Lecture 20: Related Rates, Part II (Section 3.9)

**ex.** A boat is pulled to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 feet higher than the front of the boat. The rope is being pulled through the ring at the rate of \[0.6 \text{ ft/sec}\]. How fast is the boat approaching the dock when 13 feet of rope are out?

\[
\begin{align*}
\text{Note: } y \text{ and } x & \text{ change with the time (both } y \text{ and } x \text{ decrease)} \\
0 \text{ Given: } \frac{dx}{dt} & = 0.6 \text{ ft/sec} \\
\text{Find: } \frac{dy}{dt} \text{ when } x = 13 \text{ ft.} \\
\end{align*}
\]

**2. Equation:** \[x^2 = y^2 + 5^2\]

**3. Diff:** \[ax \cdot \frac{dx}{dt} = ay \cdot \frac{dy}{dt}\]

**4. Substitute:** \[2 \cdot (13) \cdot (-0.6) = 2 \cdot (12) \cdot \frac{dy}{dt}\]

\[
\frac{dy}{dt} = -0.65 \text{ ft/sec.}
\]

(y decreases with the time.)
ex. A lighthouse is located on a small island 2 miles away from the nearest point $P$ on a straight shoreline and its light makes 10 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 2 miles from $P$?

Note: $x, \theta$ change with the time.

Given: $\frac{d\theta}{dt} = 10 \text{ rev} = 10 \cdot 2\pi \text{ rad} \text{ min}^{-1}$.

Find $\frac{dx}{dt}$ when $x = 2$ miles.

Note: since $\frac{d\theta}{dt} = 10 > 0$, $\theta$ increases with the time.

So, $x$ also increases with the time.

2. Equation: $\tan \theta = \frac{x}{2}$.

3. Diff: $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{2} \left[ \frac{dx}{dt} \right]$.

4. $\left( \frac{2 \sqrt{2}}{2} \right)^2 (20 \pi) = \frac{1}{2} \frac{dx}{dt}$.

When $x = 2$:

$$\sec \theta = \frac{1}{\cos \theta} = \frac{2 \sqrt{2}}{2} \Rightarrow \frac{dx}{dt} = 80 \pi \text{ miles/min}.$$
**ex.** A 6-foot tall man is walking away from an 18-foot high streetlight at a rate of 8 ft/sec. At what rate is the tip of his shadow changing when he is 100 feet from the light pole?

Note: $x$ and $y$ change with the time.

1. **Given:** \( \frac{dx}{dt} = 8 \text{ ft/sec} \)
   
   Find: \( \frac{dy}{dt} \) when \( x = 100 \text{ ft} \).

2. **Use similar triangles:**
   
   \[ \frac{y}{18} = \frac{y-x}{6} \]
   
   \[ 6y = 18(y-x) \]
   
   \[ 18x = 12y \]
   
   \[ 3x = 2y \]

3. **Diff.:**
   
   \[ 3 \cdot \frac{dx}{dt} = 2 \cdot \frac{dy}{dt} \]
   
   \[ 3 \cdot 8 = 2 \cdot \frac{dy}{dt} \]
   
   \[ \frac{dy}{dt} = 12 \text{ ft/sec} \]

At what rate is the **length of his shadow** changing at this same time?

Since the length of the shadow $y-x$ and both $y$ and $x$ change with the time, we let $z = y-x$.

Now, 1. **Given** \( \frac{dx}{dt} = 8 \)

Find \( \frac{dz}{dt} \) when \( x = 100 \text{ ft} \).

2. **Again, use similar triangles:**
   
   \[ \frac{y}{18} = \frac{z}{6} \]
   
   \[ 3z = y \]

3. **Diff.:**
   
   \[ 3 \cdot \frac{dz}{dt} = \frac{dy}{dt} \]

4. \[ 3 \cdot \frac{dz}{dt} = 12 \]

\[ \frac{dz}{dt} = 4 \text{ ft/sec} \]
An Economics Example

The price of a product is related to the number of items that will sell. We consider the demand function $p(x)$ where $p$ is the price per unit at which the company will sell $x$ units. This is expected to be a decreasing function. Total revenue from the sale of $x$ items is then $R(x) = xp(x)$.

**ex.** Suppose that the demand function for a product is

$$p(x) = 50 - \frac{x}{200}, \quad 0 \leq x \leq 10,000.$$ 

If revenue is increasing at the rate of $\$2400$ per week, at what rate is the production level $x$ changing with respect to time if the current weekly production level is 200 units?

Let $R$ be the revenue.

1. Given: $\frac{dR}{dt} = 2400 \, \$/$\text{week}$.

   Find: $\frac{dx}{dt}$ when $x = 200 \, \text{units}$.

2. $R = x \cdot p = x \left( 50 - \frac{x}{200} \right) = 50x - \frac{x^2}{200}$

3. $\frac{dR}{dt} = 50 \cdot \frac{dx}{dt} - \frac{2x}{200} \cdot \frac{dx}{dt}$

   $\frac{dR}{dt} = \left( 50 - \frac{x}{100} \right) \frac{dx}{dt}$

4. $2400 = \left( 50 - \frac{200}{100} \right) \frac{dx}{dt}$

   $2400 = 48 \cdot \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 50 \, \text{units/week}$. 
ex. The demand function for a certain product is given by the equation \( x^2p + 24p = 200 \), where demand \( x \) is measured in hundreds of units. At what rate is the demand for the product changing when the current weekly demand is 400 units and the price is decreasing at the rate of 25 cents weekly?

1. Given: \( \frac{dp}{dt} = -0.25 \text{ $/week} \).

2. Find: \( \frac{dx}{dt} \) when \( x = 4 \) (\( x \) is in hundreds of units, so we need to divide 400 by 100)

3. Eq: \( x^2p + 24p = 200 \) (both \( x \) and \( p \) change)

4. Diff: \( x^2 \cdot \frac{dp}{dt} + 2x \cdot p \cdot \frac{dx}{dt} + 24 \cdot \frac{dp}{dt} = 0 \).

5. \( (4)^2 (-0.25) + 2 (4) \cdot 200 \cdot \frac{dx}{dt} (5) + 24 (-0.25) = 0 \)

6. \( -4 + 40 \frac{dx}{dt} - 6 = 0 \)

\( \frac{dx}{dt} = \frac{10}{40} \)

\( \frac{dx}{dt} = \frac{1}{4} \)

i.e demand is increasing by \( \frac{1}{4} \) hundreds of units per week.

i.e \( \frac{1}{4} \times 100 = 25 \) units per week.
5. A feeding trough 36 inches long has a triangular cross section 20 inches across and 15 inches high. If the trough is being filled with grain at a rate of $24$ cubic in/min, how fast is the height of the grain changing when it is 8 inches high?

Note: Both $w$ and $h$ change with the time. (They increase)

1. Given: $\frac{dv}{dt} = 24$ cubic in/min

Find: $\frac{dh}{dt}$ when $h=8$.

2. Equation: $V =$ volume of grain inside
   \[ V = \frac{1}{2} w \cdot h \cdot 36 \]
   \[ V = 18wh. \]

3. Diff: $\frac{dv}{dt} = 18 \cdot h \cdot \frac{dw}{dt} + 18 \cdot w \cdot \frac{dh}{dt}.$

4. $24 = 18 \cdot 8 \cdot \frac{dh}{dt} + 18 \cdot \frac{w}{h} \cdot \frac{dh}{dt}.$

To find $w$ and $\frac{dw}{dt}$ when $h = 8$;

Use similar triangles:

\[ \frac{20}{15} = \frac{w}{h} \Rightarrow w = \frac{4}{3}h. \]

So, when $h = 8$; $w = \frac{4 \cdot 8}{3}$. 2

To find $\frac{dw}{dt}$; $w = \frac{4}{3}h$

\[ \frac{dw}{dt} = \frac{4}{3} \frac{dh}{dt}. \]

So, we get,

\[ 24 = 18 \cdot 8 \cdot \frac{dh}{dt} + 18 \cdot \frac{w}{h} \cdot \frac{dh}{dt} \]

\[ 24 = 2 \left( \frac{8 \cdot 8 \cdot 4}{3} \right) \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{1}{16} \text{ ft/sec} \]