Lecture 30: Areas

Let $f(x) = \sqrt{x}$ and consider the area beneath the graph of the function on $[0, 4]$.

Let $R_n$ be the sum of the areas of $n$ rectangles with equal width $\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and height $f(x_i)$, where $x_i$ is the right endpoint of the $i$th sub-interval.

$$R_n = \int f(x) \Delta x = \int f(x_i) \Delta x = \int \frac{\sqrt{x}}{x} \Delta x = \left[ \sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_i} + \ldots + \sqrt{x_n} \right] \Delta x.$$

As $n \to \infty$, what happens to our approximation?

We define

$$\text{Area} = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( \frac{4}{n} \left[ \sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_i} + \ldots + \sqrt{x_n} \right] \right) = \frac{4}{3}.$$
We now generalize this process:

**To find the area under the curve** $y = f(x)$ **on** $[a, b]$:

Divide $[a, b]$ into $n$ subintervals using partition

$$ a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b $$

This creates $n$ subintervals:

$$ [x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n] $$

Then consider $n$ rectangles, one for each subinterval:

**Width** $\Delta x = \frac{b-a}{n}$

**Height:** $f(x_i)$, where $x_i$ is the **right end point** of the $i^{th}$ sub-interval.

$$ x_i = a + i \Delta x $$

Area $A$ can be approximated by the sum of the areas of the $n$ rectangles:

$$ A \approx f(x_0) \Delta x + f(x_2) \Delta x + \cdots + f(x_i) \Delta x + \cdots + f(x_n) \Delta x $$

This sum is called a **Riemann sum**.
Summation Notation

We use summation notation to write sums in compact form:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

ex. $$\sum_{i=1}^{4} i^3 = (1)^3 + (2)^3 + (3)^3 + (4)^3 = 100.$$

ex. $$\sum_{k=2}^{5} (k^2 - 1) = (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + (5^2 - 1) = 50.$$

Now, we use summation notation to express the sum more concisely as

$$A \approx \sum_{i=1}^{n} f(x_i) \Delta x$$

Generally, if $f$ is continuous, as the number of subintervals gets larger and widths get smaller the approximation is closer to actual area. We can then define

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

Area as the limit of a Riemann sum.
\( f(x) \to [a,b] \to n:\ \Delta x = \frac{b-a}{n} \) \[
\begin{array}{l}
\ x_i = a + i \Delta x \\
\end{array}
\]

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ex.

1) Find an expression for the exact area under \( f(x) = x^2 + 1 \) from \( x = 0 \) to \( x = 3 \) as the limit of a Riemann sum with \( n \) subintervals of equal width.

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.
\]

Rewrite for \( f(x) = x^2 + 1 \) on \([0,3] \):

\[
\begin{align*}
\Delta x &= \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n} \\
\end{align*}
\]

\[
\begin{align*}
x_i &= a + i \Delta x = 0 + i \frac{3}{n} = \frac{3i}{n} \\
\end{align*}
\]

\[
\begin{align*}
f(x_i) &= f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^2 + 1 \\
\end{align*}
\]

So,

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 + 1\right] \cdot \frac{3}{n}
\]
2) Consider the following formula

\[ 1^2 + 2^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}. \]

Use it to find the exact area under \( f(x) = x^2 + 1 \) from \( x = 0 \) to \( x = 3 \) by evaluating the limit of the Riemann sum:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( \left( \frac{3i}{n} \right)^2 + 1 \right) \left( \frac{3}{n} \right)
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{9i^2}{n^2} + \frac{3}{n} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{9}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \right)
\]

\[
= \lim_{n \to \infty} \left[ \frac{9}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{9}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n \right]
\]

\[
= \frac{27}{6} + 3 = \frac{12}{1}.
\]
ex. What area is represented by

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{1}{1 + \frac{2i}{n}} \right) \left( \frac{2}{n} \right) dx
\]

\[\Delta x = \frac{2}{n} = \frac{b-a}{n}\]

\[\text{Choose: } a = 0 \quad b = 2\]

so, \([0, 2] : \checkmark\)

\[\text{Find } f(x) \text{ ??}\]

\[x_i = a + i \Delta x = 0 + i \frac{2}{n} = \frac{2i}{n}\]

\[\Psi(x_i) = \frac{1}{1 + \frac{2i}{n}} = \frac{1}{1 + x_i}\]

Goal: Identify \(f(x)\) and \([a, b]\).

\[f(x) = \frac{1}{1+x}\]

NOTE: Choose: \([1, 3] \Rightarrow a = 1, b = 3\).

\[\text{Find } \Psi(x) \text{ ??}\]

\[x_i = a + i \Delta x = 1 + i \frac{2}{n} = 1 + \frac{2i}{n}\]

\[f(x_i) = \frac{1}{1 + \frac{2i}{n}} = \frac{1}{x_i}\]

\[f(x) = \frac{1}{x}\]
Application of Area: Distance

**ex.** Suppose an object moves along a track, and its velocity in feet per second is measured every five seconds over a 20 second time interval as recorded in the following table:

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (ft/sec)</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>40</td>
<td>45</td>
</tr>
</tbody>
</table>

Estimate the distance traveled over the 20 second interval.

\[ d = \text{distance}, \quad v = \text{velocity}, \quad t = \text{time}. \quad \Rightarrow \quad v = \frac{d}{t}. \]

Use: \[ d = v \cdot t. \]

\[ d \approx 30 \cdot 5 + 36 \cdot 5 + 40 \cdot 5 + 45 \cdot 5 \]

(right end point approximation).

\[ \approx \text{??} \]

\[ d \approx 24 \cdot 5 + 30 \cdot 5 + 36 \cdot 5 + 40 \cdot 5 \]

(left end point approximation).
Find the distance traveled by an object during a certain time interval \([a, b]\) if the velocity is known at all times (and is positive).

\[
\Delta t = \frac{b-a}{n}
\]

Distance \( \approx \sum_{i=1}^{n} f(t_i) \cdot \Delta t \)

\[
\text{distance} = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \cdot \Delta t = \text{Area under } V = f(t) \text{ from } t = a \text{ to } t = b.
\]