Invent. math. 99, 567–578 (1990)

Inventiones mathematicae

Blow-up and global existence for heat flows of harmonic maps

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Summary. In this paper it is proved that the solution to the evolution problem for harmonic maps blows up in finite time, if the initial map belongs to some nontrivial homotopy class and the initial energy is sufficiently small.

1. Introduction

Let (M, g) and (N, h) be two compact Riemannian manifolds with $\dim M = n$ and $\dim N = m$. We will assume that N has no boundary and (N, h) is isometrically embedded in some Euclidean space R^k so that N is viewed as a submanifold of R^k. Harmonic maps are critical points of the energy

$$E(u) = \frac{1}{2} \int_M |du|^2 \, dV$$

defined on $C^1(M, N)$, where in local coordinates,

$$|du|^2 = g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} = g^{\alpha\beta} \left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right\rangle.$$

Here we have used the summation convention, and $\langle \cdot, \cdot \rangle$ is the inner product of R^k.

For $y \in N \subset R^k$, let $P(y)$: $R^k \to T_y N$ be the orthogonal projection onto the tangent space N at y, and let $A(y)$: $T_y N \times T_y N \to (T_y N)^\perp$ be the second fundamental form of $N \subset R^k$ at y. The tension field $\tau(u)$ of $u \in C^2(M, N)$ is defined by

$$\tau(u) \equiv P(u) \Delta u = \Delta u - A(u)(du, du),$$

with respect to the metric g, and

$$A(u)(du, du) = g^{\alpha\beta} A(u) \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right).$$

If $u \in C^1(M, N)$ is a harmonic map then it is smooth and satisfies the Euler-Lagrange equation $\tau(u)=0$, or

$$\Delta u = A(u)(du, du).$$

In their fundamental work on the existence of harmonic maps into manifolds of nonpositive curvatures, Eells and Sampson [7] initiated the study of the heat flow equation of harmonic maps, which is the following parabolic system:

$$\frac{\partial u}{\partial t} = \tau(u) = \Delta u - A(u)(du, du). \tag{1.1}$$

They proved that if the sectional curvatures of N are nonpositive, then the solution $u(t)=u(t, x)$ of (1.1) with an arbitrary initial data

$$u(0) = u_0 \in C^{2,\alpha}(M, N) \tag{1.2}$$

exists for all $t>0$. Moreover, $u(t)$ subconverges to u_∞ as $t \to +\infty$. Later, Hamilton [9] obtained a similar result in the Dirichlet case where M has a boundary ∂M and the flow is required to satisfy the boundary condition

$$u(t, \cdot)|_{\partial M} = u_0|_{\partial M}. \tag{1.3}$$

Recently, Struwe [18] and Chang [1] showed that if $\dim M = 2$, some results of Sacks and Uhlenbeck [15] and their extensions in the Dirichlet case can be obtained by the heat flow methods.

However, the study of the heat flow of harmonic maps is still quite incomplete. In general, one can only prove the solution of (1.1)–(1.2) exists locally, i.e. there exists a maximal existence interval $[0, T)$, where $T \leqq +\infty$, such that the solution exists for $t \in [0, T)$, and it cannot be extended beyond T. If $T < +\infty$, then we say that the solution "blows up in finite time," or simply "blows up." We did not even know whether blow-up can actually occur until quite recently, when Coron and Ghidaglia [3] produced such examples for heat flows of harmonic maps from R^n or S^n into S^n, for $n \geqq 3$. Their examples and methods rely heavily on the symmetries of both a S^n and the initial maps. Shortly after, Ding [4] provided more general examples of blow-up assuming only that the initial map u_0 belongs to a nontrivial homotopy class and the initial energy $E(u_0)$ is sufficiently small. However his method works only for $n = \dim M = 3$. It is one of our aims in this note to generalize Ding's result to all dimensions $n \geqq 3$. The main ingredient in this generalization is the replacement of an "elliptic" monotonicity inequality in [4] by a "parabolic" monotonicity inequality obtained by Struwe [19] (see also Chen and Struwe [2]). It turns out that using the parabolic monotonicity inequality one can also prove the maximal existence interval shrinks to zero as the initial energy $E(u_0) \to 0$.

Theorem 1.1. *Assume that M has no boundary and* $\dim M = n \geqq 3$. *Let $\mathscr{F}$ be any nontrivial homotopy class in $C(M, N)$ with*

$$E_{\mathscr{F}} = \inf\{E(u) | u \in \mathscr{F} \cap W^{1,2}(M, R^k)\} = 0.$$

There exists $\varepsilon>0$ such that if $u_0 \in \mathscr{F}$ and $E(u_0)<\varepsilon$ then the solution of (1.1)–(1.2) blows up in finite time. Moreover, if $[0, T(u_0))$ is the maximal existence interval for the solution, we have $T(u_0)\to 0$ as $E(u_0)\to 0$.

Remark 1.1. The hypothesis that $E_{\mathscr{F}}=0$ will be satisfied provided one of the following three conditions are met (1) $\pi_1(M)=0$ and $\pi_2(M)=0$; (2) $\pi_1(N)=0$ and $\pi_2(N)=0$; (3) $\pi_1(M)=0$ and $\pi_2(N)=0$ (see [20]).

It may seem strange but it is actually natural that the proof of Theorem 1.1 can easily be adapted to obtain a global existence result, which has been proved for $M=R^n$ by Struwe [19].

Corollary 1.1. *Let M be as in Theorem 1.1. For any constant $K>0$ there exists $\varepsilon=\varepsilon(K)>0$ such that if the initial map satisfies (1) $|du_0|(x)\leqq K$ for all $x\in M$, and (2) $E(u_0)<\varepsilon$, then the solution u of (1.1)–(1.2) exists for all $t>0$. Moreover, as $t\to+\infty$, $u(t)$ converges to a constant map.*

Remark 1.2. Although both Theorems 1.1 and Corollary 1.1 assume the smallness of the initial energy, there is a major difference between their assumptions, namely the initial map u_0 in Theorem 1.1 is not homotopic to constant maps while in Corollary 1.2 it is. That u_0 in Corollary 1.1 is homotopic to the constant map is actually not an assumption but a consequence of (1) and (2). In fact, for any $p>n$, we have

$$\int_M |du_0|^p\, dV \leqq K^{p-2} \int_m |du_0|^2 dV = 2K^{p-2} E(u_0) < 2K^{p-2}\varepsilon.$$

This implies that the $W^{1,p}(M, R^k)$-norm of u_0 is small if ε is small. (Note that we can always assume $\int_M u_0\, dV=0$ by choosing the origin of R^k suitably. Then the Poincaré inequality applies to give the assertion.) Since $p>n$, the Sobolev embedding theorem implies the $C^{0,\alpha}(M, R^k)$-norm of u_0 is also small, where $\alpha=1-(n/p)$. It follows that for small ε, the image of the map u_0 is contained in a contractible neighborhood of some point in N. Hence u_0 is homotopic to constant maps. This also indicates that the proof of Corollary 1.1 may as well follow the line of [11], since one can assume the image of u_0 is contained in a convex geodesic ball. In this paper we would like to get Corollary 1.1 as a consequence of Theorem 1.1.

Remark 1.3. When $\partial M\neq\emptyset$, we can prove results for the Dirichlet case similar to Theorems 1.1 and 1.2. For such a case $\mathscr{F}$ will be a homotopy class in

$$C_\phi(M, N)=\{u\in C(M, N) | u|_{\partial M}=\phi|_{\partial M}\},$$

where $\phi\in C^{2,\alpha}(M, N)$. Note that the condition $E_{\mathscr{F}}=0$ can be valid only if $\phi|_{\partial M}$ is a constant map. In such a case, the condition that $\mathscr{F}$ is nontrivial means that the unique constant map in $C_\phi(M, N)$ is not contained in $\mathscr{F}$. Since the proof for the Dirichlet case is similar to the case that M is closed, we will only remark in appropriate context on the necessary modifications in the Dirichlet case.

In the next section we present two basic lemmas. Although they have been proved before, we include their proofs here for convenience of the readers. The proofs of Theorems 1.1 and Corollary 1.1 will be given in Sect. 3.

2. Preliminary lemmas

Throughout this section we assume M has no boundary. Let $u(t)=u(t, x)$ be any solution of (1.1)–(1.2), and let $[0, T)$ be the maximal existence interval of u, where $0<T\leqq +\infty$. We will use the following notations:

$$e(u)=|du|^2,$$
$$\bar{e}(t)=\max_M e(u(t)).$$

The following lemma is essentially contained in [4].

Lemma 2.1. *There exists a $\delta>0$ depending only on the geometry of M and N such that for any $t_0\in[0, T)$ we have*

$$t_0+\delta\ln\left(1+\frac{1}{2\bar{e}(t_0)}\right)<T, \tag{2.1}$$

and

$$\bar{e}(t)\leqq\frac{1+\bar{e}(t_0)}{1-\bar{e}(t_0)\{\exp[\delta^{-1}(t-t_0)]-1\}} \tag{2.2}$$

for $t_0<t<t_0+\delta\ln\left(1+\frac{1}{\bar{e}(t_0)}\right)$.

Proof. From the equation for the evolution of $e(u)$ (see [9]) we see that

$$\frac{\partial e(u)}{\partial t}\leqq\Delta e(u)+C(e(u)+1)\, e(u), \tag{2.3}$$

where the constant C depends only on the curvatures of M and N. For any $x\in M$ with $e(u)(t, x)=\bar{e}(t)$ we have $\Delta e(u)(t, x)\leqq 0$. By (2.3),

$$\frac{\partial e(u)}{\partial t}\Big|_{(t, x)}\leqq C(\bar{e}(t)+1)\bar{e}(t)$$

for all x with $e(u)(t, x)=\bar{e}(t)$. This implies

$$D^+\bar{e}(t)\leqq C(\bar{e}(t)+1)\,\bar{e}(t) \tag{2.4}$$

where

$$D^+f(t)=\limsup_{h\to+0}\frac{f(t+h)-f(t)}{h}.$$

By the comparison theorem (see [10], pp. 26–27), we have $\bar{e}(t) \leqq y(t)$ for $t > t_0$, where y is a solution of the ordinary differential equation

$$y' = C(y+1)\,y, \qquad y(t_0) = \bar{e}(t_0).$$

If follows that

$$\bar{e}(t) \leqq \frac{\bar{e}_0 \exp[C(t-t_0)]}{1-\bar{e}_0\{\exp[C(t-t_0)]-1\}} \leqq \frac{1+\bar{e}_0}{1-\bar{e}_0\{\exp[C(t-t_0)]-1\}}$$

for $t_0 < t < t_0 + C^{-1} \ln\left(1 + \frac{1}{\bar{e}_0}\right)$, where $\bar{e}_0 = \bar{e}(t_0)$. Setting $C^{-1} = \delta$, we see that (2.2) holds. It is then easy to use the a priori estimate (2.2) to prove (2.1).

Remark 2.1. Lemma 2.1 can also be proved in the Dirichlet case, using parabolic estimates to replace the simple maximum principle in the above proof (see [4]).

The next lemma is a parabolic version of the monotonicity inequality, which can be found in [19] (for $M = R^n$) and [2] (for general cases).

In the sequel we let ρ be a positive constant less than the injectivity radius of M, and $\{x^\alpha\}$ a normal coordinate on a geodesic ball $B_\rho(p_0)$ centered at p_0 with radius ρ. If $u\colon (0, T) \times M \to N$ is a solution of (1.1), we may restrict u to any such coordinate neighborhood and regard u as a map $u\colon (0, T) \times B_\rho \equiv B_\rho(0) \subset R^n \to N$ with energy density

$$e(u) = \frac{1}{2} g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta}.$$

(Note that in fact here g is the pull back metric g on M via the exponential map at point p_0.) Moreover, there exists a positive constant $A = A(\rho)$ depending on the geometry of M, such that

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + q_{\alpha\beta}(x), \qquad g^{\alpha\beta}(x) = \delta_{\alpha\beta} + q^{\alpha\beta}(x)$$

with

$$|q_{\alpha\beta}(x)| < A r^2,$$
$$|\partial q_{\alpha\beta}(x)| < A r \tag{2.5}$$

where $r = |x| \leqq \rho$. Now, given any $t_0 \in (0, T)$, define a function $\phi(R) = \phi_{t_0}(R)$ for $R \in (0, \min\{\sqrt{t_0}, \rho\})$ by

$$\Phi(R) = \tfrac{1}{2} R^{2-n} \int_{B_\rho} |du|^2 (t_0 - R^2, x) \exp\left\{-\frac{|x|^2}{4R^2}\right\} \phi^2(x) \sqrt{g(x)}\, dx, \tag{2.6}$$

where ϕ is a smooth real function such that $\phi(x) = 1$ for $|x| \leqq \rho/2$, $\phi(x) = 0$ for $|x| \geqq \rho$ and $0 \leqq \phi(x) \leqq 1$ for all x.

Lemma 2.2. *There exists a constant $C > 0$ (depending only on M and N), such that for $0 < R_1 \leqq R_2 < \min\{\sqrt{t_0}, \rho\}$.*

$$\Phi(R_1) \leqq e^{C(R_2 - R_1)} \Phi(R_2) + C E(u_0)(R_2 - R_1). \tag{2.7}$$

Remark 2.2. For the Dirichlet case where u_0 maps the boundary ∂M into a single point $q \in N$ and the boundary condition

$$u(0, x) = u_0, \; u(t, \cdot)|_{\partial M} = q \tag{2.8}$$

is posed, one can prove a similar monotonicity inequality as follows. At a point $p_0 \in \partial M$, choose a coordinate system $\{x^\alpha\}$ so that p_0 is at the origin and $B_\rho(p_0) = \{p \in M | \text{dist}(p, p_0) < \rho\}$ corresponds to the half ball

$$B_\rho^+ = \{x \in R^n \| x| < \rho, \; x^n \geqq 0\}.$$

In such coordinates, one can define a function $\Phi^+(R)$ exactly the same as the function $\Phi(R)$, except that B_ρ is replaced by B_ρ^+. Then we have for $0 < R_1 \leqq R_2 \leqq \min\{\sqrt{t_0}, \rho\}$,

$$\Phi^+(R_1) \leqq e^{C(R_2 - R_1)} \Phi^+(R_2) + C E(u_0)(R_2 - R_1).$$

The proof is completely similar to that of Lemma 2.2. One needs only to notice that

$$v_R|_{x^n = 0} = \frac{d\bar{u}_R}{dR}|_{x^n = 0} = 0$$

due to (2.8). Hence the boundary term vanishes when integrating by parts.

3. Proofs of the theorems

Proof of Theorem 1.1. Let u be a solution of (1.1)–(1.2), with maximal existence interval $[0, T)$ and small initial energy

$$E(u_0) < \varepsilon. \tag{3.1}$$

We first prove that if ε is sufficiently small and $u_0 \in \mathscr{F}$, where $\mathscr{F}$ is a nontrivial homotopy class, then

$$\sup\{\bar{e}(t) | t \in [0, T)\} = +\infty. \tag{3.2}$$

Indeed, if (3.2) is false then we may assume there exists a constant $C > 0$ such that

$$\bar{e}(t) < C, \qquad \forall t \in [0, T). \tag{3.3}$$

If $T < +\infty$, Lemma 2.1 (with t_0 sufficiently close T) will lead to a contradiction. Hence $T = +\infty$. Then, since

$$\frac{dE(u(t))}{dt} = -\int_M |\tau(u(t))|^2 \, dV = -\|\tau(u(t))\|_2^2, \tag{3.4}$$

we have

$$\int_0^\infty \|\tau(u(t))\|_2^2 \, dt \leqq E(u_0) < \varepsilon.$$

Therefore we can find a sequence $t_i \to +\infty$ such that

$$\|\tau(u(t_i))\|_2^2 \to 0, \quad \text{as } t_i \to +\infty. \tag{3.5}$$

On the other hand, (3.3) and the local estimates for linear parabolic equations (see [12], pp. 351–355) lead to

$$\|u\|_{C^{2,\alpha}([t_i-1,t_i]\times M,N)} < C.$$

If follows that

$$\|u(t_i)\|_{C^{2,\alpha}(M,N)} < C.$$

Hence, by passing to a subsequence, if necessary, we may assume $u(t_i) \to u_\infty$ in $C^2(M, N)$. By (3.5) we have $\tau(u_\infty)=0$, i.e. u_∞ is harmonic, while from (3.4) we derive that

$$E(u_\infty) \leqq E(u_0) < \varepsilon.$$

Notice also that $u_\infty \in \mathscr{F}$ since $u(t_i) \in \mathscr{F}$ for each i. The following lemma shows that such an u_∞ cannot exist provided ε is sufficiently small and $\mathscr{F}$ is nontrivial. Therefore (3.2) holds under our assumptions.

Lemma 3.1. *There exists $\varepsilon_0 > 0$ such that if $u \in C^2(M, N)$ is harmonic and $E(u) < \varepsilon_0$, then u is a constant map.*

Proof. Suppose the lemma is untrue. Then we may assume there is a sequence of nonconstant harmonic maps u_i such that

$$E(u_i) \to 0. \tag{3.6}$$

However, in [16] Schoen attributes Theorem 2.2 to Schoen-Uhlenbeck. This Theorem uses elliptic scaling inequality and that implies global C^1 estimates. Then standard elliptic theory gives

$$\|u_i\|_{C^{2,\alpha}(M,N)} \leqq C.$$

This together with (3.6) implies that u_i subconverges in C^2-norm to some constant map. Thus we may assume $u_i(M)$ is contained in an arbitrarily small geodesic ball on N if only i is big enough. This is known to be impossible for a nonconstant harmonic map from a compact manifold without boundary (see [8]). Hence the lemma must be true.

Next we claim that if (3.1) and (3.2) hold, then there exists $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$ we have

$$T \leqq C\varepsilon^{\frac{2}{n-2}} \tag{3.7}$$

where C is a positive constant depending only on the geometry of M and N. Clearly, this will complete the proof of Theorem 1.1.

To prove the claim notice that by (3.2) we assume for some sequence $t_i \to T$

$$\bar{e}(t_i) \to +\infty,$$

$$\lambda_i^2 \equiv \ln\left(1+\frac{1}{2\bar{e}(t_i)}\right) \to 0. \tag{3.8}$$

We also require that the sequence is chosen so that

$$\bar{e}(t) \leqq \bar{e}(t_i), \qquad \forall t \in [0, t_i]. \tag{3.9}$$

Let $\rho > 0$ be a constant as in the proof of Lemma 2.2. Let $p_i \in M$ be such that

$$e(u(t_i, p_i)) = \bar{e}(t_i),$$

and let $\{x^\alpha\}$ be the normal coordinates on the geodesic ball $B_\rho(p_i)$ centered at p_i with radius ρ. In this coordinate system we may define the function $\Phi(R) = \Phi_{t_0}(R)$ as in Sect. 2. By Lemma 2.1, there exists $\delta > 0$ such that

$$t_i + \lambda_i^2 \delta < T$$

$$\bar{e}(t) \leqq 2\bar{e}(t_i) + 2, \quad \textit{for } t_i < t \leqq t_i + \lambda_i^2 \delta. \tag{3.10}$$

We choose $t_0 = t_i + \lambda_i^2 \delta$ in the definition of $\Phi = \Phi_{t_0}$.

Now we define a mapping v_i by

$$v_i(t, x) = u(t_i + \lambda_i^2 t, \lambda_i x),$$

where $t \in [-\lambda_i^{-2} t_i, \delta]$, $x \in B_{\rho \lambda_i^{-1}} \subset R^n$. The v_i satisfies the equation

$$\frac{\partial v_i}{\partial t} - \Delta_i v_i = A^i(v_i)(dv_i, dv_i) \tag{3.11}$$

on $[-\lambda_i^{-2} t_i, \delta] \times B_{\rho \lambda_i^{-1}}$, where Δ_i is the Laplacian with respect to the metric $g^i_{\alpha\beta}(x) \equiv g_{\alpha\beta}(\lambda_i x)$, while A^i means that we take the trace by g^i. It is easy to see from (3.8)–(3.10) and the definition of v_i that for sufficiently large i

$$|dv_i|^2(0, 0) = \bar{e}(t_i) \ln\left(1+\frac{1}{2\bar{e}(t_i)}\right) > \tfrac{1}{4}$$

and

$$|dv_i|^2(t, x) \leqq 4|dv_i|^2(0, 0) = 4\bar{e}(t_i) \ln\left(1+\frac{1}{2\bar{e}(t_i)}\right) < 4 \tag{3.12}$$

for $(t, x) \in [-\lambda_i^{-2} t_i, \delta] \times B_{\rho \lambda_i^{-1}} \equiv Q_i$. Set

$$e_i(t, x) = |dv_i|^2(t, x).$$

Then, similar to (2.3),

$$\frac{\partial e_i}{\partial t} \leqq \Delta_i e_i + \frac{C_i}{2}(e_i + 1)\, e_i.$$

In view of (3.12), we know that on any open set $O_i \subset Q_i$,

$$\frac{\partial e_i}{\partial t} \leqq \Delta_i e_i + C_i e_i.$$

Equivalently, $h_i \equiv e_i \exp(-C_i t)$ satisfies

$$\frac{\partial h_i}{\partial t} \leqq \Delta_i h_i.$$

Take $O_i = \left(-\min\left(\frac{\delta}{2}, \frac{\delta}{C_i}\right), \frac{\delta}{2}\right) \times B_1$. By a result of Moser (see [14], Theorem 3), there is a constant $r > 0$ such that

$$\frac{1}{4} < h_i(0,0) \leqq r \left(\frac{2}{\delta \mathrm{Vol}(B_1)} \int_{O_i} h_i^2 \, dx \, dt \right)^{\frac{1}{2}}.$$

Since $e_i \leqq 4$ and $0 \leqq h_i \leqq e_i \exp(\delta)$ in O_i and $\sqrt{g^i} > \frac{1}{2}$ for sufficiently large i, we have (with $r_1 = 128 r^2 e^{2\delta} / \delta \mathrm{Vol}(B_1)$)

$$1 \leqq r_1 \int_{O_i} |dv_i|^2 \, dV_i \, dt. \tag{3.13}$$

Now we consider the function

$$\Phi(R) = \tfrac{1}{2} R^{2-n} \int_{B_\rho} |du|^2 (t_0 - R^2, x) \, e^{-\frac{|x|^2}{4R^2}} \phi^2(x) \sqrt{g(x)} \, dx,$$

where $0 < R < \min\{\rho, \sqrt{t_0}\} \equiv R_0$. By Lemma 2.2 we have

$$\Phi(R) \leqq e^{C(R_0 - R)} \Phi(R_0) + C E(u_0)(R_0 - R) \leqq e^{C R_0} \Phi(R_0) + C \varepsilon R_0. \tag{3.14}$$

$$\Phi(R_0) \leqq \tfrac{1}{2} R_0^{2-n} \int_{B_\rho} |du|^2 (t_0 - R_0^2, x) \, dV \leqq R_0^{2-n} E(u(t_0 - R_0^2)) \leqq \varepsilon R_0^{2-n}.$$

Thus, by (3.14),

$$\Phi(R) \leqq (R_0^{2-n} e^{C R_0} + C R_0) \varepsilon, \qquad \text{for } 0 < R < R_0.$$

On the other hand, for $R^2 = \lambda_i^2 S^2$, where $\sqrt{\delta/2} < S < \sqrt{\delta + \min(\delta/2, \delta/C_i)}$, we have

$$\begin{aligned} &\lambda_i^{2-n} \int_{B_{\lambda_i}} |du(t_i + \lambda_i^2(\delta - S^2))|^2 \, dV \\ &= (R/S)^{2-n} \int_{B_{R/S}} |du|^2 (t_\circ - R^2, x) \, dV \leqq 2(2\delta)^{\frac{n-2}{2}} e^{\frac{1}{2\delta}} \Phi(R) \\ &\leqq 2(2\delta)^{\frac{n-2}{2}} e^{\frac{1}{2\delta}} (R_0^{2-n} e^{C R_0} + C R_0) \varepsilon \leqq \alpha R_0^{2-n} \varepsilon \end{aligned} \tag{3.15}$$

for some positive constant $\alpha>0$, since $e^{-\frac{r^2}{4R^2}} \geqq e^{-\frac{1}{4S^2}}$ and $\phi \equiv 1$ on $B_{R/S}$. However, direct computation shows

$$\int_{B_1} |dv_i(t)|^2 \, dV_i = \lambda_i^{2-n} \int_{B_{\lambda_i}} |du(t_i + \lambda_i^2 t)|^2 \, dV.$$

Thus, for $-\min(\delta/2, \delta/C_i) < t = \delta - S^2 < \delta/2$, (3.15) gives

$$\int_{B_1} |dv_i(t)|^2 \, dV_i \leqq \alpha R_0^{2-n} \varepsilon,$$

which together with (3.13) leads to

$$1 \leqq C R_0^{2-n} \varepsilon, \tag{3.16}$$

where $C = r_1 \alpha \delta$ depends only on M and N.

Consider first the case $\rho \leqq \sqrt{t_0}$, which implies $R_0 = \rho$. We see from (3.16) that in this case $\varepsilon \geqq \varepsilon_1 = C^{-1} \rho^{n-2}$. Hence, if $\varepsilon < \varepsilon_1$ we must have $\rho > \sqrt{t_0}$, hence $R_0 = \sqrt{t_0}$. It follows from (3.16) that

$$t_0^{\frac{n-2}{2}} \leqq C\varepsilon.$$

Since $t_0 \to T$ as $i \to \infty$, we see (3.7) holds. This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. Let u be solution of (1.1)–(1.2) with

$$(1) \quad |du_0| \leqq K \text{ on } M, \qquad (2) \quad E(u_0) < \varepsilon.$$

Let K be fixed. We are going to show that

$$\bar{e}(t) = \max_M |du(t)|^2 \leqq C, \qquad \forall t > 0, \tag{3.17}$$

provided ε is sufficiently small. Indeed, if (3.17) does not hold, we may assume the existence of a sequence $t_i \to T$ which satisfies (3.8) and (3.9). This, exactly the same as in the proof of Theorem 1.1, will imply that

$$T \leqq C \varepsilon^{\frac{2}{n-2}}, \qquad \text{if } \varepsilon < \varepsilon_1. \tag{3.18}$$

On the other hand, condition (1) and Lemma 2.1 indicates that there exists $\delta > 0$ such that

$$T > \delta \ln\left(1 + \frac{1}{2K^2}\right),$$

which contradicts (3.18) if ε is small enough. Hence, (3.17) holds provided ε is small. Once we have (3.17), the first part of the proof of Theorem 1.1 applies to show $u(t)$ subconverges to some harmonic map u_∞ as $t \to +\infty$, and $E(u_\infty) < \varepsilon$. Finally, by Lemma 3.1 u_∞ has to be constant since ε is small.

Remark 3.1. We explain the modification in the Dirichlet case, as mentioned in Remark 1.3.

First, the points p_i, at which $e(u(t_i))$ takes on its maximum, may go to the boundary. For such a case one should use the monotonicity inequality (2.7) as well as the monotonicity inequality at the boundary discussed in Remark 2.2. The former might be useful for R small, while the latter is needed for R large. One also needs an argument in [17] (Lemma 1.3). We leave the details to the reader.

Secondly, since we do not have a "boundary version" of Moser's Harnack inequality, we need a proof of the following fact:

Let $v \in C^\infty([-1, 0] \times B_1^+, N)$ be a solution to

$$\frac{\partial v}{\partial t} - \Delta v = A(v)(dv, dv), \; v(t, \cdot)|_{x^n=0} = q \in N, \tag{3.19}$$

such that

$$(1) \quad |dv|^2 \leqq 1, \quad \text{on } [-1, 0] \times B_1^+, \qquad (2) \quad \max\{|dv|^2(0, x) | x \in B_{\frac{1}{2}}^+\} = 1,$$

then there exists $C > 0$ such that

$$1 \leqq C \int_{[-1,0] \times B_1^+} |dv|^2 \, dV dt.$$

We prove this by way of contradiction. Suppose that the constant C does not exist. Then we may assume that there is a sequence $\{v_i\}$ of solutions of (3.19) which satisfies (1) and (2) such that

$$\int_{[-1,0] \times B_1^+} |dv_i|^2 \, dV dt \to 0. \tag{3.20}$$

Now, noticing condition (1) implies that we have the uniform estimates for v_i in $C^{2,\alpha}([-\frac{1}{2}, 0] \times \bar{B}_{\frac{1}{2}}^+, N)$. So we may assume $v_i \to \bar{v}$ in C^2, and $\bar{v}$ also satisfies (2). On the other hand, in view of (3.20) we should have $d\bar{v} = 0$ which contradicts (2). Therefore, the stated fact must be true. Except for the above two points, the proof for the Dirichlet case through as before.

Final Remarks. (1) Theorem 1.1 answers a question raised in 1964 by Eells and Sampson [7]. See also Eells and Lemaire [6]. problem 1.1 (page 63).

(2) Let us consider Theorem 1.1 in lower dimensions:

Case $n = 1$. The heat flow is defined for all positive time. That is a consequence of the proof in Chapter II of [7]. Full details and discussion are in [13].

Case $n = 2$ remains open. Certainly an assertion like Theorem 1.1 is not valid, for $E_{\mathscr{F}} = 0$ iff $\mathscr{F}$ is trivial when $n = 2$ [20].

(3) Case $n \geqq 3$. What geometric conditions ensure there exists an initial data in $\mathscr{F}$ for which the heat flow is defined for all positive time and converges to a harmonic map as $t \to \infty$.

Acknowledgment. The authors would like to thank Professor J. Eells for valuable discussions and many good suggestions.

References

1. Chang, K.C.: Heat flow and boundary value problem for harmonic maps. (preprint 1988)
2. Chen, Y., Struwe, M.: Existence and partial regularity for heat flow for harmonic maps. Math. Z. **201**, 83–103 (1989)
3. Coron, J.M., Ghidaglia, J.M.: Explosion en temps fini pour le flot des applications harmoniques. C.R. Acad. Sci. Paris **308**, Serie I, 339–344 (1989)
4. Ding, W.Y.: Blow-up of solutions of heat flows for harmonic maps. (preprint 1988)
5. Eells, J., Lemaire, L.: A report on harmonic maps. Bull. Lond. Math. Soc. **16**, 1–68 (1978)
6. Eells, J., Lemaire, L.: Selected topics in harmonic maps. Conf. Board Math. Sci. Am. Math. Soc. **50** (1983)
7. Eells, J., Sampson, J.H.: Harmonic mapping Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
8. Gordon, W.B.: Convex functions and harmonic maps. P.A.M.S **33**, 433–437 (1972)
9. Hamilton, R.: Harmonic maps of manifolds with boundary. Lecture Notes in Math., vol. **471**, Berlin Heidelberg New York: Springer 1975
10. Hartman, P.: Ordinary differential equations, 2nd edition, Boston, Birkhäuser, 1982
11. Jost, J.: Ein Existenzbeweis für harmonische Abbildungen, die ein Dirichletproblem lösen, mittels der Method des Wärmeflusses, Manuscr. Math. **34**, 17–25 (1981)
12. Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and quasilinear equation of parabolic type. Transl. Math. Momo. **23** Am. Math. Soc. 1968
13. Ottarsson, S.K.: Closed geodesics on Riemannian manifolds via the heat flow. J. Geom. Phys. **2**, 49–72 (1985)
14. Moser, J.: A Harnack inequality for parabolic differential equations. Comm. Pure Appl. Math. **17**, 101–134 (1964)
15. Sacks, J., Uhlenbeck, K.: The existence of minimal immersions of two spheres. Ann. Math. **113**, 1–24 (1981)
16. Schoen, R.: Analytic aspects of harmonic maps. M.S.R.I. Publications **2**, 321–358 Berlin: Springer 1984
17. Schoen R., Uhlenbeck, K.: Boundary regularities and the Dirichlet problem for harmonic maps. J. Differ. Geom. **18**, 253–268 (1983)
18. Struwe, M.: On the evolution of harmonic maps of Riemannian surfaces. Comment. Math. Helv. **60**, 558–581 (1985)
19. Struwe, M.: On the evolution of harmonic maps in higher dimensions. J. Differ. Geom. **28**, 485–502 (1988)
20. White, B.: Infima of energy functionals in homotopy classes of mappings. J. Differ. Geom. **23**, 127–142 (1986)

Oblatum 20-III-1989 & 21-VI-1989