Exam 2.

Due March 24 before class.

Let $X$ be a Polish space with a complete compatible metric $d$. Fill out the details in the proof that the group $G$ of all isometries of $(X,d)$ with pointwise convergence topology and the composition operation is a Polish group. Fix a countable dense set $D \subset X$.

1. Let $Y = X^D$ be the set of all functions from $D$ to $X$. Show that the product topology is the same as the pointwise convergence topology on $Y$. This is to say, a sequence $\langle y_n : n \in \omega \rangle$ of elements of $Y$ pointwise converges to $z \in Y$ if and only if it converges to $z$ in the product topology.

2. Let $Z \subset Y$ be the set of all maps which preserve the metric $d$ and have range dense in $X$. Show that $Z$ is $G_\delta$ and so $Z$ in the inherited topology is Polish.

3. Prove that if $\langle g_n : n \in \omega \rangle$ and $k$ are elements of $G$ then $g_n$ converges pointwise to $k$ if and only if $g_n \upharpoonright D$ converges pointwise to $k \upharpoonright D$.

4. Conclude that the map $h : G \to Z$ defined by $h(g) = g \upharpoonright D$ is a homeomorphism of the spaces with the respective pointwise convergence topologies and therefore $G$ is Polish.

5. Show that the composition and inverse operations on $G$ are continuous.