Lecture 23: Mean Value Theorem (Section 4.2)

Rolle’s Theorem

Let $f$ be a function satisfying the following:

1) $f$ is continuous on $[a, b]$.
2) $f$ is differentiable on $(a, b)$.
3) $f(a) = f(b)$.

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

The conditions of Rolle’s Theorem are necessary:
**ex.** Find the value of $c$ implied by Rolle’s Theorem for $f(x) = (x^2 - 4x)^{2/3}$ on $[0, 4]$.

1. $f$ is continuous on $[0, 4]$.
2. $f'(x) = \frac{2}{3} (x^2 - 4x)^{-\frac{1}{3}}(2x - 4) = \frac{2(2x - 4)}{3 (x^2 - 4x)^{\frac{1}{3}}}$.
   
   $f'$ exist for all $x$ except $(x^2 - 4x)^{\frac{1}{3}} = 0$.
   
   $x^2 - 4x = 0$.
   
   $x(x - 4) = 0$.
   
   $x = 0, 4$.

   so, $f$ is differentiable on $(0, 4)$.

3. $f(0) = 0$.
   
   $f(4) = (4^2 - 4 \cdot 4)^{\frac{2}{3}} = 0$.

By the Rolle’s Theorem, there exists $c$ in $(0, 4)$ such that $f'(c) = 0$.

To find $c$:

$$\frac{2(2c - 4)}{3(c^2 - 4c)^{\frac{1}{3}}} = 0.$$ 

$$2c - 4 = 0.$$ 

$$c = 2.$$
Rolle's Theorem is a special case of ...

**Mean Value Theorem:**

Let $f$ be a function that satisfies the following conditions:

1) $f$ is continuous on $[a, b]$.

2) $f$ is differentiable on $(a, b)$.

Then there is a number $c$ in the interval $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Slope of the tangent at $c$.

Slope of the secant line through two end points.
ex. Find the value of $c$ implied by the Mean Value Theorem for $f(x) = x^3 - x^2 - 2x$ on $[-1, 1]$.

① $f$ is $c^t$ on $[-1, 1]$.

② $f'(c) = 3x^2 - 2x - 2$.

$f'$ exist for all $x$ in $(-1, 1)$, so, $f$ is differentiable on $(-1, 1)$.

By the MVT, there is a $c$ in $(-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

To find $c$:

$$3c^2 - 2c - 2 = \frac{(1-1-2) - (1-1+2)}{2}$$

$$3c^2 - 2c - 2 = -1$$

$3c^2 - 2c - 1 = 0$.

$3c + 1)(c-1) = 0$ $ightarrow$ $c = -\frac{1}{3}$ or $c = 1$.

What happens if we try the function $f(x) = x + \frac{2}{x}$ on $[-1, 1]$?

$f$ is not $c^t$ at $x=0$.

so, we can't apply the MVT.
**ex.** The position of an object dropped from 800 ft is $s(t) = 800 - 16t^2$, where $t$ is in seconds. Find the **average velocity** on the time interval $[0, 5]$.

\[
\text{AV on } [0, 5] : = \frac{s(5) - s(0)}{5 - 0} = \frac{800 - 16(5^2) - 800}{5} \\
= \frac{-16 \cdot 5 \cdot 8}{5} \\
= -80 \text{ ft/sec.}
\]

Use the Mean Value Theorem to verify that at some time in the first five seconds, average velocity equals instantaneous velocity.

1. $s(t) = 800 - 16t^2$ \(\text{cts on } [0, 5]\).

2. $s'(t) = -32t$ : 
   $s(t)$ is differentiable on \((0, 5)\).

By the **MVT**, there exists $c$ in \((0, 5)\) such that

\[
s'(c) = \frac{s(5) - s(0)}{5 - 0} \\
\uparrow \\
\text{Instantaneous velocity at } c.
\]

To find $c$?

\[
-32c = -80 \\
c = \frac{-80}{-32} = \frac{20}{8} = \left(\frac{5}{2}\right)
\]
ex. Suppose that $f(3) = -2$ and $f'(x) \leq 1$ for all values of $x$. Find the largest possible value of $f(6)$.

→ Use MVT on $[3, 6]$

2. $f'(x) \leq 1 \Rightarrow f'(x)$ exist for all values of $x$.
   $\Rightarrow f$ is differentiable for all values of $x$.
   $\Rightarrow f$ is differentiable on $(3, 6)$

1. Since $f$ is differentiable, it is definitely continuous on $[3, 6]$.

By the MVT, there exists $c$ in $(3, 6)$ such that

$$f'(c) = \frac{f(6) - f(3)}{6 - 3}$$

$$= \frac{f(6) - (-2)}{3},$$

$$f'(c) = \frac{f(6) + 2}{3}.$$

$f'(x) \leq 1 \Rightarrow f'(c) \leq 1$

$$\frac{f(6) + 2}{3} \leq 1$$

$$f(6) + 2 \leq 3$$

$$f(6) \leq 1$$

i.e. $i$ is the largest possible value of $f(6)$. 
The following consequence of Mean Value Theorem will be a fundamental result as we begin our study of integration.

**Theorem:** If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\), then \( f(x) \) is constant on \([a, b]\).

Pick \( x_1 < x_2 \) in \([a, b]\) i.e. \( a < x_1 < x_2 < b \).

Use MVT on \([x_1, x_2]\),

\[ f'(c) = 0 \Rightarrow f \text{ is constant on } [x_1, x_2], \text{ and differentiable on } (x_1, x_2). \]

By the MVT, there is \( c \in (x_1, x_2) \) such that

\[ f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \]

\[ f(x_2) - f(x_1) = 0. \]

**Corollary:** If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a, b)\), then \( f - g \) is a constant on \([a, b]\).

That is, \( f(x) = g(x) + C \).

\[ f(x) = x^2 \Rightarrow f'(x) = 2x \]

\[ g(x) = x^2 + 5 \Rightarrow g'(x) = 2x \]
ex. Use the theorem to verify the identity
\[ \sin^2 x + \cos^2 x = 1. \]

Let \( f(x) = \sin^2 x + \cos^2 x. \)

\[
f'(x) = 2 \cdot \sin x \cdot \cos x + 2 \cdot \cos x \cdot (-\sin x).
\]

\[ f'(x) = 0 \quad \text{for all } x. \]

By the previous Theorem,
\[ f(x) = \sin^2 x + \cos^2 x = C \quad (a \text{ const}). \]

To find \( C \):

Plug \( x = 0 \);
\[ f(0) = 0 + 1 = 1 = C. \]

So,
\[ f(x) = \sin^2 x + \cos^2 x = 1 \quad \text{for all } x. \]