Scott Complexity and Finitely α-generated Structures

Rachael Alvir
University of Notre Dame

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What's a Scott Sentence?

Everything in this talk is motivated by a theorem of Scott's:

Scott's Isomorphism Theorem

Every countable structure can be described up to isomorphism (among countable structures) by a sentence φ of $L_{\omega_1\omega}$.

Such a sentence is called a **Scott sentence** for A.

This is exactly the kind of categoricity result which is not possible in the finitary first-order context.

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A d-Σ_α formula is a finite conjunction of Π_α and Σ_α formulas

A Measure of Internal Complexity

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The least such α, denoted **SR(A)**, is one definition of the *Scott Rank* of A, and is thought be an "internal" measure of A's descriptive complexity.

Disagreement in the Literature

Unfortunately, many non-equivalent definitions of Scott Rank exist in the literature. Antonio Montalban in "A Robuster Scott Rank" argued to standardize the following definition:

Definition (A. Montalban)

The **(Categoricity) Scott Rank** of A is the least α such that A has a $\Pi_{\alpha+1}$ Scott sentence.

Note briefly that the complexity of a Scott sentence gives an "external" measure of the structure's complexity.

The List

Montalban believed this notion was most robust, having many other conditions equivalent to it.

Theorem

The following are equivalent:

1. A has a $\Pi_{\alpha+1}$ Scott sentence.
2. The automorphism orbit of any tuple can be defined by a Σ_α formula (without parameters).

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In other words, Scott Sentences are also related to notions in computability theory and descriptive set theory.

Why just consider $\Pi_{\alpha+1}$ Scott sentences?

Theorem (A. Miller)

For $\alpha \geq 1$, if A has a Scott sentence that is Π_α and one that is Σ_α, then it has one that is d-$\Sigma_{<\alpha}$.

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The sentences of $L_{\omega_1\omega}$ can be sorted into classes $(\Pi_\alpha, \Sigma_\alpha, \Delta_\alpha)$ by complexity and arranged in a hierarchy in the natural way. Miller's result implies a unique least class in this hierarchy containing a Scott sentence for the structure.

Scott Complexity

Definition (R.A*, M. Harrison-Trainor, D. Turetsky, N. Greenberg)

The **Scott Complexity** of a structure A is the least complexity of a Scott sentence for A.

Scott Complexity is finer than Scott Rank, and just as robust.

Fact: A structure has a $\Sigma_{\alpha+2}$ Scott sentence iff there is some finite tuple $\bar{c}$ such that $(A, \bar{c})$ has a $\Pi_{\alpha+1}$ Scott sentence. The least such α is called the **parametrized Scott Rank** by Montalban.

Finitely α-generated Structures: Motivation

In previous work with Dino Rossegger, we gave sharp upper bounds on the Scott Complexity of an arbitrary scattered linear order.

A linear order A is **scattered** if there is no embedding from η into A. Equivalently, a linear order is scattered if it has a **Hausdorff rank,** which we define by induction:

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- $\mathrm{rk}_H(\mathrm{A}) = 0$ iff $A \in \{n : n \in \omega\}$
- $\mathrm{rk}_H(\mathrm{A}) = \alpha$ iff A is in the class of linear orders of the form $\Sigma_{i\in\omega} A_i$ or $\Sigma_{i\in\omega^*} A_i$ where each A_i is of lower rank, closed under finite sum.

Finitely α-generated Structures: Motivation

To give a $\Sigma_{\alpha+1}$ Scott sentence for a scattered linear order A, we had to identify the tuple $\bar{c}$ such that $(A, \bar{c})$ has a Π_α Scott sentence.

In doing so, we noticed that such a tuple acted in many ways like the generating tuple of the structure, even though the structure was not finitely generated.

How formally could one capture this intuition?

While one could call the desired tuple $\bar{c}$ "a tuple over which no other tuple of the structure is α-free," this is cumbersome.

Definition

A tuple $\bar{c}$ is said to be an α-**generator** for a structure A if:

1. the automorphism orbit of each finite tuple of A is Σ_α-definable over $\bar{c}$.

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1. the automorphism orbit of each finite tuple of A is Σ_α-definable over $\bar{c}$.
2. The ordinal α is the least such that (1) holds.

A structure A with an α-generator is called an α-**generated structure.** These are exactly the structures with Scott complexity $\Sigma_{\alpha+2}$, d-$\Sigma_{\alpha+1}$, or $\Sigma_{\alpha+1}$ for limit α.

Example: Finitely α-generated Structures

The structure $\mathbb{Z} + \mathbb{Z}$ has a Σ_4 Scott sentence; in fact, its Scott complexity is d-Σ_3.

It is not finitely generated in the language of linear orders, but is finitely generated in the language with the ordering, the predecessor, and the successor relations.

The generating tuples for $\mathbb{Z} + \mathbb{Z}$ in this expanded language are precisely the tuples which are 2-generators for $\mathbb{Z} + \mathbb{Z}$ as a linear order.

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Lemma (R.A.*)

Suppose that A is finitely α-generated by $\bar{c}$ and almost rigid, witnessed by $\bar{d}$. Let $\{\phi_{\bar{a}}(\bar{x}, \bar{c}, \bar{d}) : \bar{a} \in A\}$ be the family of $\Sigma_{<\alpha}$-formulas defining the automorphism orbits of $(A, \bar{c}\bar{d})$. In the definitional expansion which includes a relation predicate for each $\phi_{\bar{a}}$, A is finitely generated by $\bar{c}\bar{d}$.

Examples: Finitely α-generated Structures

There are examples of α-generated structures at arbitrarily large levels, even when the structure is not almost rigid.

The structure $\mathbb{Z}^\alpha + \mathbb{Z}^\alpha + \mathbb{Z}^\alpha$ has Scott Complexity d-$\Sigma_{2\alpha+1}$. The 2α-generators are precisely the ones which contain an element from each copy of $\mathbb{Z}^\alpha$.

Theorem (R.A.*)

A structure A has a d-$\Sigma_{\alpha+1}$ Scott sentence iff some α-generator has a Π_α automorphism orbit.

This is a generalization of a result on finitely generated groups obtained with Julia Knight and Charlie McCoy.

Back to Scott Rank and Scott Complexity

Theorem (R.A.*)

A structure A has a d-$\Sigma_{\alpha+1}$ Scott sentence iff some α-generator has a Π_α automorphism orbit.

r(A): The least α such that every tuple's automorphism orbit is Π_α-definable.

Corollary: A has Scott Complexity d-$\Sigma_{\alpha+1}$ iff for some $\bar{c}$, $r(A, \bar{c}) = \alpha$.

Thanks + References

Thank You!

R. A*, J. Knight, C.McCoy, "Complexity of Scott Sentences." Accepted, *Fundamenta Mathematicae*.

R.A.*, D. Rossegger, "Complexity of Scott Sentences of Scattered Linear Orders." Submitted.

R.A.*, M. Harrison-Trainor, D. Turetsky, N. Greenberg, "Scott Complexity of Countable Structures." Submitted.

R.A.*, "Finitely α-generated structures." In preparation.

A. Montalban, "A Robuster Scott Rank." 2015.