Lecture 10: Derivatives and Rates of Change

\[ m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{slope of the tangent line to the graph at } P = (a, f(a)) \]

Alternate Definition of the Slope of a Tangent Line

Let \( h = x - a \). Then \( x = a + h \).

and \( m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \)

**Def.** The derivative of a function \( f \) at a number \( a \), denoted by \( f'(a) \), is

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

if this limit exists.
**ex.** Find the slope of the tangent line to \( f(x) = \frac{x}{x+1} \) at \( x = 2 \).

\[
M = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}
\]

\[
= \lim_{x \to 2} \frac{x - \frac{2}{x+1}}{x+1 - 1}
\]

\[
= \lim_{x \to 2} \frac{\frac{3x - 2(x+1)}{3(x+1)}}{x-2}
\]

\[
= \lim_{x \to 2} \frac{x - 2}{3(x+1)} \cdot \frac{1}{x-2}
\]

\[
= \frac{1}{3(x+1)}
\]

\[
= \frac{1}{7}
\]

Slope of the tangent line at \( x = 2 \) is \( \frac{1}{7} \).
\[ q^2 - b^2 = (a-b)(a+b) \]

**NOTE:** We can use our formulas to find the slope of the tangent line for \( f(x) \) at any point \((a, f(a))\).

**ex.** Find the slope of the tangent line to \( f(x) = \sqrt{9 - 2x} \) at \( x = a \).

\[ m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \]

\[ \left( \frac{\partial}{\partial x} \right) = \lim_{x \to a} \frac{\sqrt{9-2x} - \sqrt{9-2a}}{x - a} \cdot \frac{\sqrt{9-2x} + \sqrt{9-2a}}{\sqrt{9-2x} + \sqrt{9-2a}}. \]

\[ = \lim_{x \to a} \frac{(9-2x) - (9-2a)}{(x-a)(\sqrt{9-2x} + \sqrt{9-2a})}. \]

\[ = \lim_{x \to a} \frac{2a - 2x}{(x-a)(\sqrt{9-2x} + \sqrt{9-2a})}. \]

\[ = \lim_{x \to a} \frac{-2(x-a)}{(x-a)(\sqrt{9-2x} + \sqrt{9-2a})}. \]

\[ = \frac{-2}{2\sqrt{9-2a}}. \]

\[ = \frac{-1}{\sqrt{9-2a}} \]
We can use this formula to find the slope of the tangent line to \( f(x) = \sqrt{9 - 2x} \) at a given value \( x = a \) (if the limit exists):

1) \( x = -8 \) \( \Rightarrow \) \( a = -8 \).

\[
m = f'(a) = \frac{-1}{\sqrt{9 - 2(-8)}} = \frac{-1}{\sqrt{25}} = -\frac{1}{5}.
\]

2) \( x = 0 \) \( a = 0 \)

\[
m = f'(a) = \frac{-1}{\sqrt{9 - 0}} = -\frac{1}{3}.
\]

3) \( x = \frac{9}{2} \) \( a = \frac{9}{2} \)

\[
m = f'(a) = \frac{-1}{\sqrt{9 - 2(\frac{9}{2})}} = \frac{-1}{0} \rightarrow \text{Undefined}.
\]
Velocity

Suppose an object moves along a straight line according to an equation of motion \( s = f(t) \) where \( s \) is the displacement (directed distance) from the starting point at time \( t \). We call \( f(t) \) the **position function** of the object.

**Average Velocity** on the time interval \( t = a \) to \( t = a+h \):

\[
\text{Average velocity} = \frac{f(a+h) - f(a)}{h}. \\
= \frac{(f(a+h) - f(a))}{(a+h-a)} = h.
\]

**Instantaneous Velocity** at time \( t = a \):

\[
\lim_{{h \to 0}} \frac{f(a+h) - f(a)}{h} = f'(a).
\]

A useful formula: The position of an object subject to the force of gravity only is given by the function

\[
s(t) = -16t^2 + v_0t + h_0
\]

where \( s \) is in **feet**, \( t \) is in **seconds**, \( v_0 \) is its **initial velocity**, and \( h_0 \) is its **initial position**.

The formula can also be expressed in meters/second:

\[
s(t) = -4.9t^2 + v_0t + h_0
\]

**NOTE:** We assume positive direction is upward.
**ex.** A stone is thrown upwards from the roof of a building which is 160 ft. tall with initial velocity 48 ft/sec. \(v_0 = 48\) ft/sec.

1) Find the velocity of the stone at \(t = a\) seconds.

\[
S(t) = -16t^2 + v_0 t + h_0.
\]

\[
S(t) = -16t^2 + 48t + 160
\]

**Velocity at \(t = a\):**

\[
V(a) = V'(a) = S'(a).
\]

\[
V(a) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h}
\]

\[
= \lim_{h \to 0} \frac{[-16(a+h)^2 + 48(a+h) + 160] - [-16a^2 + 48a + 160]}{h}
\]

\[
= \lim_{h \to 0} \frac{[-16(a^2 + 2ah + h^2) + 48a + 48h + 160] - [-16a^2 + 48a + 160]}{h}
\]

\[
= \lim_{h \to 0} \frac{-32ah - 16h^2 + 48h}{h} = \lim_{h \to 0} \frac{-32a - 16h + 48}{1} = 48 - 32a.
\]

2) What is the velocity of the stone after \(\frac{3}{2}\) seconds and when it hits the ground?

\[
V(a) = 48 - 32a.
\]

\[
V\left(\frac{3}{2}\right) = 48 - 32 \left(\frac{3}{2}\right) = 48 - 48 = 0.
\]

When it hits the ground,

\[
S(t) = 0.
\]

\[
-16t^2 + 48t + 160 = 0.
\]

\[
-16(t^2 - 3t - 10) = 0,
\]

\[
-16(t-5)(t+2) = 0.
\]

\[
V(5) = -112 \text{ ft/sec}.
\]
Other Rates of Change

Consider a general function $y = f(x)$. If $x$ changes from $x_1$ to $x_2$, we define the **average rate of change** of $y$ with respect to $x$ to be $x' : x_1 \rightarrow x_2$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

**Instantaneous rate of change** of $y$ with respect to $x$ at $x = a$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

**ex.** Suppose that the demand for a barrel of oil in a certain country is given by the formula $D(p) = \frac{4000}{p}$, where $D$ is the yearly demand per individual when the price per barrel is $p$ dollars.

1) Find the average rate of change in demand when the price increases from $80 to $100.

$$\frac{\Delta \text{Demand}}{\Delta \text{Price}} = \frac{\text{Change in Demand}}{\text{Change in Price}}.$$

$[80, 100]$ = $rac{D(100) - D(80)}{100 - 80}$

$= \frac{\frac{4000}{100} - \frac{4000}{80}}{20}$

$= \frac{40 - 50}{20} = -\frac{1}{2}$

The demand decreases at the rate of $-\frac{1}{2}$ as the prices increases by a $\$
2) Find the average rate of change in demand on the price interval \( p = a \) to \( p = a + h \).

\[
\frac{[q,a+h]}{[a,a+h]} \cdot \text{Rate of change} = \frac{D(a+h) - D(a)}{(a+h) - a} \\
= \frac{\left(\frac{4000}{a+h} - \frac{4000}{a}\right)}{h} \\
= \left(\frac{4000a - 4000(a+h)}{a(a+h)}\right) \cdot \frac{1}{h} \\
= \frac{4000a - 4000a - 4000h}{a(a+h)} \cdot \frac{1}{h} \\
= \frac{-4000h}{a(a+h)} \cdot \frac{1}{h} = -\frac{4000}{a(a+h)}
\]

3) Find a formula for the rate at which demand is changing when the price is \( a \) dollars.

\[
\text{Instantaneous rate of change at } p = a \\
= \lim_{h \to 0} \frac{D(a+h) - D(a)}{(a+h) - a} \\
= \lim_{h \to 0} \frac{-4000}{a(a+h)} = \frac{-4000}{a^2}
\]

The demand is decreasing at the rate of \( -\frac{4000}{a^2} \) when barrels price per barrel is \( a \).