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Existence and Partial Regularity Results for the Heat Flow for Harmonic Maps

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I. Problem and Notations

1. Let M, N be compact, smooth Riemannian manifolds of dimensions m, l with metrics g, h respectively. Since N is compact, N may be isometrically embedded into R^n for some n.

For a C^1-map $u\colon M \to N \subset R^n$, the energy of u is given by

$$E(u) = \int_M e(u)\, dM \tag{1.1}$$

with energy density

$$e(u) = \frac{1}{2} g^{\alpha\beta}(x) \frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_i}{\partial x_\beta}$$

and volume element

$$dM = \sqrt{|g|}\, dx,$$

in local coordinates on M.

Here,

$$g^{\alpha\beta} = (g_{\alpha\beta})^{-1}, \quad 1 \leqq \alpha, \beta \leqq m, \qquad |g| = \det(g_{\alpha\beta}) \tag{1.2}$$

and a summation convention is used.

u is harmonic, if E is stationary at u, i.e.

$$-\Delta_M u + \lambda\, \gamma_u = 0 \tag{1.3}$$

for some function $\lambda\colon M \to R$, where γ_u is a unit normal vector to N at u and

$$\Delta_M = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{|g|}\, g^{\alpha\beta} \frac{\partial}{\partial x_\beta} \cdot \right) \tag{1.4}$$

denotes the Laplace-Beltrami operator on M.

From (1.3) we obtain for any $x \in M$ that

$$\lambda(x) = -g^{\alpha\beta}(x) \frac{\partial u(x)}{\partial x_\beta} \cdot \frac{\partial}{\partial x_\alpha} \gamma(u(x)), \tag{1.5'}$$

for any smooth normal vector field γ with $\gamma(u(x)) = \gamma_u(x)$, and hence that λ can be estimated

$$|\lambda| \leqq C \cdot |\nabla u|^2 \tag{1.5''}$$

with a constant C depending only on the geometry of N.

One may ask how much of the topology of N is represented by harmonic maps into N. In particular: Given a smooth map $u_0 \colon M \to N$, is there a harmonic map homotopic to u_0?

A natural approach to this problem is to deform a given map $u_0 \colon M \to N$ under the "heat flow" related to the "energy" (1.1), i.e., to study the following evolution problem

$$\partial_t u - \Delta_M u + \lambda \gamma_N(u) = 0, \qquad \text{in } M \times R_+ \tag{1.6}$$

$$u(\cdot, 0) = u_0, \qquad \text{on } M. \tag{1.7}$$

In their fundamental paper [2], Eells and Sampson establish the following result:

Theorem 1.1. *Suppose the sectional curvature of N is non-positive. Then for any smooth map $u_0 \colon M \to N$ there exists a global, regular solution $u \colon M \times R_+ \to N$ to the evolution problem* (1.6–7), *and as $t \to \infty$ the functions $u(\cdot, t) \colon M \to N$ converge to a smooth harmonic map $u_\infty \colon M \to N$ homotopic to u_0.*

Simple examples show that the restriction on the curvature in general is necessary, cf. Eells-Wood [3]. However, if one relaxes the notion of "solution" slightly, in case $m = \dim M = 2$, a result analogous to Theorem 1.1 holds for arbitrary targets N, cf. [7]:

Theorem 1.2. *Suppose $m = 2$. Then for any smooth map $u_0 \colon M \to N$ there exists a global distribution solution $u \colon M \times R_+ \to N$ with finite energy $E(u(\cdot, t)) \leqq E(u_0)$ and which is regular on $M \times R_+$ with exception of at most finitely many singular points (x_k, t_k), $1 \leqq k \leqq K$, $t_k \leqq \infty$. The solution u is unique in this class.*

At a singularity $(\bar{x}, \bar{t})$ a non-constant, smooth harmonic map $u \colon S^2 \cong \overline{R^2} \to N$ separates in the sense that for sequences $R_m \searrow 0$, $t_m \nearrow \bar{t}$, $x_m \to \bar{x}$ as $m \to \infty$

$$u_m(x) \equiv u(\exp_{x_m}(R_m x), t_m) \to \bar{u} \qquad \text{in } H^{1,2}_{\mathrm{loc}}(R^2; N).$$

Finally, $u(\cdot, t)$ converges weakly in $H^{1,2}(M; N)$ to a smooth harmonic map $u_\infty \colon M \to N$ as *$t \to \infty$ suitably, and strongly if $t_k < \infty$ for all $k = 1, \ldots, K$.*

Here, $\exp_p \colon T_p M \to M$ denotes the exponential map, and

$$H^{1,2}(M; N) = \{u \in H^{1,2}(M; R^N) \mid u(M) \subset N \text{ a.e.}\}$$

denotes the Sobolev space of square integrable (L^2-)functions u: $M \to N$ with distributional derivative $\nabla u \in L^2$.

In [9] Theorem 1.2 was applied to rederive the existence results of Lemaire and Sacks-Uhlenbeck for harmonic maps of surfaces into target manifolds N with $\pi_2(N)=0$.

Note that in general the weak limit u_∞ will not be homotopic to the initial map u_0 due to change in topology at the singularities of the flow.

In higher dimensions $m>2$, the following result was obtained in [8; Theorem 6.1]:

Theorem 1.3. *Suppose u: $R^m \times R_+ \to N$ is limit of a sequence $\{u_k\}$ of regular solutions to* (1.6–7) *with uniformly finite energy $E(u_k(\cdot, t)) \leqq E_0$ for all $t \geqq 0$, all $k \in N$, in the sense that $E(u(\cdot, t)) \leqq E_0$,* a.e. *and*

$$\nabla u_k \to \nabla u \quad \textit{weakly in } L^2(Q)$$

on any compact set $Q \subset R^m \times R_+$.

Then u is regular and solves (1.6–7) *in the classical sense on a dense open set $Q \subset R^m \times R_+$ whose complement Σ has locally finite m-dimensional Hausdorff-measure with respect to the metric $\delta((x, t), (y, s)) = |x-y| + \sqrt{|t-s|}$.*

Singularities of the flow are related to harmonic maps of spheres or special solutions $u(x, t) = v(x/\sqrt{|t|})$ of (1.6–7), cp. [8, Theorem 8.1].

Moreover, for regular $u_0 \in C^1$ with small initial energy the existence of a global, regular solution u to (1.6–7) and asymptotic convergence $u(\cdot, t) \to u_\infty$ to a smooth harmonic map u_∞ can be established, cp. Mitteau [6], Struwe [8, Theorem 7.1].

Finally, by local finiteness of the m-dimensional Hausdorff measure of the singular set Σ the limit map u in Theorem 1.3 also solves (1.6–7) weakly (in the sense of (1.14) below) on $R^m \times R_+$.

However, Theorem 1.3 does not yield a general existence result for weak solutions to (1.6–7). This is the question we confront now:

Given a smooth map u_0: $M \to N$ between two (compact) manifolds M and N, is there always a global weak solution to the evolution problem (1.6–7)?

In [1], a partial answer was obtained. (R. Kohn has pointed out that Theorem 1.4 below was independently found by Keller, Rubinstein and Sternberg [4].)

Theorem 1.4. *Suppose $N = S^n \subset R^{n+1}$. Then for any $u_0 \in H^{1,2}(M; S^n)$ there exists a weak solution u: $M \times R_+ \to S^n$ with $E(u(\cdot, t)) \leqq E(u_0)$* a.e.

Theorem 1.4 is proved by approximating u with a sequence of solutions u_K: $M \times R_+ \to R^{n+1}$ to the evolution problem for the "penalized" energy functionals

$$E_K(u) = E(u) + (K/2) \int_M (|u|^2 - 1)^2 \, dM. \tag{1.8}$$

As $K \to \infty$ the "penalty term" $(K/2) \int_M (|u|^2-1)^2 \, dM$ forces the unconstrained functions u_K to approach N and the weak limit u will be a map u: $M \times R_+ \to N$,

as desired. However, in order to show that u weakly solves (1.6), the special geometry of the sphere is crucially used, and with the method of [1] Theorem 1.4 cannot be extended to more general targets.

But this is precisely the kind of difficulty that is encountered in Theorem 1.3, and adjusting the proof of Theorem 1.3 to the penalty approximation used in Theorem 1.4 as a combination and unification of the above results we obtain the following general existence and partial regularity result for the evolution problem (1.6–7).

Theorem 1.5. *For any smooth u_0: $M \to N$ there exists a global weak solution u: $M \times R_+ \to N$ of the evolution problem* (1.6–7) *which is regular off a singular set Σ of locally finite m-dimensional Hausdorff-measure* (*with respect to the metric δ*). *Moreover, u satisfies the condition $\partial_t u \in L^2(M \times R_+)$, and $E(u(\cdot, t)) \leqq E(u_0)$* a.e.

Finally, as $t \to \infty$ suitably, a sequence $u(\cdot, t)$ converges weakly in $H^{1,2}(M; N)$ to a harmonic map u_∞: $M \to N$ with energy $E(u_\infty) \leqq E(u_0)$ and which is regular off a set Σ_∞ whose $(m-2)$-dimensional Hausdorff measure is bounded by $E(u_0)$.

Let us point out that in contrast to the 2-dimensional case in dimensions $m > 2$ our methods do not yet permit to establish uniqueness of weak solutions to (1.6–7) in the class of partially regular maps.

Another open problem concerns the question whether (in dimensions $m \geqq 2$) the flow (1.6–7) actually develops singularities in finite time. Note that in the non-compact case $M = R^m$, $m > 2$, we have regularity for large time, provided the initial data have finite energy, cp. Theorem 3.1. See "Added in proof" for some recent results in this regard.

For simplicity we first consider the case $M = R^m$.

Notations. Let $z = (x, t)$ denote points in $R^m \times R$. For a distinguished point $z_0 = (x_0, t_0)$, $R > 0$ let

$$P_R(z_0) = \{z = (x, t) \mid |x - x_0| < R, |t - t_0| < R^2\}, \tag{1.9}$$

$$S_R(z_0) = \{z = (x, t) \mid t = t_0 - R^2\}, \tag{1.10}$$

and

$$T_R(z_0) = \{z = (x, t) \mid t_0 - 4R^2 < t < t_0 - R^2\}. \tag{1.11}$$

Denote the fundamental solution to the (backward) heat equation

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{(x - x_0)^2}{4(t_0 - t)}\right), \qquad t < t_0. \tag{1.12}$$

We simply write $P_R(0) = P_R$, $T_r(0) = T_r$ and $G_0(z) = G(z)$. δ denotes the parabolic distance function

$$\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|s - t|}\} \tag{1.13}$$

and the letters c, C denote generic constants.

Remark 1.6. From now on, by slight abuse of terminology, a map u: $R^m \times R_+ \to R^n$ will be called regular if $\partial_t u$, $\nabla^2 u \in L^p_{\mathrm{loc}}$ for all $p < \infty$. By the embedding

theorems, cf. [5], a regular map u and its spatial derivative ∇u will be Hölder continuous (after modification on a set of measure 0 in $R^m \times R_+$, if necessary).

Thus by iteration, using the Schauder estimates for the heat equation (see [5], p. 320, (5.9)), a regular solution to (1.6–7) will be smooth if the initial data are smooth.

A map u: $M \times R_+ \to N \subset R^n$ will be called a weak solution to (1.6–7) if $\partial_t u$, $\nabla u \in L^2_{\mathrm{loc}}$ and if

$$\int_M \int_{R_+} \left(\partial_t u\, \varphi + g^{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial \varphi}{\partial x_\beta} + \lambda \gamma_N(u)\, \varphi\right) \sqrt{|g|}\, dx\, dt = 0 \tag{1.14}$$

for all $\varphi \in C_0^\infty(M \times R_+; R^n)$. (We may suppose that the support of φ is contained in a coordinate chart for M.) By (1.5″), if u is a weak solution to (1.6–7) identity (1.14) will also hold for testing functions $\varphi \in L^\infty(M \times R_+; R^n)$ with compact support and $\nabla \varphi \in L^2$.

II. The Case $M = R^m$: Approximation

We will approximate a solution of (1.6)–(1.7) by the solutions to the heat flow for the penalized functionals (1.8). For general targets N the definition of E_K has to be modified somewhat.

Since N is smooth and compact there exists a uniform tubular neighborhood U of N of width $2\delta_N$ such that any point $p \in U$ has a unique nearest neighbor $q = \pi_N(p) \in N$, $|p - q| = \mathrm{dist}(p, N)$ and such that the projection π_N: $U \to N$ is smooth.

Let χ be a smooth, non-decreasing function such that $\chi(s) = s$ for $s \leqq \delta_N^2$ and $\chi(s) \equiv 2\delta_N^2$ for $s \geqq 4\delta_N^2$. Then the function

$$p \to \chi(\mathrm{dist}^2(p, N))$$

is everywhere differentiable and at points p with $\mathrm{dist}(p, N) \leqq 2\delta_N$ its gradient is parallel to $p - \pi_N(p)$, hence orthogonal to $T_{\pi_N(p)} N$.

For $K = 1, 2, \ldots$ thus consider the heat flow

$$\partial_t u - \Delta u + K \chi'(\mathrm{dist}^2(u, N)) \frac{d}{du}\left(\frac{\mathrm{dist}^2(u, N)}{2}\right) = 0 \quad \text{in } R^m \times R_+ \tag{2.1}$$

with initial data

$$u(x, 0) = u_0(x) \quad \text{on } R^m \tag{2.2}$$

for the penalized functionals F_K, given by

$$F_K(u) = E(u) + (K/2) \int_{R^m} \chi(\mathrm{dist}^2(u, N))\, dx.$$

For brevity, in the sequel we will omit writing the argument of χ' appearing in (2.1) explicitly.

By Galerkin's method, we know that for every $K \in \mathbb{N}$, problem (2.1)–(2.2) has a global solution $u = u_K$ satisfying

$$\nabla u \in L^\infty([0, \infty[; L^2(R^m)), \quad \chi(\mathrm{dist}^2(u, N)) \in L^\infty([0, \infty[; L^1(R^m)),$$
$$u_t \in L^2(R^m \times R_+), \tag{2.3}$$

whence also

$$\chi' \frac{d}{du}\left(\frac{\mathrm{dist}^2(u, N)}{2}\right) \in L^\infty([0, \infty[; L^2(R^m)),$$

cf. [1]. Moreover, by applying the Sobolev embedding theorem to u and on account of the a-priori estimates for the heat operator (see [5], p. 345 (9.12)–(9.13)) we find that

$$u, u_t, \nabla^2 u \in L^p_{\mathrm{loc}}(R^m \times R_+) \tag{2.4}$$

for any $p < \infty$, i.e., u is regular.

More precisely, for $u = u_K$, we have the following estimates:

Lemma 2.1 ("Energy inequality"). *Let* $u_0 = R^m \to N$ be smooth with $E(u_0) < \infty$. *Then,*

$$\sup_{t \geqq 0}\left(\int_0^t \int_{R^m} |\partial_t u|^2 \, dx \, dt + F_K(u(\cdot, t))\right) \leqq (1/2) \int_{R^m} |\nabla u_0|^2 \, dx = F_K(u_0) = E(u_0).$$

Proof. Multiply (2.1) by u_t and integrate by parts.

Lemma 2.2 ("Monotonicity formula"). *For any point* $(x_0, t_0) \in R^m \times R_+$, *the functions*

$$\Phi(R, u, K) = (1/2) R^2 \int_{S_R(z_0)} \{|\nabla u|^2 + K\chi(\mathrm{dist}^2(u, N))\} G_{z_0} \, dx \tag{2.5}$$

$$\Psi(R, u, K) = (1/2) \int_{T_R(z_0)} \{|\nabla u|^2 + K\chi(\mathrm{dist}^2(u, N))\} G_{z_0} \, dx \, dt \tag{2.6}$$

are non-decreasing for $0 < R < \sqrt{t_0/2}$.

Proof. The proof of [8, Lemma 3.2, Proposition 3.3] carries over immediately if we replace the energy density $e(u) = (1/2)|\nabla u|^2$ by

$$e_K(u) = (1/2)(|\nabla u|^2 + K\chi(\mathrm{dist}^2(u, N)). \tag{2.7}$$

For completeness, we recall the details. Note that equation (2.1) is invariant under translation $x \to x - x_0$, $t \to t - t_0$, hence we may shift $z_0 = (0, 0)$; moreover, the scaled function

$$u_R(x, t) = u(Rx, R^2 t)$$

solves (2.1) with $K_R = R^2 K$ instead of K.

Finally,

$$\Phi(R,u,K)=(1/2)\,R^2\int_{S_R}\{|\nabla u|^2+K\chi(\mathrm{dist}^2(u,N))\}\,G\,dx$$
$$=(1/2)\int_{S_1}\{|\nabla u_R|^2+K_R\chi(\mathrm{dist}^2(u_R,N))\}\,G\,dx=\Phi(1,u_R,K_R).$$

We want to prove

$$\frac{d}{dR}\Phi(R,u,K)\Big|_{R=R_1}\geqq 0$$

for any R_1 with $0<R_1<\sqrt{t_0}/2$. Reparametrizing, it suffices to consider $R_1=1$.

By the regularity of u and G, we may differentiate under the integral sign. Using (2.1) and $\nabla G=\frac{x}{2t}G$, we get

$$\begin{aligned}
&\frac{d}{dR}\Phi(R,u,K)\Big|_{R=1}\\
&=\frac{d}{dR}\Phi(1,u_R,K_R)\Big|_{R=1}\\
&=\int_{S_1}\Big\{\nabla u\cdot\nabla\Big(\frac{d}{dR}u_R\Big|_{R=1}\Big)+K\chi(\mathrm{dist}^2(u,N))\\
&\quad+K\chi'\frac{d}{du}((1/2)\,\mathrm{dist}^2(u,N))\Big(\frac{d}{dR}u_R\Big|_{R=1}\Big)\Big\}\,G\,dx\\
&=\int_{S_1}\Big(-\Delta u+K\chi'\frac{d}{du}\Big(\frac{\mathrm{dist}^2(u,N)}{2}\Big)\Big)(x\cdot\nabla u+2t\,\partial_t u)\,G\,dx\\
&\quad-\int_{S_1}\nabla u\,\nabla G(x\cdot\nabla u+2t\,\partial_t u)\,dx+\int_{S_1}K\chi(\mathrm{dist}^2(u,N))\,G\,dx\\
&=\int_{S_1}-\partial_t u(x\cdot\nabla u+2t\,\partial_t u)\,G\,dx-\int_{S_1}\frac{x\cdot\nabla u}{2t}(x\cdot\nabla u+2t\,\partial_t u)\,G\,dx\\
&\quad+\int_{S_1}K\chi(\mathrm{dist}^2(u,N))\,G\,dx\\
&=\int_{S_1}\frac{1}{2|t|}(x\cdot\nabla u+2t\,\partial_t u)^2\,G\,dx+\int_{S_1}K\chi(\mathrm{dist}^2(u,N))\,G\,dx\geqq 0,
\end{aligned}\tag{2.8}$$

as desired.

The proof of (2.6) is similar.

Lemma 2.3. *Let $Q\subset R^m\times R$ be an open set. Then, if $u=u_K$ satisfies* (2.1) *and is regular in Q, the following inequality holds in Q*

$$(\partial_t-\Delta)\,e(u)\leqq c\,e(u)^2,\tag{2.9}$$

where $C>0$ is a constant depending only on N and m and for brevity

$$e(u)=e_K(u)=(1/2)(|\nabla u|^2+K\chi(\operatorname{dist}^2(u,N))).$$

Proof. Note that if $\operatorname{dist}(u,N)\leqq 2\delta_N$ then

$$\left|K\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right|^2=K^2\operatorname{dist}^2(u,N). \tag{2.10}$$

By equation (2.1) we have

$$\begin{aligned}
&(K/2)(\partial_t-\Delta)\,\chi(\operatorname{dist}^2(u,N))\\
&\quad=K\chi'\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)(\partial_t u-\Delta u)-\nabla\left\{K\chi'\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right\}\cdot\nabla u\\
&\quad=-K^2\chi'^2\operatorname{dist}^2(u,N)-\nabla\left\{K\chi'\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right\}\cdot\nabla u,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
(\partial_t-\Delta)\left(\frac{|\nabla u|^2}{2}\right)&=\nabla(\partial_t u-\Delta u)\cdot\nabla u-|\nabla^2 u|^2\\
&=-\nabla\left\{K\chi'\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right\}\cdot\nabla u-|\nabla^2 u|^2.
\end{aligned} \tag{2.12}$$

(2.12) implies (2.9) if $\operatorname{dist}(u,N)>2\delta_N$; for the remaining case we have

$$\begin{aligned}
&(\partial_t-\Delta)\,e(u)+|\nabla^2 u|^2+K^2\chi'^2\operatorname{dist}^2(u,N)\\
&\quad=-2\nabla\left\{K\chi'\frac{d}{du}\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right\}\cdot\nabla u\\
&\quad=-2K(\chi'+\chi''\operatorname{dist}^2(u,N))\,|\nabla\operatorname{dist}(u,N)|^2-2K\chi'\operatorname{dist}(u,N)\\
&\qquad\cdot\left\{\frac{d^2}{du^2}\operatorname{dist}(u,N)\,\nabla u\cdot\nabla u\right\}\\
&\quad\leqq\tfrac{1}{2}K^2\operatorname{dist}^2(u,N)+c\,|\nabla u|^4.
\end{aligned} \tag{2.13}$$

In the last inequality, we use the fact that $\chi'\geqq 0$ and that for $u\notin N$

$$\left|\frac{d^2}{du^2}\operatorname{dist}(u,N)\right|\leqq C(N), \tag{2.14}$$

where $C(N)$ depends only on a bound for the curvature of N.

Now, (2.9) follows from (2.13) immediately.

Lemma 2.4. *There exists a constant $\varepsilon_0>0$ depending only on m and N, such that if for some $0<R<\sqrt{t_0}/2$, $z_0=(x_0, t_0)$, $u=u_K$ satisfies*

$$\Psi(R)=\Psi(R,u,K)=(1/2)\int_{T_R(z_0)}(|\nabla u|^2+K\chi(\mathrm{dist}^2(u,N)))\,G_{z_0}\,dx\,dt<\varepsilon_0, \tag{2.15}$$

then

$$\sup_{P_{\delta R}(z_0)}\{|\nabla u|^2+K\chi(\mathrm{dist}^2(u,N))\}\leqq C(\delta R)^{-2} \tag{2.16}$$

with a constant $\delta>0$ depending only on N, m, E_0 and $\inf\{R,1\}$ and an absolute constant C.

Proof. The proof of [8; Theorem 5.1] carries over almost literally. By translation invariance of problem (2.1), in order to prove Lemma 2.3 it is sufficient to consider the case that $z_0=0$ and that $u=u_K$ satisfies (2.1) and is regular on $R^m\times\,]-t_0,0]$.

Let $e(u)=e_K(u_K)$ be defined by (2.7).

Set $r_1=\delta R$, $\delta\in]0,1/2]$ to be determined in the sequel. For r, $\sigma\in]0,r_1[$, $r+\sigma<r_1$, and any $z_0=(x_0,t_0)\in P_r$ our monotonicity formula implies

$$\begin{aligned}\sigma^{-m}\int_{P_\sigma(z_0)}e(u)\,dx\,dt &\leqq c\int_{P_\sigma(z_0)}e(u)\,G_{(x_0,t_0+2\sigma^2)}\,dx\,dt\\ &\leqq c\int_{T_\sigma(t_0+2\sigma^2)}e(u)\,G_{(x_0,t_0+2\sigma^2)}\,dx\,dt\\ &\leqq c\int_{T_R(t_0+2\sigma^2)}e(u)\,G_{(x_0,t_0+2\sigma^2)}\,dx\,dt\\ &\leqq c\left(\int_{-4R^2}^{-R^2}+\int_{-R^2}^{t_0+2\sigma^2-R^2}\right)\{e(u)\,G_{(x_0,t_0+2\sigma^2)}\}\,dx\,dt\\ &\leqq c\int_{T_R}e(u)\,G_{(x_0,t_0+2\sigma^2)}\,dx\,dt.\end{aligned} \tag{2.17}$$

But on T_R, given $\varepsilon>0$, if $\delta>0$ is small enough:

$$\begin{aligned}G_{(x_0,t_0+2\sigma^2)}(x,t) &\leqq \frac{c}{(4\pi|t|)^{m/2}}\exp\left(-\frac{|x-x_0|^2}{4(t_0+2\sigma^2-t)}\right)\\ &\leqq c\exp\left(\frac{|x|^2}{4|t|}-\frac{|x-x_0|^2}{4|t_0+2\sigma^2-t|}\right)G(x,t)\\ &\leqq c\exp\left(c\,\delta^2\frac{|x|^2}{4|t|}\right)G(x,t)\\ &\leqq \begin{cases}c\,G(x,t) & \text{if } |x|\leqq R/\delta\\ c\,R^{-m}\exp(-c\,\delta^{-2}), & \text{if } |x|\geqq R/\delta\end{cases}\\ &\leqq c\,G(x,t)+c\,R^{-2}\exp((2-m)\log R-c\,\delta^{-2})\\ &\leqq c\,G(x,t)+\varepsilon R^{-2},\end{aligned} \tag{2.18}$$

where c is independent of δ and R.

Remark that $\delta \approx |\log R|^{-1/2}$ for small R and may be chosen independent of R, if $R \geqq 1$. Hence with this choice of δ, for $z_0 \in P_r$, $r+\sigma < r_1$ we may estimate

$$\sigma^{-m} \int_{P_\sigma(z_0)} e(u)\, dx\, dt \leqq c\, \Psi(R) + c\, \varepsilon E_0 \leqq c(\varepsilon_0 + \varepsilon E_0). \tag{2.19}$$

Since u is regular, there exists $\sigma_0 \in]0, r_1[$ such that

$$(r_1 - \sigma_0)^2 \sup_{P_{\sigma_0}} e(u) = \max_{0 \leqq \sigma \leqq r_1} (r_1 - \sigma)^2 \sup_{P_\sigma} e(u).$$

Moreover, there exists $(x_0, t_0) \in \overline{P_{\sigma_0}}$ such that

$$\sup_{P_{\sigma_0}} e(u) = e(u)(x_0, t_0) = e_0.$$

Set $\rho_0 = (1/2)(r_1 - \sigma_0)$. By choice of σ_0, (x_0, t_0)

$$\sup_{P_{\rho_0}(x_0, t_0)} e(u) \leqq \sup_{P_{\sigma_0 + \rho_0}} e(u) \leqq 4\, e_0.$$

Introduce

$$r_0 = \sqrt{e_0} \cdot \rho_0$$

$$v(x, t) = u\left(\frac{x}{\sqrt{e_0}} + x_0, \frac{t}{e_0} + t_0\right).$$

We claim $r_0 \leqq 1$. v solves (2.1) in P_{r_0} with $\tilde{K} = K/e_0$; moreover, v satisfies

$$e(v)(0, 0) = 1,$$

$$\sup_{P_{r_0}} e(v) \leqq 4,$$

where $e = e_{\tilde{K}}$. By Lemma 2.3, $e(v)$ satisfies

$$(\partial_t - \Delta)\, e(v) \leqq c_1\, e(v) \quad \text{in } P_{r_0}$$

with a constant c_1 depending only on m and N. Thus, if instead of $e(v)$ we consider the function $f(x, t) = \exp(-c_1 t)\, e(v)$ in P_{r_0} and if $r_0 \geqq 1$, Moser's Harnack inequality implies the estimate

$$1 = e(v)(0, 0) \leqq C \int_{P_1} e(v)\, dx\, dt.$$

But, scaling back, by (2.19) and since $\dfrac{1}{\sqrt{e_0}} + \sigma_0 \leqq \rho_0 + \sigma_0 < r_1$ we have

$$\int_{P_1} e(v)\, dx\, dt = (\sqrt{e_0})^m \int_{P_{\frac{1}{\sqrt{e_0}}}(x_0, t_0)} e(u)\, dx\, dt \leqq c(\varepsilon_0 + \varepsilon E_0)$$

and we obtain a contradiction for small ε_0, $\varepsilon>0$. Hence, $r_0 \leqq 1$.
By choice of σ_0 this implies:

$$\max_{0 \leqq \sigma \leqq r_1} (r_1-\sigma)^2 \sup_{P_\sigma} e(u) \leqq 4\rho_0^2 e_0 = 4 r_0^2 \leqq 4.$$

Hence, we may choose $\sigma=(1/2)\, r_1=(\delta/2) R$ and divide by σ^2 to complete the proof with $C=16$.

III. Passing to the Limit $K \to \infty$

From Lemma 2.1 we know that for smooth $u_0: \mathbb{R}^m \to N$ with $E(u_0)<\infty$ there exist a subsequence of $\{u_K\}$ (also denoted by u_K) and a function u defined a.e. on $R^m \times R_+$, such that as $K \to \infty$ we have

$$\nabla u_K \to \nabla u \quad \text{weakly* in } L^\infty([0,\infty[; L^2(R^m)), \tag{3.1}$$

$$\partial_t u_K \to \partial_t u \quad \text{weakly in } L^2(R^m \times R_+), \tag{3.2}$$

$$u_K \to u \quad \text{weakly in } H^{1,2}_{\mathrm{loc}}(R^m \times R_+), \tag{3.3}$$

and hence also $u_K \to u$ a.e. on $R^m \times R_+$. Moreover,

$$\operatorname{dist}(u_K, N) \to 0 \quad \text{in } L^2_{\mathrm{loc}}(R^m \times R_+). \tag{3.4}$$

It follows that

$$\partial_t u \in L^2(R^m \times R_+), \quad \nabla u \in L^\infty([0,\infty[; L^2(R^m)), \tag{3.5}$$

$$u \in N, \text{ a.e.} \tag{3.6}$$

For $M=R^m$, now u will have the properties listed in Theorem 1.5:

Theorem 3.1. *Let $u_0: \mathbb{R}^m \to N$ be smooth with $E(u_0)<\infty$ and let u be the weak limit of a sequence $\{u_K\}$ of solutions to* (2.1) *as above.*

Then the function u weakly solves the evolution problem (1.6)–(1.7). *Moreover, u is regular and solves* (1.6) *classically on a dense open set $Q_0 \subset R^m \times R_+$, whose complement Σ has locally finite m-dimensional Hausdorff-measure* (*with respect to the parabolic metric δ*). *Moreover, u satisfies $E(u(\cdot,t)) \leqq E(u_0)$,* a.e. *and $\partial_t u \in L^2(R^m \times R_+)$.*

Finally, there exists $t_0>0$ (depending on m, N and E_0) such that $\Sigma \cap (R^m \times [t_0,\infty))=\varnothing$, and, $u(t) \to u_\infty \equiv p \in N$ in $H^{1,2}_{\mathrm{loc}}$ as $t \to \infty$ suitably, where $u_\infty \equiv p$ is a constant harmonic map.

Proof. As in [8; proof of Theorem 6.1] define

$$\Sigma = \bigcap_{R>0} \{z_0 \in R^m \times R_+ \mid \liminf_{K\to\infty} \int_{T_R(z_0)} e_K(u_K)\, G_{z_0}\, dx\, dt \geqq \varepsilon_0\} \tag{3.7}$$

where $e_K(u)$ is defined by (2.7) and $\varepsilon_0>0$ is the constant determined in Lemma 2.4. The proof of the result that Σ is closed and has locally finite m-dimensional Hausdorff-measure with respect to the metric δ is the same as that of Theorem 6.1 in [8]. We only have to replace the densities $e(u_K)$ by $e_K(u_K)$ given by (2.7).

To see that u solves (1.6) off Σ some care is needed. Moreover, the argument in [1] cannot be employed.

For $z_0\notin\Sigma$, there exists $R>0$, such that

$$\int_{T_R(z_0)} e_K(u)\, G_{z_0}\, dx\, dt \leqq \varepsilon_0 \tag{3.8}$$

for an unbounded sequence $K\in N$. By Lemma 2.4, we have

$$|\nabla u_K|,\ K\,\mathrm{dist}^2(u_K, N)\leqq C, \tag{3.9}$$

uniformly in a uniform neighborhood Q of z_0. It follows that there exists a subsequence u_K such that as $K\to\infty$

$$u_K\to u \qquad \text{in } C^0_{\mathrm{loc}}(Q), \tag{3.10}$$

$$\nabla u_K\to\nabla u \qquad \text{weakly* in } L^\infty_{\mathrm{loc}}(Q). \tag{3.11}$$

Moreover, by (2.13)

$$(\partial_t-\Delta)\, e_K(u_K)+K^2\,\mathrm{dist}^2(u_K, N)\leqq C \tag{3.12}$$

in the distribution sense on Q, with a constant C independent of K. (Note that by (3.10) $\chi'(\mathrm{dist}^2(u_K, N))=1$ for sufficiently large K.) Choose a function $\varphi\in C_0^\infty(Q)$, multiply (3.12) by φ and integrate over Q. Then after integrating by parts we obtain that

$$\int_Q K^2\,\mathrm{dist}^2(u_K, N)\,\varphi\, dx\, dt\leqq\int_Q |\partial_t\varphi+\Delta\varphi|\, e_K(u_K)+c\,|\varphi|\, dx\, dt\leqq c(\varphi),$$

uniformly in K, and it follows that $K\,\mathrm{dist}(u_K, N)$ is uniformly bounded in $L^2_{\mathrm{loc}}(Q)$. But now from (2.1) also $(\partial_t-\Delta)u_K$ is uniformly bounded in $L^2_{\mathrm{loc}}(Q)$, and therefore also $\partial_t u_K$ and $\nabla^2 u_K$ are.

Hence we may assume that

$$(\partial_t-\Delta)u_K\to(\partial_t-\Delta)u \qquad \text{weakly in } L^2_{\mathrm{loc}}(Q), \tag{3.13}$$

and

$$K\,\mathrm{dist}(u_K, N)\to\bar{\lambda} \qquad \text{weakly in } L^2_{\mathrm{loc}}(Q). \tag{3.14}$$

Finally, note that there exist unit vector fields $\gamma_N^{(K)}\perp T_{p_N(u_K)}N$ such that

$$K\frac{d}{du}\left(\frac{\mathrm{dist}^2(u_K, N)}{2}\right)=K\,\mathrm{dist}(u_K, N)\cdot\gamma_N^{(K)}.$$

By (3.10) therefore there holds

$$\int_{Q'} K \frac{d}{du}\left(\frac{\operatorname{dist}^2(u_K, N)}{2}\right)\cdot \varphi \, dx\, dt \to 0, \tag{3.15}$$

for any vector field $\varphi \in L^2_{\mathrm{loc}}(Q)$ such that $\varphi(z) \in T_{u(z)}N$ a.e. on Q, and for any $Q' \Subset Q$.

For (3.13–15) it follows that $(\partial_t - \Delta)u \perp T_u N$ a.e. in Q; i.e., there exists a unit normal vector field $\gamma_N(u)$ along u and a scalar function $\lambda \in L^2_{\mathrm{loc}}(Q)$ such that there holds

$$\partial_t u - \Delta u + \lambda \gamma_N(u) = 0 \tag{3.16}$$

a.e. on Q and in the distribution sense.

Furthermore, by using (3.10), (3.16) and the regularity theory of parabolic operators, we infer that u is regular and solves (1.6)–(1.7) in the classical sense off Σ.

To see that u weakly solves (1.6) on $R^m \times R_+$, now choose any domain $Q \Subset R^m \times R_+$. Given $R > 0$ let $\{P_i = P_{R_i}(z_i)\}_{i \in J}$ be a covering of $\Sigma \cap Q$ by parabolic cylinders $P_{R_i}(z_i)$ with $R_i \leqq R$ and such that $\sum_{i \in J} R_i^m \leqq 2 \cdot H^m(\Sigma \cap Q; \delta)$, where $H^m(\Sigma \cap Q; \delta)$ denotes the m-dimensional Hausdorff-measure of $\Sigma \cap Q$ with respect to the parabolic metric δ.

Also choose a smooth cut-off function $0 \leqq \eta \leqq 1$ having support in $P_2(0)$ and identically 1 on $P_1(0)$, and for $i \in J$ denote η_i the scaled function $\eta_i(x, t) = \eta\left(\frac{x - x_i}{R_i}, \frac{t - t_i}{R_i^2}\right)$.

Let $\varphi \in C^\infty(M \times R_+; R^n)$ be an arbitrary testing function with support in Q. By (3.16) we have

$$\begin{aligned} 0 &= \int_Q (u_t - \Delta u + \lambda \gamma_N(u))\, \varphi \inf_{i \in J}(1 - \eta_i)\, dx\, dt \\ &= \int_Q \{u_t \varphi + \nabla u \nabla \varphi + \lambda \gamma_N(u)\, \varphi\} \inf_{i \in J}(1 - \eta_i)\, dx\, dt + F, \end{aligned}$$

where the term F involves error terms

$$\begin{aligned} |F| &\leqq \int_Q |\nabla u|\, |\varphi| \sup_{i \in J} |\nabla \eta_i|\, dx\, dt \\ &\leqq c \sum_{i \in J} R_i^{-1} \int_{Q_i} |\nabla u|\, dx\, dt, \end{aligned}$$

and where the family Q_i is a disjoint cover of $\bigcup_{i \in J} \operatorname{supp} |\nabla \eta_i|$ with $Q_i \subset P_{2R_i}(z_i)$ and such that

$$|\nabla \eta_i| = \max_{k \in J} |\nabla \eta_k| \quad \text{on } Q_i \text{ for any } i \in J.$$

By Hölder's and the Cauchy-Schwarz inequality therefore

$$\begin{aligned}|F| &\leqq C\sum_{i\in J} R_i^{m/2}\Big(\int_{Q_i} |\nabla u|^2\,dx\,dt\Big)^{1/2}\\ &\leqq C\Big(\sum_{i\in J} R_i^m\Big)^{1/2}\Big(\int_{\bigcup_{i\in J} Q_i} |\nabla u|^2\,dx\,dt\Big)^{1/2}\\ &\leqq c\int_{\bigcup_{i\in J} P_{2R_i(z_i)}} |\nabla u|^2\,dx\,dt.\end{aligned}$$

Passing to the limit $R\to 0$ by absolute continuity of the Lebesgue integral the error term $F\to 0$, and u weakly solves (1.6) in $R^m\times R_+$, as claimed.

Finally, using Lemma 2.2, for large t_0, $R=\sqrt{t_0}/2$, we get

$$\begin{aligned}&\int_{T_R(z_0)} e(u_K)\,G_{z_0}\,dx\,dt\\ &\quad\leqq \int_0^{3t_0/4}\int_{R^m} e(u_K)\,G_{z_0}\,dx\,dt \leqq C\,t_0^{\frac{2-m}{2}}\,E_0<\varepsilon_0\end{aligned}\tag{3.17}$$

uniformly in K, if $t_0>c(E_0/\varepsilon_0)^{\frac{2}{m-2}}$, where $E_0=E(u_0)$.

Applying Lemma 2.4 we infer the uniform decay

$$|\nabla u(x,t)|^2\leqq c/t\tag{3.18}$$

for large t, and $u(t)\to u_\infty\equiv\text{const}$, as $t\to\infty$.

IV. Compact Domains

The preceding arguments with minor changes convey to mappings $u\colon M\times R_+\to N$ on compact manifolds M.

To indicate the necessary changes denote $R_M>0$ a lower bound for the injectivity radius of the exponential map on M such that for any $R<R_M$ and any point $x_0\in M$ the geodesic ball $B_R(x_0)$ of radius R around x_0 is defined and diffeomorphic to the Euclidean ball $B_R(0)\subset R^m$ via the exponential map. Remark that in these coordinates the metric g on M is represented by a matrix $(g_{\alpha\beta})$, $1\leqq\alpha,\beta\leqq m$, with $g(0)=\text{id}$. Now, if $u=u_k\colon M\times[-T,0]\to R^n$ is a regular solution to the penalized heat equation

$$\partial_t u-\Delta_M u+K\chi'\frac{d}{du}\left(\frac{\text{dist}^2(u,N)}{2}\right)=0,\tag{4.1}$$

we may restrict u to any such coordinate neighborhood and regard u as a map $u\colon B_R(0)\times[-T,0]\subset R^m\times[-T,0]\to R^n$ with energy density

$$e(u)=e_K(u)=\frac{1}{2}\left\{g^{\alpha\beta}\frac{\partial}{\partial x_\alpha}u\frac{\partial}{\partial x_\beta}u+K\chi(\text{dist}_2(u,N))\right\}.\tag{4.2}$$

Now let G be the fundamental solution (1.12) to the backward heat equation on R^m with the flat (Euclidean) metric, as before. Also let $\varphi \in C_0^\infty(B_{R_M}(0))$ be a cut-off function $0 \leqq \varphi \leqq 1$ such that $\varphi = 1$ in a neighborhood of 0. Finally, introduce the analogues of the weighted energy integral (2.5–6)

$$\begin{aligned} \Phi(R, u, K) &= R^2 \int_{S_R} e_K(u)\, G\, \varphi^2 \sqrt{|g|}\, dx, \\ \Psi(R, u, K) &= \int_{T_R} e_K(u)\, G\, \varphi^2 \sqrt{|g|}\, dx\, dt, \end{aligned} \tag{4.3}$$

where $\sqrt{|g|}\, dx$ denotes the volume element on M.

Then we have the following estimates analogous to Lemmas 2.1–4:

Lemma 4.1 ("Energy inequality"). *Let $u = u_K$: $M \times R_+ \to R^n$ be a regular solution to the penalized equation* (4.1) *with initial value $u(\cdot, 0) = u_0 \in H^{1,2}(M; N)$. Then*

$$\sup_{t \geqq 0} \left(\int_0^t \int_M |\partial_t u|^2 \sqrt{|g|}\, dx + F_K u(\cdot, t) \right) \leqq F_K(u_0) = E(u_0),$$

where $F_K(u) = \int_M e_K(u)\, dM$ with density e_K given by (4.2).

Lemma 4.2 ("Monotonicity formula"). *Suppose $u = u_k$: $B_{R_M}(0) \times [-T, 0] \subset R^m \times [-T, 0] \to R^n$ is a regular solution to* (4.1) *with $E(u(t)) \leqq E_0$ for all $t \in [-T, 0]$. We may assume that $T \leqq R_M^2$. Then for any $0 < R \leqq R_0 \leqq R_M$ there hold the relations*

$$\Phi(R, u, K) \leqq \exp(c(R_0 - R))\, \Phi(R_0, u, K) + c E_0 (R_0 - R),$$

$$\Psi(R, u, K) \leqq \exp(c(R_0 - R))\, \Psi(R_0, u, K) + c E_0 (R_0 - R),$$

with a uniform constant c depending only on M and N.

Remark. An analogous estimate of course is valid for regular solutions u: $M \times [-T, 0] \to N$ of (1.6).

Proof. We present the proof for the function Ψ. After scaling

$$\begin{aligned} \Psi(R; u, K) = \frac{1}{2} \int_{T_1} & \left\{ g^{\alpha\beta}(R\cdot) \frac{\partial}{\partial x_\alpha} u_R \frac{\partial}{\partial x_\beta} u_R + R^2 K \chi(\operatorname{dist}^2(u_R, N)) \right\} G \\ & \cdot \varphi^2(R\cdot) \sqrt{|g|}(R\cdot)\, dx\, dt \end{aligned}$$

where $u_R(x, t) = u(Rx, R^2 t)$ as above.

Now compute, using $\frac{\partial}{\partial x_\beta} G = \frac{x_\beta}{2t} G$ as before:

$$\begin{aligned}
\frac{d}{dR}\Psi(R;u,K)\Big|_{R=1} = \int_{T_1} \Big\{ \Big(-\frac{x_\beta}{2t} g^{\alpha\beta} \frac{\partial}{\partial x_\alpha} u - \Delta_M u + K\chi' \frac{d}{du}\Big(\frac{\mathrm{dist}^2(u,N)}{2}\Big)\Big) \\
\cdot (x\cdot\nabla u + 2t\,\partial_t u) + K\chi(\mathrm{dist}^2(u,N))\Big\} G\varphi^2 \sqrt{|g|}\,dx\,dt \\
-2\int_{T_1} g^{\alpha\beta} \frac{\partial}{\partial x_\alpha} u \frac{\partial}{\partial x_\beta} \varphi (x\cdot\nabla u + 2t\,\partial_t u) G\varphi \sqrt{|g|}\,dx\,dt \\
+\frac{1}{2}\int_{T_1} x\cdot\nabla g^{\alpha\beta} \frac{\partial}{\partial x_\alpha} u \frac{\partial}{\partial x_\beta} u G\varphi^2 \sqrt{|g|}\,dx\,dt \\
+\frac{1}{2}\int_{T_1} e_K(u) G\varphi^2 \frac{x\cdot\nabla|g|}{|g|}\sqrt{|g|}\,dx\,dt \\
+\int_{T_1} e_K(u) G\varphi\, x\cdot\nabla\varphi \sqrt{g}\,dx\,dt =: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}$$

The first term may be estimated

$$\begin{aligned}
\mathrm{I} \geqq \int_{T_1} \Big\{ \frac{1}{2|t|}(x\cdot\nabla u + 2t\,\partial_t u)^2 + K\chi(\mathrm{dist}^2(u,N))\Big\} G\varphi^2\sqrt{|g|}\,dx\,dt \\
-\int_{T_1} \frac{1}{2|t|}|x|\,|g-\mathrm{id}|\,|\nabla u|\,|x\cdot\nabla u + 2t\,\partial_t u|\, G\varphi^2\sqrt{|g|}\,dx\,dt \\
\geqq \int_{T_1} \Big\{ \frac{1}{4|t|}|x\cdot\nabla u + 2t\,\partial_t u|^2 + K\chi(\mathrm{dist}^2(u,N))\Big\} G\varphi^2\sqrt{|g|}\,dx\,dt \\
-c\int_{T_1} |x|^2 |g-\mathrm{id}|^2 |\nabla u|^2 G\varphi^2\sqrt{|g|}\,dx\,dt \\
\geqq \frac{1}{4}\int_{T_1} \frac{|x\cdot\nabla u + 2t\,\partial_t u|^2}{|t|} G\varphi^2\sqrt{|g|}\,dx\,dt - c\,\Psi(1,u,K)
\end{aligned}$$

while for the remaining error terms we have

$$\begin{aligned}
|\mathrm{II}| \leqq \frac{1}{8}\int_{T_1} \frac{|x\cdot\nabla u + 2t\,\partial_t u|^2}{|t|} G\varphi^2\sqrt{|g|}\,dx\,dt + c\int_{T_1} |\nabla u|^2 G\sqrt{|g|}\,dx\,dt \\
\leqq \tfrac{1}{2} I + c\,\Psi(1,u,K) + c\,E(u_0).
\end{aligned}$$

$$|\mathrm{III}| + |\mathrm{IV}| \leqq c\,\Psi(1,u,K),$$

$$\begin{aligned}
|\mathrm{V}| \leqq \tfrac{1}{2}\Psi(1;u,K) + \int_{T_1} e_K(u) G|x\cdot\nabla\varphi|^2\sqrt{g}\,dx\,dt \\
\leqq \tfrac{1}{2}\Psi(1;u,K) + c\,E(u_0).
\end{aligned}$$

If we evaluate $\frac{d}{dR}\Psi(R_1, u, K) = R_1^{-1}\frac{d}{dR}\Psi(RR_1, u, K)\Big|_{R=1}$ at $R_1 \leqq 1$ we use the estimate $|g(x) - \mathrm{id}| \leqq c|x|$ and the fact that

$$R_1^{-1}|t|^{-1}|x|^4 G \leqq G + c \quad \text{on } T_{R_1}$$

to control the error term in I as follows:

$$\int_{TR_1} \frac{1}{2|t|}|x|^2|g-\mathrm{id}|^2|\nabla u|^2 G\varphi^2\sqrt{|g|}\,dx\,dt \leqq c\Psi(R, u, K) + cE_0.$$

The estimates for II, III, IV, and V can be handled in a similar way. From the differential inequality

$$\frac{d}{dR}\Psi(R, u, K) \geqq \frac{1}{8R}\int_{T_R} \frac{|x\cdot\nabla u + 2t\,\partial_t u|^2}{|t|} G\varphi^2\sqrt{g}\,dx\,dt - c\Psi(R, u, K) - cE_0, \quad (4.4)$$

now the claim follows.

In the Bochner-type estimate Lemma 2.3 we pick up an additional term involving the Ricci-curvature of M resulting from differentiating the metric g. This gives

Lemma 4.3. *Suppose $u = u_K$ is a regular solution of $(4.1)_K$ in an open set $Q \subset B_{R_M}(0) \times R_+ \subset R^m \times R_+$. Then there holds*

$$(\partial_t - \Delta)\, e_K(u) \leqq c(1 + e_K(u))\, e_K(u)$$

with a constant $c > 0$ depending only on M and N.

The monotonicity formula Lemma 4.2, the Bochner-type estimate Lemma 4.3 and Moser's Harnack inequality in the same way as before imply the ε-regularity theorem Lemma 2.4. However, the range of admissible radii R is restricted to the range of validity of Lemma 4.2:

Lemma 4.4. *Suppose $u = u_K$: $M \times [-T, T] \to R^n$ is a regular solution to* (4.1), *and assume that $T \leqq R_M^2$. There exists a constant $0 < \varepsilon_0 < R_M$ depending only on M and N such that if for some $0 < R < \min\{\varepsilon_0, \sqrt{T}/2\}$ the inequality*

$$\Psi(R, u, K) \leqq \varepsilon_0$$

is satisfied, then there holds

$$\sup_{P_{\delta R}} e_K(u) \leqq c(\delta R)^{-2}$$

with constants c depending only on M and N and $\delta > 0$ possibly depending in addition on $E_0 = E_k(u(\cdot, -T))$ and R.

Proof. By the restriction on R and Lemma 4.2 estimate (2.17) may be modified

$$\sigma^{-m} \int_{P_\sigma(z_0)} e_K(u)\, dx\, dt \leqq \ldots \leqq c \int_{T_R} e_K(u)\, G_{(x_0, t_0+2\sigma^2)}\, dx\, dt + cRE_0,$$

and the remainder of the proof of Lemma 2.4 carries over unchanged.

Proof of Theorem 1.5. With the aid of the modified Lemmata 4.1–4.4 the proof of the first assertions is identical with that of the analogous statements in Theorem 3.1.

For the last assertion note that by Lemma 4.1 we may choose a sequence $t_K \to \infty$ such that $u_K(\cdot\,; t_K) \to u_\infty$ weakly in $H^{1,2}(M; N)$, while

$$\int_{t_K-1}^{t_K} \int_M |\partial_t u_K|^2 \sqrt{|g|}\, dx\, dt \to 0, \tag{4.5}$$

and $\partial_t u_K(\cdot, t_K) \to 0$ in $L^2(K \to \infty)$. Define

$$\Sigma_\infty = \bigcap_{R>0} \bigcup_{0<r<R} \{x_0 \in M \mid \liminf_{K\to\infty} \int_{T_r(x_0, t_K)} e_K(u_K)\, G_{(x_0, t_K)}\, \varphi^2 \sqrt{|g|}\, dx\, dt \geqq \varepsilon_0\}$$

where ε_0 is determined as in Lemma 4.4.

Σ_∞ is closed by the same argument as given in [6] in the case of Σ.

Moreover, $H^{m-2}(\Sigma_\infty) \leqq cE_0$. To see this let $R>0$ be given and let $\{B_{R_i}(x_i)\}_{i\in J}$ be a cover of Σ_∞ by balls of radius R_i centered at $x_i \in \Sigma_\infty$. We may assume $R_i \leqq R \leqq \varepsilon_0 \leqq R_M$. By compactness and Vitali's covering lemma there is a finite subfamily $J' \subset J$ such that $B_{2R_i}(x_i) \cap B_{2R_j}(x_j) = \varnothing$ for $i, j \in J'$, $i \neq j$, and such that the collection $\{B_{10R_i}(x_i) \mid i \in J'\}$ covers Σ_∞.

Now for sufficiently small $R>0$ and sufficiently large K, estimate (2.18), the definition of Σ_∞ and our monotonicity formula Lemma 4.2 imply that for any $\varepsilon>0$ there is a constant $C(\varepsilon)$ such that for any $i \in J'$ with a suitable number $0 < r_i \leqq R_i/C(\varepsilon) \leqq R_i \leqq R$ there holds

$$\begin{aligned}
\varepsilon_0 &\leqq \int_{T_{r_i/2}(x_i, t_K)} e_K(u_K)\, G_{(x_i, t_K)}\, \varphi^2 \sqrt{|g|}\, dx\, dt \\
&\leqq 4 r_i^{-2} \int_{t_k - r_i^2}^{t_k - r_i^2/4} \Big(r_i^2 \int_{S_{r_i}(x_i, t_k)} e_K(u_K)\, G_{(x_i, t_K)}, \varphi^2 \sqrt{|g|}\, dx\Big)\, dt \\
&\leqq c \exp(c r_i)\, r_i^2 \int_{S_{r_i}(x_i, t_K)} e_K(u_K)\, G_{(x_i, t_K)}\, \varphi_2 \sqrt{|g|}\, dx + c r_i E_0 \\
&\leqq c \exp(c(R_i/C(\varepsilon) - r_i))(R_i/C(\varepsilon))^2 \int_{S_{R_i/C(\varepsilon)}(x_i, t_K)} e_K(u_K)\, G_{(x_i, t_K)}\, \varphi^2 \sqrt{|g|}\, dx + c R_i E_0 \\
&\leqq C(\varepsilon)\, R_i^{2-m} \int_{B_{R_i}(x_i)} e_K(u_K(\cdot, t_K - (R_i/C(\varepsilon))^2) \sqrt{|g|}\, dx + \varepsilon E_0 + C_0 R E_0.
\end{aligned} \tag{4.6}$$

Here we have used that for suitable $C(\varepsilon)$ like (2.18) there holds

$$G_{(x_i, t)} \leqq \varepsilon \quad \text{on} \quad S_{R_i/C(\varepsilon)}(x_i, t) \setminus B_{R_i}(x_i) \times \{t - (R_i/C(\varepsilon))^2\}.$$

By (4.6), if we choose $\varepsilon=\frac{\varepsilon_0}{3E_0}$, $R<\frac{\varepsilon_0}{3C_0E_0}$ we obtain that for any $i\in J'$ and sufficiently large K

$$\varepsilon_0 R_i^{m-2}\leqq C\int_{B_{R_i}(x_i)} e_K(u_K(\cdot,t_K-(R_i/C(\varepsilon))^2)\sqrt{|g|}\,dx.$$

Analogous to [7, Lemma 3.6] now we have

Lemma 4.5. *Suppose u_K solves* (4.1), *and let $0<2R<R_M$, $0\leqq S<T<\infty$ be given. Then*

$$\int_{B_{2R}(x_0)} e_K(u_K(\cdot,T))\sqrt{|g|}\,dx\geqq c\int_{B_R(x_0)} e_K(u_k(\cdot,S))\sqrt{|g|}\,dx$$
$$-\int_S^T\int_M |\partial_t u_K|^2\sqrt{|g|}\,dx\,dt-c\cdot\left(\frac{T-S}{R^2}E_0\cdot\int_S^T\int_M |\partial_t u_K|^2\,dx\,dt\right)^{1/2}$$

where $E_0=E(u_0)$, and c only depends on M and N.

For completeness, a proof is given at the end of this section.

Choosing $R=R_i$, $S=t_K-(R_i/C(\varepsilon))^2$, $T=t_K$ by (4.5) we thus obtain that for any $i\in J'$ for sufficiently large K

$$\varepsilon_0 R_i^{m-2}\leqq C\int_{B_{2R_i}(x_i)} e_K(u_K(\cdot,t_K))\sqrt{|g|}\,dx.$$

J' being finite, we may choose K such that this estimate holds simultaneously for all $i\in J'$. Upon adding, by Lemma 4.1 therefore

$$\sum_{i\in J'} R_i^{m-2}\leqq C\int_{\bigcup_{i\in J'} B_{2R_i}(x_i)} e_K(u_K(\cdot,t_K))\sqrt{|g|}\,dx\leqq CE_K(u_K(\cdot,t_K))\leqq CE_0.$$

Passing to the limit $R\to 0$ we hence obtain that

$$H^{m-2}(\Sigma_\infty)\leqq CE_0,$$

as desired.

Conversely, for $x_0\notin\Sigma_\infty$ there exists $R_0>0$ such that for a sequence $K_m\to\infty$ with $t_m=t_{K_m}$ we have

$$\int_{T_{R_0}(x_0,t_m)} e_{K_m}(u_{K_m})\,G_{(x_0,t_m)}\,\varphi^2\sqrt{|g|}\,dx\leqq\varepsilon_0.$$

By Lemma 4.2

$$\nabla u_{K_m},\ K_m\operatorname{dist}^2(u_{K_m},N)\leqq C,$$

uniformly on parabolic cylinders $P_{\delta R_0}(x_0,t_m)$.

Moreover, by assumption

$$\partial_t u_{K_m}(\cdot,t_m)\to 0\quad\text{in } L^2.$$

Therefore, as in the proof of Theorem 3.1 $u_{K_m}(\cdot, t_m)$ converges weakly in $H^{2,2}_{\mathrm{loc}}(B_{R_0}(x_0))$ to a regular harmonic map $u_\infty \in H^{2,2}_{\mathrm{loc}}(B_{R_0}(x_0); N)$.

Since we may exhaust $M \backslash \Sigma_\infty$ by countably many such neighborhoods $B_{R_0}(x_0)$, we may assume that $u_K(\cdot, t_K)$ converges weakly in $H^{2,2}$ locally off Σ_∞. Hence the map u_∞ is a classical solution to the harmonic map equation off Σ_∞.

Finally, by the Hausdorff-dimension estimate for Σ_∞ above, u_∞ also is weakly harmonic as a map $u_\infty \in H^{1,2}(M; N)$.

The proof is complete.

Proof of Lemma 4.5. Let $\varphi \in C_0^\infty(B_{2R}(x_0)$ with $|\nabla \varphi| \leqq C/R$ and $\varphi \equiv 1$ on $B_R(x_0)$. Multiply (4.1) by $\partial_t u_K \varphi^2$ and integrate over $M \times [S, T]$. Upon integrating by parts there results

$$\begin{aligned}
&\int_S^T \frac{d}{dt}\left(\int_M e_K(u_K)\varphi^2 \sqrt{|g|}\, dx + \int_S^T \int_M |\partial_t u_K|^2 \varphi^2 \sqrt{|g|}\, dx\, dt\right. \\
&\quad \geqq -\int_S^T \int_M |\nabla u_K| |\nabla \varphi| |\partial_t u_K| |\varphi| \sqrt{|g|}\, dx\, dt \\
&\quad \geqq -\frac{C}{R}\left(\int_S^T \int_M |\nabla u_K|^2 \sqrt{|g|}\, dx\, dt \int_S^T \int_M |\partial_t u_K|^2 \varphi^2 \sqrt{|g|}\, dx\, dt\right)^{1/2}.
\end{aligned}$$

The Lemma now follows from Lemma 4.1 and the estimate

$$\int_S^T \int_M |\nabla u_K|^2 \sqrt{|g|}\, dx\, dt \leqq (T-S) \sup_{S \leqq t \leqq T} F_K(u_K(\cdot, t) \leqq (T-S) E_0.$$

Remark. Analogous to [6; Theorem 8.1] by (4.4) singularities of the flow u can be related to harmonic spheres or to solution $v(x, t) = w(x/\sqrt{|t|})$ of the heat flow (1.6) on R^m.

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Added in Proof. In a recent note J.-M. Coron [10] has pointed out that (even partially regular) weak solutions to (1.6–7) satisfying the energy inequality need not be unique. Moreover, Coron and Ghidaglia [11] have been able to construct examples of smooth solutions to (1.6–7) for $M = \mathbb{R}^m$ or S^m and $N = S^m$ which develop singularities in finite time.

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