Pre-Lecture 27: Optimization (Section 4.7)

To find the absolute extrema of the function $f$ using Calculus:

- **Extreme Value Theorem** (Lecture 22)

If $f$ is continuous on a closed interval $[a, b]$, then $f$ have both a minimum and maximum on the interval.

1. **at critical points** or
2. **end points**.

What happen if the interval in not closed and bounded?

- **First Derivative Test for Absolute Extreme Values**

If $c$ is the **only** critical number of $f$ on the interval $I$ and there is a relative extremum at $c$ (i.e. $f'$ changes sign at $c$), then $f(c)$ is an absolute extreme value on $I$. 

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**Graphs:**

- $f'$ with $c$ as a critical point, showing $f$ increasing before and decreasing after $c$.
- $f'$ with $c$ as a critical point, showing $f$ decreasing before and increasing after $c$. 

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**Labels:**

- Abs max
- Abs min.
ex. Find the extreme values of \( f(x) = x\sqrt{4-x^2} \) on \([-1, 2]\).

\[
\begin{align*}
\text{ex.} & \quad \text{Find the extreme values of } f(x) = xe^{-x} \text{ on } (0, \infty). \\
\Rightarrow \quad f'(x) &= e^{-x} + xe^{-x} \Rightarrow (1-x) = 0 \Rightarrow x = 1.
\end{align*}
\]

unbounded not closed.

ONLY ONE C.IN on \((0, \infty)\)

at \(x = 1\), we have

\[
\begin{align*}
\text{Abs max: } & f(1) = e^{-1}.
\end{align*}
\]
$xy^3$ gets maximum when $y = 15$

\[ x = \frac{60 - 3(15)}{2} \]
\[ = \frac{15}{2}. \]

Two numbers are $15, \frac{15}{2}$. 
ex. Find two nonnegative numbers so that the product of the first and the cube of the second is a maximum if the sum of twice the first and three times the second is 60.

Let \[ x = 1^{st} \# \]
\[ y = 2^{nd} \# \]

\[ \text{max: } p = x \cdot y^3 \text{ (Primary function)} \]

\[ 2x + 3y = 60 \text{ (Constraint)} \]

4 Solve for \( x \) or \( y \) using the constraint.

\[ x = \frac{60 - 3y}{2} \]

\[ p = \left( \frac{60 - 3y}{2} \right) \cdot y^3 \cdot y^3 = 30y^3 - 3y^4. \]

5 \[ p^1 = 90y^2 - 6y^3 = 6y^2(15 - y) \]

CN: \[ \Rightarrow p^1 = 0 \rightarrow y = 0, y = 15. \]

To find the domain of \( p \):

\[ x \geq 0 \rightarrow \frac{60 - 3y}{2} \geq 0 \rightarrow 30 - \frac{3y}{2} \geq 0 \rightarrow 30 \geq \frac{3y}{2} \rightarrow 20 \geq y \]

\[ y > 0 \text{ [0, 20]} \]

\[ p(0) = 0 \text{ min. } \]

\[ p(15) = 30 \cdot (15)^3 - \frac{3}{2} (15)^4 \text{ max. } \]

\[ p(20) = 0 \text{ max. } \]
**ex.** A closed rectangular box is to be constructed with a surface area of 48 square feet so that its length is twice the width. What dimensions will maximize the volume of the box? What is the maximum volume?

Maximize: \( V = (2x) \cdot (x) \cdot h = 2x^2 h. \)

**Primary function**

Constraint: \( SA = 48. \)

\( 2x \cdot 2x \cdot x + 2x \cdot h + 2(2x) \cdot h = 48. \)

\( 4x^2 + 6xh = 48.\)

Solve for \( h. \)

\( h = \frac{48 - 4x^2}{6x}.\)

\( V = 2x^2 \cdot \left( \frac{48 - 4x^2}{6x} \right) = \frac{x}{3} \left( \frac{48 - 4x^2}{6x} \right) = \frac{16x - 4x^3}{3}.\)

\( V = 16x - \frac{4x^3}{3}. \rightarrow V' = 16 - 4x^2 = 4(4 - x^2)\)

\( CN: V' = 0 \rightarrow x = \pm 2. \rightarrow x = 2.\)

\( V' = \text{DNE} \rightarrow \text{No sols}.\)

**i.e.** Volume is highest when \( x = 2.\)

\( \text{so,} \ h = \frac{48 - 4(2^2)}{6(2)} = \frac{8}{3}.\)

\( \text{max. volume} \ e = 2 \cdot 2^2 \cdot \frac{8}{3} = \frac{64}{3} \) ft³.
ex. Find the coordinates of the point on the curve \( y = x^2 \) closest to the point \( \left( 2, \frac{1}{2} \right) \).

Minimize 

\[
D = \sqrt{(x-2)^2 + (y - \frac{1}{2})^2}
\]

Constraint: \( y = x^2 \).

Minimize: 

\[
D = \sqrt{(x-2)^2 + (x^2 - \frac{1}{2})^2}
\]

Minimize \( D^2 = (x-2)^2 + (x^2 - \frac{1}{2})^2 \).

Domain \( (-\infty, \infty) \).

\[D' = 2(x-2) \cdot 1 + 2(x^2 - \frac{1}{2}) \cdot 2x\]

\[= 4x^3 + 2x - 2x - 1 = 4x^3 - 4 = 4(x^3 - 1)\]

\[D' = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1\]

\[D' \text{ ONE} \Rightarrow \text{No solution}\]

Domain: \( (-\infty, \infty) \). We have only one critical number:

\[\begin{array}{c|c|c}
\text{D}^2 & - & + \\
\hline
0 & 1 & \\
\end{array}\]

Abs min \( \text{min} \) for \( D \) at \( x = 1 \)

Min: Distance is \( \text{min} \) occurs when \( x = 1 \).

At the point \( (1, 1^2) = (1, 1) \)

Minimum Distance from the graph \( y = x^2 \) to \( (1, \frac{1}{2}) \) is \( \sqrt{1^2 + \left(1 - \frac{1}{2}\right)^2} = \sqrt{\frac{5}{2}} \).
**ex.** The management of a local Target store has decided to enclose an 800 square foot area outside the building for the garden display. One side will be formed by an external wall of the store, two sides will be constructed of pineboards costing $6 per foot and the side opposite the store will be constructed of fencing that costs $3 per foot. What dimensions of the enclosure will minimize the cost? Let \( x \) be the length of the side with fencing.

\[
\text{Original: minimizing cost,}
\quad C = 3x + 12y.
\]

**Constraint:** \( xy = 800 \) (area).

\[
y = \frac{800}{x}.
\]

**Minimize:** \( C(x) = 3x + 12 \cdot \frac{800}{x} = 3x + \frac{9600}{x} \).

**Domain:** \( (0, \infty) \)

\[\overbrace{C'} = \frac{dC}{dx}
\]

\[
C'(x) = 3 - \frac{9600}{x^2} = \frac{3x^2 - 9600}{x^2}.
\]

\[\overbrace{C''} = \frac{d^2C}{dx^2}
\]

\[
c''(x) = \frac{19200}{x^3}.
\]

So the cost is minimum when \( x = 40\sqrt{2} \) ft.

\[y = \frac{800}{x} = 10\sqrt{2} \text{ ft}.
\]
ex. When fresh lemonade drinks sell for $4.00 each at a college stadium concession stand, an average of 3000 will sell on a hot day. The concessions manager has observed that when he raises the price by 20 cents, an average of 300 fewer will sell during a game. If the manager has fixed costs of $1250 per game and variable costs are 80 cents per unit, find the price of the drink that will maximize his profit.

Let \( x \) = # units sold

\[ p = \text{unit price}. \]

Max. profit = \( P = R - C = \frac{x^2}{3000} - (1250 + 0.8x) \)

Constraint: \( (x, p) \rightarrow (3000, \$4) \)

\[ (2700, \$4.2) \]

**Max profit:** \( P = -2x^2 + 2 + 4 = -2x + 6 \)

\[ P = \frac{-2x}{3000} + 2 + 4 = -2 \times \frac{x}{3000} + 6 \]

\[ P = \frac{x}{3000} \]

\[ -2x + 6 \geq 0 \]

\[ x \leq 900 \]

\[ 0 \leq x \leq 900 \]

\[ 2700 \leq x \leq 3000 \]

\[ \text{C.N.: } P' = -\frac{2}{3000}x + 5.2 \]

\[ = 0 \rightarrow x = 3900 \]

\[ \text{2nd D.T.: } P'' = -\frac{2}{3000} < 0 \]

\[ \text{i.e., Profit is at highest when } x = 3900. \]

Drice of the drink = \( -\frac{2(3900)}{2000} \)