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PARTIAL REGULARITY FOR A SELECTIVE SMOOTHING FUNCTIONAL FOR IMAGE RESTORATION IN BV SPACE*

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Abstract. In this paper we study the partial regularity of a functional on BV space proposed by Chambolle and Lions [*Numer. Math.*, 76 (1997), pp. 167–188] for the purposes of image restoration. The functional is designed to smooth corrupted images using isotropic diffusion via the Laplacian where the gradients of the image are below a certain threshold ϵ and retain edges where gradients are above the threshold using the total variation. Here we prove that if the solution $u \in BV$ of the model minimization problem, defined on an open set Ω, is such that the Lebesgue measure of the set where the gradient of u is below the threshold ϵ is positive, then there exists a nonempty open region E for which $u \in C^{1,\alpha}$ on E and $|\nabla u| < \epsilon$, and $|\nabla u| \geq \epsilon$ on $\Omega \backslash E$ almost everywhere. Thus we indeed have smoothing where $|\nabla u| < \epsilon$.

Key words. bounded variation, selective smooothing, image processing, image restoration, noise removal, partial regularity

AMS subject classifications. 49J40, 35K65

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1. Introduction.

In the last decade PDE and variational method based diffusion models have grown significantly to tackle the problems of image restoration, reconstruction, and inpainting. The challenging aspect of these problems is to design methods which can filter selectively the noise without losing significant features.

Total variation (TV) based regularization, as first proposed by Rudin, Osher, and Fatemi [17], has proved to be an invaluable tool for preserving edges while reconstructing an image. This method has been studied extensively in [1, 7, 4, 20, 2, 5, 19, 16] and a sequence of papers in the book of [2]. The definition of the total variation seminorm for $u \in L^1(\Omega)$, given by

$$TV(u) = \sup \left\{ \int_\Omega u \, \mathrm{div}(\varphi) dx : \varphi \in C_0^1(\Omega, \mathbf{R}^n), |\varphi| \leq 1 \right\},$$

does not require differentiability or even continuity of u. Thus images with discontinuities are allowed as solutions in the space of $BV(\Omega)$, which is the space of the functions $u \in L^1(\Omega)$ with $TV(u) < \infty$. Moreover, the diffusion resulting from minimizing TV norm is strictly orthogonal to the gradient of the image, and tangential to the edges. This is important for preserving edges while image is smoothed. However, TV-based denoising sometimes causes a *staircasing* effect [6, 7, 8]. The restored image by this regularization can consequently be blocky and even contain false edges.

To overcome this problem and make the filter self-adjustable in order to reap the benefits of isotropic smoothing and TV based regularization, Chambolle and Lions [7]

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proposed minimizing the following energy functional for image restoration:

$$\frac{1}{2a}\int_{|\nabla u|<a}|\nabla u|^2dx+\int_{|\nabla u|\geq a}\left(|\nabla u|-\frac{a}{2}\right)+\frac{1}{2}\int_\Omega (u-I)^2dx,$$

where I is an observed noisy image, and we want to recover an image u from I, which is related to I by

$$I=u+noise.$$

Using the above functional we then expect to have isotropic diffusion where the image is more uniform ($|\nabla u|<a$) and feature preservation via TV-based diffusion where the boundaries of features are present (the locations where the image gradients most likely have high magnitude: $|\nabla u|\geq a$). It has been shown numerically in [7] that this model is successful in restoring images where homogeneous regions are separated by distinct edges. The purpose of this paper is to prove this mathematically. Our partial regularity results for $a=1$ (without loss of generality) below indicates that the restored image through this model is smooth on the region with smaller gradients. The edges appear at the points where the gradient is larger.

More precisely, consider the problem

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)}\left\{\int_\Omega \varphi(Du)+\frac{1}{2}\int_\Omega (u-I)^2dx\right\}, \tag{1.1}$$

where φ is the C^1 convex function defined on $\mathbf{R}^n$

$$\varphi(p)=\begin{cases}\frac{1}{2}|p|^2 & \text{if } |p|<1,\\ |p|-\frac{1}{2} & \text{if } |p|\geq 1,\end{cases}$$

$\Omega\subset\mathbf{R}^n$ is a bounded domain with Lipschitz boundary, and $I\in L^\infty(\Omega)\cap BV(\Omega)$ is given.

For $u\in BV(\Omega)$ the gradient of u is a measure Du; it can be decomposed into its absolutely continuous and singular parts with respect to Lebesgue measure, that is,

$$Du=\nabla u\,dx+D^s u.$$

See [10] for a complete discussion. Then we define ([12] or [9])

$$J(u)\equiv\int_\Omega\varphi(Du)\equiv\int_\Omega\varphi(\nabla u)dx+\int_\Omega |D^s u|$$

with

$$\int_\Omega |D^s u|\equiv\int_\Omega d|D^s u|=|D^s u|(\Omega).$$

It is important to note ([21] or [12]) that the functional J can also be written as

$$J(u)=\sup_{\phi\in C_0^1(\Omega,\mathbf{R}^n)}\left\{-\int_\Omega\left(\frac{1}{2}|\phi|^2+u\,\mathrm{div}(\phi)\right)dx:|\phi(x)|\leq 1\ \forall x\in\Omega\right\}.$$

Using this, we see that the functional J is lower semicontinuous with respect to convergence in $L^1(\Omega)$. Then by a standard argument we can show that there is a unique solution $u \in BV(\Omega) \cap L^2(\Omega)$ to (1.1). Now we are interested as to whether this solution $u \in BV$ is smooth on the region where $|\nabla u| < 1$. If so, it shows that the denoising governed by (1.1) smoothes out lower gradients while preserving the boundaries of features, which are the discontinuities in an image.

We now state the two main partial regularity results of this paper.

THEOREM 1.1. *If u is the solution to* (1.1)*, then for any given $0 < \mu < 1$ there exist positive constants ϵ_0 and κ_0 depending only on n and μ such that if*

$$\frac{1}{|B_r|}\int_{B_r(a)} |Du - l| \le \epsilon_0$$

holds for some ball $B_r(a) \subset\subset \Omega$ and for some $l \in \mathbf{R}^n$, with

$$rC\left(1 + \|I\|_{L^\infty(\Omega)}\right) < \kappa_0 \quad \textit{and} \quad |l| < 1 - 2\mu,$$

for some constant C depending only on n and Ω, then

$$|D^s u|(B_{r/2}(a)) = 0 \quad \textit{and} \quad |\nabla u| < 1 - \mu \quad \textit{on} \quad B_{r/2}(a)$$

and u solves

$$-\Delta u = I - u \quad \textit{on} \quad B_{r/2}(a).$$

Hence $u \in C^{1,\alpha}(B_{r/2}(a))$ for any $\alpha < 1$.

THEOREM 1.2. *Let u be as in Theorem* (1.1)*. If $\mathcal{L}^n(\{|\nabla u| < 1\}) > 0$, then there exists a nonempty open region E on which u is $C^{1,\alpha}$, $|\nabla u| < 1$, and u solves*

$$-\Delta u = I - u \quad \textit{on} \quad E.$$

In addition we have $|\nabla u| \ge 1$ a.e. on $\Omega \setminus E$.

It is actually straightforward to show that Theorem 1.2 is a direct consequence of Theorem 1.1. Thus from Theorem 1.2, we do indeed have smoothing where $|\nabla u| < 1$.

Here we should point out that partial regularity results were obtained in [3] for minimizers in $BV(\Omega)$ of functionals of the form $\int_\Omega (F(x, Du) + G(x, u))$, where $F(x, p)$ is a convex function in p with $c_1|p| \le F(x, p) \le c_2(1 + |p|)$ for all $p \in \mathbf{R}^n$, F is locally Hölder continuous in x, and $G(x, z)$ satisfies Hölder continuity conditions in both x and z. In our case, $G(x, z) = 1/2(z - I(x))^2$ with only the stated assumption on I, and therefore their results cannot directly be applied in our case. Moreover, our approach is quite different from theirs and can be applied to more general cases.

The partial regularity results for the flow associated with the minimization problem (1) is also discussed in [15] for more general φ. However, these hold only for $\Omega \subset \mathbf{R}^n$ for $n = 1$ and $n = 2$. We also apply some different techniques to get our results.

2. Proof of Theorems 1.1 and 1.2. First we will show that the solution u to (1.1) is in $L^\infty(\Omega)$. To prove this we could consider the time evolution problem corresponding to (1.1), prove an L^∞ bound for the time-dependent solution $u(x, t)$, and then consider the time-asymptotic limit u, which is the solution to (1.1). We would then conclude that $u \in L^\infty(\Omega)$. The following, however, provides a proof of this without having to consider the time evolution of (1.1).

LEMMA 2.1. *If u is the solution to* (1.1)*, then $u \in L^\infty(\Omega)$. In fact, we have $\|u\|_{L^\infty(\Omega)} \le \|I\|_{L^\infty(\Omega)}$.*

Proof. Let φ_ϵ be defined on $\mathbf{R}^n$ by

$$\varphi_\epsilon(p) = \begin{cases} \dfrac{1}{2}|p|^2 & \text{if } |p| < 1, \\ \dfrac{1}{1+\epsilon}|p|^{1+\epsilon} + \left(\dfrac{1}{2} - \dfrac{1}{1+\epsilon}\right) & \text{if } |p| \geq 1 \end{cases}$$

for $\epsilon > 0$, and consider the following minimization problem:

$$\min_{u \in W^{1,1+\epsilon}(\Omega) \cap L^2(\Omega)} \left\{ \int_\Omega \varphi_\epsilon(\nabla u) + \frac{1}{2}\int_\Omega (u - I)^2 dx \right\}.$$

By standard methods, there is a unique solution u_ϵ to this problem. We follow a standard truncation argument where we fix ϵ and $t \geq 0$ and let $v = \min(u_\epsilon, t)$. Noting that $v \in W^{1,1+\epsilon}(\Omega) \cap L^2(\Omega)$ with

$$\nabla v = \begin{cases} \nabla u_\epsilon & \text{if } u_\epsilon < t, \\ 0 & \text{if } u_\epsilon \geq t, \end{cases}$$

we have

$$\int_\Omega \varphi_\epsilon(\nabla u_\epsilon) + \frac{1}{2}\int_\Omega (u_\epsilon - I)^2\, dx \leq \int_\Omega \varphi_\epsilon(\nabla v) + \frac{1}{2}\int_\Omega (v - I)^2 dx, \tag{2.1}$$

and thus after subtracting

$$\int_{\{u_\epsilon \geq t\}} \varphi_\epsilon(\nabla u_\epsilon)\, dx + \int_{\{u_\epsilon \geq t\}} (u_\epsilon - I)^2 dx \leq \int_{\{u_\epsilon \geq t\}} (t - I)^2 dx.$$

Hence

$$\int_{\{u_\epsilon \geq t\}} (u_\epsilon - I)^2 dx \leq \int_{\{u_\epsilon \geq t\}} (t - I)^2 dx.$$

But setting $t = \|I\|_{L^\infty(\Omega)}$ we see that if ess sup $u_\epsilon > t$, then

$$\int_{\{u_\epsilon \geq t\}} (t - I)^2 dx < \int_{\{u_\epsilon \geq t\}} (u_\epsilon - I)^2 dx,$$

which contradicts the above, hence ess sup $u_\epsilon \leq \|I\|_{L^\infty(\Omega)}$. Applying a similar argument to $v = \max(u_\epsilon, -t)$ for $t = \|I\|_{L^\infty(\Omega)}$ we get ess inf $u_\epsilon \geq -\|I\|_{L^\infty(\Omega)}$ and thus $\|u_\epsilon\|_{L^\infty(\Omega)} \leq \|I\|_{L^\infty(\Omega)}$. Furthermore, letting $v = 0$ in (2.1) we see that u_ϵ is bounded in $W^{1,1+\epsilon}(\Omega) \cap L^2(\Omega) \subset BV(\Omega) \cap L^2(\Omega)$ independent of ϵ. Thus there is a $\tilde{u} \in BV(\Omega) \cap L^2(\Omega)$ and a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, such that $u_\epsilon \to \tilde{u}$ strongly in $L^1(\Omega)$, $u_\epsilon \rightharpoonup \tilde{u}$ weakly in $L^2(\Omega)$, and $u_\epsilon \to \tilde{u}$ almost everywhere (a.e.) in Ω. Letting $\epsilon \to 0$ in (2.1), noting that $\varphi(p) \leq \varphi_\epsilon(p)$ for all p, $\int_\Omega \varphi_\epsilon(\nabla v) \to \int_\Omega \varphi(\nabla v)$, lower semicontinuity of the functional $\int_\Omega \varphi(\nabla u)$ defined on $BV(\Omega)$, and weak lower semicontinuity of the second term on the left-hand side, we get

$$\int_\Omega \varphi(\nabla \tilde{u}) + \frac{1}{2}\int_\Omega (\tilde{u} - I)^2 dx \leq \int_\Omega \varphi(\nabla v) + \frac{1}{2}\int_\Omega (v - I)^2 dx$$

for all $v \in W^{1,1+\epsilon}(\Omega) \cap L^2(\Omega)$. We now note [12] that for any $v \in BV(\Omega) \cap L^2(\Omega)$ there exists a sequence v_n in $C^\infty(\overline{\Omega})$ such that

$$\int_\Omega \varphi(\nabla v_n) dx \to \int_\Omega \varphi(\nabla v)$$

and $v_n \to v$ in $L^1(\Omega)$, and since $v \in L^2(\Omega)$ from the construction of v_n [12] we can also take $v_n \to v$ in $L^2(\Omega)$. Therefore we see that the above holds for all $v \in BV(\Omega) \cap L^2(\Omega)$ as well. Hence $\tilde{u}$ solves (1.1). By uniqueness, $\tilde{u} = u$. By the uniform L^∞ bound for u_ϵ and the convergence of u_ϵ to u a.e. in Ω we have $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq \|I\|_{L^\infty(\Omega)}$ ∎

Throughout the rest of the paper, we fix $\mu > 0$ and unless otherwise stated, all constants depend at most on n, μ, u, Ω, φ, and possibly I.

We begin with a local lower bound estimate for any BV function u and C^1 function h with gradient strictly less than 1.

Lemma 2.2. *Let $u \in BV(B_r(a))$ for $B_r(a) \subset\subset \Omega$ and $h \in C^1(\overline{B}_r(a))$ with*

$$\sup_{B_r(a)} |\nabla h| \leq 1 - \mu;$$

then

$$\int_{B_r(a)} \varphi(Du) - \int_{B_r(a)} \varphi(\nabla h) dx \geq \mu \int_{B_r(a)} |D^s u| + \int_{B_r(a)} \nabla(u-h) \cdot \nabla h dx$$
$$+ \int_{B_r(a)} D^s u \cdot \nabla h + \frac{\mu^2}{2} \int_{B_r(a) \cap \{|\nabla u| \geq 1\}} |\nabla u| dx$$
$$+ \frac{1}{2} \int_{B_r(a) \cap \{|\nabla u| < 1\}} |\nabla(u-h)|^2 dx.$$

Proof. Where $|\nabla u| \geq 1$, we have

$$\begin{aligned}
&\varphi(\nabla u) - \varphi(\nabla h) - \nabla(u-h) \cdot \nabla h \\
&= |\nabla u| - \frac{1}{2} + \frac{1}{2}|\nabla h|^2 - \nabla u \cdot \nabla h \\
&\geq \frac{1}{2}(2|\nabla u| - 1 - 2|\nabla u||\nabla h| + |\nabla h|^2) \\
&= \frac{1}{2}(2|\nabla u| - 1 - |\nabla h|)(1 - |\nabla h|) \geq \frac{\mu^2}{2}|\nabla u|.
\end{aligned}$$

Where $|\nabla u| < 1$, we have

$$\varphi(\nabla u) - \varphi(\nabla h) - \nabla(u-h) \cdot \nabla h = \frac{1}{2}|\nabla(u-h)|^2.$$

We now obtain the lemma by using

$$\int_{B_r(a)} |D^s u| \geq \int_{B_r(a)} D^s u \cdot \nabla h + \int_{B_r(a)} |D^s u|(1 - |\nabla h|),$$

the assumption on h, and the above estimates. ∎

We now fix $B_{2r}(a) \subset\subset \Omega$. Let v be a Lipschitz function defined on $B_{2r}(a)$ and assume there exists an $l \in \mathbf{R}^n$ with $|l| \leq 1 - 2\mu$, such that $\sup_{B_{2r}(a)} |\nabla v - l| \leq \beta^{2\delta}$ for $\delta > 0$ and $0 < \beta < 1$ to be chosen later. Also let $\overline{v}$ be defined by $\overline{v}(x) = v(x) - l \cdot x$. Let η_ϵ be the usual mollifier on $\mathbf{R}^n$ and denote $\overline{v}_\beta = \eta_{r\beta} * \overline{v}$ and $v_\beta = \eta_{r\beta} * v$. We then have the following estimates from [18]:

$$\sup_{B_r(a)} |\nabla v_\beta - l| = \sup_{B_r(a)} |\nabla \overline{v}_\beta| \leq \beta^{2\delta}, \tag{2.2}$$

$$\sup_{B_r(a)} |v_\beta - v| = \sup_{B_r(a)} |\overline{v}_\beta - \overline{v}| \le r\beta \sup_{B_r(a)} |\nabla \overline{v}_\beta| \le r\beta^{1+2\delta}, \tag{2.3}$$

$$\begin{aligned} & r^\delta \sup_{B_r(a)} |x-y|^{-\delta} |\nabla v_\beta(x) - \nabla v_\beta(y)| \\ & \le c_1 r^\delta \sup_{B_r(a)} |\nabla v - l| \sup_{x' \ne y'} |x'-y'|^{-\delta} |\eta_1((r\beta)^{-1}x') - \eta_1((r\beta)^{-1}y')| \\ & \le c_2 \beta^{2\delta}\beta^{-\delta} = c_2\beta^\delta. \end{aligned} \tag{2.4}$$

Now for any $\tilde{r} \in [\frac{r}{2}, r]$ there exists a unique solution [11] $w \in H^1(B_{\tilde{r}}(a)) \cap C^{1,\delta}(\overline{B}_{\tilde{r}}(a))$ with $\delta \in (0,1)$ for the problem

$$-\Delta w = I - w \quad \text{on } B_{\tilde{r}}(a), \quad w = v_\beta \quad \text{on } \partial B_{\tilde{r}}(a). \tag{2.5}$$

Lemma 2.3. *For $I \in L^\infty(\Omega)$, the solution w to (2.5) satisfies*

$$\|w\|_{L^\infty(B_{\tilde{r}}(a))} \le \|v_\beta\|_{L^\infty(\partial B_{\tilde{r}}(a))} + \|I\|_{L^\infty(\Omega)}. \tag{2.6}$$

$$\sup_{B_{\tilde{r}}(a)} |\nabla w - l| \le c_3(\beta^\delta + r(\|I\|_{L^\infty(\Omega)} + \|v_\beta\|_{L^\infty(\partial B_{\tilde{r}}(a))})) \quad \textit{for any } l \in R^n. \tag{2.7}$$

$$\begin{aligned} \sup_{x,y \in B_{\tilde{r}/2}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^{1/2}} \le c_4 \Bigg(& \frac{1}{r^{n+1/2}} \int_{(\partial B_{\tilde{r}}(a))} |v_\beta| \, d\mathcal{H}^{n-1} \\ & + r^{1/2}(\|I\|_{L^\infty(\Omega)} + \|v_\beta\|_{L^\infty(\partial B_{\tilde{r}}(a))}) \Bigg). \end{aligned} \tag{2.8}$$

Proof. Estimate (2.6) is from Theorem 8.16 in [11]. To prove (2.7) and (2.8), we decompose w as $w = w_1 + w_2$, such that

$$-\Delta w_1 = I - w \quad \text{on } B_{\tilde{r}}(a), \quad w_1 = 0 \quad \text{on } \partial B_{\tilde{r}}(a) \tag{2.9}$$

and

$$-\Delta w_2 = 0 \quad \text{on } B_{\tilde{r}}(a), \quad w = v_\beta \quad \text{on } \partial B_{\tilde{r}}(a). \tag{2.10}$$

Let $\tilde{w}_2 \equiv w_2 - v_\beta$. Then $\tilde{w}_2$ solves

$$-\Delta \tilde{w}_2 = -\operatorname{div}(\nabla v_\beta - l) \quad \text{on } B_{\tilde{r}}(a), \quad \tilde{w}_2 = 0 \quad \text{on } \partial B_{\tilde{r}}(a), \tag{2.11}$$

for any $l \in R^n$. Representing the solution of (2.9) using Green's function, i.e., $w_1(x) = \int_{B_{\tilde{r}}(a)} \Gamma(x-y)(I-w)(y)dy$, where Γ is the fundamental solution of Laplace's equation, it is not difficult to get

$$\|\nabla w_1\|_{L^\infty(B_{\tilde{r}}(a))} \le cr\|I - w\|_{L^\infty(B_{\tilde{r}}(a))}, \tag{2.12}$$

where c is independent of r.

Moreover, by the Stobolev imbedding theorem, Theorem 9.9 in [11], and (2.6),

$$\begin{aligned} \|\nabla w_1\|_{C^{0,1/2}(B_{\tilde{r}}(a))} & \le c\|w_1\|_{W^{2,2n}(B_{\tilde{r}}(a))} \le c\|I - w\|_{L^{2n}(B_{\tilde{r}}(a))} \\ & \le cr^{1/2}\|I-w\|_{L^\infty(B_{\tilde{r}}(a))} \le cr^{1/2}(\|v_\beta\|_{L^\infty(\partial B_{\tilde{r}}(a))} + \|I\|_{L^\infty(\Omega)}). \end{aligned} \tag{2.13}$$

Next we shall estimate w_2. Multiplying both sides of (2.11) by $\tilde{w}_2$, and integrating over $B_{\tilde{r}}(a)$, carrying out a simple computation, and using (2.2), we have for any $l \in R^n$,

$$\int_{B_{\tilde{r}}(a)} |\nabla w_2 - l|^2 dx \le c \int_{B_{\tilde{r}}(a)} |\nabla v_\beta - l|^2 dx \le c r^n \beta^{4\delta}, \tag{2.14}$$

where $c > 0$ is a constant independent of r.

Furthermore, applying Theorem 8.16 and 8.33 (with a rescaling argument) in [11] to (2.11), we get the following estimates:

$$\|\tilde{w}_2\|_{L^\infty(B_{\tilde{r}})} \le c \|\nabla v_\beta - l\|_{L^\infty(B_{\tilde{r}})} \tag{2.15}$$

and

$$r^\delta [D\tilde{w}_2]_{C^{0,\delta}(B_{\tilde{r}})} \le c(\|\tilde{w}_2\|_{L^\infty(B_{\tilde{r}})} + \|\nabla v_\beta - l\|_{L^\infty(B_{\tilde{r}})} + r^\delta [Dv_\beta]_{C^{0,\delta}(B_{\tilde{r}})}), \tag{2.16}$$

where $c > 0$ is a constant independent of r. Inserting (2.15) into (2.16), and using (2.2) and (2.4), it yields

$$\begin{aligned} r^\delta [Dw_2]_{C^{0,\delta}(B_{\tilde{r}})} &\le (r^\delta [D\tilde{w}_2]_{C^{0,\delta}(B_{\tilde{r}})} + r^\delta [Dv_\beta]_{C^{0,\delta}(B_{\tilde{r}})}) \\ &\le c(\|\nabla v_\beta - l\|_{L^\infty(B_{\tilde{r}})} + r^\delta [Dv_\beta]_{C^{0,\delta}(B_{\tilde{r}})}) \le c\beta^\delta. \end{aligned} \tag{2.17}$$

Now we can estimate $\sup_{B_{\tilde{r}}(a)} |\nabla w_2 - l|$. Denoting $|B_{\tilde{r}}(a)|^{-1} \int_{B_{\tilde{r}}(a)} f dx$ by $(f)_{B_{\tilde{r}}(a)}$, and using (2.14) and (2.17), we get

$$\begin{aligned} \sup_{B_{\tilde{r}}(a)} |\nabla w_2 - l| &\le \sup_{B_{\tilde{r}}(a)} \{|\nabla w_2 - (\nabla w_2)_{B_{\tilde{r}}(a)}| + |(\nabla w_2)_{B_{\tilde{r}}(a)} - l|\} \\ &\le r^\delta [Dw_2]_{C^{0,\delta}(B_{\tilde{r}})} + |B_{\tilde{r}}(a)|^{-1/2} \left(\int_{B_{\tilde{r}}(a)} |\nabla w_2 - l|^2 \right)^{1/2} dx \le c\beta^\delta; \end{aligned} \tag{2.18}$$

here we used (2.14) and (2.17) in the last inequality.

We then have from (2.6) and (2.18)

$$\begin{aligned} \sup_{B_{\tilde{r}}(a)} |\nabla w - l| &\le \sup_{B_{\tilde{r}}(a)} |\nabla w_2 - l| + \sup_{B_{\tilde{r}}(a)} |\nabla w_1| \\ &\le c_3(\beta^\delta + r(\|I\|_{L^\infty(B_{\tilde{r}}(a))} + \|v_\beta\|_{L^\infty(\partial B_{\tilde{r}}(a))})). \end{aligned}$$

Hence (2.7) is proved. To prove (2.8) we represent w_2 by the Poisson's formula on the ball $B_{\tilde{r}}(a)$, i.e.,

$$w_2(x) = \frac{\tilde{r}^2 - |x|^2}{n\alpha_n r} \int_{\partial B_{\tilde{r}}(a)} \frac{v_\beta(y)}{|x-y|^n} dS_y, \quad x \in B_{\tilde{r}}(a),$$

where α_n represents the volume of n dimensional unit ball. A direct computation leads to the estimate

$$\sup_{B_{\tilde{r}/2}(a)} |D^2 w_2| \le c r^{-n-1} \int_{\partial B_{\tilde{r}}(a)} |v_\beta(y)| dS_y,$$

where $c > 0$ dependents only on n. Then we have

$$\sup_{x,y\in B_{\tilde r/2}(a)} \frac{|\nabla w_2(x)-\nabla w_2(y)|}{|x-y|^{1/2}} \le \left(\sup_{x,y\in B_{\tilde r/2}(a)} |D^2 w_2|\right)|x-y|^{1/2} \le \frac{c}{r^{n+1/2}}\int_{\partial B_{\tilde r}(a))} |v_\beta|\, d\mathcal{H}^{n-1}. \tag{2.19}$$

Now (2.8) immediately follows from (2.13) and (2.19). ∎

LEMMA 2.4. *Suppose there is a* $v \in C^{0,1}(B_{2r}(a))$ *and* $l \in \mathbf{R}^n$ *with* $|l| \le 1-2\mu$, $\sup_{B_{2r}(a)} |\nabla v - l| \le \beta^{2\delta}$, *and* $\sup_{B_{2r}(a)} |v| \le C_u$, *where* C_u *is a constant depending only on* u. *Let* $v_\beta, \tilde r$, *and* w *be as in the previous discussion. Then there exists constants* c_5 *and* c_6 *such that if* $\beta \le c_5$ *and* $r(C_u + \|I\|_{L^\infty(\Omega)}) \le c_6$, *then*

$$\begin{aligned}\int_{B_{\tilde r}(a)} \varphi(Du) - \int_{B_{\tilde r}(a)} \varphi(\nabla w)dx &\ge \int_{\partial B_{\tilde r}(a)} (u-v_\beta)\frac{\partial w}{\partial n}\, d\mathcal{H}^{n-1} \\ &+ \int_{B_{\tilde r}(a)} (u-w)(I-w)dx + \mu\int_{B_{\tilde r}(a)} |D^s u| + \frac{\mu^2}{2}\int_{B_{\tilde r}(a)\cap\{|\nabla u|\ge 1\}} |\nabla u| dx \\ &+ \frac{1}{2}\int_{B_{\tilde r}(a)\cap\{|\nabla u|<1\}} |\nabla(u-w)|^2 dx \\ &\ge \int_{\partial B_{\tilde r}(a)} (u-v_\beta)\frac{\partial w}{\partial n}\, d\mathcal{H}^{n-1} + \frac12\int_{B_{\tilde r}(a)} (w-I)^2 dx - \frac12\int_{B_{\tilde r}(a)} (u-I)^2 dx \\ &+ \mu\int_{B_{\tilde r}(a)} |D^s u| + \frac{\mu^2}{2}\int_{B_{\tilde r}(a)\cap\{|\nabla u|\ge 1\}} |\nabla u| dx \\ &+ \frac12\int_{B_{\tilde r}(a)\cap\{|\nabla u|<1\}} |\nabla(u-w)|^2 dx.\end{aligned}$$

Proof. From (2.7)–(2.8), the definition of v_β, and the assumption on l we see that

$$\begin{aligned}\sup_{B_{\tilde r(a)}} |\nabla w| &\le \sup_{B_{\tilde r(a)}} |\nabla w - l| + |l| \\ &\le c_3(\beta^\delta + r(\|v\|_{L^\infty(\partial B_{\tilde r}(a))} + \|I\|_{L^\infty(\Omega)})) + 1 - 2\mu \\ &\le c_3(\beta^\delta + r(C_u + \|I\|_{L^\infty(\Omega)})) + 1 - 2\mu.\end{aligned}$$

Later, v will be chosen (see, for instance, [14]) to be a Lipschitz approximation of u so that $\|v\|_{L^\infty(B_{2r}(a))}$ can be bounded by a constant C_u depending only on u. Now choose c_5 and c_6 such that $\beta^\delta \le c_5$ and

$$r(C_u + \|I\|_{L^\infty(\Omega)}) \le c_6$$

imply

$$c_3(\beta^\delta + r(C_u + \|I\|_{L^\infty(\Omega)})) \le \mu.$$

Thus

$$\sup_{B_{\tilde{r}(a)}} |\nabla w| \leq 1 - \mu. \tag{2.20}$$

The conditions of Lemma 2.2 now hold for $h = w$. Substituting in w for h in the inequality in Lemma 2.2, integrating by parts, and using Young's inequality for $(u - w)(I - w) = -(u - I)(w - I) + (I - w)^2$ the lemma is proved. ∎

LEMMA 2.5. *If the function $u \in BV(\Omega)$ is solution to* (1.1), *then*

$$\int_{B_r} \varphi(Du) - \int_{B_r} \varphi(Dw) \leq 1/2 \int_{B_r} (w - I)^2 dx$$
$$-1/2 \int_{B_r} (u - I)^2 dx + \int_{\partial B_r} |Tw - Tu| d\mathcal{H}^{n-1}$$

for any $w \in BV(B_r)$, $B_r \subset\subset \Omega$. Here T denotes the trace operator on BV.

Proof. Let $w \in BV(B_r)$ and define

$$\zeta = \begin{cases} w - u & \text{on } B_r, \\ 0 & \text{in } \Omega \backslash \overline{B}_r. \end{cases}$$

Then since u is a solution we have letting $v = u + \zeta$ in (1.1) and using Theorem 1 of section 5.4 in [10],

$$\int_{\Omega} \varphi(Du) + 1/2 \int_{\Omega} (u - I)^2 dx \leq \int_{B_r} \varphi(Dw) + \int_{\partial B_r} |Tw - Tu| d\mathcal{H}^{n-1}$$
$$+ \int_{\Omega \backslash \overline{B}_r} \varphi(Du) + 1/2 \int_{B_r} (w - I)^2 dx + 1/2 \int_{\Omega \backslash B_r} (u - I)^2 dx.$$

Hence

$$\int_{\overline{B}_r} \varphi(Du) + 1/2 \int_{B_r} (u - I)^2 dx \leq \int_{B_r} \varphi(Dw) + 1/2 \int_{B_r} (w - I)^2 dx$$
$$+ \int_{\partial B_r} |Tw - Tu| d\mathcal{H}^{n-1}. \qquad ∎$$

We use the above lemma, Lemma 2.4, and estimates (2.2)–(2.4) to obtain the following inequality for the solution u to (1.1).

LEMMA 2.6. *Let v, l be as in Lemma* 2.4 *with*

$$r(C_u + \|I\|_{L^\infty(\Omega)}) \leq c_6,$$

w as in (2.5), *and u a solution to* (1.1). *Then*

$$\int_{B_{\tilde{r}(a)}} |D^s u| + \int_{B_{\tilde{r}(a)} \cap \{|\nabla u| \geq 1\}} |\nabla u| dx + \int_{B_{\tilde{r}(a)} \cap \{|\nabla u| < 1\}} |\nabla(u - w)|^2 dx$$
$$\leq c_7 \int_{\partial B_{\tilde{r}(a)}} |u - v| \, d\mathcal{H}^{n-1} + c_8 r^n \beta^{1+2\delta},$$

where u and v on $\partial B_{\tilde{r}}(a)$ is understood in the sense of trace.

Proof. By the previous lemma with w from (2.5) and Lemma 2.4 we have

$$\int_{\partial B_{\tilde r}(a)} |u - v_\beta|\, d\mathcal{H}^{n-1} \geq \int_{B_{\tilde r}(a)} \varphi(Du) + \frac{1}{2}\int_{B_{\tilde r}(a)} (u-I)^2 dx$$

$$- \int_{B_{\tilde r}(a)} \varphi(\nabla w) dx - \frac{1}{2}\int_{B_{\tilde r}(a)} (w-I)^2 dx$$

$$\geq \int_{\partial B_{\tilde r}(a)} (u - v_\beta)\frac{\partial w}{\partial n}\, d\mathcal{H}^{n-1} + \mu \int_{B_{\tilde r}(a)} |D^s u| + \frac{\mu^2}{2}\int_{B_{\tilde r(a)}\cap\{|\nabla u|\geq 1\}} |\nabla u| dx$$

$$+\frac{1}{2}\int_{B_{\tilde r(a)}\cap\{|\nabla u|<1\}} |\nabla (u-w)|^2 dx.$$

The lemma is thus proved by using (2.20) and the estimate for $|v - v_\beta|$ from (2.3). ∎

The following first variational formula is from Hardt and Kinderlehrer [12]: if u is a solution to (1.1), then

$$\int_\Omega \sigma \cdot \nabla \zeta dx + \int_\Omega \sigma \cdot \xi |D^s u| = -\int_\Omega (u - I)\zeta dx, \tag{2.21}$$

where ζ is any function in $BV_0(\Omega)$ with $D^s\zeta << |D^s u|$, ξ is the Radon–Nikodym derivative of $D^s\zeta$ with respect to $|D^s u|$, and $\sigma \in L^1(\Omega)$ is the stress tensor defined by

$$\sigma(u) = \begin{cases} \varphi_P(\nabla u) & \text{in } \Omega_a, \\ D^s u/|D^s u| & \text{in } \Omega_s. \end{cases}$$

Here $D^s u/|D^s u|$ denotes the Radon–Nikodym derivative of $D^s u$ with respect to $|D^s u|$ and $\Omega = \Omega_a \cup \Omega_s$ is the decomposition of Ω with respect to the mutually singular measures $\mathcal{L}^n$ and $|D^s u|$. Clearly $|\sigma(u)| \leq 1$. Note that $\sigma(u)$ depends only on u. In the sequel we will write σ instead of $\sigma(u)$ and write the left-hand side of (2.21) as

$$\int_\Omega \sigma \cdot D\zeta.$$

We may also note that if

$$\int_\Omega \sigma \cdot D\zeta = -\int_\Omega (u - I)\zeta dx$$

holds for arbitrary $\zeta \in BV(\Omega)$ for some u where σ is defined as above, then u solves (1.1). In fact, for arbitrary $v \in BV(\Omega)$ we take $\zeta = v - u$, noting that by convexity of φ we have $\varphi(\nabla v) - \varphi(\nabla u) \geq \nabla(v-u)\cdot\varphi_P(\nabla u)$ on Ω_a and that on Ω_s we have

$$\int_{\Omega_s} |D^s v| - \int_{\Omega_s} |D^s u| \geq \int_{\Omega_s} D^s(v-u)\cdot\frac{D^s u}{|D^s u|}.$$

The proof of the lemma below is based on [13], with some necessary modifications.

LEMMA 2.7. *Suppose u is a solution to our minimization problem, $B_{2r}(a) \subset\subset \Omega$, $v \in C^{0,1}(B_{2r}(a))$ with $\sup_{B_{2r}(a)} |\nabla v| \leq 1 - \mu$, and*

$$\mathcal{L}^n(\{u \neq v\} \cap B_\rho(a)) \leq \frac{1}{2}|B_\rho| \quad \textit{for all} \quad r \leq \rho \leq 2r;$$

then there exists positive constants c_9 and c_{10} such that if

$$\mathcal{L}^n(\{u \neq v\} \cap B_{2r}(a)) \leq c_9 r^n,$$

then

$$\|u - v\|_{L^\infty(B_r(a))} \leq c_{10} \left(\mathcal{L}^n(\{u \neq v\} \cap B_{2r}(a))\right)^{\frac{1}{n}}.$$

Proof. First we note that the function φ satisfies $|p| - \lambda \leq \varphi(p) \leq |p|$ for all $p \in \mathbf{R}^n$, some $\lambda > 0$. By convexity of φ we have $\varphi(p) \leq \varphi_P(p) \cdot p + \varphi(0)$ for all $p \in \mathbf{R}^n$. Hence we have

$$|Du| = |\nabla u| dx + |D^s u| \leq \varphi(\nabla u) dx + |D^s u| + \lambda dx$$

$$\leq \varphi_P(\nabla u) \cdot \nabla u dx + |D^s u| + (\lambda + \varphi(0)) dx = \sigma \cdot Du + \lambda dx.$$

Let $\theta : \mathbf{R} \to \mathbf{R}$ be a bounded, increasing, piecewise differentiable function with $\theta'(t) \leq 1$ for almost all t. Let $0 < \rho < h$ and

$$\eta(x) = \begin{cases} 1 & \text{in } B_\rho(a), \\ (h-\rho)^{-1}(h - |x-a|) & \text{in } B_h(a) \backslash B_\rho(a), \\ 0 & \text{in } \Omega \backslash B_h(a). \end{cases}$$

Now apply the first variational formula to $\zeta = \eta\theta(u - v)$ to get

$$\int_{B_h(a)} \eta\sigma \cdot D[\theta(u-v)] = (h-\rho)^{-1} \int_{B_h(a)\backslash B_\rho(a)} \sigma \cdot \frac{x-a}{|x-a|} \theta(u-v) dx$$

$$- \int_{B_h(a)} \eta\theta(u-v)(u-I) dx. \tag{2.22}$$

To obtain a lower bound for $\eta\sigma \cdot D[\theta(u - v)]$ we use the above properties of φ. We have $D[\theta(u - v)] = \theta'(u - v)D(u - v)$ and hence by noting the bound of $|\nabla v|$

$$\int_{B_\rho(a)} |D[\theta(u-v)]| \leq \int_{B_\rho(a)} \theta'(u-v)|Du| + \int_{B_\rho(a)} \theta'(u-v)$$

$$\leq \int_{B_\rho(a)} \theta'(u-v)\varphi(Du) + \int_{B_\rho(a)} (\lambda+1)\theta'(u-v)$$

$$\leq \int_{B_\rho(a)} \theta'(u-v)\sigma \cdot Du + \int_{B_\rho(a)} (\lambda+1)\theta'(u-v)$$

$$= \int_{B_\rho(a)} \theta'(u-v)\sigma \cdot D(u-v) + \int_{B_\rho(a)} \theta'(u-v)\sigma \cdot Dv$$

$$+ \int_{B_\rho(a)} (\lambda+1)\theta'(u-v) \leq \int_{B_h(a)} \eta\sigma \cdot D[\theta(u-v)] + \int_{B_h(a)} C_\lambda \theta'(u-v) \tag{2.23}$$

for some constant C_λ depending only on λ. Inserting (2.22) into (2.23), and noting the L^∞ bound for u, we get

$$\int_{B_\rho(a)} |D[\theta(u-v)]|$$

$$\leq (h-\rho)^{-1} \int_{B_h(a)\setminus B_\rho(a)} |\theta(u-v)|dx + C_\lambda |\operatorname{supp} \eta\theta(u-v)|$$

$$+ 2\|I\|_{L^\infty(\Omega)} \int_{B_h(a)} |\theta(u-v)|dx.$$

For $0 < k < s$ we choose θ as

$$\theta(t) = \begin{cases} 0 & \text{for } t \leq k, \\ t-k & \text{for } k < t < s, \\ s-k & \text{for } t \geq s. \end{cases}$$

Now let $A(k,h) \equiv B_h \cap \{u - v > k\}$. Clearly $\operatorname{supp}[\eta\theta(u-v)] \subset A(k,h)$. Thus

$$\int_{B_\rho(a)} |D[\theta(u-v)]|$$

$$\leq ((h-\rho)^{-1} + 2\|I\|_{L^\infty(\Omega)}) \int_{B_h(a)} |\theta(u-v)|dx + C_\lambda |A(k,h)|.$$

By assumption, $|A(0,\rho)| \leq \frac{1}{2}|B_\rho(a)|$ for $r \leq \rho \leq 2r$. Thus we see that

$$\frac{\mathcal{L}^n\{\{\theta(u-v) = 0\} \cap B_\rho(a)\}}{|B_\rho(a)|} \geq \frac{1}{2}.$$

We can then apply the isoperimetric inequality for $s > k > 0$ to get

$$(s-k)|A(s,\rho)|^{\frac{n-1}{n}} \leq \left(\int_{B_\rho(a)} |\theta(u-v)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}}$$

$$\leq c_{11} \int_{B_\rho(a)} |D[\theta(u-v)]|$$

$$\leq c_{12}((h-\rho)^{-1} + \|I\|_{L^\infty(\Omega)}) \int_{B_h(a)} |\theta(u-v)|dx + c_{13}|A(k,h)|.$$

So since $h \leq 2r$ we get

$$(s-k)|A(s,\rho)|^{\frac{n-1}{n}} \leq c_{14}(h-p)^{-1} \int_{B_h(a)} |\theta(u-v)|dx + c_{14}|A(k,h)|.$$

And since

$$\int_{B_h(a)} |\theta(u-v)|dx \leq (s-k)|A(k,h)|,$$

we arrive at

$$|A(s,\rho)|^{\frac{n-1}{n}} \le c_{14}((h-p)^{-1} + (s-k)^{-1})|A(k,h)|$$

for every $r \le \rho < h \le 2r$ and $s > k > 0$. We now apply Lemma 2.1 in [13] to obtain the upper bound.

The lower bound for $u - v$ is obtained by using a similar argument for $0 < k < s < \infty$,

$$\tilde{\theta}(t) = \begin{cases} 0 & \text{for } t \ge -k, \\ -t-k & \text{for } -s < t < -k, \\ s-k & \text{for } t \le -s, \end{cases}$$

and $\tilde{A}(k,h) \equiv B_h \cap \{u - v < -k\}$. The lemma then follows by again applying Lemma 2.1 in [13]. ∎

Now define the energy function

$$\Phi(r,l,x) = \frac{1}{|B_r|}\left\{\int_{B_r(x)\cap\{|\nabla u|\ge 1\}} |\nabla u| dx + \int_{B_r(x)\cap\{|\nabla u|<1\}} |\nabla u - l|^2 dx + \int_{B_r(x)} |D^s u|\right\}.$$

The following theorem provides a decay estimate for Φ.

THEOREM 2.8. *If u solves (1.1) with $B_r(a) \subset\subset \Omega$, $l_1 \in \mathbf{R}^n$ with $|l_1| \le 1 - \mu$, then there exist positive constants ω, ϵ, κ, c_{37}, c_{38}, and c_{39} such that*

$$\Phi(4r, l_1, a) \le \epsilon$$

and

$$r \le \kappa$$

implies

$$\Phi(\omega r, l_2, a) \le \frac{1}{2}\Phi(4r, l_1, a) + c_{37} r,$$

where

$$|l_1 - l_2| \le c_{38}\Phi(4r, l_1, a)^{\frac{1}{2}} + c_{39} r.$$

Proof. For fixed $\lambda > 0$, define

$$R^\lambda \equiv \{x \in B_{2r}(a) \,|\, \Phi(\rho, l_1, x) \le \lambda \text{ for all } 0 < \rho \le 2r\}.$$

By Vitali's covering theorem, there exist disjoint balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$ such that

$$B_{2r}(a) \backslash R^\lambda \subset \cup_{i=1}^\infty B_{5r_i}(x_i)$$

and $\Phi(r_i, l_1, x_i) \ge \lambda$. Then we have

$$\mathcal{L}^n(B_{2r}(a) \backslash R^\lambda) \le 5^n \sum_{i=1}^\infty |B_{r_i}(x_i)| \le \frac{5^n}{\lambda}|B_{4r}(a)|\Phi(4r, l_1, a).$$

Let $g(x) = u(x) - l_1 \cdot x$. By Poincarè's inequality we have for $x \in R^\lambda$ and $0 < \rho \le 2r$

$$\frac{1}{|B_\rho|}\int_{B_\rho(x)} |g(y) - \overline{g}_{x,\rho}|dy \le \frac{c_{15}}{\rho^{n-1}}\int_{B_\rho(x)} |Dg|$$

$$\le \frac{c_{15}}{\rho^{n-1}}\left\{2\int_{B_\rho(x)\cap\{|\nabla u|\ge 1\}} |\nabla u|dx + \int_{B_\rho(x)} |D^s u|\right.$$

$$\left. + |B_\rho|^{1/2}\left(\int_{B_\rho(x)\cap\{|\nabla u|<1\}} |\nabla u - l_1|^2 dx\right)^{1/2}\right\}$$

$$\le c_{16}\rho\Phi(\rho, l_1, x)^{1/2} \le c_{16}\lambda^{1/2}\rho,$$

where $\overline{g}_{x,\rho} = \frac{1}{|B_\rho|}\int_{B_\rho(x)} g(y)dy$. Then

$$|\overline{g}_{x,\rho/2^{k+1}} - \overline{g}_{x,\rho/2^k}| \le \frac{1}{|B_{\rho/2^{k+1}}|}\int_{B_{\rho/2^{k+1}}(x)} |g(y) - \overline{g}_{x,\rho/2^k}|dy$$

$$\le 2^n \frac{1}{|B_{\rho/2^k}|}\int_{B_{\rho/2^k}(x)} |g(y) - \overline{g}_{x,\rho/2^k}|dy \le c_{17}\rho\lambda^{1/2}/2^k.$$

Since $g(x) = \lim_{\rho\to 0} \overline{g}_{x,\rho}$ for $\mathcal{L}^n$ a.e. $x \in R^\lambda$,

$$|g(x) - \overline{g}_{x,\rho}| \le \sum_{k=1}^{\infty} |\overline{g}_{x,\rho/2^{k+1}} - \overline{g}_{x,\rho/2^k}| \le c_{17}\rho\lambda^{1/2}.$$

For x, $y \in R^\lambda$ with $|x - y| \le 2r$, set $\rho = |x - y|$. Then

$$|\overline{g}_{x,\rho} - \overline{g}_{y,\rho}| \le \frac{1}{|B_\rho(x)\cap B_\rho(y)|}\int_{B_\rho(x)\cap B_\rho(y)} |\overline{g}_{x,\rho} - g(z)| + |g(z) - \overline{g}_{y,\rho}|dz$$

$$\le c_{18}\frac{1}{B_\rho}\left(\int_{B_\rho(x)} |g(z) - \overline{g}_{x,\rho}|dz + \int_{B_\rho(y)} |g(z) - \overline{g}_{y,\rho}|dz\right) \le c_{19}\lambda^{1/2}\rho.$$

So by combining the above, we have

$$|g(x) - g(y)| \le c_{20}\lambda^{1/2}\rho = c_{20}\lambda^{1/2}|x - y|$$

for $\mathcal{L}^n$ a.e. x, $y \in R^\lambda \subset B_{2r}(a)$. Let $\lambda = c_{20}^{-2}\beta^{4\delta}$, so that

$$|u(x) - l_1 \cdot x - u(y) + l_1 \cdot y| = |g(x) - g(y)| \le \beta^{2\delta}|x - y|,$$

and let v be a Lipschitz function defined on $B_{2r}(a)$ such that

$$v = u \text{ on } R^\lambda, \text{ and } \sup_{B_{2r}(a)} |\nabla v - l_1| \le \beta^{2\delta}. \tag{2.24}$$

Such a v exists by a standard extension for a Lipschitz function. Also note that for this choice of v we have $\sup_{B_{2r}(a)} |v| \le C_u$. With the choice of λ, and by choosing

$$\beta = \Phi(4r, l_1, a) \text{ and } \delta = \frac{1}{8(n+1)},$$

we can estimate the size of the nonzero set of $u - v$ as

$$\mathcal{L}^n(B_{2r}(a) \cap \{u \neq v\}) \leq c_{21} r^n \beta^{-4\delta} \Phi(4r, l_1, a) \leq c_{21} r^n \Phi(4r, l_1, a)^{1-4\delta}.$$

We made the choice of δ so that $(1-4\delta) \cdot \frac{n+1}{n} = 1 + \frac{1}{2n} > 1$. Now choose $\tilde{r} \in [\frac{1}{2}r, r]$ so that both

$$\int_{\partial B_{\tilde{r}(a)}} |u - v| d\mathcal{H}^{n-1} \leq \frac{3}{r} \int_{B_{\tilde{r}(a)}} |u - v| dx$$

and

$$\int_{\partial B_{\tilde{r}(a)}} |u - \overline{u}_{a,r} - l_1 \cdot (x-a)| d\mathcal{H}^{n-1} \leq \frac{3}{r} \int_{B_{\tilde{r}(a)}} |u - \overline{u}_{a,r} - l_1 \cdot (x-a)| dx$$

are satisfied. By the choice of $\tilde{r}$,

$$\int_{\partial B_{\tilde{r}(a)}} |u - v| d\mathcal{H}^{n-1} \leq \frac{3}{r} \|u - v\|_{L^\infty(B_r(a))} \cdot \mathcal{L}^n \{B_r(a) \cap \{u \neq v\}\}.$$

Choose $r(C_u + \|I\|_{L^\infty(\Omega)}) \leq c_6$. By Lemma 2.7, for $\Phi(4r, l_1, a) \leq c_{22}$, we have

$$\frac{1}{r} \|u - v\|_{L^\infty(B_r(a))} \leq c_{10} \frac{1}{r} (\mathcal{L}^n(B_{2r}(a) \cap \{u \neq v\}))^{1/n}.$$

Thus

$$\frac{1}{r^n} \int_{\partial B_{\tilde{r}(a)}} |u - v| d\mathcal{H}^{n-1} \leq c_{23} \Phi(4r, l_1, a)^{1+\frac{1}{2n}}.$$

We now apply Lemma 2.6 to the above, using the estimate for the boundary integral of $u - v$, to obtain

$$\begin{aligned} &\int_{B_{r\omega}(a)} |D^s u| + \int_{B_{r\omega(a)} \cap \{|\nabla u| \geq 1\}} |\nabla u| dx + \int_{B_{r\omega(a)} \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 dx \\ &\quad \leq c_{24} r^n \left(\Phi(4r, l_1, a)^{1+\frac{1}{2n}} + \Phi(4r, l_1, a)^{1+\frac{1}{4(n+2)}} \right) \end{aligned} \tag{2.25}$$

for any $\omega \leq 1/2$. Let $l_2 \equiv \nabla \omega(a)$. By using the gradient estimate, (2.7)–(2.8), for ω, the choice of $\tilde{r}$, the definition of v_β, the above bound for v, and Poincarè's inequality,

$$\begin{aligned} |l_1 - l_2| &\leq \frac{1}{|B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}(a)}} |v_\beta - \overline{u}_{a,r} - l_1 \cdot (x-a)| d\mathcal{H}^{n-1} + c_{25} r(\|I\|_{L^\infty(\Omega)} + C_u) \\ &\leq \frac{1}{|B_{\tilde{r}}|} \int_{\partial B_{\tilde{r}(a)}} |v_\beta - u| + |u - \overline{u}_{a,r} - l_1 \cdot (x-a)| d\mathcal{H}^{n-1} + c_{25} r(\|I\|_{L^\infty(\Omega)} + C_u) \\ &\leq c_{26} \Phi(4r, l_1, a) + \frac{c_{27}}{r^n} \int_{B_r(a)} |Du - l_1| + c_{25} r(\|I\|_{L^\infty(\Omega)} + C_u). \end{aligned}$$

By the Hölder inequality, we obtain $|l_1 - l_2| \leq c_{28} \Phi(4r, l_1, a)^{1/2} + c_{25} r(\|I\|_{L^\infty(\Omega)} + C_u)$. The last term on the left side of inequality (2.25) satisfies

$$\int_{B_{r\omega}(a) \cap \{|\nabla u| < 1\}} |\nabla (u - w)|^2 dx \geq \int_{B_{r\omega}(a) \cap \{|\nabla u| < 1\}} \frac{1}{2} |\nabla u - l_2|^2 - |\nabla w - l_2|^2 dx.$$

Thus by (2.25) and the above inequality,

$$(2.26)\quad |B_{r\omega}|\Phi(r\omega, l_2, a) \le c_{29} r^n \Phi(4r, l_1, a)^{1+\frac{1}{4(n+2)}} + c_{30}\int_{B_{r\omega}} |\nabla w - l_2|^2 dx.$$

To estimate the last term, we again use the estimates for the gradient of w. Note that

$$\sup_{x,y\in B_{r/4}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^{1/2}} \le c_4 \frac{1}{r^{n+1/2}} \int_{\partial B_{\tilde{r}(a)}} |v_\beta - \overline{u}_{a,r} - l_1 \cdot (x-a)| d\mathcal{H}^{n-1} + c_4 r^{1/2}(\|I\|_{L^\infty(\Omega)} + C_u).$$

Therefore, similar to the estimate for $|l_1 - l_2|$, we have

$$\sup_{x,y\in B_{r/4}(a)} \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^{1/2}} \le c_{31}(r^{-1/2}\Phi(4r, l_1, a)^{1/2} + r^{1/2}(\|I\|_{L^\infty(\Omega)} + C_u)).$$

Using this we then have

$$\int_{B_{r\omega}(a)} |\nabla w - l_2|^2 dx \le c_{32}(r\omega)^n\{\omega\Phi(4r, l_1, a) + r\Phi(4r, l_1, a)^{1/2}(\|I\|_{L^\infty(\Omega)} + C_u) + r^2(\|I\|_{L^\infty(\Omega)} + C_u)^2\}$$

$$\le c_{33}(r\omega)^n\{\omega\Phi(4r, l_1, a) + r(\|I\|_{L^\infty(\Omega)} + C_u)\}.$$

Hence by combining the above with (2.26) we arrive at

$$\Phi(r\omega, l_2, a) \le c_{34}\omega^{-n}\Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} + c_{35}\omega\Phi(4r, l_1, a) + c_{36} r(\|I\|_{L^\infty(\Omega)} + C_u).$$

Choose $\omega < 1/4$ so small so that $c_{35}\omega < 1/4$, and again restrict $\Phi(4r, l_1, a)$ so that $c_{34}\omega^{-n}\Phi(4r, l_1, a)^{1+\frac{1}{4(n+1)}} < 1/4$. This now proves the theorem. ∎

We now prove Theorem 1.1 using an iteration argument (see, for example, [14] or [3]).

Proof. Assume that $\frac{1}{|B_r|}\int_{B_r(a)} |Du - l_1| \le \epsilon_0$ for some $l_1 \in \mathbf{R}^n$ with $|l_1| \le 1 - 4\mu$ and for any r with $r \le \kappa$. For each $x \in B_{r/2}(a)$ we have

$$\Phi(r/2, l_1, x) \le 2^n\Phi(r, l_1, a) \le c_{40}\frac{1}{|B_r|}\int_{B_r(a)} |Du - l_1| \le c_{40}\epsilon_0.$$

We will use Theorem 2.8 iteratively. Choose ϵ_0 so small so that $c_{40}\epsilon_0 \le \epsilon$ and restrict

r so that $c_{37}r \le r/2$. Assume $|l_{j-1}| < 1 - 2\mu$ and

$$\Phi\left(\left(\frac{\omega}{4}\right)^{j-1}\frac{r}{2}, l_j, x\right) \le \left(\frac{1}{2}\right)^{j-1}\Phi\left(\frac{r}{2}, l_1, x\right)$$
$$+\sum_{i=1}^{j-1}\left(\frac{1}{2}\right)^{j-1}\omega^{j-i-1}c_{41}r \text{ for } j = 2, \dots, k.$$

We need to show $\Phi\big(\left(\frac{\omega}{4}\right)^{k-1}\frac{r}{2}, l_k, x\big) \le \epsilon$ and $|l_k| < 1 - 2\mu$ in order to continue the inductive step. Since $\omega < 1/2$,

$$\sum_{i=1}^{k-1}\left(\frac{1}{2}\right)^{i-1}\omega^{k-i-1} \le \left(\frac{1}{2}\right)^{k-2}(k-1) \le c_{42}\left(\frac{1}{2}\right)^{k/2}$$

for all k. By further restricting r, we have

$$\Phi\left(\left(\frac{\omega}{4}\right)^{k-1}\frac{r}{2}, l_k, x\right) \le \epsilon.$$

Note that

$$|l_k| \le \sum_{j=1}^{k-1}|l_{j+1} - l_j| + |l_1|$$
$$\le \sum_{j=1}^{k-1}\left\{c_{38}\Phi\left(\left(\frac{\omega}{4}\right)^{j-1}\frac{r}{2}, l_j, x\right)^{1/2} + c_{39}\left(\frac{\omega}{4}\right)^{j-1}r\right\} + 1 - 4\mu$$
$$\le c_{38}\sum_{j=1}^{k-1}\left\{\left(\frac{1}{2}\right)^{(j-1)/2}\Phi(\frac{r}{2}, l_1, x)^{1/2} + c_{41}^{1/2}\left(\frac{1}{2}\right)^{j/4}c_{37}^{1/2}r^{1/2}\right\}$$
$$+c_{39}r\sum_{j=1}^{k-1}\left(\frac{\omega}{4}\right)^{j-1} + 1 - 4\mu$$
$$\le c_{42}\Phi\left(\frac{r}{2}, l_1, x\right)^{1/2} + c_{42}r^{1/2} + 1 - 4\mu.$$

So by restricting ϵ_0 and r again, we see that $|l_k| < 1 - 2\mu$. Thus we may continue the iterative step indefinitely, giving

$$\lim_{k\to\infty}\left(\Phi\left(\frac{\omega}{4}\right)^k\frac{r}{2}, l_{k+1}, x\right) = 0 \text{ for all } x \in B_{r/2}(a).$$

Thus

$$\lim_{\rho\to 0}\frac{1}{|B_\rho|}\left(\int_{B_\rho(x)}|D^s u| + \int_{B_\rho(x)\cap\{|\nabla u|\ge 1\}}|\nabla u|dx\right) = 0$$

for all $x \in B_{r/2}(a)$. We then have (see, for instance, [10]) $|D^s u|\left(B_{r/2}(a)\right) = 0$ with $|\nabla u| \le 1-\mu < 1$ a.e. on $B_{r/2}(a)$. By (2.21), u also satisfies the stated equation. ∎

Using Theorem 1.1, we can now easily prove Theorem 1.2.

Proof. Assume that u is a minimizer of (1.1) and that $\tilde{E} = \{|\nabla u| < 1\}$ has positive Lebesgue measure. From standard measure theory (see, for example, [10]),

$$\lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r(x)} |D^s u| = 0 \tag{2.27}$$

for $\mathcal{L}^n$-a.e. $x \in \tilde{E}$. Also, since $|\nabla u| \in L^1(\Omega)$,

$$\lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r(x)} |\nabla u(y) - \nabla u(x)| dx = 0 \tag{2.28}$$

for $\mathcal{L}^n$-a.e. $x \in \tilde{E}$ by Lebesgue's differentiation theorem. Now let E be the set of all points of $\tilde{E}$ for which both (2.27) and (2.28) hold. Clearly $\mathcal{L}^n(\tilde{E}\backslash E) = 0$, $|\nabla u| < 1$ on E, and both (2.27) and (2.28) hold at each point of E. For each fixed $x \in E$, there exists some $\mu_x > 0$ such that

$$|\nabla u(x)| < 1 - 2\mu_x.$$

Then (2.27) and (2.28) combined with Theorem 1.1 show that there exists an r_x such that

$$|D^s u|(B_{r_x}(x)) = 0 \text{ and } |\nabla u| < 1 - \mu_x \text{ on } B_{r_x}(x)$$

and $u \in C^{1,\alpha}(B_{r_x}(x))$, giving $B_{r_x}(x) \subset E$ in particular. Thus E is an open set in Ω with the required properties. ∎

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