> erty" having the expected meaning from above: Suppose that U is a measure in L[U], and the critical point  $\kappa$  is as small as possible, and  $0^{\dagger}$  does not exist. Then either L[U] has the covering property, or else there is a Prikry sequence C such
- that the model L[U, C] has the covering property. Furthermore the model L[U, C] is unique, though the sequence C is unique only up to finite changes. The first publication of this work was in a handwritten note, [Dodd and Jensen,
- <sup>36</sup> ] titled A Modest Remark. The title referred not to the covering lemma, but to an alleged theorem claiming that the existence of a measurable cardinal was inconsistent with ZF. The error was quickly discovered independently by several
- <sup>39</sup> parties in widespread location: Jensen had incorrectly assumed that a cardinal  $\kappa$  of cofinality  $\omega$  could be collapsed onto  $\omega_1$  without also collapsing  $\kappa^+$ . Jensen had, as he admitted, written and hurriedly distributed the note without consulting
- <sup>42</sup> its putative coauthor Dodd. In spite of the error, this note provided a valuable early exposition of Dodd and Jensen's theory of the core model. The formal publication of the theory came in three papers: [Dodd and Jensen, 1981] gave the

basic construction of  $K^{\text{DJ}}$  and the two papers [Dodd and Jensen, 1982a; Dodd and Jensen, 1982b] gave the proof of the covering lemma for  $K^{\text{DJ}}$  and L[U] respectively.

<sup>3</sup> In addition Dodd wrote a book, [Dodd, 1982], giving an exposition of  $K^{DJ}$  and looking towards extensions of this model.

They showed that, just as the existence of a nontrivial embedding  $i: L \to L$ <sup>6</sup> implies that  $0^{\#}$  exists, the existence of a nontrivial embedding  $i: K \to K$  implies that there is a model L[U] with a measurable cardinal. Where the critical point of an embedding  $i: L \to L$  is always a Silver indiscernible, however, Jensen showed

- [[Citation?]] that the critical point of an embedding  $i: K \to K$  may be smaller than the measurable cardinal in L[U].
- The most important immediate consequence of the Dodd-Jensen core model theory is that a failure of the Singular Cardinal Hypothesis entails the failure of the hypothesis for  $K^{\text{DJ}}$ , that is, that  $0^{\dagger}$ , the sharp for a model L[U] with a measure, exists. The proof of this implication was the same as the proof using L: if  $0^{\dagger}$  does
- <sup>15</sup> not exist, then there is a model of one of the forms  $K^{DJ}$ , L[U] or L[U, C] which has the covering property. Since all of these models satisfy GCH, this implies that SCH holds in V.
- <sup>18</sup> An important later consequence was Jensen's extension in [Donder *et al.*, 1981] of Shoenfield's theorem [Shoenfield, 1961] that any model M of set theory containing  $\omega_1$  is absolute for  $\Sigma_2^1$  formulas. Jensen's theorem stated: Assume  $a^{\#}$  exists for
- <sup>21</sup> every real a, and that there is no inner model with a measurable cardinal. Then K is absolute for  $\Sigma_3^1$  formulas. His proof actually showed that the same is true of any model  $M \supseteq K$ .
- Another consequence of the Dodd-Jensen covering lemma was a generalization, due to Mitchell, of Kunen's theorem in [Kunen, 1970] that all Jónsson cardinals in L[U] are Ramsey: Mitchell [Mitchell, 1979b] showed that all Jónsson cardinals
- <sup>27</sup> in V are Ramsey in  $K^{\text{DJ}}$ . It followed that a minimal model for a Ramsey cardinal could be constructed by taking the least initial segment of  $K^{\text{DJ}}$  which satisfies that there is a Ramsey cardinal. This was generalized by Jensen [Donder *et al.*,
- <sup>30</sup> 1981] to show that any  $\delta$ -Erdős, and even any  $\delta$ -Jónsson cardinal, is  $\delta$ -Erdős in  $K^{\text{DJ}}$ . Thus the core model provides minimal models for these cardinals.
- Before going on, a word about terminology is in order. At this point we have three covering lemmas for three different models: for the model L if  $0^{\#}$  does not exist, for the Dodd-Jensen core model  $K^{\text{DJ}}$  if  $0^{\#}$  exists but L[U] does not, and for L[U] if it exists but  $0^{\dagger}$  does not. In the current terminology any of these models
- <sup>36</sup> would, under the appropriate hypothesis, be called the *core model*. Much of the rest of this chapter is concerned with the continuing effort to extend this theory to encompass larger cardinals.
- It will be useful here to consider some of the properties of the core model as described so far in the article in order to get some sense of what can be expected of larger core models, both those in the remainder of this history and those to
- <sup>42</sup> come in the future. One such property is that it is structured in a hierarchy, with each step in the hierarchy containing only sets which self evidently must be in any model containing the previous steps of the hierarchy together with a measurable

cardinal. Thus the result is guaranteed to be the smallest possible model of set theory containing all of the ordinals and every fragment of a measurable cardinal.

- Following this intuition, the model which is today called the *core model*, and is denoted by the symbol K, can be defined as a goal, though not mathematically as the minimal model of ZFC containing all of the large cardinal structure *existing in the universe* (cf. [Mitchell, 2009], section 5).
- Some of the other properties of the core model K, whether L,  $K^{\text{DJ}}$  or L[U], described so far provide evidence that the core model constructions described thus
- <sup>9</sup> far satisfy this definition, provided that  $0^{\dagger}$  does not exist. The core model is rigid, in the sense that there is no nonidentity embedding  $i: K \to K$ , although this sense of rigidity is weakened in the case of L[U] by the existence of embeddings
- <sup>12</sup>  $i: L[U] \to M$  given by iterated ultrapowers. The core model is absolute: in the case K = L or  $K = K^{DJ}$  this is true in the strong sense that  $K^M = K \cap M$  for any uncountable model M of set theory, but again the statement in the case K = L[U]<sup>15</sup> is weaker as  $K^M$  may be an iterated ultrapower of L[U].

Finally, the covering lemma shows that the model K is close to V, though it is perhaps surprising that the closeness involved can be regarded as instances of large

cardinal structure. Again, the statement of the covering lemma is significantly weakened in the case K = L[U].

## 4.2 More Measures

- <sup>21</sup> The obvious next step was to incorporate the new core model theory into Mitchell's model  $L[\mathcal{U}]$  for a sequence of measures, thereby extending the core model construction to include any number of measures. Mitchell had been attempting to do so
- even before Dodd and Jensen completed their work on  $K^{\text{DJ}}$ . Now, with notes available on the Dodd-Jensen techniques, he was able to proceed.
- Mitchell called his model  $K(\mathcal{F})$ ; however we will write  $K[\mathcal{F}]$ , using square <sup>27</sup> brackets to agree with the standard notation for relative constructibility. The sequence  $\mathcal{F}$  was a sequence of filters in V, which was intended to become a sequence of ultrafilters in  $K[\mathcal{F}]$ ; however it is noteworthy that neither the definition of  $K[\mathcal{F}]$
- <sup>30</sup> nor the fine structure analysis made any use of the structure of  $\mathcal{F}$ : what is defined is really a relativized core model, built about an arbitrary set  $\mathcal{F}$ .<sup>4</sup>
- The model  $K[\mathcal{F}]$  was defined to be  $L[\mathcal{F}, \mathcal{M}]$ , where  $\mathcal{M}$  was the class of structures called  $\mathcal{F}$ -mice. The part of this work which was published in [Mitchell, 1984a] ended at the point where fine structure was to be introduced. It proved that the model  $K[\mathcal{F}]$  was the union of the class of  $\mathcal{F}$ -mice, and that if  $\mathcal{F}$  was strong —
- that is, it was a coherent iterable sequence of measures in  $K[\mathcal{F}]$  then  $K[\mathcal{F}]$  was a model of ZFC + GCH + the set of reals has a  $\Delta_3^1$  well-ordering + every normal ultrafilter is a member the sequence  $\mathcal{F}$ .
- <sup>39</sup> The rest of this paper was never published, though it had some distribution as the manuscript [Mitchell, 1985]. Section 4, which analyzed the fine structure of

 $<sup>^4</sup>Of$  course this statement assumes (as will we, unless stated otherwise) that  ${\cal F}$  is not a proper class.

the model, took up more than half of this manuscript, 65 out of about 110 double-spaced pages. Much of this was a straightforward extension of the Dodd-Jensen

- <sup>3</sup> fine structure of L[U], as extended to sequences of measures by using Mitchell's techniques from  $L[\mathcal{U}]$ . An  $\mathcal{F}$ -mouse M was a structure  $J_{\alpha}[\mathcal{G}]$ , where  $\mathcal{G}$  has  $\mathcal{F}$  as an initial segment and, above that, a coherent sequence of measures corresponding to
- <sup>6</sup> the single measure in the Dodd-Jensen mice. Some technical adaptations were, of course, needed to accommodate new sets being directly defined from the members of the sequence, and from the fact of working above the arbitrary set  $\mathcal{F}$ . The major
- complication involved the criterion of equal comparison: a new complement was needed to the use of mice, as described in section 2.3. We will return to this point when we look at the Dodd-Jensen lemma, which is a more general alternative to
   the technical trick used in this paper.

Only at this point was it possible to prove the crucial observation that if  $\mathcal{F}$  is strong then  $K[\mathcal{F} \upharpoonright \alpha] \subseteq K[\mathcal{F}]$ , and in particular that the definition of  $K[\mathcal{F}]$  in the <sup>15</sup> case that  $\mathcal{F}$  is a proper class,  $K[\mathcal{F}] = \bigcup_{\alpha \in \mathrm{ON}} K[\mathcal{F} \upharpoonright \alpha]$ , makes sense.

- Perhaps the most important change from Dodd and Jensen's analysis of  $K^{\text{DJ}}$ is that the fine structure analysis took on a new role: it enabled a sequence  $\mathcal{F}$ to be defined by recursion. More specifically, the main theorem of section 4 was: Suppose  $\mathcal{F}$  is a sequence of filters such that for each  $(\alpha, \beta)$  in the domain of  $\mathcal{F}$ , the filter  $\mathcal{F}(\alpha, \beta)$  (i) is a normal ultrafilter of order  $\beta$  on  $K[\mathcal{F} \upharpoonright (\alpha, \beta)]$  and (ii) is
- <sup>21</sup> countably complete. Then  $\mathcal{F}$  is strong in the sense that each filter  $\mathcal{F}(\alpha,\beta)$  in the sequence  $\mathcal{F}$  is an ultrafilter in  $K[\mathcal{F}]$ . This was used to define a preliminary version of the core model, which is now known as  $K^{c}$ , the countably complete core <sup>24</sup> model, by setting  $K^{c}$  equal to the model  $K[\mathcal{F}]$  where  $\mathcal{F}$  is the maximal sequence
- of countably complete measures, defined recursively by adding a filter  $\mathcal{F}(\alpha,\beta)$  whenever it satisfies the criterion above.
- <sup>27</sup> A full covering lemma, in the style of that in the case that the core model is equal to L or  $K^{\text{DJ}}$ , or even L[U], is not possible for these models. Mitchell instead used what the reader may recognize as Jensen's first version of the covering <sup>30</sup> lemma in [Jensen, 1974b]: he said that an inner model M satisfies the *weak cov*-
- <sup>30</sup> lemma in [Jensen, 1974b]: he said that an inner model M satisfies the weak covering lemma if, for every sufficiently large strong limit cardinal  $\lambda$ , the successor  $(\lambda^+)^M$  of  $\lambda$  as evaluated in M is the same as the successor  $\lambda^+$  as calculated in V.
- <sup>33</sup> Mitchell used recursion to define what came to be known as the countably complete (later, with Steel's work, countably certified) core model  $K^{c} = K[\mathcal{F}]$ , by requiring that each filter  $\mathcal{F}(\alpha, \beta)$  on the sequence  $\mathcal{F}$  be a countably complete  $K[\mathcal{F}\uparrow(\alpha, \beta)]$ -
- <sup>36</sup> ultrafilter. The countable completeness ensured that all iterated ultrapowers were well-founded, and this in turn allowed the use of fine structure to show that each filter  $\mathcal{F}(\alpha,\beta)$  was still an ultrafilter in the full model  $K^c$ . Mitchell's proof that
- <sup>39</sup>  $K^c$  has the weak covering property was essentially the same as Dodd and Jensen's proof that  $K^{DJ}$  has the covering property, here applied to  $\kappa = (\lambda^+)^{K^c}$  which has, if it is smaller than  $\lambda^+$  of V, cofinality less than  $\lambda$ .
- <sup>42</sup> One important consequence of the weak covering lemma is that it provides a new technique for satisfying the criterion of equal comparison: a model  $K[\mathcal{F}]$  which satisfies the weak covering property is *universal* in the sense that when  $K[\mathcal{F}]$  is

compared, using iterated ultrapowers, with any other model  $K[\mathcal{F}']$  then the model  $K[\mathcal{F}]$  will compare at least as long as  $K[\mathcal{F}']$ : the final model in the iteration of

- $K[\mathcal{F}']$  will always be an initial segment of the final model in the iteration of  $K[\mathcal{F}]$ . In particular, if  $K[\mathcal{F}']$  is also universal then the final models of the two iterations will be equal.
- <sup>6</sup> The weak covering lemma not only showed that  $K^c$  is universal; it also provided a criterion for the transitive collapse of an elementary submodel  $X \prec K^c$  to be universal. This criterion is a counterpart to the particular case of condensation
- <sup>9</sup> for L that any elementary subset of L which contains a proper class of ordinals is isomorphic to L. For  $K[\mathcal{F}]$  it was not enough to assume that X is a proper class: Mitchell defined X to be *thick* if, for a stationary class of singular cardinals  $\lambda$  the
- order type of  $X \cap (\lambda^+)$  is equal to  $\lambda^+$ . It follows from the weak covering lemma that not only is  $K^c$  universal, but also the transitive collapse of any thick  $X \prec K^c$ is universal. This fact was used in Section 7 to obtain the true core model, which
- <sup>15</sup> was written  $K[\mathcal{F}_{\mathrm{M}}]$  but today would be written simply as K. The sequence  $\mathcal{F}_{\mathrm{M}}$ was defined recursively in the same way as  $\mathcal{F}^{\mathrm{c}}$ , but with countable completeness replaced with the requirement that  $\mathrm{Ult}(K[\mathcal{G}], \mathcal{F}(\alpha, \beta))$  be well-founded whenever
- <sup>18</sup>  $\mathcal{G}$  is a strong sequence which extends  $\mathcal{F} \upharpoonright (\alpha, \beta)$ . This fact was used to show that  $K[\mathcal{F}_{\mathrm{M}}]$  is iterable by showing that it is an elementary submodel of  $K^{\mathrm{c}}$ .
- An addendum, dated July 1985, stated a fact which Steel had pointed out as <sup>21</sup> missing from the manuscript, though it had been widely used: If there is no inner model with a cardinal  $\kappa$  such that  $o(\kappa) = \kappa^{++}$  then every embedding  $i: K[\mathcal{F}_{\mathrm{M}}] \to N$ into a well-founded model N is an iterated ultrapower by measures in  $K[\mathcal{F}_{\mathrm{M}}]$ .
- An additional technique which came into use at this time is the Dodd-Jensen lemma which, as was pointed out earlier, serves as an important complement to the use of mice in the verification that the criterion of equal comparison is
- <sup>27</sup> satisfied. The lemma states that an iterated ultrapower is minimal among all  $\Sigma_0$ -elementary embeddings mapping into the same final model; more specifically: Suppose  $i: M \to M'$  is an iterated ultrapower of the premouse M, and  $\sigma: M \to M'$
- is a  $\Sigma_0$ -elementary embedding. Then range( $\sigma$ ) is cofinal in M', and  $\sigma(\alpha) \ge i(\alpha)$ for all ordinals  $\alpha$  in M. A typical application occurs in the fine structure analysis: in this case  $\sigma = j \circ \pi$ , where  $\pi \colon M \to J_{\nu}[\mathcal{U}]$  arises from the transitive collapse of
- a Skolem hull, and j and i are iterated ultrapowers on M and  $J_{\alpha}[\mathcal{U}]$  arising from the comparison of these models. The model M can be treated as a mouse, and since its projectum is not moved by the iterated ultrapower i on M this implies
- that the last model M' of this iteration is not a proper initial segment of the last model of the iteration j. It is not immediately clear, however, that the embedding j does not move the projectum, and hence the mouse criterion cannot be used to
- <sup>39</sup> rule out the possibility that the last model of the iteration j is a proper initial segment of M'. The Dodd-Jensen lemma does rule this out, since i is an iterated ultrapower and hence minimal.
- <sup>42</sup> The author's memory is that the discovery of the Dodd-Jensen lemma came was a result of a message from Jensen pointing out a hole in [Mitchell, 1985]. The author, after some thought, filled the hole, and Dodd and Jensen turned

the argument in the correction into a general lemma. The author remembers doubting an application the general statement and being told that the proof was

<sup>3</sup> his own. Some doubt—perhaps not fatal—may be thrown on this story by the fact that [Mitchell, 1985] does contain a quite different and apparently valid argument covering the same question.

## <sup>6</sup> Consequences of the covering lemma for sequences of measures

The unpublished article [Mitchell, 1985] concluded with a number of applications of the weak covering lemma in K: Any of the following assertions imply the existence of an inner model of  $\exists \kappa \ o(\kappa) = \kappa^{++}$ : (i)  $\kappa$  and  $\kappa^{+}$  are both weakly compact, (ii)  $\kappa$  is  $\kappa^{+}$  strongly compact, (iii)  $\kappa$  is measurable and  $\kappa^{+} > (\kappa^{+})^{K[\mathcal{F}]}$ , (iv)  $\kappa$  is measurable and  $2^{\kappa} > \kappa^{+}$ , (v) there is a  $\kappa$ -complete ultrafilter U on  $\kappa$  such that

<sup>12</sup>  $i^{U}(\kappa) = \kappa^{+}$  (in particular, the Axiom of Determinacy holds), (vi) every  $\kappa$ -complete filter can be extended to a  $\kappa$ -complete ultrafilter, or (vii) there is a  $\kappa^{+}$ -saturated ideal on a successor cardinal  $\kappa$ .

- These results relied on consequences of the weak covering lemma, and in particular on one which did not originally appear, at least in its strongest form, in [Mitchell, 1985]: If there is no model of  $\exists \kappa o(\kappa) = \kappa^{++}$  then every elementary embedding  $i: K[\mathcal{F}_{\mathrm{M}}] \to N$  into a well-founded model N is an iterated ultrapower by measures in  $\mathcal{F}$ 
  - submax.
- The arguments from K used in this proof seemed much stronger than the core model K itself, and it was expected that, once larger core models had been constructed, the same arguments would yield a stronger conclusion. This turned out
- <sup>24</sup> to be true for all but clause (iii): A more closer at this look led Mitchell to suggest  $\exists \kappa \, o(\kappa) = \kappa^{++}$  was the exact large cardinal strength of the failure of GCH at a measurable cardinal. This was later confirmed by work of Woodin (using a slightly stronger assumption) and Gitik [Gitik, 1989].

Of the the topics one might have expected to see in [Mitchell, 1985], two were omitted. One was the weak covering lemma for the case of cardinals  $\kappa$  which are not countably closed, that is, such that  $\lambda^{\omega} \geq \kappa$  for some  $\lambda < \kappa$ , and the other was the Singular Cardinal Hypothesis, or, indeed, any version of the covering lemma beyond the weak covering lemma. The first of these was the less consequential,

- <sup>33</sup> but the longest to resolve: the proof for  $\kappa$  which are not  $\omega$  closed does not require any techniques except [Mitchell, 1985], together with those used for L by Jensen in [Jensen, 1974c]; however no published, or even manuscript, exposition of
- the extension of the covering lemma for sequences of measures to the non-closed case appeared until [Mitchell and Schimmerling, 1995] (for a much more powerful version of the core model).
- <sup>39</sup> Applying this model to SCH was much more troublesome. The most complicated case of the Dodd-Jensen covering lemma is that in which  $0^{\dagger}$  does not exist and there is a model L[U] with a measurable cardinal along with a Prikry sequence
- 42 C for that measure. Then the model L[U, C] provides a covering set. The set C is

not quite unique — this is a change from previous versions of the covering lemma — but the model L[U, C] is unique. Mitchell showed in [Mitchell, 1984b] that a

- straightforward generalization of this is false in general. If  $\lambda$  is an inaccessible limit of a set I of measurable cardinals in V, then there is a generic extension V[G] in which each  $\kappa \in I$  is made singular by Prikry sequence over V, but there is
- 6 no sequence of Prikry sequences which is uniform: for any sequence  $\langle C_{\kappa} : \kappa \in I \rangle$ of Prikry sequences there is a sequence of sets  $\langle A_{\kappa} : \kappa \in I \rangle$  in the ground model, with each set  $A_{\kappa}$  a member of the measure on  $\kappa$ , such that  $\{\kappa \in I : C_{\kappa} \nsubseteq A_{\kappa}\}$  is
- unbounded in  $\kappa$ .

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In this model, there is a uniform sequence of Prikry sequences for witnesses for each subset of I which is bounded in  $\kappa$ , and this proved to be the general situation. Mitchell defined a notion of a system of indiscernibles which generalized,

- in a natural way, the notion of a Prikry sequence, and he proved a version of the covring lemma stating that for any uncountable set X in V, there is a system C
- of indiscernibles for K such that X is contained in a set  $Y \in K[\mathcal{C}]$  of the same cardinality. However the system  $\mathcal{C}$  is, in contrast to the case in the Dodd-Jensen covering lemma for L[U, C], essentially local: it depends on the set X, and its size
- <sup>18</sup> is at most that of X. Furthermore, while there is a sense in which the choice of  $\mathcal{C}$  is, on its domain, unique up to finite variations, this is not enough to directly generalize the Jensen proof that the nonexistence of  $0^{\#}$  implies SCH.
- <sup>21</sup> This version of the covering lemma was published in [Mitchell, 1987], which included several of its consquences. The simplest of these gave a converse to Magidor's construction in [Magidor, 1978]: Suppose that there is no inner model
- satisfying  $\exists \kappa \, o(\kappa) = \kappa^{++}$ , and that  $\lambda$  is a singular cardinal of uncountable cofinality  $\delta$  which is regular in K. Then  $o(\lambda) \geq \delta$  in K. An additional result in [Mitchell, 1987] showed that if there is a model of ZF in which the closed unbounded ultra-
- <sup>27</sup> filter on  $\omega_1$  is an ultrafilter then  $\kappa$  is a (weak) repeat point, a result which Mitchell later showed (in unpublished work) is best possible. A final result considered the model L[i], constructed from an embedding  $i: V \to M$ : it was shown that if there
- is no inner model of  $\exists \kappa \, o(\kappa) = \kappa^{++}$  then the model L[j] contains an iterate of the core model K, and is equal to an iterate of K if the embedding is an ultrapower embedding  $i: V \to \text{Ult}(V, U)$ .
- <sup>33</sup> Several succeeding papers yielded further consequences: notably [Mitchell, 1991], which characterized the ways (assuming no inner model of  $\exists \kappa o(\kappa) = \kappa^{++}$ ) in which a cardinal could become singular. Not until [Mitchell, 1992b] did he publish the
- <sup>36</sup> partial results he obtained on the strength of SCH, and by this time Gitik's [Gitik, 1991] had already appeared, giving the final solution to this question: If SCH fails, then there is an inner model satisfying  $\exists \kappa o(\kappa) = \kappa^{++}$ . His proof of this result
- <sup>39</sup> required deep techniques from Shelah's pcf theory in addition to Mitchell's results. As related above, he had already strengthened a result of Woodin to prove that the existence of such a model is sufficient to obtain a forcing extension in which  $\kappa$ <sup>42</sup> is measurable and  $2^{\kappa} = \kappa^{++}$ , and hence one in which SCH failed.

At about the same time, Mitchell [Mitchell, 1992a] generalized Jensen's absoluteness result mentioned earlier: If there is no model of  $\exists \kappa o(\kappa) = \kappa^{++}$  and  $a^{\#}$ 

exists for every real a, then every model M containing K, or an iterated ultrapower of K, is  $\Sigma_3^1$  absolute. But, as we shall see, Steel shortly afterward had a stronger result, which was actually published earlier.

## 5 EXTENDER MODELS: WOODIN CARDINALS AND BEYOND)

## 5.1 Moving Beyond Measurable Cardinals

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- <sup>6</sup> With the core model understood up to  $\exists \kappa \, o(\kappa) = \kappa^{++}$ , both Mitchell and the team of Dodd and Jensen turned their attention to larger cardinals. The smallest of these was a  $\mu$ -measurable cardinal, defined by Mitchell in [Mitchell, 1979a] to
- <sup>9</sup> be a cardinal  $\kappa$  such that there exists an embedding  $i: V \to M$  with  $\kappa = \operatorname{crit}(i)$ such that the induced ultrafilter  $\mu = \{ x \subseteq \kappa : \kappa \in i(x) \}$  is a member of M. Such an embedding cannot be obtained from an ultrapower using a normal ultrafilter,
- <sup>12</sup> but it can be obtained by using a nonnormal ultrafilter on  $V_{\kappa}$ , namely  $W = \{x \subseteq V_{\kappa} : \mu \in i(x)\}$ . The next natural step beyond a  $\mu$ -measurable cardinal is a  $\mathcal{P}^2(\kappa)$  strong cardinal, that is, a cardinal  $\kappa$  with an embedding  $i: V \to M$  such
- that  $\mathcal{P}^2(\kappa) \subseteq M$ . Such an embedding cannot be represented by a ultrafilter on  $V_{\kappa}$ , since any such ultrafilter would be a member of M. Thus significant progress beyond  $\exists \kappa o(\kappa) = \kappa^{++}$  depended on the invention of a more flexible representation for large cardinal embeddings.
- Such a representation was discovered independently by Jensen and by Mitchell. Of these two, it was Jensen's formulation which has become standard. This notion, which Jensen called an *extender*, is the same construction as was used to extend the
- collapse embedding in the proof of the covering lemma. An embedding  $i: V \to M$ , with critical point  $\kappa$  is said to be an *ultrapower by a*  $(\kappa, \lambda)$ -*extender* if crit $(i) = \kappa$ <sup>24</sup> and  $M = \{i(f)(a) : a \in [\lambda]^{<\omega}\}$ . In this case the  $(\kappa, \lambda)$ -extender E such that
- M = Ult(V, E) is as a sequence of ultrafilters  $\langle E_a : a \in [\lambda]^{<\omega} \rangle$ , with  $E_a = \{x \in \kappa^{|a|} : a \in i(x)\}$ . The embedding *i* is then a direct limit of the ultrapower embeddings  $i_a : V \to \text{Ult}(v, E_a)$ , with the image of  $[f]_{E_a}$  in Ult(V, E) being written
- embeddings  $i_a: V \to \text{Ult}(v, E_a)$ , with the image of  $[f]_{E_a}$  in Ult(V, E) being written as  $[f]_{E,a}$ .

Mitchell's construction in [Mitchell, 1979a] was effectively the same, but complicated by using, instead of all ordinals less than  $\lambda$ , only the generators of the embedding *i*, that is, the set of ordinals  $\alpha < \lambda$  such that  $\alpha \neq i(f(a))$  for any  $a \in [\alpha]^{<\omega}$ . This decision to separate out the generators proved to be unnecessary and undesirable: for example, the question of which ordinals are generators depends on the power set of  $\kappa$ , and hence may differ between an embedding *i*:  $V \to M$ and the restriction of that embedding to an inner model.

- Jensen's notion of an extender had a side effect of eliminating a significant complication in Mitchell's models  $L[\mathcal{U}]$ : the need for coherence functions. If  $\alpha < \beta < o(\kappa)$  then the requirement that  $\mathcal{U}(\kappa, \alpha) \triangleleft \mathcal{U}(\kappa, \beta)$  entails that there is
- <sup>39</sup> a function  $f \in {}^{\kappa}\kappa$  such that  $\alpha = [f]_{\mathcal{U}(\alpha,\beta)}$ . Mitchell's core model construction, for example, required special care to ensure that no measure  $\mathcal{U}(\kappa,\beta)$  was used in the construction until all of its coherence functions had appeared. If this same model

is recast by representing the measures as extenders then this becomes unnecessary: the measure  $U = \mathcal{U}(\kappa, \beta)$  has the single generator  $\kappa$  and is most conservatively

- <sup>3</sup> represented as a  $(\kappa, \kappa + 1)$  extender; however it can be equivalently represented as a  $(\kappa, \lambda)$  extender for any  $\lambda > \kappa$ . If  $\lambda \ge \beta$  then, since  $\alpha = i(\mathbf{id})(\alpha)$ , where  $\mathbf{id}$  is the identity function. Thus  $\alpha = [\mathbf{id}]_{\mathcal{U}(\kappa,\beta),\{\alpha\}}$  for all  $\alpha < \beta$ , so the identity function is
- <sup>6</sup> the only coherence function needed:

## **Overlapping Extenders and Iteration Trees**

The use of extenders instead of normal ultrafilters made it straightforward to define models  $L[\mathcal{E}]$  large enough to contain a *strong cardinal*, that is, a cardinal  $\kappa$  such that for all  $\lambda$  there is an embedding  $i: V \to M$  with critical poing  $\kappa$  such that  $V_{\lambda} \subseteq M$ . Beyond this a new problem arose. Recall that one of the three

- criteria given in section 2.3 for the success of a comparison by iterating the least differences was that generators not be moved. The use at that point of the term "moving generators" was somewhat anachronistic: because the only generator of a
- <sup>15</sup> normal ultrafilter U is its critical point, the coherence of the sequence of ultrafilters was enough to ensure that no generators were moved. For a  $(\kappa, \lambda)$ -extender E, in contrast, every ordinal in the interval  $[\kappa, \lambda)$  is a generator, so that the principle of
- <sup>18</sup> no moving generators would seem to require that if E is used in an iteration, then the rest of the iteration cannot involve any  $(\kappa', \lambda')$ -extender E' with  $\kappa' < \lambda$ . Up to a strong cardinal, this difficulty does not occur; however it is easy to see that <sup>21</sup> having two strong cardinals is more than enough to make it unavoidable.

A secondary problem at this level involves the indexing of the sequence. The indexing  $\mathcal{U}(\kappa,\beta)$  used by Mitchell relied on the fact that given two measures on

- <sup>24</sup> different cardinals, the one on the larger cardinal is stronger. Again, this is not a problem up to a strong cardinal; however if  $\kappa$  is a strong cardinal, and there is also an extender E' with critical point  $\kappa' > \kappa$ , then there is an extender E on  $\kappa$
- such that E' is in a member of the codomain Ult(V, E) of the embedding  $i^E$ , so that  $E' \triangleleft E$ .

Baldwin [Baldwin, 1986] proposed one solution to both problems. He relied on a different, carefully defined indexing of the extenders, designed so that, although the iteration did have moving generators, there was guaranteed to be a closed unbounded set of stages of the iteration where the generators of the extender used

<sup>33</sup> were not later moved. This was sufficient to ensure the success of the comparison. The apparent problem with Baldwin's idea is that there did not seem to be a clear scheme for finding a suitable indexing as one moved to larger cardinals.

Mitchell proposed a second possibility. The iterated ultrapowers up to this point had been linear: they looked like

$$L[\mathcal{E}] = M_0 \xrightarrow{i_{0,1}} M_1 \xrightarrow{i_{1,2}} \cdots M_{\omega} \xrightarrow{i_{\omega,\omega+1}} M_{\omega+1} \xrightarrow{i_{\omega+1,\omega+2}} \cdots M_{\theta}$$
(5)

<sup>36</sup> where  $\theta$  is the length of the iteration. In this construction, each model  $M_{\alpha+1}$  is the ultrapower of the previous model:  $M_{\alpha+1} = \text{Ult}(M_{\alpha}, E_{\alpha})$  where  $E_{\alpha}$  is the least extender in the sequence  $i_{0,\alpha}(\mathcal{E})$  of  $M_{\alpha} = L[i_{0,\alpha}(\mathcal{E})]$  which is not in the corresponding sequence on the other side of the comparison. In the new construction, the extender  $\mathcal{E}_{\gamma}$  is chosen in the same way, but  $M_{\alpha+1}$  may be an ultrapower by an

- the extender  $\mathcal{E}_{\gamma}$  is chosen in the same way, but  $M_{\alpha+1}$  may be an ultrapower by an earlier model in the sequence:  $M_{\alpha+1} = \text{Ult}(M_{\alpha^*}, E_{\alpha})$  where  $\alpha^*$  is the least ordinal such that  $i^{E_{\alpha}}$  moves generators of the extender  $E_{\alpha^*}$  used to define  $M_{\alpha^*+1}$ . This
- <sup>6</sup> gives a tree ordering, defined by taking the ordinal  $\alpha^*$  to be the immediate predecessor of  $\alpha + 1$ . If  $M_{\gamma}$  and  $M_{\gamma'}$  are two models of the iteration, then the iteration provides an embedding  $i_{\gamma,\gamma'} \colon M_{\gamma} \to M_{\gamma'}$  between them if and only if  $\gamma$  precedes
- γ' in the tree. Any branch of this tree looks like a linear iteration, except that the extenders used to construct the embeddings along the branch are typically taken from models in other branches, rather than from the models to which they are applied.

The resulting *iteration trees* satisfy the criterion of no moving generators, and they can be used in much the same way as linear iterations: in particular, a <sup>15</sup> comparison between two models  $L[\mathcal{E}]$  and  $L[\mathcal{E}']$ , using the process of iteration by the least difference but organized as a tree iteration, will terminate if it can be carried out far enough.

<sup>18</sup> This task of verifying that this process can be carried out "far enough" — the *iterability* problem — proved to be far more difficult than for linear iterations. There was some question even at successor stages: Since the extender  $E_{\alpha}$  used

- <sup>21</sup> to define  $M_{\alpha+1}$  is not a member of  $M_{\alpha^*}$ , why does  $\text{Ult}(M_{\alpha^*}, E_{\alpha})$  even exist, and if it does, why is it well-founded? The primary difficulty, however comes a limit stages. Suppose that the iteration has been carried out for  $\nu$  stages, where
- <sup>24</sup>  $\nu$  is a limit ordinal. If *b* is any branch of the tree  $(\nu, \prec)$  then  $i_{\alpha,\alpha'}$  is defined for any  $\alpha < \alpha'$  in *b*, so we can define  $M_b$  to be the direct limit of the system  $(\langle M_{\alpha} : \alpha \in b \rangle, \langle i_{\alpha,\alpha'} : \alpha \prec \alpha' \in b \rangle)$ . If there is a branch *b* which is cofinal in the
- <sup>27</sup> tree such that  $M_b$  is well-founded then the model  $M_b$  is a possible choice for  $M_{\nu}$ , but it is not evident that there should be any such branch. Indeed it is not clear that the tree even has a cofinal branch b, let alone a well-founded one: one could
- imagine a worst case in which  $\alpha^* = 0$  for all  $\alpha > 0$ , so that the tree has length  $\omega$ , but every branch has length 2.

When Mitchell introduced the notion of iteration trees at the end of a talk at the 1979 Association of Symbolic Logic meeting in Biloxi, Mississippi he confidently asserted that they would soon yield a model for a supercompact cardinal; however further investigations were discouraging. A proof of iterability seemed to

- <sup>36</sup> be difficult even for cardinals not far above a strong cardinal; at a supercompact cardinal it seemed that iterability provably failed—a conclusion later confirmed by Neeman's discovery [Neeman, 2004] of an infinite iteration tree with no branches
- <sup>39</sup> of length 4. Since there didn't seem to be any very interesting large cardinal properties between measurability and supercompactness, Mitchell largely stopped work on the problem.

## Woodin Cardinals

It is not clear whether Steel discovered the notion of iteration trees entirely independently, as he remembers, or whether he took a hint from comments at the end of Mitchell's 1979 talk (of which not even an abstract seems now to exist). In any case he and Woodin continued to work on the problem, but little progress was made for the first five years.

The breakthrough came from Foreman, Magidor and Shelah [Foreman *et al.*, 1988] and did not immediately involve inner models at all; indeed it contradicted

- some of the aims of the inner model program. One of the results in this paper was that if it is consistent that there is a supercompact cardinal, then it is consistent that the filter of nonstationary sets on  $\omega_1$  is saturated. This in turn implies that,
- <sup>12</sup> in a generic extension, there is an embedding  $i: V \to M$  which, if it were in V rather than in a generic extension, would be much stronger than a supercompact embedding, and this contradicted an implicit premise of the inner model program:
- that any such embedding would extend a large cardinal embedding from the ground model.

Woodin and Shelah [Woodin, 1988], building on the techniques and results of

- <sup>18</sup> [Foreman *et al.*, 1988], showed if there is a supercompact cardinal then in the model  $L(\mathbb{R})$  (i) all sets of reals are Lebesgue measurable, (ii) all sets of reals have the property of Baire, and (iii) the partition property  $\omega \to (\omega)_2^{\omega}$  holds. As
- <sup>21</sup> a consistency result this was not strong: all three are true in Solovay's model [Solovay, 1970] arising from the Levy collapse of a inaccessible cardinal to become  $\omega_1$  (with (iii) having been added to Solovay's results by Mathias [Mathias, 1977]).
- <sup>24</sup> However the fact that, given a supercompact cardinal, these consequences were outright true in  $L(\mathbb{R})$  was striking, especially in view of the fact that all are consequences of the Axiom of Determinacy.
- Attempts to weaken the hypothesis ultimately led to Woodin's definition of what came to be called a *Woodin cardinal*: a cardinal δ is said to be Woodin if for all functions f: δ → δ there is an embedding i: V → M with critical point κ < δ</li>
  such that V<sub>i(f)(κ)</sub> ⊆ M. The last theorem in [Woodin, 1988] stated that if there is a model with a Woodin cardinal, together with a sharp for that model, then every
- $\Sigma_3^1$  set of reals is Lebesgue measurable. An important tool in dealing with Woodin cardinals was a modification, invented by Woodin, of the forcing of [Foreman *et al.*, 1988]. The new forcing, which he called the *stationary tower*, came in two forms. The first, which was
- used for results such as that stated in the last paragraph, collapsed cardinals to make the Woodin cardinal  $\omega_1$ . The other left  $\delta$  as a Woodin cardinal and defined, in the generic extension V[G], an elementary embedding  $i^G \colon V \to M$  with the
- <sup>39</sup> remarkable property that  $(V_{\delta})^M = V_{\delta}[G]$ . This forcing ruled out a core model  $L[\mathcal{E}]$  with the expected properties which has a single Woodin cardinal: The forcing collapses many successors of singular cardinals, so  $L[\mathcal{E}]$  could not satisfy the
- weak covering lemma. Furthermore  $M = L[i^G(\mathcal{E})]$ , which would evidentially be the core model in V[G], contained mice which were not in the ground model  $L[\mathcal{E}]$

and were not iterable enough to compare with the original model  $L[\mathcal{E}]$ . If the core model was not changed by small forcing, then these 'mice' should not be in the core model; however there could not be any grounds in  $L[\mathcal{E}][G]$  to exclude them

without going beyond  $V_{\delta}$  of that model.

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The introduction of Woodin cardinals spurred the development of inner models in two ways: first, it gave as a goal an important new large cardinal property which was more accessible than supercompactness, and, second, it helped to explain some of the difficulties which had been encountered. Martin and Steel [Martin

- <sup>9</sup> and Steel, 1988; Martin and Steel, 1989] proved that if there is a model with n Woodin cardinals with a measurable cardinal above them then all  $\Pi_{n+1}^1$  sets are determined. They also used a result of Woodin to prove that if there is a model
- <sup>12</sup> with  $\omega$  many Woodin cardinals with a measurable cardinal above them, then the Axiom of Determinacy holds in  $L(\mathbb{R})$ . The proof involved first developing the theory of iteration trees far enough to show that no countable iteration tree, using
- <sup>15</sup> countably complete extenders, could have ill-founded limits on all branches with the ill-foundedness witnessed by a continuous function on the nodes. Thus, by constructing a tree having such a witness to ill-foundedness on all but one of its
- <sup>18</sup> branches, they could ensure that the remaining branch did have a well-founded limit. They then used the Woodin cardinal to find a special type of iteration tree, which they called an *alternating chain*, consisting of just two branches, each of
- <sup>21</sup> length  $\omega$ . They constructed the tree so that the well-foundedness of one of the branches implied the existence of a strategy, and they ensured that the limit on that branch was well-founded by constructing the other branch to be ill-founded.
- In the search for inner models for a Woodin cardinal, one key fact was observed by Martin and Steel: Suppose that M is an extender mode, and  $\mathcal{T}$  is an iteration tree on M having two distinct branches b and c with well-founded limits  $M_b$  an  $M_c$ .
- <sup>27</sup> Then there is model with a Woodin cardinal. In fact if  $\delta$  is the supremum of the lengths of the extenders used in  $\mathcal{T}$ , then  $\delta$  is Woodin with respect to any function  $f \in M_b \cap M_c$ . In particular  $\delta$  is Woodin in  $L[\mathcal{E}]$ , where  $\mathcal{E} = \mathcal{E}^{M_b} \upharpoonright \delta = \mathcal{E}^{M_c} \upharpoonright \delta$ .

This meant that, in order to reach a minimal model with a Woodin cardinal, it would be enough to define a sequence £ by recursion in such a way that, so long as it did not reach a Woodin cardinal, every iteration tree had a branch with
well-founded limits. Their strategy for doing so was to assume one had an ill-

founded tree, take an elementary substructure to obtain a countable model with a ill-founded tree, and then define maps from models in this iteration tree into V<sup>36</sup> which commuted with the tree embeddings, thus obtaining a contradiction since

- the union of the range of the maps would be ill-founded. Using this strategy, they succeeding in proving [Martin and Steel, 1994] that *if there are n Woodin*
- <sup>39</sup> cardinals, for  $n < \omega$ , then there is a proper class model  $M_n$  satisfying ZFC + there are n Woodin cardinals +  $\mathbb{R}$  has a  $\Sigma_{n+2}^1$  well-order. This result was essentially best possible in the sense that by their previous result the existence of n Woodin <sup>42</sup> cardinals plus a measurable cardinal above them implies that  $\Pi_{n+1}^1$  determinacy
  - holds, which implies that there is no  $\Delta_{n+2}^1$  well-ordering of the reals.

This Martin-Steel model does have a serious weakness, however. The models

introduced in this paper have the form  $L[\mathcal{E}]$ , where  $\mathcal{E}$  is a sequence of extenders. The comparison process for these models, however, does not use ultrapowers of the

- <sup>3</sup> models themselves, but instead relies on ultrapowers in a larger model constructed from a sequence  $\mathcal{F}$  of *background extenders*. For each extender E in the sequence  $\mathcal{E}$  there is a background extender F in the sequence  $\mathcal{F}$ . The extender F extends
- *E*, however it is substantially stronger than *E*: it is an extender in *V*, not just  $L[\mathcal{E}]$ , and if *E* is a  $(\kappa, \lambda)$ -extender then *F* is satisfies  $V_{\lambda+2} \subseteq \text{Ult}(V, F)$ . Because of this use of external iterations, many of the structural properties of
- <sup>9</sup> their model were unclear: for example, they were not able to show that GCH holds in their model, although they did show, using a result of Woodin, that it satisfies CH.
- <sup>12</sup> In the summer of 1986 Woodin discovered the second of the forcing orders associated with a Woodin cardinal, the extender algebra. This forcing goes back to the class forcing of Vopěnka [Vopěnka and Hájek, 1972], by which any set is
- <sup>15</sup> generic, by a class forcing, over any given class model of set theory. Let M be a model of set theory with a Woodin cardinal  $\delta$  such that the Woodinness of  $\delta$  is witnessed by extenders in M which are iterable in the universe V (though perhaps
- <sup>18</sup> not in M). The extender algebra is a forcing of size  $\delta$ , defined in M, with the property that, for any set x in V, there is an iterated ultrapower  $i: M \to M^*$  of M such that x is  $M^*$ -generic over i(P). Thus, for example, if there is a supercompact
- <sup>21</sup> cardinal in V then there is, for each  $\lambda$ , a generic extension of M which has a  $\lambda$ -supercompact cardinal even if the model M has no more than one Woodin cardinal. The apparent contradiction is avoided because the model M does not
- <sup>24</sup> know the strategy for iterating itself, and hence the argument cannot be carried out inside M. Indeed this forcing can be used to show that there is no fully iterable model of the form L[A], with A a set, having a Woodin cardinal: any such model
- <sup>27</sup> would have, in a generic extension, a model L[A'] having the same property with A' contained in the smallest measurable cardinal of L[A].

This does not exclude the possibility there is a a strategy in V for iterating the

- $_{30}$  model M, but it does imply that an iteration of the model M will not, in general, be fully internal to M even if it is internal in the sense that all extenders involved are members of the appropriate model of the iteration tree, and even if (as in the
- case of the genericity iteration of the last paragraph) the iteration is defined inside M. Of course this is not always such a difficulty as it may appear, since in most cases an iteration tree is used which is not defined inside M. The iteration trees
- used to compare two models M and N, for example, require knowledge of both models and hence cannot be carried out inside of either.

## 5.2 Fine Structural Extender Models

- <sup>39</sup> Encouraged by the work of Martin and Steel, and prodded by work of Baldwin, Mitchell returned to work on inner models with extenders in 1987. Since he intended that the end result was a core model, he assumed from the start that the <sup>42</sup> models would include mice as well as extenders.
- <sup>12</sup> models would include mice as well as extenders.

One complication from the presence of mice came from the need for mixed iteration trees, that is, iterations involving several models. A comparison of two <sup>3</sup> models M and N by means of iterating the least difference will sometimes reach a stage where first difference between the models  $M_{\alpha}$  and  $N_{\alpha}$  lies in the mice they contain, rather than in their extenders. If, say, the mouse  $M_{\alpha}^*$  is in  $M_{\alpha}$  but <sup>6</sup> not in  $N_{\alpha}$ , then the tree could only continue by substituting  $M_{\alpha}^*$  for  $M_{\alpha}$ . Since there is no embedding from M into  $M_{\alpha}^*$ , the final argument needs to consider cases

depending on whether the branch through the final tree at some point drops to a 9 mouse  $M^*_{\alpha}$ .

The opposite situation is also possible. Recall that, as was seen previously in the case of the Dodd-Jensen lemma, mice can be used to ensure equal comparison only if the embedding does not move the projectum of the mouse. The presence of extenders presents another way in which this can fail: a mouse M can include a  $(\kappa, \lambda)$ -extender E with  $\rho^M < \lambda$ , so that E lies above the projectum  $\rho^M$  of M, but has critical point  $\kappa < \rho^M$  so that the embedding  $i^E$  moves the projectum. In order

- has critical point  $\kappa < \rho^M$  so that the embedding  $i^E$  moves the projectum. In order to avoid this problem, a comparison in which the initial model is a mouse uses a full class model to "back up" the iteration: if an extender with critical point less
- than the projectum of the mouse is called for, then it will be applied to the full class model instead of to the mouse. The criterion of equal comparison can then be satisfied in one of two ways. The well-founded branch will either originate at the
- <sup>21</sup> mouse or will, as a result of the backing up operation, originate at the model L[E]. If the well-founded branch of the iteration originates at the the mouse then equal comparison would be ensured by using the mouse as in previous models, while if the well-founded branch ends below the class model then equal comparison can be
- verified using universality or  $\phi$ -minimality arguments applied to this model. The other innovation was suggested by Baldwin and greatly simplified the pre-

<sup>27</sup> sentation of the model. Recall that Mitchell's original core model  $K(\mathcal{F})$  had the form  $L[\mathcal{F}, \mathcal{M}]$  where  $\mathcal{F}$  was the sequence of ultrafilters of the model and  $\mathcal{M}$  was a class of mice. This was awkward at the time, and promised to become more <sup>30</sup> so in the newer models, as the mice would be expected to mimic the full core

model and hence would need to themselves contain mice. In the new presentation the model had the form  $L[\mathcal{E}]$ , being constructed entirely from extenders  $\mathcal{E}_{\gamma}$  in the <sup>33</sup> sequence  $\mathcal{E}$ . However these extenders were not necessarily extenders on the full model  $L[\mathcal{E}]$ ; instead  $\mathcal{E}_{\gamma}$  was an extender on the model  $J_{\gamma}[\mathcal{E} \upharpoonright \gamma]$  constructed up to that point. The result was that the mice in the model  $L[\mathcal{E}]$  were just the initial

<sup>36</sup> segments  $M = J_{\alpha}[\mathcal{E}]$  of  $L[\mathcal{E}]$ .

30

As a consequence, the Condensation Property, Lemma 1, would (apart from some technical complications which were realized later) literally hold in these models: a sufficiently closed substructure of a model  $J_{\alpha}[\mathcal{E}]$  would be isomorphic to  $J_{\alpha'}[\mathcal{E}]$  for some  $\alpha' \leq \alpha$ .

Mitchell did not write any detailed notes on this work, and did not attempt to prove that there was an iterable model of this form. However he did present this work in a seminar at UCLA in 1988, after which Steel adapted the techniques from his work with Martin in [Martin and Steel, 1994]. The result, published in [Mitchell and Steel, 1994], proved that if there is a Woodin cardinal, then there is an inner model  $L[\mathcal{E}]$  which has a Woodin cardinal, satisfies GCH and  $\Diamond$ , and has a  $\Delta_4^1$  well-ordering of the reals.

Steel later extended this work in [Steel, 1993] to models with arbitrarily many Woodin cardinals, and Neeman [Neeman, 2002] has defined such extender models

- <sup>6</sup> as far as a Woodin limit of Woodin cardinals. This is the best result known giving, from the assumption of a large cardinal, an iterable extender model  $L[\mathcal{E}]$  for that (or even a smaller) cardinal. However if one is willing to add appropriate
- <sup>9</sup> iterability hypotheses to the large cardinal assumptions then this construction can be carried out as far as, and even somewhat beyond, a superstrong cardinal, that is, a cardinal  $\kappa$  such that there is an embedding  $i: V \to M$  with critical point  $\kappa$ <sup>2</sup> such that  $V_{i(\kappa)} \subseteq \kappa$ . Doing so relies on a modification, due to Jensen and using
- a suggestion of Sy Friedman, of the construction of  $L[\mathcal{E}]$ . Suppose that E is a  $(\kappa, \lambda)$ -extender to be added to the sequence  $\mathcal{E}$ , where E cannot be represented as
- an extender of length less than  $\lambda$ . The construction of [Mitchell and Steel, 1994] adds E as  $\mathcal{E}_{\gamma}$  where  $\gamma = \lambda^+$  in  $L[\mathcal{E} \upharpoonright \lambda^+]$ , because this value of  $\gamma$  was the least choice so that the mapping  $E \mapsto \gamma$  would be one-to-one. The Jensen construction sets
- $\gamma = i^{E}(\kappa^{+})$ , which avoids some anomalies in the construction but makes it more difficult to define a suitably coherent sequence  $\mathcal{E}$ . This approach was introduced in [Jensen, 1997], and a detailed exposition is given in the last part of the book [Zeman 2009]
- $_{21}$  [Zeman, 2002].

Most applications of the models  $L[\mathcal{E}]$  rely on core model techniques, which will be discussed in the next subsection; however one striking application can be described here. This history started with the development of Jensen's definition of the principle  $\Box_{\kappa}$  and the proof that it holds in L. This result had been extended a number of times to larger models: by Solovay to L[U], to  $K^{\text{DJ}}$  by Welch [Welch,

- <sup>27</sup> 1979], and for arbitrary sequences with measures of order 0 by Wylie [Wylie, 1989]. Schimmerling [Schimmerling, 1995] extended this to a model with  $\exists \kappa o(\kappa) = \kappa^{++}$ and proved weaker square principles for larger models. Finally Schimmerling and
- <sup>30</sup> Zeman, working first independently and then together, completed this program. Jensen had defined a *subcompact cardinal* to be a cardinal  $\kappa$  such that for each  $B \subseteq \kappa^+$  there is a  $\mu < \kappa$ ,  $A \subseteq \mu^+$  and an elementary embedding  $j: (H_{\mu^+}, \in, A) \to$
- <sup>33</sup>  $(H_{\kappa^+}, \in, B)$ . This is a property somewhat stronger than superstrong, but weaker then  $\kappa^+$ -supercompact, and Jensen had proved that  $\Box_{\kappa}$  fails for any subcompact cardinal. In [Schimmerling and Zeman, 2001; Schimmerling and Zeman, 2004],
- Schimmerling and Zeman proved a partial converse: that  $\Box_{\kappa}$  holds in any extender model  $L[\mathcal{E}]$  for any cardinal  $\kappa$  which is not subcompact.

## 5.3 $L[\mathcal{E}]$ as a core model

- <sup>39</sup> The models presented in [Mitchell and Steel, 1994] gave the form for a core model. Like the core models previously developed by Dodd and Jensen and by Mitchell, and unlike L[U], they were built up recursively from below using mice, and so could
- <sup>42</sup> present an apparently minimal model for any cardinal smaller than a Woodin car-

dinal. Furthermore all of the previously described core models could be presented as a model in the new form  $L[\mathcal{E}]$ . In the case of the Mitchell's core model for sequences of measures, and arguably even for the Dodd-Jensen core model  $K^{\text{DJ}}$ ,

- sequences of measures, and arguably even for the Dodd-Jensen core model  $K^{DJ}$ , doing so yielded a great gain in clarity. However, [Mitchell and Steel, 1994] did not present a core model construction: they relied on the Martin-Steel technique
- <sup>6</sup> for the proof of iterability, and that required starting with extenders in the real world, called background extenders. Indeed, the background extenders had to be stronger than the extenders in the model  $L[\mathcal{E}]$ . Thus, like the model L[U], they
- <sup>9</sup> would not give a model with a Woodin cardinal unless a Woodin cardinal was already known to exist in the universe. A core model construction, in contrast, should expose the latent large cardinal structure even if the large cardinal has <sup>12</sup> been destroyed by, for example, a forcing extension collapsing cardinals.

Before proceeding, it will be useful to compare the effect of Woodin cardinals in an inner model with that of measurable cardinals. The presence of measurable cardinals changed the theory in two major respects: (i) models were required to be iterable, rather than merely well-founded, and (ii) the covering lemma had to be adapted to allow for the possible presence of Prikry sequences.

- In the case of measurable cardinals, iterability can be ensured by using only countably complete measures. Furthermore, if M is a model with only measurable cardinals for which all countable iterations are well-founded, then M us iterable
- <sup>21</sup> for arbitrary iterations. Hence any model M containing  $\omega_1$  is iterable in V if and only if  $M \models$  "I am iterable" — indeed any iterated ultrapower in V can be embedded into an iterated ultrapower which is defined inside M.
- None of these observations are true for extenders. Two important conjectures assert that the first is true for countable iteration trees: If  $\mathcal{T}$  is a countable iteration tree, using only extenders which are countably complete in the model in which they
- <sup>27</sup> appear, then the *cofinal branch hypothesis* (CBH) asserts that  $\mathcal{T}$  has a cofinal branch with a well-founded limit, and the *unique branch hypothesis* (UBH) asserts that it has at most one such branch. Surprisingly, and fortunately,  $\omega_1$ +1-iterability, that is, iterability for trees of length at most  $\omega_1$ , is sufficient for much of inner
- model theory.

This increased complexity of the concept of iterability does not actually begin at a Woodin cardinal; in fact it is a consideration in any model with overlapping extenders. However it is mitigated, short of a Woodin cardinal, by the Martin-Steel result in [Martin and Steel, 1994] that any iteration tree with two well-founded

<sup>36</sup> branches of length  $\delta$  induces a model in which  $\delta$  is Woodin. This means that, except at a Woodin cardinal, the property of an iteration tree having a wellfounded branch is not changed by homogeneous forcing, such as a Levy collapse.

<sup>39</sup> Clause (ii) above, the effect of Prikry forcing on the covering lemma at a measurable cardinal, is paralleled by a much larger effect at a Woodin cardinal: as was pointed out earlier, the stationary tower forcing witnesses that a model with a Waadim cardinal & more barry tower forcing witnesses that a model with

<sup>42</sup> a Woodin cardinal  $\delta$  may not have even the weak covering property below  $\delta$ .

This goal of eliminating the need for background extenders in V was reached, or nearly reached, by Steel [Steel, 1996] in 1990. The key observation was that a

much weakened background extender could be used: an extender on a cardinal  $\kappa$  would only need background extenders for  $\kappa$ -sized restrictions of the extender, and

- $_{3}$  such background extenders could be obtained by assuming the presence of a single measurable cardinal in V. By adapting the recursive construction of [Mitchell and Steel, 1994] to this weaker notion of background extenders, he defined a model of
- <sup>6</sup> height Ω which he called  $K^c$ , corresponding to Mitchell's countably closed core model  $K^c$ ; in this case Steel took the superscript 'c' to mean "certified." He proved, assuming there is no inner model with a Woodin cardinal, that this model
- satisfies what he called a "cheapo" covering lemma: if  $\mu$  is the given measure on  $\kappa$ , then  $K^{c}$  is a model of height  $\kappa$  such that the set of cardinals  $\lambda < \kappa$  such that  $\lambda^{+} = (\lambda^{+})^{K^{c}}$  is a member of  $\mu$ . This version of the weak covering theorem implies that  $K^{c}$  is universal (as a model of height  $\Omega$ ), and enabled him to define the true
- <sup>12</sup> that  $K^c$  is universal (as a model of height  $\Omega$ ), and enabled him to define the true core model K as a elementary submodel of  $K^c$ . As with Mitchell's core model for sequences of measures, this proof gave an alternate direct definition of K, as
- <sup>15</sup> a model  $L[\mathcal{E}]$  where the members  $\mathcal{E}_{\gamma}$  of the sequence are chosen by induction on  $\gamma$ ; however the characterization of which extenders to add to the sequence so as to ensure the final model is iterable is somewhat more delicate than in the case of Mitchell's model.
- <sup>18</sup> Mitchell's model.

Among the applications given in [Steel, 2002], Steel showed that if there is a measurable cardinal  $\Omega$  then the existence of a model with a Woodin cardinal follows

- <sup>21</sup> from the existence of a presaturated ideal on  $\omega_1$ , and the existence of a tree on  $V_{\Omega}$  with two cofinal well-founded branches implies the existence of a model with two Woodin cardinals. In other applications, the assumption of a measurable cardinal
- $_{24}$  could be dispensed with because it was implied by the hypothesis, either in V or in an inner model. Thus the existence of a model with a Woodin cardinal follows from Martin's Maximum, from the assumption that every set of reals is weakly
- <sup>27</sup> homogeneous, or (using an unpublished result of Woodin) from the existence of a strongly compact cardinal. Steel also reproved Woodin's result that a model with a Woodin cardinal follows from  $\Delta_2^1$  determinacy plus the assumption that  $a^{\#}$
- exists for all reals a, and in 1993 he extended a result of Jensen by showing that if there are two measurable cardinals in V, and there no Woodin cardinals in  $K^c$ , then  $K^c$  is  $\Sigma_3^1$  correct.
- The results of [Steel, 1996] raised several obvious questions. Is the measurable cardinal really necessary? Does Steel's core model satisfy the weak covering property? Is there a core model for more than a single Woodin cardinal?
- The first question was still unanswered at the end of the century. Although it was easy to see that the full strength of a measurable cardinal was not necessary, it did not seem possible to fully eliminate the need for some large cardinal in V.
- <sup>39</sup> This has continued to be an active field of research during the first decade of the current century, and it seems likely that this problem has been solved by work of Jensen and Steel.
- <sup>42</sup> The second question was answered by Mitchell, Schimmerling and Steel [Mitchell et al., 1997], who showed that if there is no model with a Woodin cardinal, then  $(\kappa^+)^K = \kappa^+$  for all countably closed singular cardinals. The assumption of count-

able closure was eliminated shortly afterward by Mitchell and Schimmerling in [Mitchell and Schimmerling, 1995]. Another answer was given by Schimmerling

and Woodin in Schimmerling and Woodin, 2001: Suppose that  $L[\mathcal{E}]$  is an iterable 3 extender model with no measurable cardinals and no extenders of superstrong type. Then either a sharp exists for  $L[\mathcal{E}]$ , or else  $cf((\kappa^+)^{L[\mathcal{E}]}) \geq |\kappa|$  whenever  $|\kappa|$  is a

countably closed cardinal. Note that although  $L[\mathcal{E}]$  does not have any measurable 6 cardinals, it may have submodels  $L[\mathcal{E} \upharpoonright \gamma]$  with Woodin or stronger cardinals.

Steel tackled the third question, of getting larger core models, in [Steel, 2002]. He succeeded in defining a model  $K^{c}$  as in [Steel, 1996], having the 'cheapo' 9 covering lemma, so long as there is no model of a non-tame mouse, that is, no mouse having a Woodin cardinal  $\delta$  and also an extender  $\mathcal{E}_{\gamma}$  with  $\operatorname{crit}(\mathcal{E}_{\gamma}) < \delta \leq \gamma$ . This construction of  $K^{c}$  required only  $\omega_{1}$  + 1-iterability, that is, every suitable 12 iteration tree of length at most  $\omega_1$  has a branch. In order to obtain the model

K, however, he needed to show that the model was  $\Omega + 1$ -iterable, that is, that any iteration tree of height at most the measurable cardinal  $\Omega$  had a well-founded branch. He was successful enough in this to show that K exists, provided that there

is no model with infinitely many Woodin cardinals, and used this to show that the existence of an  $\Omega$ -iterable premouse of height  $\Omega$  with infinitely many Woodin 18 cardinals is equivalent to any of the following three statements: (i) the first-order theory of  $L(\mathbb{R})$  is unchanged by small forcing, (ii) the Axiom of Determinacy Is

true in  $L(\mathbb{R})$  after any small forcing, and (iii) no small forcing extension adds an uncountable sequence of distinct reals in  $L(\mathbb{R})$ . This was a result that had been reached independently, and slightly different methods, by Woodin. Steel also

showed in [Steel, 2002] that a failure of the Unique Branch Hypothesis for trees of 24 arbitrary length, that is, the existence of a nonoverlapping iteration tree  $\mathcal{T}$  with two distinct well-founded branches, implies that there is an inner model with  $\omega$ many Woodin cardinals. 27

One of the obstacles to extending the construction  $K^{c}$  beyond the restriction to tame mice in Steel, 2002 was removed by Neeman and Steel Neeman and Steel. 1999]. The proof of the Dodd-Jensen lemma, which is heavily used in establishing 30 the basic fine structure, depends on the fact that iteration trees have at most one well-founded branch, and this is not known to be true for the trees needed for larger models. In this paper Neeman and Steel gave a variant of the Dodd-Jensen 33 lemma, which they called the *weak Dodd-Jensen lemma*, which does not require unique branching but which is strong enough for the construction of  $K^{\rm c}$ . With this result, iterability is the only gap that remains to be filled in order to construct  $K^{\rm c}$  for any cardinal up to a superstrong cardinal. And retta, Neeman and Steel [Andretta *et al.*, 2001] used the weak Dodd-Jensen lemma to show that  $K^{c}$  exists provided that there is no non-domestic mouse, that is, that there is no sharp for 39

a model with a proper class of Woodin cardinals and a proper class of strong cardinals. This is currently the best result known on the existence of  $K^{\rm c}$ .

## 6 EPILOGUE

The end of the twentieth century is the cutoff date for this volume, but it is far from being a natural end point for the subject of this chapter. The idea of trying

- 3 to continue the exposition to the present, however, is countered by the observation that the present is unlikely to be a better ending point, and that stopping at the
- end of the century yields the benefit of a few years perspective. However it seems desirable to mention two recent advances which give at least partial answers to major questions raised in this history, and which promise to be major subjects of 9
- investigation in the future.

One of these questions concerns the elimination of the measurable cardinal which was needed by Steel in Steel, 1996 to obtain a core model up to a Woodin

- cardinal. After progress was made by several people, the problem appears to have 12 been solved by the use of a new technique of Jensen, which he calls stacks of mice. The technique was introduced in a paper by Jensen, Schimmerling, Schindler and
- Steel [Jensen et al., 2009], which applies the technique to obtain a model with a 15 nondomestic mouse from any of several hypotheses, including the failure of both  $\square_{\kappa}$  and  $\square(\kappa)$  at a countably closed cardinal  $\kappa \geq \omega_3$ . The construction of K up to
- a Woodin cardinal is due to Jensen and Steel, and is currently unpublished. 18 The other major advance is due to Woodin, who has discovered a framework which gives inner models for essentially every larger cardinal property which has
- been considered, including properties involving nontrivial embeddings  $i: V_{\lambda} \to V_{\lambda}$ . 21 He avoids a problem, which Steel had called *moving spaces* and which seemed to show that an extender model  $L[\mathcal{E}]$  could not contain cardinals larger than a
- supercompact cardinal, by defining a modified extender model  $L[\mathcal{E}]$  in which the 24 critical points of the extenders in the sequence  $\mathcal{E}$  are bounded, and then showing that the desired embeddings with larger critical point are constructed from  $\mathcal{E}$ . This
- work is as yet unpublished. 27

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