Lecture 18: Rates of Change
 '"Physics

\textbf{ex.} Suppose the position of a particle is given by

\[ s = f(t) = 2t^3 - 15t^2 + 24t, \]

where \( t \) is measured in seconds and \( s \) in feet.

1) Find the velocity of the particle at any time \( t \).

\[ v = \frac{ds}{dt} = 6t^2 - 30t + 24 \]

2) Find the velocity at \( t = 3 \) seconds.

\[ v(3) = 6(3)^2 - 30(3) + 24 = -12 \text{ ft/sec} \]

3) When is the particle at rest? \( v = 0 \).

\[ v(t) = 0 \quad 6t^2 - 30t + 24 = 0 \]

\[ 6(t^2 - 5t + 4) = 0 \]

\[ 6(t-4)(t-1) = 0 \]

\[ t = 1, 4. \]

4) When is the particle moving in a positive direction? \( v > 0 \)

\[ v(t) > 0 \]

\[ 6t^2 - 30t + 24 > 0 \]

\[ 6(t-4)(t-1) > 0 \]

\[ t \in (0, 1) \cup (4, \infty) \]

\( \text{or: } 0 < t < 1 \text{ and } t > 4 \)

\[ \text{negative direction } v < 0. \]
5) Draw a diagram to represent the particle’s motion.

6) Find the total distance the particle moves in the first six seconds.

\[ s(t) = 2t^3 - 15t^2 + 24t. \]

\[ s(0) = 0, \quad s(1) = 2 - 15 + 24 = 11, \quad s(4) = 2(4)^3 - 15(4)^2 + 24(4) = -16, \quad s(6) = 36. \]

Total distance = \[ |s(1) - s(0)| + |s(4) - s(1)| + |s(6) - s(4)| = 11 + 27 + 52 = 90 \text{ ft}. \]

Acceleration = rate of change of the velocity.

7) Find the acceleration of \( s(t) = 2t^3 - 15t^2 + 24t \) at any time \( t \).

\[ a(t) = \frac{d}{dt} v(t) = \frac{d}{dt} \left( 6t^2 - 30t + 24 \right) \]

\[ a(t) = 12t - 30. \]
8) When is the particle speeding up and when is it slowing down?

1. When the object is moving forward: \( v \geq 0 \)
   - If \( a(t) > 0 \); \( v(t) \) is increasing: speeding up.
   - If \( a(t) < 0 \); \( v(t) \) is decreasing: slowing down.

2. When the object is moving backward: \( v < 0 \)
   - If \( a(t) > 0 \); \( v(t) \) is decreasing but decreasing in the absolute: slowing down.
   - If \( a(t) < 0 \); \( v(t) \) is decreasing (increasing in the absolute value): speeding up.

\[ v(t) = 6t^2 - 30t + 24, \]

\[ a(t) = 12t - 30, \]

\( t \approx 5/2 \).

Speeding up: \( 1 < t < 5/2 \) and \( 4 < t \).

Slowing down: \( 0 < t < 1 \) and \( 5/2 < t < 4 \).
Economics

The total cost of producing \( x \) units of a product is called the **cost function** \( C(x) \).

Average rate of change of cost as the number of items produced increases from \( x_1 \) to \( x_2 \):

\[
AV_{[x_1,x_2]} = \frac{A\Delta C(x)}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}
\]

**Marginal Cost** = rate at which the cost changes as one additional item is produced.

\[
\frac{C(x+1) - C(x)}{1} \approx C'(x), \quad \text{for large } x.
\]

**ex.** Suppose the cost function for a certain product is given by \( C(x) = 1000 + 25x - 0.1x^2 \).\textsuperscript{4} - cost for of producing \( x \) items.

1) Find the total cost of producing 100 items.

\[
C(100) = 1000 + 25(100) - 0.1(100)^2 = 2500 \text{ $}.
\]

2) Estimate the marginal cost at the production level of 100 items.

\[
C'(100) \approx \frac{C(101) - C(99)}{2} \approx 25 - 0.2 \times 100 = 5 \text{ $}.
\]

M: \( C'(x) = 25 - 0.2x \).

\[
C'(100) = 25 - 0.2(100) = 5 \text{ $}. \quad (\text{approximated cost})
\]
3) Find the actual cost of producing the 101st item given 
\[ C(101) = 1000 + 25(101) - 0.1(101)^2 = 2504.90. \]

\[ C(100) = 2500 \] $.

Actual cost: \[ C(101) - C(100) = 4.90 \] $.

Now suppose that the unit price \( p \) at which \( x \) items will sell can be modeled by the **demand function**

\[ p(x) = -0.3x + 125, \quad 0 \leq x \leq 400. \]

4) Find the **revenue** from the sale of \( x \) items.

\[
R = p(x) \cdot x = (-0.3x + 125) \cdot x
= -0.3x^2 + 125x
\]

5) Find the profit function, \( P(x) \), which gives the profit from the sale of \( x \) items.

\[
P(x) = \text{Revenue} - \text{Cost}.
= R(x) - C(x)
= (-0.3x^2 + 125x) - (1000 + 25x - 0.1x^2)
= -0.2x^2 + 100x - 1000
\]

6) Estimate the marginal profit when 50 items are sold.

\[
P'(x) = -0.4x + 100.
\]

\[
P'(50) = -0.4(50) + 100 = 80 \] $.

Note: \[ P(51) - P(50) = 3579.80 - 3500 = 79.80 \] $.
Chemistry

Consider a chemical reaction $A + B \rightarrow C$, where $A$ and $B$ are reactants and $C$ is the product. The concentration of product $C$ in moles per liter is denoted $[C](t)$.

Average rate of reaction of the product $C$ over $[t_1, t_2]$ is

$$\frac{\Delta [C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

Instantaneous rate of reaction $= \lim_{\Delta t \to 0} \frac{\Delta [C]}{\Delta t} = \frac{d[C]}{dt}$

**ex.** Assume that the initial concentration of $A$ and $B$ have the same value $[A] = [B] = 2$ moles/L, and $[C](t) = \frac{16t}{8t + 1}$ moles/L.

1) Find the rate of reaction at time $t$, $R(t)$.

$$R(t) = \frac{d}{dt} \frac{[C](t)}{dt} = \frac{(8t+1) \cdot 16 - 16t \cdot (8)}{(8t+1)^2} = \frac{16}{(8t+1)^2}$$
2) Find the rate of reaction at $t = 0$, 1, and 2 seconds. Include units.

$$ R(t) = \frac{16}{(8+t+1)^2} $$

$t=0$: $R(0) = 16 \text{ mol} / \text{L} / \text{sec}$.

$t=1$: $R(1) = \frac{16}{81} \text{ mol} / \text{L} / \text{sec}$.

$t=2$: $R(2) = \frac{16}{289} \text{ mol} / \text{L} / \text{sec}$.

3) What happens to $[C](t)$ and $R(t)$ at $t \to \infty$. Does this make sense?

$$ [C](t) = \frac{16t}{8+t+1} \xrightarrow{t \to \infty} \frac{16}{8} = 2 $$

$$ R(t) = \frac{16}{(8+t+1)^2} \xrightarrow{t \to \infty} 0 $$