Something Simple

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July 11, 2005

Let κ be inaccessible. Define the forcing $\mathbb{P}$ as follows:

- A condition is a finite set p of countable models $M \prec H_\kappa$ with the compatibility condition: if $M, N \in p$ then either
  - $M \cap N \in M$, [And $M \cap N \in p$].
  - $M \cap N = H_\theta$ for some $\theta \in M$ of uncountable cofinality,
  - or $M \subseteq N$.
- $p \leq q$ if $p \supseteq q$.

Lemma. $\mathbb{P}$ *collapses all cardinals between* ω_1 *and* κ.

Definition. If $\mathbb{Q}$ is a forcing notion, and M is a model, then a *strongly* $M, \mathbb{Q}$*-generic condition* is a condition $q \in \mathbb{Q}$ such that

$$q \Vdash \dot{G} \cap M \text{ is } V\text{-generic.}$$

Equivalently, q is strongly M-generic if there is a function $q' \mapsto q'|M$ for all $q' \leq q$ such that every condition $r \leq q'|M$ in $M \cap \mathbb{Q}$ is compatible with q'.

Lemma. *In the forcing $\mathbb{P}$,*

- *If $M \prec H_\kappa$ is countable then $\{M\}$ is strongly M-generic.*
- *If $\theta < \kappa$ and $\mathrm{cf}(\theta) > \kappa$ then $\emptyset$ is strongly H_θ-generic.*

For either $X = M$ or $X = H_\theta$, the witnessing function $p \mapsto p|X$ is

$$p|X = \{N \cap X : N \in p \ \& \ N \cap X \in X\}$$

By the same argument as for proper forcing, this implies that $\mathbb{P}$ is ω_1-presaturated and κ-presaturated. Hence these cardinals (and all larger cardinals) are preserved.

Thus $\kappa = \omega_2^{V[G]}$.

No ω_2-Aronszajn Trees

- Assume to the contrary $\dot{T}$ is a name for a $\kappa = \omega_2$-Aronszajn tree T.
- By weak compactness, pick $H_\theta \prec_{\Pi^1_1} (H_\kappa, \dot{T})$ with θ inaccessible.
- Then $G \cap H_\theta$ is a generic subset of $\mathbb{P} \cap H_\theta$, $T \restriction \theta \in V[G \cap H_\theta]$, and there are no branches of $T \restriction \theta$ in $V[G \cap H_\theta]$.
- There is a branch b in $V[G]$; let $\dot{b}$ be a name for this branch. Ever initial segment (and hence every countable restriction) of b is in $V[G \cap H_\theta]$.
- We will show that $b \in V[G \cap H_\theta]$ — **Contradiction!**

We need to show that $\dot{b}$ is really a $P \cap H_\theta$-term.

Suppose the contrary: then there is a dense set of conditions r such that for some $r_1, r_2 \leq r$ and $\xi < \theta$ and x we have

- $r_2 | H_\theta \leq r_1 | H_\theta$
- $r_1 \Vdash \dot{b}(\xi) = \check{x}$ and $r_2 \Vdash \dot{b}(\xi) \neq \check{x}$

- Pick a countable $M \prec H_\kappa$ such that everything relevant is a member of M, and let q be strongly M-generic.
- Since M is countable, $b \restriction M \in V[H_\theta \cap G]$.
  Let $q \Vdash \dot{b} \restriction M = \dot{\sigma}$ for some $\mathbb{P} \cap H_\theta$-term $\dot{\sigma}$.
- Working in M and using the last observation, find $r_1, r_2 \leq q|M$ in $\mathbb{P} \cap M$ forcing conflicting information about $\dot{b}$.
- Since $q \Vdash \dot{b} \restriction M = \dot{\sigma}$, the conditions $r_1 \wedge q$ and $r_2 \wedge q$ force the same conflicting information about $\dot{\sigma}$.
- Since $\dot{\sigma}$ is a $\mathbb{P} \cap H_\theta$-term, the conditions $(r_1 \wedge q)|H_\theta$ and $(r_2 \wedge q)|H_\theta$ force the same conflicting information about $\dot{\sigma}$.
- But $(r_2 \wedge q)|H_\theta = r_2|H_\theta \wedge q|H_\theta = r_1|H_\theta \wedge q|H_\theta = (r_1 \wedge q)|H_\theta$.

Contradiction!

More cardinals?

There are two possible approaches to a model with no ω_2 or ω_3 trees.

- Do it all at once with finite conditions.
- Do it in two steps.

Suppose that we have $\kappa < \lambda$ and want them to become ω_2 and ω_3.

- The first alternative would use as conditions finite sets containing models $M \prec H_\lambda$ such that M is either countable or of size less than κ, with $M \cap \kappa$ transitive.
- The compatibility conditions would be complicted.
- The proof would involve a long and somewhat doubtful case analysis.
- It would almost certainly not work for three cardinals.
- Nevertheless I will mention on Thursday an application where this construction might possibly be useful.

Doing it in two steps

First, note that if κ is regular and $\lambda > \kappa$ is inaccessible then (assuming GCH) we can use the same construction to get a model with no κ^{++}-Aronszajn trees.

The conditions are sets of size less than κ containing models $M \prec H_\lambda$ of size κ with $M \cap \kappa$ transitive.

The condition for compatibility of these models is the same as before.

Theorem (Abraham, 1980). *Assume that κ is λ-supercompact and λ is weakly compact. Then there is a generic extension in which there are no Aronszajn trees on ω_2 or ω_3.*

Like Uri Abraham's 1980 proof, we try an iteration of the forcings, but need to modify it.

We will write the forcing for our new proof as $\mathbb{P} \star \tilde{\mathbb{Q}}$.

A member of the forcing is a pair (p, q) with $p \in \mathbb{P}$ and $q \in \tilde{\mathbb{Q}}$.

- $\mathbb{P}$ is the same forcing as before, giving no ω_2-Aronszajn trees.
- $\tilde{\mathbb{Q}}$ is essentially a subset of the terms for members of the same forcing for no ω_3-Aronszajn trees, as defined in $V^{\mathbb{P}}$.

More precisely, $q \in \tilde{\mathbb{Q}}$ is a $\mathbb{P}$-term $\dot{a}$ such that

1. $\Vdash \dot{a}$ is a countable collection of sets $X \in V$ such that $X \prec H_\lambda$, $|X| < \kappa$, and $\theta_X = X \cap \kappa$ is an inaccessible cardinal smaller than κ.
2. If $p \Vdash X, Y \in \dot{a}$ then
   (a) If $\theta_Y \geq \theta_X$ then $X \cap Y = X \cap H_\tau$ for some inaccessible τ in X$\cup\{\lambda\}$.
   (b) If $\theta_Y < \theta_X$ then $X \cap Y \in X$ and $p|\theta_X \Vdash Y \in \dot{a}$.
3. $\dot{a}$ has countable support: there is a countable set $m \in V$ so that $X \in \dot{a}^G$ if and only if there is $p \in m \cap G$ such that $p \Vdash X \in \dot{a}$.

The ordering is defined by $(p', \dot{a}') \leq (p, \dot{a})$ if and only if $p' \leq p$ and $p' \Vdash \dot{a}' \subset \dot{a}$.