$I[\omega_2]$ and Finite Forcing

Bill Mitchell
University of Florida

Institute for Mathematical Sciences
National University of Singapore
July 14, 2005

Definition 1. If $A = \{a_\xi : \xi < \kappa\}$ then an ordinal $\nu < \kappa$ is *approchable via* A if there is $c \subset \nu$ such that $\nu = \bigcup c$, $\mathrm{otp}(c) = \mathrm{cf}(\nu)$, and $c \cap \beta \in \{a_\xi : \xi < \nu\}$ for all $\beta < \nu$.

A set S is approachable if there is a sequence A such that the set of ordinals $\nu \in B$ which are not approachable via A is nonstationary.

Theorem 2. *It is equiconsistent with the existence of a κ^+-Mahlo cardinal that every approachable subset of* $\mathrm{Cof}(\omega_1)$ *is nonstationary.*

A κ^+-Mahlo cardinal is needed

Define B_α to be the set of cardinals $\lambda < \kappa$ which are $f_\alpha(\lambda)$-Mahlo in L, where $[f_\alpha]_{\mathrm{NS}} = \alpha$.

Suppose that $\kappa = \omega_2$ is not κ^+-Mahlo in L, and let α be least such that $\mathrm{Cof}(\omega_1) \setminus B_\alpha$ is stationary. (Assume that $\alpha = \gamma + 1$.)

Then pick a club $D \subset B_\gamma \cup \mathrm{Cof}(\omega)$, and for each $\nu \in \mathrm{Cof}(\omega_1) \setminus B_\alpha$ pick a club $E_\nu \subset \nu \setminus B_\gamma$ with $E_\nu \in L$.

For $\nu \in \lim(D) \cap (\mathrm{Cof}(\omega_1) \setminus B_\alpha)$ set $c_\nu = D \cap E_\nu$.

Now $D \cap E_\nu \cap \mathrm{Cof}(\omega_1) = \emptyset$, so c_ν contains no members of cofinality ω_1. It is unbounded in ν, so $\mathrm{otp}(c_\nu) = \omega_1$.

Each initial segment $c_\nu \cap \beta$ is in $L[D \cap \beta]$.

- Hence every member of $\lim(D) \cap (\mathrm{Cof}(\omega_1) \setminus B_\alpha)$ is approachable.

The Strategy

- Add a club subset of $B^*_\alpha = B_\alpha \cup \mathrm{Cof}(\omega)$ for each $\alpha < \kappa^+$.
- Work gently, so as not to add other sets of approachable ordinals.

Adding one club set $D \subset B^*$

Assume $\lambda \in B$ implies $H_\lambda \prec H_\kappa$ and λ is inaccessible. We define a forcing $\mathbb{P}$ to add a closed unbounded subset of B^*.

The conditions are finite sets, which may contain as members:

1. Ordinals $\lambda \in B^*$
   - which means that $\lambda \in D$.
2. Half-open intervals $(\eta', \eta]$, with $\eta' < \eta < \kappa$
   - which means that $(\eta', \eta] \cap D = \emptyset$.
3. Countable models $M \prec H_\kappa$ like those in the talk Monday
   - which makes M is strongly M-generic.

The members of a condition must satisfy some compatability conditions. . .

1. If $\{\lambda, (\eta', \eta]\} \subset p$ then $\lambda \notin (\eta', \eta]$.

2. If $\{\lambda, M\} \subset p$ and $\lambda < \sup(M)$ then $\min(M \setminus \lambda) \in B^*$.

3. If $\{(\eta', \eta], M\} \subset p$ then either $(\eta', \eta] \in M$ or $(\eta', \eta] \cap M = \varnothing$.

4. If $\{M, N\} \subset p$ then

   (a) Either $M \cap N \in M$ or $M \cap N = M \cap H_\tau$ for some $\tau \in (B \cap M) \cup \{\kappa\}$.

   (b) If $\lambda < \sup(M \cap N)$, $\lambda \in M$, and $\sup(N \cap \lambda) < \sup(M \cap \lambda)$ then $\lambda \in B$. Furthermore there are only finitely many such ordinals λ.

Lemma 3. *If p is any condition then*

- *The set p' obtained by adding to p all the ordinals which were required by the previous slide to be in B is still a condition.*
- *As is the set p'' obtained by adding to p' the ordinals* $\sup(M)$ *for $M \in p$ and* $\sup(M \cap \lambda)$ *for each $M, \lambda \in p'$.*
- *Furthermore, if $\lambda \notin p''$ then there is $(\eta', \eta]$ with $\eta' < \lambda \leq \eta$ so that $p'' \cup \{(\eta', \eta]\}$ is a condition.*

Corollary 4. *If $G \subset \mathbb{P}$ is generic then the set*

$$D = \{\lambda < \kappa : \exists p \in G \ \lambda \in p\}$$

is a closed and unbounded subset of B^*.

Lemma 5. *1. If $\tau \in B$ then the condition $\{\tau\}$ is strongly H_τ-generic.*

2. The condition $\{M\}$ is strongly M-generic.

- *For (1), we set $p|H_\tau = p \cap H_\tau \cup \{N \cap H_\tau : N \in p\}$.*
- *For (2), we set*

$$
\begin{aligned}
p|M = p \cap M \\
&\cup \{N \cap M : N \in p \ \& \ M \cap N \in M\} \\
&\cup \{\lambda : \lambda \in B \text{ is needed for compatibility of } M \\
&\qquad \text{with some other requirement in } p\}
\end{aligned}
$$

Corollary 6. *The cardinals ω_1, κ, and all larger cardinals are preserved, and κ becomes ω_2 in the generic extension.*

Lemma 7. *If $G \subset \mathbb{P}$ is generic then*

1. *$\kappa \setminus B^*$ is nonstationary in $V[G]$.*
2. *$\{\lambda \in B : B \cap \lambda$ is nonstationary$\}$ is approachable in $V[G]$.*
3. *Any stationary subset of B in the ground model remains stationary in $V[G]$.*
4. *No stationary subset of $\{\lambda \in B : B \cap \lambda$ is stationary$\}$ is approachable in $V[G]$.*

Adding a club subset of *each* set B^*_α

Bookkeeping

We assume $\square_\kappa$, and use it with a minimal walk construction to define sets $A_{\alpha,\lambda}$ for $\alpha < \kappa^+$ and $\lambda < \kappa$ so...

- $A_{\alpha,\lambda} \subset \alpha$ and $|A_{\alpha,\lambda}| = |\lambda|$.
- If $\lambda < \lambda'$ then $A_{\alpha,\lambda} \subseteq A_{\alpha,\lambda'}$.
- If λ is a limit ordinal then $A_{\alpha,\lambda} = \bigcup_{\lambda'<\lambda} A_{\alpha,\lambda'}$.

- If $\alpha' \in A_{\alpha,\lambda} \cup \lim(A_{\alpha,\lambda})$ then $A_{\alpha',\lambda} = A_{\alpha,\lambda} \cap \alpha'$.

The first use of these sets $A_{\alpha,\lambda}$ is for the definition of canonical functions f_α such that $[f_\alpha]_{\mathrm{NS}} = \alpha$.

$$f_\alpha(\lambda) = \mathrm{otp}(A_{\alpha,\lambda})$$

for all $\alpha < \kappa^+$ and $\lambda < \kappa$.

These will be used, for example

- In the definition of $B_\alpha = \{\lambda < \kappa : \lambda \text{ is } f_\alpha(\lambda)\text{-Mahlo}\}$.
- If $\alpha' < \alpha$ and $\alpha' \in A_{\alpha,\lambda}$ then $D_\alpha \setminus \lambda \subset D_{\alpha'}$.
- If α is a limit ordinal then $D_\alpha = \{\lambda < \kappa : \forall \alpha' \in A_{\alpha,\lambda}\ \lambda \in D_{\alpha'}\}$.

- And, crucially, to keep the countable models straight...

We'll call M a countable model if M is countable, $M \prec (H_{\kappa^+}, \vec{C})$ (where $\vec{C}$ is the $\square_\kappa$ sequence) and

$$\lim(C_\alpha) \cap M \text{ is cofinal in } \alpha = \sup(M).$$

Note that this implies that $M \cap \kappa^+$ is determined by $M \cap \kappa$ and $\sup(M)$.

The sets $A_{\alpha,\lambda}$ determine functions $\pi_\alpha \colon \kappa \to \alpha$. Because of the coherence property of $A_{\alpha,\lambda}$ the sets M will satisfy

$$M \cap \kappa^+ = \pi_\alpha``(M \cap \kappa).$$

The conditions in this forcing (in the first approximation) are finite sets, which may contain as members:

1. Pairs $(\alpha, \lambda) \in B^*$
   - which means that $\lambda \in D_\alpha$.
2. Pairs $(\alpha, (\eta', \eta])$, with $\eta' < \eta < \kappa$
   - which means that $(\eta', \eta] \cap D_\alpha = \emptyset$.
3. Countable models $M \prec H_{\kappa^+}$
   - which (roughly) act like the set $M \cap H_\kappa$ in the previous forcing, for each $\alpha \in M \cap \kappa^+$.

Compatibility of members of a condition is worked out similarly to the one club forcing, keeping in mind that the club sets D_α are supposed to be continuously diagonally decreasing.

It turns out that the third component of the conditions is too simple: the models M are replaced with pairs (M, b) where b is a finite sequence of ordinals I call "proxies".

This leads to a serious new technical complication in the proof of strong genericity for the models M. Other than that the arguments are similar to those of the one club forcing.

In particular, the argument use for lemma 10 shows that for any sequence $A = \langle a_\xi : \xi < \kappa \rangle$ in the extension there is an α and a club set $E \in V$ so that no member of $D_\alpha \cap E \cap \mathrm{Cof}(\omega_1)$ is approchable via A.

Questions

- Can this be done for two consecutive cardinals, such at both ω_2 and ω_3?

  (Note that the use of $\square_\kappa$ raises a difficulty in doing this).

- Can anything be done at the successor of a singular cardinal (or even the sucessor of a regular cardinal)?