Iterating Forcing using Models as Side Conditions

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1. Aims

This work is designed as a first step towards answering some problems involving $I[\lambda^+]$.

I proved in [Mit07a]:

Theorem 1. *If there is a κ^+-Mahlo cardinal then there is a generic extension in which there is no stationary $S \subset \mathrm{Cof}(\omega_2)$ such that $S \in I[\omega_2]$.*

This, together with results of Shelah, raises several questions:

Question 2.

1. *Can we get, for any regular λ, a model in which $I[\lambda^+]\restriction \mathrm{Cof}(\lambda) \subseteq \mathrm{NS}_{\lambda^+}$?*
2. *Can we get, for singular λ, a model in which for each regular μ such that $\omega < \mu < \lambda$, a stationary set $S \subseteq \mathrm{Cof}(\mu)$ which is not in $I[\lambda^+]$.*
3. *Can the conclusion of Theorem 1 hold for two successive cardinals, say $I[\omega_2]$ and $I[\omega_3]$?*

I believe that I can answer item 1 affirmatively for λ of the form μ^+ where $\mu^{<\mu} = \mu$ and $2^\mu = \mu^+$. This talk discusses a step towards answering item 3 by showing that the use of models as side conditions can be applied simultaneously at ω_2 and ω_3. I expect that this will enable a full positive answer to item 3. This is likely to be a small, but necessary, step towards a positive answer to item 2, or to item 1 with λ a limit cardinal.

2. Models as side conditions

The conditions for the $I[\omega_2]$ model are finite sets which contain two types of "requirements":

1. Specifications towards a nested sequence of new club sets of the κ^+-Mahlo cardinal κ:

2. The "side conditions" — countable models $M \prec H_{\kappa^+}$.

The members of a condition p must satisify certain compatibility requirements. In order to simplify the discussion we will consider the simplified forcing with conditions containing only models, and with models $M \prec H_\kappa$ instead of $M \prec H_\kappa$. For this forcing we only need one compatibility condition: if p is a condition and $M, M' \in p$ then either $M \cap M' \in M$ or else $M \cap M'$ is an initial segment of M.

If κ is Mahlo then this forcing gives a model in which $\kappa = \omega_2$ and there are no special ω_2-Aronszajn trees; if κ is weakly compact then the resulting model has no ω_2-Aronszajn trees.

Definition 3. If $\mathbb{P}$ is a forcing and M a model, then a condition $p \in \mathbb{P}$ is *strongly $M, \mathbb{P}$-generic* if $p \Vdash$ "$\dot{G} \cap M$ is a V-generic subset of $\mathbb{P} \cap M$."

Equivalently: there is a function $r \mapsto r|M$ such that if $r \leq p$ then $r|M \in \mathbb{P} \cap M$, and every $q \leq r$ in $\mathbb{P} \cap M$ is compatible with r. The condition p is said to be *tidily* strongly $M, \mathbb{P}$-generic if $\mathbb{P}$ has meets and the function $r \mapsto r|M$ satisifies $(r \wedge r')|M = r|M \wedge r'|M$ whenever $r \leq p$, $r' \leq p$, and $r \wedge r \neq \mathbf{0}$.

The forcing under consideration satisfies:

Lemma 4. *1. If $M \in p$ then p is tidily strongly M-generic.*

2. If $H_\delta \prec H_\kappa$ and δ is inaccessible then any condition p is tidily strongly H_δ-generic.

The first statement implies that $\mathbb{P}$ is proper, and hence preserves ω_1. The two together imply that the forcing has the following property: If δ is as in item 2, and $x \in V[G]$ is a subset of δ such that $x \cap \eta \in V[G \cap H_\delta]$ for all $\eta < \delta$, then $x \in V[G \cap H_\delta]$.

3. "Iterating" this forcing

The forcing $\mathbb{P}_2$ can readily be extended to any cardinal $\lambda^+ > \omega_2$ where $\lambda = \mu^+$ for a regular cardinal μ such that $\mu^{<\mu} = \mu$ and $2^\mu = \mu^+ = \lambda$. The forcing in this case uses as conditions sets of size less than μ containing models $M \prec H_\kappa$ of size μ with $\mu \subset M$. We would like to do this for two consecutive cardinals. Let $\kappa_2 < \kappa_3$ be Mahlo cardinals; we will discuss a forcing $\mathbb{P}_2 \star \mathbb{P}_3$ which makes $\kappa_2 \mapsto \omega_2$ and $\kappa_2 \mapsto \omega_3$ and which has the effect of performing the forcing above for both κ_2 and κ_3.

Note that a straightforward iterated forcing $\mathbb{P}_2 * \dot{\mathbb{P}}_3$ does not work — forcing with $\mathbb{P}_2$ makes $2^\omega = \omega_2$, while $\mathbb{P}_3$ requires $2^\omega = \omega_1$. Also, $\mathbb{P}_2 \times \mathbb{P}_3$ almost certainly does not work, as $\mathbb{P}_3$ will undo the work of $\mathbb{P}_2$.

The forcing $\mathbb{P}_2$ will be as described: Let $\mathcal{M}_2$ be the set of countable $M \prec H_{\kappa_2}$. Then $\mathbb{P}_2$ is the set of finite pairwise compatible subsets of $\mathcal{M}_2$.

The set $\mathcal{M}_3$ of models for $\mathbb{P}_3$ is the set of models $M \prec H_{\kappa_3}$ of size less than κ_2 such that $\omega_1 \subset M$ and ${}^{<\delta^M}M \subset M$, where $\delta^M = \kappa_2 \cap M \in \kappa_2$. Notice that if G_2 is a generic subset of $\mathbb{P}_2$, then $M[G_2 \cap M] \prec H_{\kappa_3}[G_2]$ will be a member of $\mathcal{M}_3^{V[G_2]}$, which would be the set of models we would use if were were trying to form $\mathbb{P}_2 * \dot{\mathbb{P}}_3$.

The combined forcing $\mathbb{P}_2 \star \mathbb{P}_3$ will have conditions of the form $(p, (\dot{q}, X))$ where

1. $p \in \mathbb{P}_2$.
2. $\dot{q}$ is a $\mathbb{P}_2$-term for a countable subset of $\mathcal{M}_3$.
3. X can be ignored for now (and until X is introduced we write conditions in $\mathbb{P}_2 \star \mathbb{P}_3$ as $(p, \dot{q})$

Aside from the properties of $\mathbb{P}_2$, we have four important properties which $\mathbb{P}_2 \star \mathbb{P}_3$ should satisfy:

1. Strong genericity — if $M \in \mathcal{M}_3$ and $p \Vdash M \in \dot{q}$ then $(p, \dot{q})$ forces that $\dot{G} \cap M$ is a V-generic subset of $(\mathbb{P}_2 \star \mathbb{P}_3) \cap M$. (Strong genericity for H_δ, with δ inaccessibile and $\kappa_2 < \delta < \kappa_3$, is straightforward. Of course $\mathbb{P}_2 \star \mathbb{P}_3$ will not have strongly M-generic conditions for $M \in \mathcal{M}_2$.)
2. $\mathbb{P}_3$ is countably closed: if $(p, \dot{q}_1) \geq (p, \dot{q}_2) \geq \dots$ then $\dot{q} = \bigcup_n \dot{q}_n$ is a condition and $(p, \dot{q}_n) \geq (p, \dot{q})$ for each n.
3. If $(p', \dot{q}') \leq (p, \dot{q})$ then there is $\dot{q}''$ so that $(p, \dot{q}'') \leq (p, \dot{q})$ and $(p', \dot{q}'') \leq (p', \dot{q}')$.
4. Suppose that $\delta < \kappa_2$ is inaccessible and $H_\delta \prec H_{\kappa_2}$. Further suppose that $\dot{q}_0$ and $\dot{q}_1$ are terms such that $(p, \dot{q}_i) \leq (p, \dot{q})$ for $i = 0, 1$, and that p forces that $\dot{q}_0|M = \dot{q}_1|M$ for all $M \in \dot{q}$ with $\delta^M \leq \delta$. Then there are terms $\dot{q}''$ and $p_0, p_1 \leq p$ in $\mathbb{P}_2$ such that (i) $p_0|M = p_1|M$, (ii) $(p, \dot{q}'') \leq (p, \dot{q})$, and (iii) $(p_i, \dot{q}'') \leq (p, \dot{q}_i)$.

[1]

[1] Do I actally want this? — Write $\mathrm{Mrest}_\delta(\dot{q}) = \bigcup\{\, M' : \delta^{M'} \leq \delta \;\;\&\;\; \exists p' \in G_2 \;\; p \Vdash M' \in \dot{q} \,\}$.

The next definition is needed so that we have any hope of satisfying the first property.

Definition 5. A model M *captures a term* $\dot{q}$ *below* $p \in \mathbb{P}_2$ if $p \Vdash M \in \dot{q}$ and there is a $\mathbb{P}_2 \cap M$-term $\dot{\tau}$ such that $p \Vdash \dot{q} = \dot{\tau}$.

Equivalently, M captures $\dot{q}$ below p if $p \Vdash M \in \dot{q}$ and for all $r \leq p$ and $M' \in M$, $r \Vdash M' \in \dot{q}$ if and only if $r|M \wedge p \Vdash M' \in \dot{q}$.

We require of a condition $(p, \dot{q})$ that it satisfy *dense capturing* — for each model $M \in \mathcal{M}_3$, there is a dense set of conditions $r \leq p$ such that either $r \Vdash M \notin \dot{q}$ or else M captures $\dot{q}$ below r.

Dense capturing will not be preserved by countable unions, so in order to satisfy the second property the definition of ordering will require that *capture is preserved* — if $(p', \dot{q}') \leq (p, \dot{q})$ and M captures $\dot{q}$ below some $r \leq p'$ then M also captures $\dot{q}'$ below r.

The third property is needed to make use of the countable closure of $\mathbb{P}_3$. In particular it is used to show that $\mathbb{P}_2 \star \mathbb{P}_3$ is proper: suppose that $M \prec H_\tau$ is a countable model with τ large enough that $\mathbb{P}_2 \star \mathbb{P}_3 \in M$. Then $M \cap H_{\kappa_2} \in \mathcal{M}_2$; Let $(p, \dot{q})$ be a condition with $M \cap H_{\kappa_2} \in p$, and $\dot{q} \in M$. Now use property 2 to find a condition $\dot{q}' \leq q$ so that $\{ \dot{q}'' \in M : \dot{q}'' \leq \dot{q}' \}$ is M-generic for the partial order defined by $(\dot{q}_0 \leq \dot{q}_1$ if $(p|M, \dot{q}_0) \leq (p|M, \dot{q}_1)$. The third property then implies that $(p, \dot{q}')$ is a $M, \mathbb{P}_2 \star \mathbb{P}_3$-generic condition.

The preservation of capturing is not enough to make the third condition true. The next definition fills this gap:

Definition 6. We say that a triple (M, p, p') is a $\dot{q}$-*linkage* if $p_2|M = p_1|M$, M captures $\dot{q}$ below either p and p', and the witnessing term $\dot{\tau}$ is the same in each case.

As a special case, a triple (M, p, p) is a $\dot{q}$-linkage if and only if M captures $\dot{q}$ below p.

Other linkages arise when a model $M \in \mathcal{M}_3$ captures $\dot{q}$ below both p and p', where p and p' are conditions such that $p|M = p'|M$ and $p \wedge p' \neq \mathbf{0}$. If the captures are witnessed by $\dot{\tau}$ and $\dot{\tau}'$, respectively, then we must have that $p|M \Vdash \dot{\tau} = \dot{\tau}'$. Further linkages can be added to the ends to create longer chains.

This is a problem in the following sort of situation: Suppose that $(p', \dot{q}') \leq (p, \dot{q})$ and that p_0, p_1 and p_2 are conditions below p such that $p_i|M$ is the same for all, and M captures $\dot{q}$ below any p_i. Suppose further that the four adjacent pairs of conditions in the chain (p', p_0, p_1, p_2, p') are compatible, and the other two pairs are incompatible. Then $(M, p' \cap p_0, p' \cap p_2)$ is a $\dot{q}$ linkage,

and since it is implies by captures of $\dot{q}$, preservation of capture implies that it must be a $\dot{q}''$ linkage for any $(p, \dot{q}'') \leq (p, \dot{q})$. However preservation of capture below p, as would be required for the extension $(p', \dot{q}') \leq (p, \dot{q})$ in the third requirement, is not enough to insure that it is a $\dot{q}'$-linkage. Thus we require that linkages, as well as captures, are preserved when $(p', \dot{q}') \leq (p, \dot{q})$.

This completes the description of the forcing, except for one problem: the forcing, as described so far, does not have meets. Suppose $\dot{q}_0$ and $\dot{q}_1$ are two compatible terms, and assume for simplicity that $\dot{q}_0 \cup \dot{q}_1$ preserves all links holding in either $\dot{q}_0$ or $\dot{q}_1$. If the forcing has meets, then $(\emptyset, \dot{q}_0) \wedge (\emptyset, \dot{q}_1)$ must be equal to $(\emptyset, \dot{q}_0 \cup \dot{q}_1)$, since any $\dot{a}'$ extending both terms must also contain $\dot{q}_0 \cup \dot{q}_1$. However it is possible to choose $\dot{q}_0$ and $\dot{q}_1$ so that there is a $\dot{q}_0 \cup \dot{q}_1$-linkage ℓ which is not a $\dot{q}_0$-linkage or a $\dot{q}_1$-linkage. In this case we can find extensions $\dot{q}'_i \supset \dot{q}_i$ for $i = 0, 1$ which violate the linkage ℓ in such a way that ℓ is not a $\dot{q}'$-linkage for any $\dot{q}' \supseteq \dot{q}'_0 \cup \dot{q}'_1$. Then $(\emptyset, \dot{q}'_0 \cup \dot{q}'_1)$ is a common extension of $(\emptyset, \dot{q}_0)$ and $(\emptyset, \dot{q}_1)$ which is not compatible with $(\emptyset, \dot{q}_0 \cup \dot{q}_1)$.

The coordinate X in a condition $(p, (\dot{q}, X))$, which has been ignored to this point, avoids this problem. It is a countable set of $\dot{q}$-linkages which is large enough to witness dense capturing, and which is closed under the chains of linkages described earlier. Here "witnesses dense capturing" means that for any $p' < p$ and any $M \in \mathcal{M}_3$ such that $p' \Vdash M \in \dot{q}$ there is a capture—that is a linkage (M, r, r)—in X such that $r \wedge p' \neq \mathbf{0}$.

Now the ordering on $\mathbb{P}_2 \star \mathbb{P}_3$ is defined by $(p', (\dot{q}', X')) \leq (p, (\dot{q}, X))$ if $p' \leq p$, $p \Vdash \dot{q}' \supseteq \dot{q}$, and $X' \supseteq X$. Since each member of X' is a $\dot{q}'$-linkage, this ensures that the linkages in X are preserved by the extension.

To be more explicit about the use of the set X, we make another definition.

Definition 7. We define the *link closure* $\mathcal{L}_X(\dot{q})$ to be the smallest term $\dot{q}' \supseteq \dot{q}$ such that X is a set of $\dot{q}'$-links.

In other words, whenever $(M, r, r') \in X$ and $(p, M') \in \dot{q}$ for some $M' \in \mathcal{M}_3 \cap M$ and some condition p such that $p \wedge r \neq \mathbf{0}$, then $((p \wedge r) | M \wedge r', M') \in \dot{q}'$.

In general, the members of the set denoted by the term $\dot{q}'$ need not be compatible, but we do have the following observation:

Proposition 8. *Suppose that $\dot{q}'' \supseteq \dot{q}$ is a term denoting a compatible set such that X is a set of $\dot{q}''$-linkages. Then $\mathcal{L}_X(\dot{q})$ is a term denoting a set of pairwise compatible models, and $\dot{q}'' \supseteq \mathcal{L}_X(\dot{q}) \supseteq \dot{q}$.*

Thus the meet of two compatible terms $(p_0, (\dot{q}_0, X_0))$ and $(p_1, (\dot{q}_1, X_1))$ is given by $(p_0 \wedge p_1, (\mathcal{L}_{X_0 \cup X_1}(\dot{q}_0 \cup \dot{q}_1), X_0 \cup X_1))$.

The link closure is also used for the proof of the last two of the four properties. For the third property, suppose that $(p', (\dot{q}', X')) \leq (p, (\dot{q}, X))$. We can assume that p is a support of $\dot{q}'$ in the sense that $r \leq p'$ whenever there is M such that $r \Vdash M \in \dot{q}'$. Then the required term $\dot{q}''$ is $\mathcal{L}_X(\dot{q}') \cup \dot{q}$. An easy argument shows that $\dot{q}'' = \mathcal{L}_{X \cup X'}(\dot{q} \cup \dot{q}')$ and that $\dot{q}''$ is as required.

Similarly, for the fourth property we are given $(p, \dot{q})$ and $(p, \dot{q}_i)$ for $i = 0, 1$. We use the fact that $\mathbb{P}_2$ is a member of any $M \in \mathcal{M}_3$ to show that there are incompatible $p_i \leq p$ for $i = 0, 1$ such that $p_i \setminus p$ is a member of any model $M' \in \dot{q}$ with $\delta^{M'} > \delta$. Now let q'_i be q_i restricted to p_i, that is, $p_i \Vdash \dot{q}'_i = \dot{q}_i$ and $r \Vdash \dot{q}'_i = \emptyset$ whenever $r \wedge p_i = \mathbf{0}$. Then it can be shown that the hypotheses imply that $\dot{q}'' = \dot{q} \cup \mathcal{L}_{X_0}(\dot{q}'_0) \cup \mathcal{L}_{X_1}(\dot{q}'_1)$ is as required.[2]

Conditions 3 and 4 are based on the hypothesis of the main result of [Mit07b]. They are used in order to show that if $M \in \mathcal{M}_3$ and $p \Vdash M \in \dot{q}$, and if x is a subset of $\delta = \delta^M$ in the extension $V[G_2 \star G_3]$ such that every initial part of x is in $V[(G_2 \star G_3) \cap M]$, then $x \in V[(G_2 \star G_3) \cap \mathrm{Mrest}_\delta(\dot{q})]$.

If the main result of [Mit07b] held, as stated in that paper, for this forcing then we would have the conclusion $x \in V[(G_2 \star G_3) \cap M]$ instead of $x \in V[(G_2 \star G_3) \cap \mathrm{Mrest}_\delta(\dot{q})]$. I don't know whether the stronger conclusion holds, and I also don't know of any way to use the weaker conclusion to show that there are no Aronszajn trees on κ_2, as Uri Abraham does in [Abr83] starting with the assumption that κ_3 is supercompact. However the proof that δ is not collapsed by $(\mathbb{P}_2 \star \mathbb{P}_3) \cap M$ does seem to extend to show that it is not collapsed by $(\mathbb{P}_2 \star \mathbb{P}_3) \cap \mathrm{Mrest}$. This is sufficient to get a model with no special ω_2- or ω_3-Aronszajn trees, and will probably be sufficient to prove that the forcing for the $I[\omega_2]$ and $I[\omega_3]$ property has the desired properties.

References

[Abr83] Uri Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Ann. Pure Appl. Logic*, 24(3):213–230, 1983.

[2] For the more general forcing in which models in $\mathcal{M}_2$ are $M \prec H_{\kappa_2^+}$ this leads to a problem, which can be solved by having $\mathcal{M}_2$ have as models elementary substructures of both H_{κ_2} and $H_{\kappa_2^+}$. I've noticed that this is probably also needed with $\mathcal{M}_3$ for other reasons. In this case we would require $M \cap H_{\kappa_2} \dot{:} q$ whenever $M \prec H_{\kappa_2^+} \in \dot{q}$. (Actually the same effect can probably be reached without the extra models.

[Mit07a] William J. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. submitted to *Transactions of the American Mathematical Society*, `arXiv:math.LO/0407225`, 2007?

[Mit07b] William J. Mitchell. On the hamkins approximation property. to appear in the Proceedings of the *Baumgartner Festival*, held 2003 at Dartmouth University, in the *Annals of Pure and Applied Logic*, 2007?