[32] Rokhlin, V., *Complex topological characteristics of real algebraic curves*, Russian Math. Surveys 33(5), 1978, pp. 85–98.
[33] Seifert, H., *Algebraische approximation von Manningfaltigkeiten*, Math. Z. 41, 1936, pp. 1–17.
[34] Thom, R., *Sur le cut-locus d'une variété plongée*, J. Differential Geom. 6, 1972, pp. 577–586.
[35] Thorpe, J., *Elementary Topics in Differential Geometry*, Springer-Verlag, Berlin, 1979.
[36] Timan, A. F., *Theory of approximations of functions of a real variable*, Pergamon Press, Oxford, 1963.
[37] Vainstein, A. G., Efremovitch, V. A., and Loginov, E. A., *On the skeleton of a Riemann manifold with an edge*, Russian Math. Surveys 33(3), 1978, 181–182.
[38] Vershik, A. M., *Topology of the convex polytopes manifolds, the manifold of the projective configurations of a given combinatorial type and representations of lattices*, in *Topology and Geometry—Rohlin Seminar*, ed. by O. Ya. Viro, Lecture Notes in Math., N 1346, Springer-Verlag, Berlin, 1988, pp. 557–581.
[39] Viro, O., *Progress in the topology of real algebraic varieties over the last six years*, Russian Math. Surveys 41(3), 1986, pp. 55–82.
[40] Volodin, I. A., Kuznetsov, V. E., and Fomenko, A. T., *The problem of discriminating algorithmically the standard three-dimensional sphere*, Russian Math. Surveys 29(5), 1974, pp. 71–172.
[41] Wallace, A. D., *Algebraic approximations of manifolds*, Proc. London Math. Soc. 7 1957, pp. 196–210.
[42] Whitney, H., *Geometric Integration Theory*, Princeton University Press, Princeton, 1957.

Received August 1993.

Partial Regularity for Weak Heat Flows into Spheres

YUNMEI CHEN
University of Florida

JIAYU LI
International Centre for Theoretical Physics, Trieste

AND

FANG-HUA LIN
Courant Institute

Abstract

In this paper we show that a weak heat flow of harmonic maps from a compact Riemannian manifold (possibly with boundary) into a sphere, satisfying the monotonicity inequality and the energy inequality, is regular off a closed set of m-dimensional Hausdorff measure zero.

1. Introduction

The global existence of weak solutions to the heat flow of harmonic maps between compact Riemannian manifolds was first shown by Chen and Struwe in [8]; see also [4] and [16]. Their specially constructed weak solutions are smooth off a relatively small and closed subset of space-time. Recently, two of us (see [6]) have generalized the main results of [8] to the case where the domains are compact Riemannian manifolds with boundary. On the other hand, it was shown by Coron in [11] some time ago that the weak solutions to the heat flow of harmonic maps are highly nonunique in general, if the dimension of the domain is greater than two. Therefore, a natural question is the uniqueness and the regularity of the weak heat flows in a suitable class. In [7] it is proved that the weak solutions obtained in [8] and [6] are unique and that they coincide with the classical solution provided that the latter exists. These results show the remarkable similarities between the theory of the heat flow of harmonic maps and the theory for Navier-Stokes equations; cf. [21] and references therein.

In this paper, motivated by the work of Caffarelli-Kohn-Nirenberg, [9], we study the question of partial regularity for a suitable class of weak solutions. Though it is very likely that the main results of this paper can be extended to more general situations, for simplicity we shall assume that the target manifold of maps is a Euclidean sphere. This suitable class of weak solutions can be defined as those weak solutions which satisfy both the generalized energy inequality (see (2.7) below) and the monotonicity inequality (see (2.1) below). It is clear from [9] that the generalized energy inequality (2.7) is crucial. On the other hand, as

Communications on Pure and Applied Mathematics, Vol. XLVIII, 429–448 (1995)

CCC 0010-3640/95/040429-20

the nonlinearity of the equations for heat flow of harmonic maps is somewhat greater than that of Navier-Stokes equations, we may need several assumptions in addition to (2.7). Inspired by the works of Helein, [15], Evans, [12], and Bethuel, [1], on the corresponding stationary case, it becomes obvious that this additional assumption should be the energy monotonicity inequality. Our main result is the following

THEOREM 1.1. *Let M be an m-dimensional compact, smooth, Riemannian manifold. Assume that $u \in H^{1,2}_{loc}(M \times \mathbb{R}^+, S^{n-1})$ is a global weak solution of heat flow of harmonic maps from M into S^{n-1} with $E(u_0) < \infty$. If u satisfies the monotonicity inequalities (2.1) and the energy inequality (2.7), then there exists an open subset V of $M \times \mathbb{R}_+$ such that $u \in C^\infty(V, S^{n-1})$ and $H^m((M \times \mathbb{R}^+) \setminus V) = 0$, where H^m denotes m-dimensional Hausdorff measure with respect to the parabolic metric $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$.*

The proof of this theorem is analogous to the proof of the partial regularity for harmonic maps given by Evans in [12] — it uses the energy inequality, the parabolic monotonicity inequality, parabolic Hardy and BMO spaces, and Morrey's lemma of parabolic type.

M. Feldman, [13], proved a similar result in June 1993 while our paper was still in its infancy. He considered the "stationary" class of weak solutions. As was shown in [13], the"stationary" weak solutions satisfy both the generalized energy inequality (2.7) and the monotonicity inequality (2.1). Since there are substantial differences in many technical details — in particular, in the application of the duality between the parabolic BMO and Hardy spaces — we think that this paper may have independent interest. Moreover, a version of partial regularity at the boundary ∂M, whenever ∂M is suitable smooth (say, C^2), is also shown (see Theorem 5.1 below).

For simplicity we shall present our proofs in the case where $M = \mathbb{R}^m$. The general cases can be done in the same manner. Here we shall consider the weak solutions of

$$\frac{\partial u}{\partial t} - \Delta u = |\nabla u|^2 u, \qquad x \in \mathbb{R}^m, \qquad t > 0, \tag{1.1}$$

$$u(x,0) = u_0, \qquad x \in \mathbb{R}^m, \tag{1.2}$$

2. The Monotonicity Inequality and the Energy Inequality

The monotonicity inequality for the regular solution of (1.1)–(1.2) has been obtained by Struwe in [23]. For the needs of this paper we give an alternative form which is just a slight modification of [23].

We adopt the same notation as in [23]. Denote by $z = (x,t)$ a point in $M \times \mathbb{R}$. For a distinguished point $z_0 = (x_0, t_0)$, $r > 0$, let

$$\begin{aligned} B_r(x_0) &= \{x \in \mathbb{R}^m \mid |x - x_0| < r\}, \\ P_r(z_0) &= \{z = (x,t) \in \mathbb{R}^m \times \mathbb{R} \mid |x - x_0| < r, \ |t - t_0| < r^2\}, \text{ and} \\ T_r(z_0) &= \{z = (x,t) \in \mathbb{R}^m \times \mathbb{R} \mid t_0 - 4r^2 < t < t_0 - r^2\}. \end{aligned}$$

Denote the fundamental solution to the (backward) heat equation $(\frac{\partial}{\partial t} + \Delta) f(x,t) = 0$ on $\mathbb{R}^m \times \mathbb{R}$ by

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\Big(-\frac{(x-x_0)^2}{4(t_0 - t)}\Big), \qquad t < t_0.$$

If $z_0 = (0,0)$, we simply put $G = G_{(0,0)}$, $T_r = T_r(0,0)$, $P_r = P_r(0,0)$, and $B_r = B_r(0)$. We denote by δ the parabolic distance function

$$\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}.$$

Let $\beta > 0$ be any fixed constant. For any $z_1 \in \mathbb{R}^m \times \mathbb{R}_+$, define, for $R \in (0, \sqrt{t_1}/2\beta)$,

$$\Psi_\beta(R, u, z_1) = \frac{1}{2} \int_{T_{\beta R}(z_1)} |\nabla u|^2 G_{z_1} \varphi_\beta^2 \, dx dt,$$

where $\varphi_\beta(x) = \varphi((x - x_1)/\beta)$ and $\varphi \in C_0^\infty(B_{1/2}(0))$ is a cut-off function such that $0 \leqq \varphi \leqq 1$ and $\varphi \equiv 1$ on $B_{1/4}(0)$.

LEMMA 2.1. *There exists a constant $C > 0$, depending only on m, such that for any $z_1 \in \mathbb{R}^m \times \mathbb{R}_+$, $0 < R_1 \leqq R_2 \leqq \min(\sqrt{t_1}/2\beta, 1/4)$, we have*

$$\begin{aligned} \Psi_\beta(R_1, u, z_1) &\leqq e^{C(R_2 - R_1)} \Psi_\beta(R_2, u, z_1) \\ &\quad + C(R_2 - R_1)\beta^{-m} \int_{P_{\sqrt{t_1}/2}(z_1)} |\nabla u|^2 \, dx dt. \end{aligned} \tag{2.1}$$

Proof: As in the derivation for the monotonicity inequalities in [23], we may shift to $z_1 = (0,0)$ and let $u_R(x,t) = u(Rx, R^2 t)$. Then

$$\begin{aligned} \frac{d}{dR}\Psi_\beta(R, u, (0,0)) &= \frac{d}{dR}\Psi_\beta(1, u_R, (0,0)) \\ &= \int_{T_\beta} \nabla u_R \cdot \nabla \frac{du_R}{dR} G \varphi_\beta^2(Rx) \, dx dt \\ &\quad + \int_{T_\beta} |\nabla u_R|^2 G \varphi_\beta(Rx)(\nabla\varphi)\Big(\frac{Rx}{\beta}\Big)\frac{x}{\beta} \, dx dt = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.2}$$

Integrating by parts, from (1.1) we get

$$\text{(2.3)}\qquad \begin{aligned} I &= 2\int_{\Gamma_\beta} \frac{R}{|t|}\left(\frac{du_R}{dR}\right)^2 G\varphi_\beta^2(Rx)\,dxdt \\ &\quad - 2\int_{\Gamma_\beta} \nabla u_R \cdot \frac{du_R}{dR} G\varphi_\beta(Rx)(\nabla\varphi)\left(\frac{Rx}{\beta}\right)\frac{R}{\beta}\,dxdt = I' + I'' \end{aligned}$$

with $I' > 0$. We have

$$\text{(2.4)}\qquad \begin{aligned} I'' &\leq I'/2 + C\int_{T_\beta} |t|\,|\nabla u_R|^2 G |\nabla\varphi|^2\left(\frac{Rx}{\beta}\right)\frac{R}{\beta^2}\,dxdt \\ &\leq I'/2 + C\int_{-4\beta^2}^{\beta^2}\int_{\beta/4R \leq |x| \leq \beta/2R} |\nabla u_R|^2 GR\,dxdt \\ &\leq I'/2 + C\int_{-4\beta^2}^{\beta^2}\int_{\beta/4R \leq |x| \leq \beta/2R} |\nabla u|^2(Rx, R^2t) GR^3\,dxdt \\ &\leq I'/2 + C\int_{-4\beta^2R^2}^{\beta^2R^2}\int_{|x|\leq\beta/2} \beta^{-m}R^{2-m}\exp\left(-\frac{1}{256R^2}\right)|\nabla u|^2\,dxdt \\ &\leq I'/2 + C\beta^{-m}\int_{P_{\beta/2}} |\nabla u|^2\,dxdt\,. \end{aligned}$$

Similarly,

$$\text{(2.5)}\qquad II \leq \Psi_\beta(R, u, (0,0)) + C\beta^{-m}\int_{P_{\beta/2}} |\nabla u|^2\,dxdt\,.$$

From (2.2)–(2.5), we infer that

$$\frac{d}{dR}\Psi_\beta(R, u, (0,0)) \geq -\Psi_\beta(R, u, (0,0)) - C\beta^{-m}\int_{P_{\beta/2}} |\nabla u|^2\,dxdt\,,$$

which implies (2.1). The proof of this lemma is completed.

For $z \in \mathbb{R}^m \times \mathbb{R}_+$ and $0 < r < \sqrt{t}$ define

$$E(r, u, z) = r^{-m}\int_{P_r(z)} |\nabla u|^2\,dyds\,.$$

From Lemma 2.1 we have the following inequality for $E(r, u, z)$.

LEMMA 2.2. *There exists a constant $K > 0$, depending only on m, such that*

$$\text{(2.6)}\qquad E(r, u, z) \leq KE(r_1, u, z_1)$$

for $z \in P_{ar_1}(z_1)$ and $0 < r < br_1$, where a and b are two positive constants satisfying $a + 2b \leq 1/2$.

Proof: By Lemma 2.1, we have

$$\begin{aligned} E(r, u, z) &\leq C\int_{(t+2r^2)-4r^2}^{(t+2r^2)-r^2}\int_{B_r(x)} |\nabla u|^2 G_{(x,t+2r^2)}\phi_{r_1}^2(y)\,dyds \\ &\leq C\Psi_{r_1}(r/r_1, u, (x, t+2r^2)) \\ &\leq C\Psi_{r_1}(1/4, u, (x, t+2r^2)) + Cr_1^{-m}\int_{P_{r_1/2}(x,t+2r^2)} |\nabla u|^2\,dyds \\ &\leq Cr_1^{-m}\int_{P_{r_1/2}(x,t+2r^2)} |\nabla u|^2\,dyds \\ &\leq Kr_1^{-m}\int_{P_{r_1}(z_1)} |\nabla u|^2\,dyds\,, \end{aligned}$$

completing the proof of the lemma.

The regular solution u of (1.1)–(1.2) also satisfies the following energy inequality: For $\varphi \in C_0^\infty(\mathbb{R}^m)$ and $0 < t_1 < t_2 < \infty$, we have

$$\text{(2.7)}\qquad \begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}^m} \varphi^2|\partial_t u|^2\,dxdt + \int_{\mathbb{R}^m} \varphi^2|\nabla u(\cdot, t_2)|^2\,dx \\ &\leq \int_{\mathbb{R}^m} \varphi^2|\nabla u(\cdot, t_1)|^2\,dx + 4\int_{t_1}^{t_2}\int_{\mathbb{R}^m} |\nabla\varphi|^2|\nabla u(\cdot, t)|^2\,dxdt\,. \end{aligned}$$

By the energy inequality we can partially control $\partial_t u$.

LEMMA 2.3. *There exists a constant $C > 0$, depending only on m, such that*

$$\text{(2.8)}\qquad r^{2-m}\int_{P_{r/2}(z)} |\partial_t u|^2\,dyds \leq Cr^{-m}\int_{P_r(z)} |\nabla u|^2\,dyds\,,$$

for $z \in \mathbb{R}^m \times \mathbb{R}_+$ and $0 < r \leq \sqrt{t}$.

Proof: Via Fubini's theorem we may choose $\alpha \in (1/2, 7/8)$ such that

$$\text{(2.9)}\qquad \int_{B_r(x)} |\nabla u|^2(y, t - \alpha^2r^2)\,dy \leq Cr^{-2}\int_{P_r(z)} |\nabla u|^2\,dyds\,.$$

Choose a smooth cut-off function $\varphi \in C_0^\infty(\mathbb{R}^m)$ such that $\varphi = 1$ in $B_{ar}(x)$, $\varphi = 0$ outside $B_r(x_0)$, $0 \leq \varphi \leq 1$, and $|\nabla\varphi| \leq C/r$. It follows from (2.7) and (2.9) that

$$\begin{aligned} r^{2-m}\int_{P_{r/2}(z)} |\partial_t u|^2\,dyds &\leq Cr^{2-m}\int_{P_{ar}(z)} |\partial_t u|^2\,dyds \\ &\leq Cr^{2-m}\int_{B_r(x)} |\nabla u|^2(y, t-\alpha^2r^2)\,dy + Cr^{-m}\int_{P_r(z)} |\nabla u|^2\,dyds \\ &\leq Cr^{-m}\int_{P_r(z)} |\nabla u|^2\,dyds\,. \end{aligned}$$

Let $u_\beta(x,t) = u(x_\beta + \beta x, t_\beta + \beta^2 t)$ be the scaling function of u. We will show the corresponding forms of (2.1), (2.6), and (2.8) for u_β.

For any $z_0 \in \mathbb{R}^m \times (-t_\beta/\beta^2, +\infty)$, define, for $0 < R < \sqrt{t_\beta + \beta^2 t_0}/2\beta$,

$$\Psi(R, u_\beta, z_0) = \frac{1}{2}\int_{T_R(t_0)} |\nabla u_\beta|^2 G_{z_0}\varphi^2(x - x_0)\,dx\,dt\,.$$

where φ is the same function as that in the $\Psi_\beta(R, u, z_1)$. It is easy to verify that $\Psi(R, u_\beta, z_0) = \Psi_\beta(R, u, z_1)$, where $x_1 = x_\beta + \beta x_0$ and $t_1 = t_\beta + \beta^2 t_0$. Therefore, Lemma 2.1 implies the following monotonicity formulas for u_β:

Proposition 2.4. *There exists a constant $C > 0$, depending only on m, such that for $z_0 \in \mathbb{R}^m \times (-t_\beta/\beta^2, +\infty)$ and $0 < R_1 \leqq R_2 \leqq \min\left(\sqrt{t_\beta + \beta^2 t_0}/2\beta, 1/4\right)$, we have*

$$\begin{aligned}\Psi(R_1, u_\beta, z_0) \quad &\leqq \quad e^{C(R_2 - R_1)}\Psi(R_2, u_\beta, z_0)\\ &+ C(R_2 - R_1)\int_{P_{1/2}(z_0)} |\nabla u_\beta|^2\,dx\,dt\,.\end{aligned} \tag{2.10}$$

As a consequence of Lemma 2.2 (taking $r_1 = \beta$, $z_1 = z_\beta$, $a = 7/16$, and $b = 1/32$), we have the following

Proposition 2.5. *There exists a constant $C > 0$, depending only on m, such that*

$$E(r, u_\beta, z) \leqq CE(1, u_\beta, (0,0)) \tag{2.11}$$

for $z \in P_{7/16}$ and $0 < r < 1/32$.

Remark 2.6. The inequality (2.8) also holds for u_β.

Remark 2.7. It is easy to see that (2.1) is essentially the same as the monotonicity inequalities in [23] and [8]. The regular solution of (1.1) satisfies (2.1) and the weak solution obtained by Chen-Struwe, [8], satisfies (2.1) off a singular set of codimension greater than or equal to two.

3. An Improved Monotonicity Inequality

In this section we shall establish an improved monotonicity formula, which plays an important role on the proof of Theorem 1.1.

Theorem 3.1. *Under the same assumptions as in Theorem 1.1, there exist $\varepsilon_0 > 0$ and $0 < \sigma \leqq 1/4$ such that $E(r, u, z) < \varepsilon_0$ implies $E(\sigma r, u, z) \leqq (1/2)E(r, u, z)$ for $z \in \mathbb{R}^m \times \mathbb{R}_+$, $0 < r < \sqrt{t}$.*

Proof: We argue by contradiction. If it were false, then there would exist $z_k \in \mathbb{R}^m \times \mathbb{R}_+$ and $0 < r_k < \sqrt{t_k}$, such that $E(r_k, u, z_k) = \lambda_k^2 \to 0$ as $k \to \infty$, but $E(\sigma r_k, u, z_k) > (1/2)\lambda_k^2$ for any $\sigma > 0$. Define

$$u_k(x,t) = u(x_k + r_k x, t_k + r_k^2 t), \qquad a_k = \frac{1}{\mathrm{Vol}\,(P_{1/2})}\int_{P_{1/2}} u_k\,dx\,dt\,, \text{ and}$$

$$v_k(x,t) = \frac{u_k(x,t) - a_k}{\lambda_k}\,.$$

Then

$$E(\sigma, v_k, (0,0)) = \lambda_k^{-2}E(\sigma r_k, u, z_k) > 1/2 \tag{3.1}$$

for any $\sigma > 0$.

We first show that we can choose a subsequence (which we also denote by v_k) such that

$$\nabla v_k \to \nabla v_\infty \text{ weakly in } L^2(P_1, \mathbf{R}^{m+n}) \tag{3.2}$$

$$\partial_t v_k \to \partial_t v_\infty \text{ weakly in } L^2(P_{1/2}, \mathbf{R}^n)\,. \tag{3.3}$$

It is obvious that

$$\int_{P_1} |\nabla v_k|^2\,dx\,dt = 1\,, \tag{3.4}$$

which implies (3.2). Furthermore, from (2.8) we have

$$\int_{P_{1/2}} |\partial_t v_k|^2\,dx\,dt \leqq C\int_{P_1} |\nabla v_k|^2\,dx\,dt \leqq C\,, \tag{3.5}$$

which leads to (3.3). Moreover, by using the Poincaré inequality with (3.4) and (3.5), we have

$$\int_{P_{1/2}} |v_k|^2\,dx\,dt \leqq C\,.$$

Therefore, there exists a subsequence of v_k, also denoted by v_k, such that

$$v_k \to v_\infty \text{ strongly in } L^2(P_{1/2}, \mathbf{R}^n)\,. \tag{3.6}$$

Let $\omega : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ be any smooth map with compact s[illegible] $P_{1/2}$. Define $\omega_k(y,s) = \omega((y - x_k)/r_k, (s - t_k)/r_k^2)$. Since u is a weak [illegible] we have

$$\int_{t_k - r_k^2/4}^{t_k + r_k^2/4}\int_{\mathbb{R}^m} u_t\omega_k\,dy\,ds + \int_{t_k - r_k^2/4}^{t_k + r_k^2/4}\int$$

$$= \int_{t_k - r_k^2/4}^{t_k + r_k^2/4}\int_{\mathbb{R}^m} |\nabla u|^2 u \cdot \omega_k$$

Rescaling this identity we obtain

$$\int_{-1/4}^{1/4}\int_{\mathbb{R}^m}(v_k)_t\omega\,dxdt+\int_{-1/4}^{1/4}\int_{\mathbb{R}^m}\nabla v_k\cdot\nabla\omega\,dxdt = \lambda_k\int_{-1/4}^{1/4}\int_{\mathbb{R}^m}|\nabla v_k|^2(a_k+\lambda_k v_k)\omega\,dxdt\,. \tag{3.7}$$

Since $|a_k+\lambda_k v_k| = 1$, letting $k\to\infty$ we have from (3.2)–(3.3) and (3.6) that

$$\int_{-1/4}^{1/4}\int_{\mathbb{R}^m}(v_\infty)_t\omega\,dxdt+\int_{-1/4}^{1/4}\int_{\mathbb{R}^m}\nabla v_\infty\cdot\nabla\omega\,dxdt = 0\,. \tag{3.8}$$

Therefore, v_∞ is a smooth solution of the equation $(\frac{\partial}{\partial t}-\Delta)v_\infty = 0$, and we have

$$\|\nabla v_\infty\|_{L^\infty(P_{\sigma})}\leqq C\int_{P_{1/2}} v_\infty^2\,dxdt\,. \tag{3.9}$$

Next we would like to prove that

$$\nabla v_k \longrightarrow \nabla v_\infty \text{ strongly in } L^2(P_{1/4},\mathbb{R}^{m+n})\,.$$

If this was proved, letting $k\longrightarrow\infty$ in (3.1) one could show that

$$1/2\leqq \sigma^{-m}\int_{P_\sigma}|\nabla v_\infty|^2\,dxdt\leqq C\sigma^2\,.$$

This leads to a contradiction when σ is sufficiently small.

Now we prove the following proposition:

PROPOSITION 3.2. *The sequence ∇v_k converges to the limit ∇v_∞ strongly in $L^2(P_{1/4},\mathbb{R}^{m+n})$.*

Let δ be the parabolic distance, equivalent to ρ defined in [2] and [3] with respect to the group

$$A_r = \begin{pmatrix} rI_m & 0\\ 0 & r^2\end{pmatrix},$$

and let μ be Lebesgue measure on $\mathbb{R}^{m+1}$. We consider the metric space $X = (\mathbb{R}^{m+1},\delta,\mu)$. Clearly X satisfies the hypotheses in [22], Chapter I. For $z=(x,t)\in\mathbb{R}^{m+1}$ and $r>0$, denote $P_r(z)=\{z'=(y,s)\in\mathbb{R}^{m+1}\,|\,\delta(z',z)<r\}$.

We set

$$f_\#(z) = \sup_{r>0}\frac{1}{\mu(P_r(z))}\int_{P_r(z)}|f-(f)_{z,r}|\,dyds\,.$$

Define

$$\mathrm{BMO}(X,\mathbb{R}^n) = \{f\in L_{\mathrm{loc}}(X,\mathbb{R}^n)\,|\,f_\#(z)\in L^\infty(X)\}\,,$$

and define the norm of f to be $\|f\|_{\mathrm{BMO}} = \|f_\#\|_{L^\infty(X)}$, where we take $(f)_{z,r} = \mu(P_r(z))^{-1}\int_{P_r(z)} f\,dyds$.

Choose a cut-off function $\zeta\in C_0^\infty(\mathbb{R}^{m+1})$ such that $\zeta = 1$ in $P_{1/4}$, $\zeta = 0$ in $\mathbb{R}^{m+1}\setminus P_{3/8}$, and $0\leqq\zeta\leqq 1$.

LEMMA 3.3. *The sequence ζv_k is bounded in $\mathrm{BMO}(X,\mathbb{R}^n)$.*

Proof: For any point $z_0\in P_{7/16}$ and radius $0<r<1/64$, from (2.11) (take $u_\beta = u_k$) we have

$$(2r)^{-m}\int_{P_{2r}(z_0)}|\nabla v_k|^2\,dxdt\leqq C\int_{P_1}|\nabla v_k|^2\,dxdt\leqq C\,. \tag{3.10}$$

Furthermore, from (2.8) (for u_k) we get

$$r^{2-m}\int_{P_r(z_0)}|\partial_t v_k|^2\,dxdt\leqq C(2r)^{-m}\int_{P_{2r}(z_0)}|\nabla v_k|^2\,dxdt\leqq C\,. \tag{3.11}$$

Therefore, we have from (3.10)–(3.11) that

$$\mu(P_r(z_0))^{-1}\int_{P_r(z_0)}|v_k-(v_k)_{z,r}|\,dxdt\leqq C\,, \tag{3.12}$$

where $(v_k)_{z,r} = \mu(P_r(z_0))^{-1}\int_{P_r(z_0)} v_k\,dxdt$.

The John-Nirenberg inequality (see [22], Chapter III) implies that

$$v_k \text{ is bounded in } L^p(P_{7/16},\mathbb{R}^n)\,,\qquad 1\leqq p<\infty\,. \tag{3.13}$$

Since ζ is smooth, we have

$$|(\zeta v_k)_{z,r}-\zeta(v_k)_{z,r}|\leqq Cr\mu(P_r(z_0))^{-1}\int_{P_r(z_0)}|v_k|\,dxdt$$

on $P_r(z_0)$, for all r and z_0.

If $z_0\in P_{13/32}$, $0<r<1/64$, we have from (3.12)–(3.13) that

$$\begin{aligned}
\mu(P_r(z_0))^{-1}&\int_{P_r(z_0)}|\zeta v_k-(\zeta v_k)_{z,r}|\,dxdt\\
&\leqq \mu(P_r(z_0))^{-1}\int_{P_r(z_0)}|v_k-(v_k)_{z,r}|\,dxdt + Cr\mu(P_r(z_0))^{-1}\int_{P_r(z_0)}|v_k|\,dxdt\\
&\leqq C+\frac{C}{r^{m+1}}\int_{t_0-r^2}^{t_0+r^2}\int_{B_r(x_0)}|v_k|\,dxdt\\
&\leqq C+\frac{C}{r^{m+1}}\left(\int_{t_0-r^2}^{t_0+r^2}\int_{B_r(x_0)}|v_k|^{m+2}\,dxdt\right)^{1/(m+2)} r^{(m+2)(1-1/(m+2))}\leqq C\,.
\end{aligned}$$

Since $\zeta = 0$ outside $P_{3/8}$, the same inequality holds for $z_0 \in \mathbb{R}^{m+1} \setminus P_{13/32}$ and $0 < r < 1/64$. The lemma follows.

We define

$$b_{k,l}^{ij} = v_{k,x_l}^j (a_k^i + \lambda_k v_k^i) - v_{k,x_l}^i (a_k^j + \lambda_k v_k^j)$$

for $1 \leq i, j \leq n$, $1 \leq l \leq m$, $k = 1, 2, \ldots$.

LEMMA 3.4. *For each function $\phi \in H^{1,2}(P_{1/2}) \cap L^\infty(P_{1/2})$ with compact support*

$$\int_{P_{1/2}} \phi_{x_l} \cdot b_{k,l}^{ij}\, dx dt = \int_{P_{1/2}} \phi \left[v_{k,t}^j (a_k^i + \lambda_k v_k^i) - v_{k,t}^i (a_k^j + \lambda_k v_k^j) \right] dx dt .$$

Proof:

$$\begin{aligned} \int_{P_{1/2}} \phi_{x_l} \cdot b_{k,l}^{ij}\, dx dt &= \int_{P_{1/2}} \left(v_{k,x_l}^j ((a_k^i + \lambda_k v_k^i)\phi)_{x_l} - v_{k,x_l}^i ((a_k^j + \lambda_k v_k^j)\phi)_{x_l} \right) dx dt \\ &= \lambda_k \int_{P_{1/2}} |\nabla v_k|^2 ((a_k^j + \lambda_k v_k^j)(a_k^i + \lambda_k v_k^i) \\ &\quad - (a_k^i + \lambda_k v_k^i)(a_k^j + \lambda_k v_k^j))\phi\, dx dt \\ &\quad + \int_{P_{1/2}} \phi (v_{k,t}^j (a_k^i + \lambda_k v_k^i) - v_{k,t}^i (a_k^j + \lambda_k v_k^j))\, dx dt . \end{aligned}$$

The last identity follows from (3.7) by setting $\omega = \phi(a_k^i + \lambda_k v_k^i)$ and $\omega = \phi(a_k^j + \lambda_k v_k^j)$.

We define the parabolic Hardy space $H^1(\mathbb{R}^{m+1})$ associated with the group

$$A_r = \begin{pmatrix} r I_m & 0 \\ 0 & r^2 \end{pmatrix}$$

as follows; see [2] and [3]. Choose $\phi \in C_0^\infty(P_1)$ satisfying $\int_{\mathbb{R}^{m+1}} \phi\, dx dt = 1$. Suppose $f(x,t) \in L^1(\mathbb{R}^{m+1})$. We set

$$f^*(x,t) = \sup_{r>0} \left| \frac{1}{r^{m+2}} \int_{\mathbb{R}^{m+1}} f(y,s) \phi\left(\frac{x-y}{r}, \frac{t-s}{r^2} \right) dy ds \right| .$$

We define

$$H^1(\mathbb{R}^{m+1}) = \{ f \in L^1(\mathbb{R}^{m+1}) \mid f^*(x,t) \in L^1(\mathbb{R}^{m+1}) \}$$

and define the norm of f to be $\|f\|_{H^1} = \|f^*\|_{L^1}$. The space $H^1(\mathbb{R}^{m+1})$ does not depend on the choice of ϕ and for any two choices of ϕ the resulting norms are equivalent.

LEMMA 3.5. *For each $1 \leq i, j \leq n$, the sequence $(\zeta v_k^i)_{x_l} b_{k,l}^{ij}$ is bounded in $H^1(\mathbb{R}^{m+1})$.*

Proof: Let ϕ be any smooth function with support in P_1 satisfying the equality $\int_{\mathbb{R}^{m+1}} \phi\, dx dt = 1$. Choose $z = (x,t) \in \mathbb{R}^{m+1}$, $r > 0$, and set $\phi_r(y,s) = \phi((x-y)/r, (t-s)/r^2)$. By Lemma 3.4, we have

$$\begin{aligned} \frac{1}{r^{m+2}} \int_{\mathbb{R}^{m+1}} ((\zeta v_k^i - (\zeta v_k^i)_{z,r})\phi_r)_{x_l} \cdot b_{k,l}^{ij}\, dy ds &= \frac{1}{r^{m+2}} \int_{P_r(z)} ((\zeta v_k^i - (\zeta v_k^i)_{z,r})\phi_r)(v_{k,t}^j (a_k^j + \lambda_k v_k^j) \\ &\quad - v_{k,t}^j (a_k^i + \lambda_k v_k^i))\, dy ds . \end{aligned}$$

Since $|(\phi_r)_{x_l}| \leq C/r$, we have

$$\begin{aligned} \left| \frac{1}{r^{m+2}} \int_{\mathbb{R}^{m+1}} (\zeta v_k^i)_{x_l} \cdot b_{k,l}^{ij} \phi_r\, dy ds \right| &\leq \frac{C}{r^{m+2+1}} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|\, |b_{k,l}^{ij}|\, dy ds \\ &\quad + \frac{1}{r^{m+2}} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|\, (|v_{k,t}^i| + |v_{k,t}^j|)\, dy ds , \end{aligned} \tag{3.14}$$

where $(\zeta v_k^i)_{z,r} = \mathrm{Vol}(P_r(z))^{-1} \int_{P_r(z)} (\zeta v_k^i)\, dy ds$.

We consider the first term on the right of the last inequality. Suppose $p, q > 1$, $p^{-1} + q^{-1} = 1$. Then

$$\begin{aligned} \frac{1}{r^{m+2+1}} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|\, |b_{k,l}^{ij}|\, dy ds &\leq \frac{C}{r} \left(\frac{1}{\mu(P_r(z))} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|^p\, dy ds \right)^{1/p} \\ &\quad \cdot \left(\frac{1}{\mu(P_r(z))} \int_{P_r(z)} |b_{k,l}^{ij}|^q\, dy ds \right)^{1/q} \\ &\leq C \left(\left(\frac{1}{\mu(P_r(z))} \int_{P_r(z)} |\nabla (\zeta v_k^i)|^{p^*}\, dy ds \right)^{1/p^*} \right. \\ &\quad \left. + \left(\frac{r^{p^*}}{\mu(P_r(z))} \int_{P_r(z)} |(\zeta v_k^i)_t|^{p^*}\, dy ds \right)^{1/p^*} \right) \\ &\quad \cdot \left(\frac{1}{\mu(P_r(z))} \int_{P_r(z)} |b_{k,l}^{ij}|^q\, dy ds \right)^{1/q} . \end{aligned} \tag{3.15}$$

where $p^* = p(m+2)/(m+2+p)$.
For $0 \leqq \delta < m+2$ we define the fractional maximal operator M_δ by

$$M_\delta f(z) = \sup_{r>0} \mu(P_r(z))^{(\delta/(m+2))-1} \int_{P_r(z)} |f|\, dyds\,.$$

When $\delta = 0$, M_δ is the Hardy-Littlewood maximal operator, and we simply denote it by M. Obviously (see [18], [19], [20]), we have

$$\|M_\delta f\|_{q'} \leqq C\|f\|_{p'} \tag{3.16}$$

if $(q')^{-1} = (p')^{-1} - \delta(m+2)^{-1}$, $1 < p' \leqq q' < \infty$.
Now, (3.15) implies that

$$\begin{aligned}
&\frac{1}{r^{m+2+1}} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|\, |b_{k,l}^{ij}|\, dyds \\
&\leqq C\Big(M(|\nabla(\zeta v_k^i)|^{p^*})\Big)^{1/p^*} \Big(M(|b_{k,l}^{ij}|^q)\Big)^{1/q} \\
&\quad + C\Big(M_{p^*}(|(\zeta v_k^i)_t|^{p^*})\Big)^{1/p^*} \Big(M(|b_{k,l}^{ij}|^q)\Big)^{1/q} \\
&\leqq C\bigg(\Big((M(|\nabla(\zeta v_k^i)|^{p^*})\Big)^{2/p^*} + \Big(M(|b_{k,l}^{ij}|^q)\Big)^{2/q} \\
&\quad + \Big(M_{p^*}(|(\zeta v_k^i)_t|^{p^*})\Big)^{\alpha/p^*} + \Big(M(|b_{k,l}^{ij}|^q)\Big)^{\beta/q}\bigg)\,,
\end{aligned}$$

where $1/\alpha + 1/\beta = 1$, $\alpha, \beta > 1$.
We choose $2(m+2)/(m+4) < \gamma < 2$, $\alpha = \gamma(m+2)/(m+2-\gamma) > 2$, $2 > \beta = \gamma(m+2)/((\gamma-1)(m+2)+\gamma) > 2(m+2)/(m+4)$. We choose p, q such that $2 < p < 2(m+2)/m$, $2(m+2)/(m+4) < q < \beta$; then $p^* < 2$. So it follows from (3.16) that

$$\begin{aligned}
\Big\|(M(|\nabla(\zeta v_k^i)|^{p^*}))^{2/p^*}\Big\|_{L^1} &\leqq C\|\nabla(\zeta v_k^i)\|_{L^2}^2\,, \\
\Big\|(M(|b_{k,l}^{ij}|^q))^{2/q}\Big\|_{L^1} &\leqq C\|b_{k,l}^{ij}\|_{L^2}^2\,, \\
\Big\|(M_{p^*}(|(\zeta v_k^i)_t|^{p^*}))^{\alpha/p^*}\Big\|_{L^1} &\leqq C\|(\zeta v_k^i)_t\|_{L^r}^{\alpha} \leqq C\|(\zeta v_k^i)_t\|_{L^2}^{\alpha}\,, \text{ and} \\
\Big\|(M(|b_{k,l}^{ij}|^q))^{\beta/q}\Big\|_{L^1} &\leqq C\|b_{k,l}^{ij}\|_{L^\beta}^{\beta} \leqq C\Big(\|b_{k,l}^{ij}\|_{L^2}^2 + 1\Big)\,.
\end{aligned}$$

Similarly we consider the second term on the right side of (3.14) and we have

$$\begin{aligned}
&\frac{1}{r^{m+2}} \int_{P_r(z)} |\zeta v_k^i - (\zeta v_k^i)_{z,r}|\, (|v_{k,t}^j| + |v_{k,l}^j|)\, dyds \\
&\leqq C\Big(M(|\nabla(\zeta v_k^i)|^{p^*})\Big)^{1/p^*} \Big(M_q((|v_{k,t}^j| + |v_{k,l}^j|)^q)\Big)^{1/q} \\
&\quad + C\Big(M_{p^*}(|(\zeta v_k^i)_t|^{p^*})\Big)^{1/p^*} \Big(M_q((|v_{k,t}^j| + |v_{k,l}^j|)^q)\Big)^{1/q}
\end{aligned}$$

$$\begin{aligned}
&\leqq C\bigg(\Big((M(|\nabla(\zeta v_k^i)|^{p^*})\Big)^{\beta/p^*} + \Big(M_q((|v_{k,t}^j| + |v_{k,l}^j|)^q)\Big)^{\alpha/q} \\
&\quad + \Big(M_{p^*}(|(\zeta v_k^i)_t|^{p^*})\Big)^{\beta/p^*} + \Big(M_q((|v_{k,t}^j| + |v_{k,l}^j|)^q)\Big)^{\alpha/q}\bigg)\,.
\end{aligned}$$

We choose $2 < p < \beta(m+2)/(m+2-\beta)$, $1 < q < 2$; then $p^* < \beta$. So

$$\begin{aligned}
\Big\|(M(|\nabla(\zeta v_k^i)|^{p^*}))^{\beta/p^*}\Big\|_{L^1} &\leqq C(\|\nabla(\zeta v_k^i)\|_{L^2}^2 + 1)\,, \\
\Big\|\Big(M_q((|v_{k,t}^j| + |v_{k,l}^j|)^q)\Big)^{\alpha/q}\Big\|_{L^1} &\leqq C\Big\|(|v_{k,t}^j| + |v_{k,l}^j|)\Big\|_{L^2}^{\alpha}\,, \\
\Big\|\Big(M_{p^*}(|(\zeta v_k^i)_t|^{p^*})\Big)^{\beta/p^*}\Big\|_{L^1} &\leqq C\Big(\|(\zeta v_k^i)_t\|_{L^2}^2 + 1\Big)\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\|(\zeta v_k^i)_\alpha \cdot b_{k,l}^{ij}\|_{H^1(\mathbb{R}^{m+1})} &\leqq C\bigg(\int_{P_{1/2}} |\nabla v_k|^2\, dxdt \\
&\quad + \Big(\int_{P_{1/2}} |v_k|^2\, dxdt\Big)^{\alpha/2} + \Big(\int_{P_{1/2}} |\partial_t v_k|^2\, dxdt\Big)^{\alpha/2} \\
&\quad + \int_{P_{1/2}} |v_k|^2\, dxdt + \int_{P_{1/2}} |\partial_t v_k|^2\, dxdt + 1\bigg) \leqq C\,.
\end{aligned}$$

by (3.14).

Now, we can prove Proposition 3.2. By (3.7) and (3.8) we have

$$\begin{aligned}
&\int_{-1/4}^{1/4} \int_{\mathbb{R}^m} (v_k - v_\infty)_t \omega\, dxdt + \int_{-1/4}^{1/4} \int_{\mathbb{R}^m} (\nabla v_k - \nabla v_\infty) \cdot \nabla\omega\, dxdt \\
&= \lambda_k \int_{-1/4}^{1/4} \int_{\mathbb{R}^m} |\nabla v_k|^2 (a_k + \lambda_k v_k)\omega\, dxdt
\end{aligned} \tag{3.17}$$

for $\omega \in H_0^{1,2}(P_{1/2}, \mathbb{R}^n) \cap L^\infty(P_{1/2}, \mathbb{R}^n)$.
Set $\omega = \zeta^2(v_k - v_\infty)$. Since $\zeta = 0$ on $\mathbb{R}^{m+1} \setminus P_{3/8}$, the left-hand side of (3.17) is

$$\begin{aligned}
L_k = &\int_{P_{3/16}} \zeta^2 |\nabla v_k - \nabla v_\infty|^2\, dxdt \\
&+ 2\int_{P_{3/16}} \zeta(v_k - v_\infty)(\nabla v_k - \nabla v_\infty)\nabla\zeta\, dxdt \\
&+ \int_{P_{3/16}} \zeta^2 (v_k - v_\infty)_t (v_k - v_\infty)\, dxdt\,.
\end{aligned}$$

By (3.2)-(3.3) and (3.6), we have

$$L_k = \int_{P_{1/4}} |\nabla v_k - \nabla v_\infty|^2\, dxdt + o(1)$$

as $k \to \infty$. Using Helein's trick, the right-hand side of (3.17) reads

$$\begin{aligned} R_k &= \int_{P_{7/16}} \zeta^2 |\nabla v_k|^2 (a_k + \lambda_k v_k)(v_k - v_\infty)\,dxdt \\ &= \lambda_k \int_{P_{7/16}} \zeta^2 v^j_{k,x_i}((a^i_k + \lambda_k v^i_k) v^j_{k,x_i} - v^i_{k,x_i}(a^j_k + \lambda_k v^j_k))(v^i_k - v^i_\infty)\,dxdt \\ &= \lambda_k \int_{P_{7/16}} \zeta^2 v^j_{k,x_i} b^{ij}_{k,l}(v^i_k - v^i_\infty)\,dxdt \\ &= \lambda_k \int_{P_{7/16}} (\zeta v^j_k)_{x_l} b^{ij}_{k,l}(\zeta(v^i_k - v^i_\infty))\,dxdt \\ &\quad - \lambda_k \int_{P_{7/16}} v^j_k \zeta_{x_l} b^{ij}_{k,l} \zeta (v^i_k - v^i_\infty)\,dxdt \\ &= \lambda_k (R^1_k + R^2_k)\,. \end{aligned}$$

Since v_k is bounded in $L^4(P_{7/16}, \mathbb{R}^n)$ and $b^{ij}_{k,l}$ is bounded in $L^{4/3}(P_{7/16}, \mathbb{R}^n)$, we have $\sup_k |R^2_k| < \infty$. By the parabolic H^1-BMO duality (see Theorem 2.5 of [3] and the remark which follows the proof) one has

$$\sup_k |R^1_k| \leq C \sum_{ij} \sup_k \|\zeta(v^i_k - v^i_\infty)\|_{BMO} \|(\zeta v^j_k)_{x_l} b^{ij}_{k,l}\|_{H^1} < \infty\,.$$

Thus, $R_k = O(\lambda_k) = o(1)$ as $k \to \infty$. This completes the proof of the proposition.

4. The Proof of Theorem 1.1

First we give a lemma which is the parabolic version of Morrey's lemma; see [17].

LEMMA 4.1. *Suppose that $\nabla u, \partial_t u \in L^2(P_R(z_0))$, and*

$$r^{-m} \int_{P_r(z)} (|\nabla u|^2 + r^2 |\partial_t u|^2)\,dyds \leq L(r/R)^{2\mu}\,, \quad r \in (0, R/2)\,, \quad \mu \in (0,1)\,,$$

for any $z \in P_{R/2}(z_0)$. Then, $u \in C^\mu(P_{R/2}(z_0))$ and

$$|u(x_1, t_1) - u(x_2, t_2)| \leq CL\left\{(|x_1 - x_2| + \sqrt{|t_1 - t_2|})/R\right\}^\mu\,,$$

for $|x_1 - x_2| + \sqrt{|t_1 - t_2|} \leq R/2$, $C = C(m, \mu) > 0$.

Proof: Consider $u \in C^1(P_R(z_0))$. Let $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2)$ be given and let $\rho = (|x_1 - x_2| + \sqrt{|t_1 - t_2|})/2$, $z^* = (z_1 + z_2)/2$.

For each $z \in P_\rho(z^*)$, one has

$$\begin{aligned} u(x,t) - u(x_1,t_1) &= (x^\alpha - x^\alpha_1)\int_0^1 \frac{\partial u}{\partial x^\alpha}(x_1 + s(x - x_1), t_1 + s^2(t - t_1))\,ds \\ &\quad + 2s(t - t_1)\int_0^1 \frac{\partial u}{\partial t}(x_1 + s(x - x_1), t_1 + s^2(t - t_1))\,ds\,. \end{aligned}$$

Then, following Morrey's arguments in Theorem 3.5.2 of [17], we get

$$\frac{1}{\operatorname{Vol} P_\rho(z^*)} \int_{P_\rho(z^*)} |u(x,t) - u(x_1,t_1)|\,dxdt \leq CL(\rho/R)^\mu\,.$$

Now, Lemma 4.1 follows easily.

Next, we shall prove our main result.

Proof of Theorem 1.1: Define

$$V = \{z \in \mathbb{R}^m \times \mathbb{R}_+ \mid E(r, u, z) < \varepsilon_0/K^2 \text{ for some } r \in (0, \sqrt{t})\}\,,$$

where $K > 0$ is the constant in (2.6). It is easy to see that V is open. The fact that $H^m((M \times \mathbb{R}^+) \setminus V) = 0$ follows from a standard covering argument; see [14], Lemma 11.

For any $z_0 \in V$, there exists $r_0 \in (0, \sqrt{t_0})$ such that $E(r_0, u, z_0) < \varepsilon_0/K^2$. Now we claim that for any $z \in P_{r_0/8}(z_0)$ and $r \in (0, r_0/64)$,

$$(4.1)\qquad (r/2)^{-m} \int_{P_{r/2}(z)} (|\nabla u|^2 + r^2 |\partial_t v_k|^2)\,dyds \leq C(r/r_0)^\theta\,,$$

for some $\theta \in (0,1)$.

Indeed, for any $z \in P_{r_0/8}(z_0)$ and $r \in (0, r_0/64)$, by Lemma 2.2 (take $a = b = 1/8$ in the first inequality and $a = 1/8$, $b = 5/32$ in the second inequality) we have

$$(4.2)\qquad E(r, u, z) \leq KE(r_0/8, u, z) \leq K^2 E(r_0, u, z_0) < \varepsilon_0\,.$$

Therefore, Theorem 3.1 implies that there exists $0 < \sigma \leq 1/4$ such that for $z \in P_{r_0/8}(z_0)$, $0 < r < r_0/64$,

$$(4.3)\qquad E(\sigma r, u, z) \leq (1/2)E(r, u, z)\,.$$

Let $k \geq 1$ be an integer such that $r \in [\sigma^k r_0/64, \sigma^{k-1} r_0/64)$ and let $\theta = \ln 2/\ln \sigma^{-1}$. Then, it follows from Lemma 2.2 (again take $a = b = 1/8$), (4.3), and (4.2) that

$$(4.4)\qquad \begin{aligned} E(r, u, z) &\leq KE(\sigma^{k-1} r_0/8, u, z) \leq K2^{-k+1} E(r_0/8, u, z) \\ &\leq 2\varepsilon_0 2^{-k} \leq C\varepsilon_0 (r/r_0)^\theta\,. \end{aligned}$$

Now (4.1) follows from (4.4) and (2.8). Lemma 4.1 and (4.1) imply that $u \in C^{\theta/2}(P_{r_0/64}(z_0))$. Then we also have that $u \in C^\infty(P_{r_0/128}(z_0))$; see, e.g., [13]. The proof of Theorem 1.1 is completed.

5. Partial Regularity at the Boundary

In this section M is either a compact, m-dimensional Riemannnian manifold with C^2 boundary ∂M or simply a C^2 domain in $\mathbb{R}^m$. In either case we let ρ be a suitable small positive constant such that for any $p_0 \in \partial M$, one can choose a coordinate system $\{x_\alpha\}$ in such a way that the set $B^M_\rho(p_0) = \{p \in M \mid \mathrm{dist}_M(p, p_0) < \rho\}$ corresponds to the unit half ball $B^+_\rho = \{x \in \mathbb{R}^m \mid |x| < \rho,\ x_m \geqq 0\}$. Thus equation (1.1) reduces to

$$\frac{\partial u}{\partial t} - \Delta u = |\nabla u|^2 u\,, \quad x \in M\,, \quad t > 0\,, \tag{5.1}$$

where Δ and ∇ are the Laplacian and gradient, respectively, with respect to a suitable metric g on M.

By an argument in [8], we can assume $M = \mathbb{R}^m_+$ and g, for simplicity, to be the Euclidian metric on $\mathbb{R}^m_+$. We let

$$u(x,t)|_{x_m=0} = u_0(x',0) \in C^2(\mathbb{R}^{m-1})\,. \tag{5.2}$$

and put

$$\begin{aligned} u^*(x,t) &= u(x,t) && \text{for } (x,t) \in \mathbb{R}^m \times \mathbb{R}_+\,, && x_m \geqq 0\,; \\ u^*(x,t) &= u_0(x',0) && \text{for } (x,t) \in \mathbb{R}^m \times \mathbb{R}_+\,, && x_m < 0\,. \end{aligned} \tag{5.3}$$

As in (2.7), we consider the weak solution u satisfying

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}^m_+} \varphi^2 |\partial_t u|^2\,dxdt + \int_{\mathbb{R}^m_+} \varphi^2 |\nabla u(\cdot,t_2)|^2\,dx \\ &\quad \leqq \int_{\mathbb{R}^m_+} \varphi^2 |\nabla u(\cdot,t_1)|^2\,dx + 4\int_{t_1}^{t_2}\int_{\mathbb{R}^m_+} |\nabla\varphi|^2 |\nabla u(\cdot,t)|^2\,dxdt \end{aligned} \tag{5.4}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^m)$, $0 < t_1 < t_2 < \infty$. Note that (5.4) valid for the classical solutions of (5.1)–(5.2).

Next, we define $T_r^+ = T_r \cap \{(x,t) \mid x_m \geqq 0\}$, $P_r^+ = P_r \cap \{(x,t) \mid x_m \geqq 0\}$ and, for $\beta > 0$,

$$\Psi^+_\beta(R,u,z_1) = \frac{1}{2}\int_{T^+_{R}(z_1)} |\nabla u|^2 G_{z_1}\varphi_\beta^2\,dxdt\,,$$

where φ_β is the same as that in $\Psi_\beta(R,u,z_1)$; see Section 2.

Motivated by [5] and [6], we shall assume that the following monotonicity is valid for the weak solutions u of (5.1)–(5.2) under consideration.

$$\begin{aligned} \Psi^+_\beta(R_1,u,z_1) \leqq\ & e^{C(R_2^\varepsilon - R_1^\varepsilon)}\Psi^+_\beta(R_2,u,z_1) \\ &+ C(R_2 - R_1)\beta^{-m}\int_{P^+_{\beta/2}(z_1)} |\nabla u|^2\,dxdt \\ &+ C(R_2 - R_1)\beta^{2-m}\int_{B^+_{\beta/2}(x_1)} |\nabla_{x'}^2 u_0(x',0)|^2\,dx \end{aligned} \tag{5.5}$$

for any $z_1 \in \{x_m = 0\}\times\mathbb{R}_+$, $0 < R_1 \leqq R_2 \leqq \min(\sqrt{t_1}/(2\beta), 1/4)$, where $0 < \varepsilon < 1$ with $C > 0$ depending only on m.

The inequality (5.5) is true for any regular solution of (5.1)–(5.2). Indeed, shift to $z_1 = (0,0)$, let $u_R(x,t) = u(Rx, R^2 t)$, and calculate

$$\begin{aligned} \frac{d}{dR}\Psi^+_\beta(R,u,(0,0)) &= \frac{d}{dR}\Psi^+_\beta(1,u_R,(0,0)) \\ &= \int_{T_1^+} \nabla u_R \cdot \nabla\frac{du_R}{dR} G\varphi^2_\beta(Rx)\,dxdt \\ &\quad + \int_{T_1^+} |\nabla u_R|^2 G\varphi_\beta(Rx)(\nabla\varphi)\Big(\frac{Rx}{\beta}\Big)\frac{x}{\beta}\,dxdt = \mathrm{I} + \mathrm{II}\,. \end{aligned} \tag{5.6}$$

By an argument similar to the proof of Lemma 2.1, we have

$$\mathrm{II} \leqq \Psi^+_\beta(R,u,(0,0)) + C\beta^{-m}\int_{P^+_{\beta/2}} |\nabla u|^2\,dxdt\,. \tag{5.7}$$

The first term is given by

$$\begin{aligned} \mathrm{I} = &\int_{T_1^+} \nabla u_R \cdot \nabla\Big(\frac{du_R}{dR} - x'\cdot\nabla_{x'}u_0(Rx',0)\Big)G\varphi^2_\beta(Rx)\,dxdt \\ &+ \int_{T_1^+} \nabla u_R \cdot \nabla\big(x'\cdot\nabla_{x'}u_0(Rx',0)\big)G\varphi^2_\beta(Rx)\,dxdt = \mathrm{I}' + \mathrm{I}''\,. \end{aligned}$$

Integrating by parts, as in [5] and [6] (see also the proof of Lemma 2.1 above), we have

$$\begin{aligned} \mathrm{I}' = &\int_{T_1^+} \frac{R}{2|t|}\Big(\frac{du_R}{dR}\Big)^2 G\varphi^2_\beta(Rx)\,dxdt \\ &+ \int_{T_1^+} \frac{R}{2t}\frac{du_R}{dR}\big(x'\cdot\nabla_{x'}u_0(Rx',0)\big)G\varphi^2_\beta(Rx)\,dxdt \\ &+ \int_{T_1^+} |\nabla u_R|^2 u_R\big(x'\cdot\nabla_{x'}u_0(Rx',0)\big)G\varphi^2_\beta(Rx)\,dxdt \\ &- 2\int_{T_1^+} \nabla u_R \cdot \frac{du_R}{dR}G\varphi_\beta(Rx)(\nabla\varphi)\Big(\frac{Rx}{\beta}\Big)\frac{R}{\beta}\,dxdt \\ &+ 2\int_{T_1^+} \nabla u_R \cdot \nabla\varphi\Big(\frac{Rx}{\beta}\Big)\big(x'\cdot\nabla_{x'}u_0(Rx',0)\big)G\varphi_\beta(Rx)\frac{2R}{\beta}\,dxdt\,. \end{aligned}$$

The same estimates used in the proof of Lemma 2.1 in [5] (see also [6]) yield that

$$\begin{aligned} \mathrm{I}' \leqq\ & \frac{\Psi^+_\beta(1,u_R,(0,0))}{R^\varepsilon} + C\beta^{-m}\int_{P^+_{\beta/2}} |\nabla u|^2\,dxdt \\ &+ C\beta^{2-m}\int_{B^+_{\beta/2}} |\nabla_{x'}u_0(x',0)|^2\,dx\,. \end{aligned} \tag{5.8}$$

In the same way, I'' can be estimated by

$$I'' \leq \frac{\Psi_{\beta}^{+}(1, u_R, (0,0))}{R^{\prime}} + C\beta^{2-m} \int_{B_{1/2}^{\prime}} |\nabla_{x'}^{2} u_0(x',0)|^2 \, dx \, . \tag{5.9}$$

Now (5.5) follows from (5.6)–(5.9) together with an integration.

Theorem 5.1. *Suppose that u is a global weak solution of (5.1)–(5.2) with $E(u_0) < \infty$ and suppose also that u satisfies the energy inequality (5.4) and the monotonicity inequality (5.5) on $\bar{M} \times \mathbb{R}^{+}$. Then, there is an open set V of $\bar{M} \times \mathbb{R}^{+}$ such that $u \in C^{\infty}(V, S^{n-1})$ and $H^m((\bar{M} \times \mathbb{R}^{+}) \setminus V) = 0$. Here, H^m denotes m-dimensional Hausdorff measure with respect to the parabolic metric $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$.*

Let

$$E^{+}(r,u,z) = r^{-m} \int_{P_r^{+}(z)} |\nabla u|^2 \, dx dt \, .$$

As in the argument of Section 3 and Section 4 above, in order to prove this theorem, it suffices to show the following:

Lemma 5.2. *Under the same assumptions as those in Theorem 5.1, there exist $\varepsilon_0 > 0$, $K_0 > 0$, and $0 < \sigma \leq 1/4$ such that*

$$E^{+}(r,u,z) + K_0 r^{2-m} \int_{B_r^{\prime}(z_0)} |\nabla^2 u_0(x',0)|^2 \, dy < \varepsilon_0 \tag{5.10}$$

implies

$$E^{+}(\sigma r, u, z) \leq \frac{1}{2} E^{+}(r,u,z) + K_0 r^{2-m} \int_{B_r^{\prime}(z_0)} |\nabla^2 u_0(x',0)|^2 \, dy \, , \tag{5.11}$$

for all $z \in \{x_m = 0\} \times \mathbb{R}_+$, $0 < r < \sqrt{t}$.

The proof of this lemma follows the same argument as the proof of Theorem 3.1. Here we just give a brief sketch.

Sketch of the Proof of Lemma 5.2: We argue by contradiction. Suppose that the conclusion of Lemma 5.2 fails. Then we can construct a blow-up sequence $\{v_k\}$ as in the proof of Theorem 3.1, with

$$\lambda_k^2 = E(r_k, u, z_k) + K_{0k} r_k^{2-m} \int_{B_{r_k}^{\prime}(z_0)} |\nabla^2 u_0(x',0)|^2 \, dy \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty \, .$$

Since we may assume $K_{0k} \longrightarrow \infty$, the limit v_∞ of $\{v_k\}$ satisfies

$$\begin{aligned} \left(\tfrac{\partial}{\partial t} - \Delta\right) v_\infty &= 0 \qquad \text{in } P_{1/2}^{+} \, , \\ v_\infty &= 0 \qquad \text{on } \{x_m = 0\} \cap P_{1/2}^{+} \, . \end{aligned}$$

The contradiction will again follow if it can be shown that

$$v_k \longrightarrow v_\infty \quad \text{strongly in } H^1(P_{1/2}^{+}) \, .$$

For this purpose we simply consider $u^*(x,t)$ defined in (5.3) instead of u. Accordingly, v_k is replaced by v_k^*, etc.

The energy inequality (5.4) is obviously valid for $u^*(x,t)$. Since $u_0(x',0)$ is in C^2, the inequality (5.5) is also valid for $u_0(x',0)$. Using these facts, it is not hard to see that (3.10) and (3.11) also hold for v_k^*. Then, the rest of the argument follows as in the proof of Theorem 3.1.

Note that by Lemma 5.2 and an argument similar to that in Section 4, (5.10) implies

$$E^{+}(r,u,z) \leq C\left(E^{+}(1,u,z) + \|u_0\|_{C^2(\partial M)}\right) r^{\theta} \, ,$$

for some $\theta \in (0,1)$. Then, Theorem 5.1 follows as in Section 4.

Acknowledgements. The first author was partially supported by National Science Foundation Grant No. DMS-9123532. The third author was partially supported by National Science Foundation Grant No. DMS-9149555.

Bibliography

[1] Bethuel, F., *On the singular set of stationary harmonic maps*, Manuscripta Math. 78, 1993, pp. 417–443.

[2] Calderón, A., and Torchinsky, A., *Parabolic maximal functions associated with a distribution*, Adv. Math. 16, 1975, pp. 1–64.

[3] Calderón, A., and Torchinsky, A., *Parabolic maximal functions associated with a distribution II*, Adv. Math. 24, 1977, pp. 101–171.

[4] Chen, Y., *Weak solutions to the evolution problems of harmonic maps*, Math. Z. 201, 1989, pp. 69–74.

[5] Chen, Y., *Dirichlet problems for heat flows of of harmonic maps in higher dimensions*, Math. Z. 208, 1991, pp. 557–565.

[6] Chen, Y., and Lin, F. H., *Evolution of harmonic maps with Dirichlet boundary conditions*, Comm. Anal. Geom. 1, 1993, pp. 327–346.

[7] Chen, Y., and Lin, F. H., *Remarks on approximate harmonic maps*, Comment. Math. Helv., in press.

[8] Chen, Y., and Struwe, M., *Existence and partial regularity for heat flow for harmonic maps*, Math. Z. 201, 1989, pp. 83–103.

[9] Caffarelli, L., Kohn, R., and Nirenberg, L., *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. 35, 1982, pp. 771–831.

[10] Coifman, R., Lions, P.-L., Meyer, Y., and Semmes, S., *Compacité par compensation et espaces de Hardy*, C. R. Acad. Sci. Paris Sér. I Math. 309, 1989, pp. 945–949.

[11] Coron, J. M., *Nonuniqueness for the heat flow of harmonic maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire 7, 1990, pp. 335–344.

[12] Evans, L. C., *Partial regularity for stationary harmonic maps into spheres*, Arch. Rational Mech. Anal. 116, 1991, pp. 101–113.

[13] Feldman, M., *Partial regularity for harmonic maps of evolution into spheres*, Comm. Partial Differential Equations 19, 1994, pp. 761–790.

[14] Giaquinta, M., and Giusti, E., *Partial regularity for the solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. 97, 1973, pp. 253–266.
[15] Hélein, F., *Regularité des applications faiblement harmoniques entre une surface et une sphere*, C. R. Acad. Sci. Paris Ser. I Math. 311, 1990, pp. 519–524.
[16] Keller, J. B., Rubinstein, J., and Sternberg, P., *Reaction-diffusion processes and evolution to harmonic maps*, SIAM J. Appl. Math. 49, 1989, pp. 1722–1733.
[17] Morrey, C. B., *Multiple Integrals in the Calculus of Variations*, Springer Verlag, Berlin, Heidelberg, New York, 1966.
[18] Muckenhoupt, B., and Wheeden, R., *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. 192, 1974, pp. 251–275.
[19] Sawyer, E. T., *A characterization of a two-weight norm inequality for maximal operators*, Studia Math. 75, 1982, pp. 1–11.
[20] Sawyer, E. T., *Two weight norm inequalities for certain maximal and integral operators*, pp. 102–127 in: *Harmonic Analysis*, F. Ricci and G. Weiss, eds., Lecture Notes in Mathematics No. 908, Springer-Verlag, Berlin, New York, 1982.
[21] Serrin, J., *The initial value problem for the Navier-Stokes equations*, pp. 69–98 in: *Nonlinear Problems*, R. E. Langer, ed., University of Wisconsin Press, Madison, 1963.
[22] Strömberg, J.-O., and Torchinsky, A., *Weighted Hardy Spaces*, Lecture Notes in Mathematics No. 1381, Springer-Verlag, Berlin, New York, 1989.
[23] Struwe, M., *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. 28, 1988, pp. 485–502.

Received November 1993.
Revised February 1994.

On the Existence of Infinitely Many Periodic Solutions to Some Problems of n-Body Type

PIETRO MAJER
Università di Pisa

AND

SUSANNA TERRACINI
Politecnico di Milano

Abstract

We prove the existence of infinitely many periodic solutions with prescribed period to a class of problems of n-body type.

1. Introduction

In this paper we present an approach, based on Ljusternik-Schnirelman variational theory on manifolds with boundary, to finding an infinite number of periodic solutions to some n-body systems. We consider systems of n-particles interacting according to Newton's law

(1.1) $$-m_i\ddot{u}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \nabla V_{ij}(t, u_i - u_j)\,, \qquad i = 1, \dots, n\,,$$

where $V_{ij} \in \mathscr{C}^1((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}; \mathbb{R})$ ($k \geq 3$) are T-periodic in time and satisfy the following assumptions ($i, j = 1, \dots, n;\ i \neq j$):

(V1) $$V_{ij}(x,t) \leqq 0 \qquad \forall\, x \in \mathbb{R}^k \setminus \{0\}\,.$$

(V2) there exist $\rho_0 > 0$ and a function $U \in \mathscr{C}^1(\mathbb{R}^k \setminus \{0\}; \mathbb{R})$ such that $\lim_{x\to 0} U(x) = -\lim_{x\to 0} V_{ij}(x,t) = +\infty$ and
$$-V_{ij}(x,t) \geqq |\nabla U(x)|^2\,, \qquad \forall\, x,\ 0 < |x| < \rho_0\,, \text{ and}$$

(V3) $$\lim_{|x|\to+\infty} \frac{\nabla V_{ij}(x,t)\cdot x}{|\nabla V_{ij}(x,t)|\,|x|} = +\infty \qquad \text{uniformly in } t\,,$$

where $\nabla V_{ij}(x,t)$ denotes $\left(\frac{\partial}{\partial x_1} V_{ij}(x,t), \dots, \frac{\partial}{\partial x_k} V_{ij}(x,t)\right) \in \mathbb{R}^k$.

Communications on Pure and Applied Mathematics, Vol. XLVIII, 449–470 (1995)
 CCC 0010-3640/95/040449-22