Geometric set theory

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Preface

We wish to present to the reader a fresh and exciting new area of mathematics: geometric set theory. The purpose of this research direction is to compare transitive models of set theory with respect to their extensional agreement and definability. It turns out that many fracture lines in descriptive set theory, analysis, and model theory can be efficiently isolated and treated from this point of view. A particular success is the comparison of various $\Sigma^1_2$ consequences of the Axiom of Choice in unparalleled detail and depth.

The subject matter of the book was rather slow in coming. The initial work, restating Hjorth’s turbulence in geometric terms and isolating the notion of a virtual quotient space of an analytic equivalence relation, existed in rudimentary versions since about 2013 in unpublished manuscripts of the second author. The joint effort [51] contained some independence results in choiceless set theory similar to those of the present book, but in decidedly suboptimal framework. It was not until the January 2018 discovery of balanced Suslin forcing that the flexibility and power of the geometric method fully asserted itself. The period after that discovery was filled with intense wonder–passing from one configuration of models of set theory to another and testing how they separate various well-known concepts in descriptive set theory, analysis, and model theory. At the time of writing, geometric set theory seems to be an area wide open for innumerable applications.

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Chapter 1

Introduction

1.1 Outline of the subject

Geometric set theory is the research direction which studies transitive models of set theory with respect to their extensional agreement. It turns out that numerous existing concepts in descriptive set theory and analysis have intuitive and informative geometric restatements, and the geometric point of view makes it possible to readily isolate many new concepts and generalize old ones. Surprising parallels appear between areas that to date did not have a discernible interface.

The oldest part of the subject deals with Borel equivalence relations on Polish spaces. Here, given an equivalence $E$ on a space $X$ and a configuration $\{M_i: i \in I\}$ of transitive models of set theory, one asks whether it is possible for an $E$-equivalence class to have representatives in some models of the configuration and fail to be represented in others. The answer to this question greatly varies with the nature of the equivalence relation and the configuration in question, and the resulting differences can be used with great effect to prove non-reducibility and ergodicity theorems for various Borel equivalence relations. One notable success in this area is a reformulation of Hjorth’s turbulence in geometric terms, which is much more practicable than the original definition and allows endless generalizations and variations entirely divorced from the original context of Polish group actions.

Geometric set theory really started to blossom after it was applied in a seemingly entirely different direction: the independence results in choiceless Zermelo–Fraenkel (ZF) set theory. This is an area which saw a surge of results in early 1970’s after which the interest in it waned. Both our purpose and methodology are quite different from the early practitioners though. We study almost exclusively ZF independence results between $\Sigma^1_2$ consequences of the Axiom of Choice which are intimately connected to various contemporary concerns of descriptive set theory and analysis. A very detailed structure appears, with some fracture lines running in parallel to existing combinatorial,
algebraic, or analytic concepts and other fracture lines showing up in places quite unexpected. The main geometric motif is the following. Given a \( \Sigma^1_2 \) sentence \( \phi = \exists A \subset X \psi(A) \), a consequence of the Axiom of Choice in which \( X \) is a Polish space and \( \psi \) is a formula quantifying only over elements of Polish spaces, and given a configuration \( \{ M_i: i \in I \} \) of transitive models of set theory with choice, is there a set \( A \subset X \) such that in all (or many) models \( M \) in the configuration, \( A \cap M \in M \) and \( A \cap M \) is a witness to \( \phi \) in the model \( M \)? The answer to this question is surprisingly varied and successful in separating consequences of the Axiom of Choice of this syntactical complexity.

The models of ZF we use for the independence results are nearly exclusively of the same form: they are extensions of the symmetric Solovay model by a simply definable \( \sigma \)-closed forcing. As a result, they are all models of DC, the Axiom of Dependent Choices. DC is instrumental for developing the basic concepts of mathematical analysis and therefore highly desirable; at the same time, it is an axiom which is difficult to obtain in the symmetric models of 1970’s. If suitable large cardinals are present, essentially identical effects can be obtained in generic extensions of the model \( L(\mathbb{R}) \) which have a very strong claim to canonicity. In addition, given \( \Sigma^1_2 \) sentences \( \phi_0, \phi_1 \), there is usually a quite canonical choice for a forcing which should generate a model of \( \text{ZF+DC+}\neg\phi_0+\phi_1 \). The whole analysis of the forcing in question takes place in \( \text{ZFC} \), using central concepts of the fields related to the \( \Sigma^1_2 \) sentences in question.

We conclude this brief outline of the area with the observation that the mass of natural questions that either seem to be treatable using the geometric method, or are generated by the inner workings of the geometric method, is entirely overwhelming. This book cannot be more than a relatively brief introduction to the subject. In many directions, our efforts faded due to sheer exhaustion and not because of insurmountable obstacles. We hope to motivate the reader to explore the enormous, inviting expanses of geometric set theory.

1.2 Equivalence relation results

The first group of results deals with the simplest geometric concern. If \( E \) is an analytic equivalence relation on a Polish space, and \( V[H_0], V[H_1] \) are mutually generic extensions of \( V \), is it possible to find an \( E \)-class which is represented in both \( V[H_0] \) and \( V[H_1] \) but not in \( V \)? Kanovei [39, Chapter 17], continuing the seminal work of Hjorth, defined an analytic equivalence \( E \) to be pinned just in case the answer is negative.

In Chapter 2, we develop the notion of the virtual quotient space for a given analytic equivalence relation on a Polish space. A virtual \( E \)-class is one which is represented in some mutually generic extensions of \( V \); from a different angle, it is a class which may be represented only in some generic extension, but it does not depend on the choice of the generic filter. A formal definition uses the notion of an \( E \)-pin and an equivalence on the class of \( E \)-pins, which naturally extends the analytic equivalence relation \( E \) into the transfinite domain–Definitions 2.1.1 and 2.1.4. The virtual \( E \)-quotient space is then the class of all \( E \)-pins up to
1.2. EQUIVALENCE RELATION RESULTS

their equivalence. Every Borel structure on the $E$-quotient space has a natural virtual version on the virtual $E$-quotient space. Virtual structures appear in many places in this book. The main theorem governing their behavior:

**Theorem 1.2.1.** (Proposition 2.2.5) Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $\mathcal{M}$ be a Borel structure on the $E$-quotient space $X/E$. The natural map from the $E$-quotient space to the virtual $E$-quotient space is a $\Pi_1$-elementary embedding from $\mathcal{M}$ to its virtual version.

To size up the virtual $E$-quotient space, in Definition 2.5.1 we isolate cardinal invariants $\kappa(E)$ and $\lambda(E)$ of an equivalence relation $E$: $\lambda(E)$ is the cardinality of the virtual space, while $\kappa(E)$ is its forcing-theoretic complexity. The cardinals $\kappa(E), \lambda(E)$ respect the Borel reducibility order of equivalence relations, and as such serve as valuable tools for non-reducibility results. Their key feature: for Borel equivalence relations $E$, these cardinals are well-defined and obey an explicit upper bound. In particular, the virtual $E$-quotient space is a set, as opposed to a proper class as occurs for some analytic equivalence relations.

**Theorem 1.2.2.** (Theorem 2.5.6) Let $E$ be a Borel equivalence relation on a Polish space $X$. Then $\kappa(E), \lambda(E) < \beth_1$.

As a result, the well-known Friedman–Stanley theorem [24] on nonreducibility of a jump $E^+$ of a Borel equivalence relation $E$ to $E$ is immediately translated to the Cantor theorem on the cardinality of the powerset: $\lambda(E^+) = 2^{\lambda(E)} > \lambda(E)$—Example 2.5.5. Other similar conceptual and brief nonreducibility results appear as well. The cardinal invariants $\kappa(E)$ and $\lambda(E)$ can attain many exotic and informative values, and inequalities between them can be manipulated by forcing. Most of Section 2.5 is devoted to examples of such manipulation. However, the main underlying concept is absolute, which justifies the parlance used throughout the book:

**Theorem 1.2.3.** (Corollary 2.6.3) Let $E$ be a Borel equivalence relation on a Polish space $X$. The statement “$E$ is pinned” is absolute among all generic extensions.

We stress that the above result holds within the context of ZFC. If one drops the Axiom of Choice but holds on to Dependent Choices, the class of Borel unpinned equivalence relations will shrink; in the symmetric Solovay model, it shrinks to the minimal possible extent: only the canonical unpinned relation $\mathbb{R}$ and the equivalence relations above it in the reducibility order are unpinned, see below. If one drops even the Axiom of Dependent Choices, the class of Borel unpinned equivalence relations becomes more erratic. We conjecture that it may expand to its largest possible extent: to the class of equivalence relations which are not reducible to $\mathcal{F}_\sigma$ equivalence relations.

As is usual in the realm of equivalence relations, we aim to prove various dichotomies.

**Theorem 1.2.4.** (Theorem 2.7.9) Suppose that there is a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. Exactly one of the following occurs:
1. $\kappa(E), \lambda(E) < \infty$;

2. $E_{\omega_1}$ is almost Borel reducible to $E$.

**Theorem 1.2.5.** (Corollary 2.7.13) In the symmetric Solovay model: let $E$ be a Borel equivalence relation on a Polish space $X$. Exactly one of the following occurs:

1. $E$ is pinned;

2. $E$ is Borel reducible to $E$.

In Chapter 3, we deal with a more sophisticated geometric concern. Let $E$ be an analytic equivalence relation on a Polish space $X$. Is it possible to find two generic extensions $V[H_0]$ and $V[H_1]$ of $V$ such that $V[H_0] \cap V[H_1] = V$ in which there is an $E$-class represented in both extensions $V[H_0]$ and $V[H_1]$, but not in $V$? Note that we do not ask the extensions to be mutually generic. In an initial approach, we develop a fruitful method for constructing separately Cohen-generic extensions $V[H_0]$ and $V[H_1]$ such that $V[H_0] \cap V[H_1] = V$. We define (Definition 3.1.4) a practical and easily checked notion of independence of continuous open maps between Polish spaces using a notion of a walk reminiscent of Hjorth’s method of turbulence [39, Section 13.1] and prove:

**Theorem 1.2.6.** (Theorem 3.1.5) Let $X$ be a Polish space and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ be continuous open maps to Polish spaces. The following are equivalent:

1. the maps $f_0, f_1$ are independent;

2. the Cohen forcing $P_X$ of nonempty open subsets of $X$ forces $V[f_0(x_{gen})] \cap V[f_1(x_{gen})] = V$.

As one of the corollaries, we obtain a characterization of Hjorth’s turbulence in geometric terms. This characterization is much more practical than the original statement for anyone familiar with the forcing relation.

**Theorem 1.2.7.** (Theorem 3.2.2) Let $\Gamma$ be a Polish group acting on a Polish space $X$ with all orbits meager and dense. The following are equivalent:

1. the action is generically turbulent;

2. $V[x] \cap V[\gamma \cdot x] = V$ whenever $\gamma \in \Gamma$ and $x \in X$ are mutually generic points.

To support a broad extension of ergodicity results due to Hjorth and Kechris [42, Theorem 12.5], we develop the classes of placid and virtually placid equivalence relations (the latter including all equivalence relations classifiable by countable structures) and prove:
Theorem 1.2.8. (Theorem 3.3.5) Let $E$ be the orbit equivalence relation on a Polish space $X$ resulting from a turbulent Polish group action. Let $F$ be an analytic, virtually placid equivalence relation on a Polish space $Y$ and let $h: X \to Y$ be a Borel homomorphism of $E$ to $F$. Then there is an $F$-class whose $h$-preimage is comeager in $X$.

It turns out that there is a version of turbulence for measure which leads to many of the same ergodicity results. It uses the notion of concentration of measure quite close to the abstract whirly actions on measure algebras isolated in [26].

Theorem 1.2.9. (Theorem 3.6.2) Let $\Gamma$ be a Polish group continuously acting on a Polish space $X$ with an invariant Borel probability measure and an invariant ultrametric. Suppose that the action has concentration of measure. Then $V[x] \cap V[\gamma \cdot x]$ whenever $x \in X$ is a random point and $\gamma \in \Gamma$ is a generic point mutually generic with $x$.

As usual with the forcing reconceptualizations of various notions in descriptive set theory, we have to show that the resulting notions are suitably absolute and evaluate their complexity—Theorem 3.5.6.

Chapter 4 addresses infinite configurations of transitive models of ZFC. The most important case is in which there is an infinite inclusion-descending sequence $\langle M_n : n \in \omega \rangle$ of models. Under suitable, commonly satisfied coherence assumptions (Definition 4.2.1 or 4.3.1), the intersection $\cap_n M_n$ is a model of ZF or even ZFC. We spend some time developing a flexible technology of constructing nontrivial coherent sequences as in Theorem 4.3.6. The breakthrough achieved in this chapter is the theorem showing that similar configurations detect the distinction between equivalence relations induced as orbit equivalences of continuous actions of Polish groups, and those which are not reducible to orbit equivalence relations:

Theorem 1.2.10. (Theorem 4.3.7) Let $\langle M_n : n \in \omega \rangle$ be a choice-coherent sequence of models of ZFC. Let $E$ be an orbit equivalence relation of a continuous Polish group action with a code in the intersection model $\cap_n M_n$. If a virtual $E$-class has a representative in every model $M_n$, then it has a representative in the intersection model $\cap_n M_n$.

Recall that the canonical equivalence relation not reducible to an orbit equivalence relation is $E_1$. It is not difficult to construct a choice-coherent sequence of models $\langle M_n : n \in \omega \rangle$ and an $E_1$-class which is represented in each model on the sequence but not in the intersection model $\cap_n M_n$—Example 4.3.11.

### 1.3 Independence: by topic

Our concerns in the part of the book that deals with independence results in ZF+DC can be grouped into several large areas.

**Cardinalities of quotient spaces.** Mirroring the traditional concerns of descriptive set theory, we work on cardinalities of quotient spaces of Borel equiv-
alence relations. Given Polish spaces $X, Y$ and Borel equivalence relations $E, F$ on each, descriptive set theorists study the question when there can be a Borel function $h : X \to Y$ which reduces $E$ to $F$. This line of work has been very successful in the last two decades.

In the ZF+DC context, the $E$- and $F$-quotient spaces can have distinct cardinalities, and the existence of a Borel reduction implies the inequality $|E| \leq |F|$, where we abuse the notation to write $|E|$ for the cardinality of the $E$-quotient space. On the other hand, the nonexistence of a Borel reduction is often connected with the possibility that ZF+DC cannot prove the cardinal inequality $|E| \leq |F|$. A number of our results deal with the question whether a given $\Sigma^1_2$ statement is consistent with ZF+DC plus a statement of the type $|E| \not\leq |F|$ for some benchmark Borel equivalence relations $E, F$ identified in Section 1.8.

The most commonly considered quotient space cardinal inequality is $|E_0| > |2^\omega|$ and we spend a great deal of energy proving that various $\Sigma^1_2$ statements are consistent with it.

**Theorem 1.3.1.** The following statements are separately consistent with ZF+DC and $|E_0| > |2^\omega|$:

1. ([16], Corollary 9.2.5) There is a nonprincipal ultrafilter on $\omega$;
2. (Corollary 9.2.23) there is a discontinuous homomorphism of $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$;
3. (Corollary 9.2.12) given a Borel equivalence relation $E$ on a Polish space, the $E$-quotient space can be linearly ordered;
4. given a pinned Borel equivalence relation $E$, $|E| \leq |E_0|$;
5. (Corollary 11.3.12) given a countable Borel equivalence relation $E$, $E$ is the orbit equivalence relation of a (discontinuous) action of $\mathbb{Z}$.

It is an interesting question whether some of the statements above can be combined; in general, we have

**Question 1.3.2.** Are there $\Sigma^1_2$-statements $\phi_0, \phi_1$ such that each of them is consistent with ZF+DC and $|E_0| > |2^\omega|$, while their conjunction implies in ZF+DC that $|E_0| \leq |2^\omega|$ holds?

We give a similar treatment to certain other well-known quotient cardinal inequalities. For example,

**Theorem 1.3.3.** The following statements are separately consistent with ZF+DC and $|E_1| \not\leq |F|$ for any orbit equivalence relation $F$ of a continuous Polish group action:

1. (Corollary 9.4.8) There is a nonprincipal ultrafilter on $\omega$;
2. (Corollary 9.4.11) there is a discontinuous homomorphism of $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$;
3. (Corollary 9.4.22) given a Borel graph $\Gamma$ on a Polish space, $\Gamma$ contains a maximal acyclic subgraph;

4. (Corollary 9.4.10) given a Borel equivalence relation $E$ on a Polish space, the $E$-quotient space can be linearly ordered;

5. (Corollary 9.4.15) given a pinned orbit equivalence relation $E$ of a continuous Polish group action, $E$ has a transversal.

**Theorem 1.3.4.** The following statements are separately consistent with ZF+DC and $|E| \not\leq |F|$ for any orbit equivalence relation $E$ of a turbulent Polish group action and an analytic equivalence relation $F$ classifiable by countable structures:

1. (Corollary 9.3.9) Given a Polish vector space $X$ over a countable field, $X$ has a basis;

2. (Corollary 9.3.10) given a Borel graph $G$ on a Polish space, $G$ contains a maximal acyclic subgraph;

3. (Corollary 9.3.13) given a pinned equivalence relation $G$ classifiable by countable structures, $G$ has a transversal.

There are many interesting open questions left in the area of quotient space cardinalities. For one, we do not know whether the various independence results we obtain can be in general combined in the spirit of Question 1.3.2. There are also more specific questions. Can the cardinalities of countable Borel equivalence relations be manipulated in the context of ZF+DC? For example, are there countable Borel equivalence relations $E, F$ such that both inequalities $|E| < |F|$ and $|F| < |E|$ are consistent with ZF+DC? In ZF+DC, does the existence of a nonprincipal ultrafilter on $\omega$ introduce new provable inequalities between quotient space cardinalities?

**Locally countable structures.** One particularly popular subject of descriptive set theory in the last quarter-century has been the study of countable Borel equivalence relations and the various structures on their equivalence classes. Using the methods of this book, one can identify many new fracture lines in this subject; various selection principles can be meaningfully stratified by the ZF+DC provability of implications between them. The following is a sampling of the results we obtain.

**Theorem 1.3.5.** (Corollary 11.3.17) Let $G$ be an analytic graph on a Polish space, with uncountable Borel chromatic number. Given a locally finite bipartite Borel graph $H$ satisfying the Marks–Unger condition, it is consistent that ZF+DC holds, $H$ has a perfect matching and the chromatic number of $G$ is uncountable.

**Theorem 1.3.6.** Let $G$ be an analytic hypergraph with all hyperedges finite, on a Polish space $X$, with uncountable Borel chromatic number. The following are consistent with ZF+DC plus the statement that the chromatic number of the hypergraph $G$ is uncountable:
1. (Corollary 11.3.12) given a countable Borel equivalence relation \( E \) on a Polish space with all classes infinite, \( E \) is the orbit equivalence relation of a (discontinuous) action of \( \mathbb{Z} \);

2. (Corollary 11.3.10) given a locally finite acyclic Borel graph \( \Gamma \) on a Polish space, the chromatic number of \( \Gamma \) is \( \leq 3 \);

3. (Corollary 11.4.7) given a pinned Borel equivalence relation \( E \) on a Polish space, \(|E| \leq |E_0|\).

An important fracture line among Borel locally countable graphs appears at the level of the diagonal Hamming graph \( \mathbb{H}_{<\omega} \) which is the product of cliques on \( 2, 3, 4 \ldots \):

**Theorem 1.3.7.** The following are consistent with ZF+DC plus the statement that the chromatic number of \( \mathbb{H}_{<\omega} \) is uncountable:

1. (Corollary 11.5.8) there is a nonprincipal ultrafilter on \( \omega \);

2. (Corollary 11.5.10) given a locally finite bipartite Borel graph \( \Gamma \) satisfying the Hall’s marriage condition, \( \Gamma \) has a perfect matching;

3. (Corollary 11.5.9) given a Borel equivalence relation \( E \) on a Polish space \( X \), the \( E \)-quotient space is linearly ordered;

4. (Corollary 11.4.3) given a pinned Borel equivalence relation \( E \) on a Polish space, \(|E| \leq |2^\omega|\).

Note that in the case of the last consistency result for the case \( E = E_0 \), the countable uniformization has to fail, as the uncountable chromatic number of \( \mathbb{H}_{<\omega} \) implies among other things that \( E_0 \) has no selector.

Another fracture line occurs at the level of the Hamming graph \( \mathbb{H}_\omega \) on \( \omega^\omega \) which is the product of infinitely many cliques on \( \omega \):

**Theorem 1.3.8.** (Corollary 11.3.24) It is consistent with ZF+DC that the chromatic number of the diagonal Hamming graph \( \mathbb{H}_{<\omega} \) is countable while the chromatic number of the Hamming graph \( \mathbb{H}_\omega \) on the Baire space is uncountable.

**Model theory.** Given a pre-geometry on a Polish structure (for example, the pre-geometry of linear span on a Polish vector space), the axiom of choice yields a maximal independent set (a basis for the vector space). This broad stroke erases many fine combinatorial distinctions between the various pre-geometries. In the choiceless context, interesting distinctions become visible.

In our context, it is more convenient to work with Borel matroids, a combinatorial concept dual to pre-geometries. Matroids from a broad class give rise to balanced forcings (Theorem 6.2.32), providing theorems of the following kind:

**Theorem 1.3.9.** ([33], Example 6.2.34 and Corollary 9.1.2) It is consistent with ZF+DC that there is a transcendence basis for the reals over \( \mathbb{Q} \) and there is no uncountable sequence of distinct reals.
One traditional fracture line between matroids leads between the modular ones (like the matroid of linearly independent sets) and the non-modular ones (like the matroid of algebraically independent sets). We develop an abstract notion of modularity in Definition 6.2.11 and, improving on earlier work of [66, 9], we prove:

**Theorem 1.3.10.** (Corollary 12.2.12) It is consistent with ZF+DC that there is a Hamel basis but no transcendence basis for the reals.

Matroid theorists also distinguish between linear matroids (like the matroid of linearly independent sets) and graphoids (the matroids of finite acyclic subsets of a given graph). This distinction gives rise to the following theorem:

**Theorem 1.3.11.** Let $G$ be a Borel graph on a Polish space $X$. It is consistent with ZF+DC that $G$ contains a maximal acyclic subgraph and there is no Hamel basis for the reals.

The proof takes an unexpected turn: we prove that in ZF+DC, existence of a Hamel basis implies that $E_1$ has a complete countable section, in particular $|E_1| \leq |E_2|$—Corollary 9.4.31. At the same time, existence of a maximal acyclic subgraph of any Borel graph $G$ is consistent with ZF+DC+$|E_1| \nsubseteq |E|$ for any orbit equivalence relation $E$—Corollary 9.4.22.

In another direction connecting model theory with choiceless ZF+DC arguments, we study theories which can have models on the various quotient spaces. In a typical development, given a Borel equivalence relation $E$ on a Polish space $X$ and a Fraissé class $F$ of relational structures with strong amalgamation, we can add a $F$-structure (a structure all of whose finite induced substructures are in $F$) on the $E$-quotient space in a highly controlled way. This results in many theorems of the following kind:

**Theorem 1.3.12.** (Corollary 13.2.5) Let $E$ be a Borel equivalence relation on a Polish space $X$. It is consistent with ZF+DC that there is a tournament on $X/E$ while there is no linear ordering on $2^\omega/E_0$.

**Combinatorics.** A challenging part of set theory deals with chromatic numbers of various natural Borel graphs and hypergraphs. Non-locally countable graphs require an entirely different approach. For a sample result, for each number $n > 0$ consider the graph $G_n$ on $\mathbb{R}^n$ connecting points of rational Euclidean distance. The work of Schmerl [67] shows that in ZF, the existence of a transcendence basis for the reals over the rationals implies that these graphs all have countable chromatic number. We can prove the following:

**Theorem 1.3.13.** It is consistent with ZF+DC that the chromatic number of $G_1$ is countable while the chromatic number of $G_2$ is uncountable.

In fact, it turns out that the existence of a Hamel basis for $\mathbb{R}$ implies in ZF+DC that $G_1$ has countable chromatic number, while it is consistent with ZF+DC plus the statement that $G_2$ has uncountable chromatic number. The analysis
of the graphs $G_n$ becomes more difficult as the number $n$ increases, and the following attractive question is left open for all $n > 1$:

**Question 1.3.14.** Let $n > 1$ be a number. Is it consistent with ZF+DC that the chromatic number of $G_n$ is countable while the chromatic number of $G_{n+1}$ is uncountable?

Hypergraphs of arity higher than 2 present enormous challenges. To illustrate the possibilities, let $R$ be the hypergraph on $2^{\omega} \times 2^{\omega}$ of arity 4 whose edges are vertices of rectangles in the $2^{\omega} \times 2^{\omega}$ plane. In ZFC, the statement that the chromatic number of $R$ is countable is equivalent with the continuum hypothesis. Without choice, we get

**Theorem 1.3.15.** It is consistent with ZF that the chromatic number of $R$ is countable while there is no uncountable sequence of distinct reals.

In a rather disconcerting turn of events, the Axiom of Dependent Choices fails in our model, and we do not know how to get the consistency of the above statement with DC.

**Ultrafilters.** The methods of the book can separate various types of ultrafilters on $\omega$ and other combinatorial objects in the context of ZF+DC.

**Theorem 1.3.16.** (Corollary 9.2.5, Proposition 9.2.24 and Corollary 12.2.9) In ZF+DC, there are no provable implications between the existence of a Hamel basis and existence of a nonprincipal ultrafilter on natural numbers.

Improving [29], we have

**Theorem 1.3.17.** (Corollary 13.1.12) In ZF+DC, the existence of a nonprincipal ultrafilter on natural numbers does not imply the existence of one which is disjoint from the summable ideal.

We isolate several ways of adding an ultrafilter to the symmetric Solovay model in a controlled way, providing ultrafilters with various interesting partition properties: Ramsey ultrafilters and stable ordered union ultrafilters are just two examples. However, the comparison of the various models obtained as well as a classification of ultrafilters appearing in each model seems to be beyond our skill. Many other combinations of implications remain unresolved. For example, we do not know how to construct a model of ZF+DC in which there is a nonprincipal ultrafilter on $\omega$, $|E_1| \leq |E_0|$ and yet $|E_0| \not\leq |2^\omega|$.

**Uniformization.** Uniformization problems belong to the guiding lights of descriptive set theory. One of the most notorious versions of it is countable to one uniformization, the statement that every subset of the plane with nonempty countable vertical sections contains the graph of a total function. In Chapter 10, greatly improving the methods and results of [46], we develop a satisfactory criterion which guarantees that various strong uniformization principles (including the countable-to-one uniformization as the humblest special case) hold in nearly
all models under study. Separating the various uniformization principles seems
to be a difficult task beyond the techniques of the present book.

**Limitations of the method.** The balanced forcing offers a very flexible and
powerful method for obtaining independence results in ZF+DC set theory. How-
ever, certain desirable types of independence results are outside of its reach.
Certain basic limitations are encapsulated in the following theorems.

**Theorem 1.3.18.** (Corollary 9.1.2) *In every balanced extension of the sym-
metric Solovay model, there is no \( \omega_1 \) sequence of distinct Borel sets of bounded
Borel rank.*

**Theorem 1.3.19.** (Theorem 14.1.1) *In every balanced extension of the sym-
metric Solovay model, there is no maximal almost disjoint family of subsets of
\( \omega \).*

**Theorem 1.3.20.** (Theorem 14.2.1 simplified) *In every balanced extension of
the symmetric Solovay model, there is no linearly ordered, unbounded set of
elements of \( \omega^\omega \) in the modulo finite domination ordering.*

Thus, the balanced forcing cannot be used to prove for example the consistency
of the possibility that \( \aleph_1 \leq |E_1| \) but \( \aleph_1 \nleq |E_0| \). For consistency results of this
type, one has to reach to the weakly balanced forcing of Section 5.3.

### 1.4 Independence: by model

One appealing way to present the work in this book is to consider several more
or less canonical generic extensions of the Solovay model and list the statements
that we know hold in them. There are also test problems that we do not
know how to resolve. The reader needs to keep in mind that all the models
are balanced extensions of the Solovay model \( W \) and as such obey the general
limitations described at the end of the previous section. The list below includes
only models that we find of immediate interest for an introduction; there are
innumerable other options.

**The Ramsey ultrafilter model.** Consider the poset \( P(\omega) \) modulo finite and
the associated extension of the Solovay model \( W \). The generic filter is identified
with a Ramsey ultrafilter, and well-known pre-existing results [20, 46] show that
in a suitable sense, every Ramsey ultrafilter is generic over \( W \) for the poset \( P(\omega) \)
modulo finite. The resulting model was investigated in numerous papers such as [51, 16].

**Theorem 1.4.1.** *In the model \( W[U] \), the following statements hold:*

1. ([16], Corollary 9.2.5) \( |E_0| \nleq |2^{\omega}| \);
2. (Corollary 9.4.8) \( |E_1| \nleq |F| \) for any orbit equivalence relation \( F \);
3. The $E_0$- and $E_1$-quotient spaces are linearly orderable (Theorem 13.2.8), while the $E_2$- and $F_2$-quotient spaces (Corollary 13.1.12 and Corollary 13.1.5) do not carry even any tournament;

4. (Example 14.4.7) If $E$ is a Borel equivalence relation and $A$ is a subset of the $E$-quotient space then either $|A| \leq \aleph_0$ or $|2^\omega| \leq |A|$;

5. (Corollary 13.1.12) Every nonprincipal ultrafilter on $\omega$ has nonempty intersection with the summable ideal;

6. (Example 10.5.9) Countable-to-one uniformization.

One can consider many variations, adding an ultrafilter which is not Ramsey. We study the cases of a stable ordered union ultrafilter and certain other ultrafilters. It is challenging to discern between the resulting models by sentences which do not mention ultrafilters.

There are many open questions about the Ramsey ultrafilter model. We do not know how to classify the ultrafilters on $\omega$ in it. We do not know if there can be Borel equivalence relations $E, F$ such that $|E| \nleq |F|$ holds in $W$ and $|E| \leq |F|$ holds in $W[U]$. In particular, we do not know if this can occur for countable Borel equivalence relations $E, F$ or in the situation where $E$ is the orbit equivalence relation of a turbulent action of a Polish group and an equivalence $F$ classifiable by countable structures.

**The Hamel basis model.** Consider the poset $P$ of countable sets of reals which are linearly independent over the rationals. The generic filter yields a Hamel basis $B \subset \mathbb{R}$. It does not seem to be easy to provide a simple criterion which would guarantee that a given Hamel basis is $P$-generic over the Solovay model. Still, we understand the theory of the model $W[B]$ fairly well:

**Theorem 1.4.2.** In the model $W[B]$, the following statements hold:

1. (Proposition 9.2.24) $E_0$ has a transversal;

2. (Corollary 9.4.31) $E_1$ has a complete countable section;

3. (Corollary 12.2.12) There is no transcendence basis for $\mathbb{R}$;

4. (Corollary 12.2.9) There is no nonatomic probability measure on $\omega$;

5. (Example 10.5.2) Countable-to-one uniformization.

One can consider a different Polish vector space in place of $\mathbb{R}$ and add a basis to it; a natural example is $\mathcal{P}(\omega)$ viewed with the symmetric difference operation as a vector space over the binary field. The conclusions of the above theorem except for (2) remain in force. We do not know how to discern between the resulting models in general.
If suitable large cardinals exist, one can find a Hamel basis generic over the model $L(R)$, independently of the size or structure of the continuum—Example 12.4.5. The model $L(R)[B]$ inherits all features quoted in Theorem 1.4.2.

**Models with a linear ordering.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Consider the poset $P$ of linear orderings of countable subsets of $X/E$, ordered by reverse extension. The generic filter yields a linear ordering $\leq$ on the $E$-quotient space. The resulting model is a very humble extension of the symmetric Solovay model.

**Theorem 1.4.3.** In the model $W[\leq]$, the following statements hold:

1. (Corollary 9.2.12) $|E_0| > |2^\omega|$;
2. (Corollary 9.4.10) $|E_1| \not\leq |F|$ for any orbit equivalence relation $F$;
3. (Corollary 14.3.17) there are no discontinuous homomorphism between Polish groups;
4. (Example 10.5.8) countable-to-one uniformization.

Other features may depend on the nature of the equivalence relation $E$. For example, if $E$ is pinned, then in the model $W[\leq]$ one cannot find a linear ordering or even a tournament on the $\mathbb{F}_2$-quotient space—Corollary 13.1.6. If $E$ is classifiable by countable structures, then in the model $W[\leq]$ cannot find a linear ordering on quotient spaces of turbulent group action orbit equivalence relations—Corollary 13.1.9. Many questions remain open. In particular, it is not clear whether in the model $W[\leq]$ there is a cardinal inequality between quotient spaces which does not appear already in $W$. If suitable large cardinals exist and $E$ is pinned, one can find a linear ordering of the $E$-quotient space generic over the model $L(R)$, independently of the size or structure of the continuum—Example 12.4.7.

One can add other structures to quotient spaces. In particular, one can add a tournament instead of a linear ordering. In the resulting model, the $\mathbb{F}_0$-quotient space cannot be linearly ordered—Corollary 13.2.5.

**The transversal models.** Let $E$ be a pinned Borel equivalence relation on a Polish space $X$. Consider the poset $P$ of countable subsets of $X$ consisting of pairwise $E$-unrelated elements, ordered by reverse inclusion. The generic filter yields an $E$-transversal $T \subset X$. The study of the resulting model is fairly straightforward with our methods:

**Theorem 1.4.4.** In the model $W[T]$, the following statements hold:

1. (Theorem 12.3.2) the Open Coloring Axiom;
2. (Corollary 12.2.7) there is no nonatomic probability measure on $\omega$;
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3. (Corollary 12.2.13) there is no transcendence basis for \( \mathbb{R} \);

4. (Corollary 14.3.7) there is no discontinuous homomorphism between Polish groups;

5. (Example 10.5.6) countable-to-one uniformization.

Other features depend on the equivalence relation \( E \) in a rather predictable way. Thus, if \( E \) is classifiable by countable structures, then in \( W[T] \), \(|F| \leq |G| \) holds whenever \( F \) is an orbit equivalence relation of a turbulent group action and \( G \) is an equivalence relation classifiable by countable structures. If \( E \) is reducible to an orbit equivalence relation, then in \( W[T] \), \(|E_1| \leq |G| \) holds whenever \( G \) is an orbit equivalence relation of a turbulent group action. If suitable large cardinals exist, one can find a transversal generic over the model \( L(\mathbb{R}) \), independently of the size or structure of the continuum. The model \( L(\mathbb{R})[T] \) inherits all features quoted in Theorem 1.4.4.

1.5 Independence: by preservation theorem

For a practitioner, it is probably most useful to categorize the results we obtain by the technical preservation theorem for balanced forcing that leads to them. The spectrum of preservation theorems for balanced forcing rivals the proper forcing technology. All of the classes considered below (with the exception of nested balanced and weakly balanced forcings) are closed under countable, full support product.

In the broad class of balanced forcings, a number of preservation theorems are possible. The balanced extensions of symmetric Solovay model do not add any well-ordered sequences of elements of the Solovay model–Theorem 9.1.1. They do not contain any inequalities of the form \(|E^+| \leq |E| \) where \( E \) is a Borel equivalence relation and \( E^+ \) is its Friedman–Stanley jump–Corollary 9.1.5. Certain more sophisticated objects are missing from these extensions, such as maximal almost disjoint families–Theorem 14.1.1. To overcome these limitations, one has to reach for the weakly balanced forcings. They do not add any sets of ordinals. A prominent example of a weakly balanced forcing which is not balanced is one which adds a maximal almost disjoint family with a certain type of approximations.

Compactly balanced forcings are those for which the space of balanced classes is naturally organized into a compact Hausdorff space–Definition 9.2.1. The most prominent example is the poset \( \mathcal{P}(\omega) \) modulo finite–Example9.2.4. The extensions of the symmetric Solovay model by compactly balanced forcings satisfy \(|\mathcal{E}_0| > 2^{2^\omega}\)–Theorem 9.2.2. It is even true that the chromatic number of a certain Hamming-type graph remains uncountable–Theorem 11.5.2.

Placid forcings are those in which placid virtual conditions exist: these are virtual conditions \( \bar{p} \) such that for any pair of separately generic extensions \( V[H_0], V[H_1] \) such that \( V[H_0] \cap V[H_1] = V \) and for any conditions \( p_0 \in V[H_0] \) and \( p_1 \in V[H_1] \) such that \( p_0, p_1 \) are both stronger than \( \bar{p} \), the conditions \( p_0, p_1 \)
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are compatible—Definition 9.3.1. Thus, the mutual genericity required in the definition of balanced forcing is replaced by the weaker demand $V[H_0] \cap V[H_1] = V$. The most prominent example is the poset adding a Hamel basis to the reals with countable approximations—Example 9.3.8. The extensions of the symmetric Solovay model by placid forcings do not contain any inequalities $|E| \leq |F|$ where $E$ is an orbit equivalence relation of a turbulent Polish group action, and $F$ is an equivalence relation classifiable by countable structures—Theorem 9.3.5.

There are many other effects (such as no nonprincipal ultrafilters on $\omega$) which are in fact consequences of preservation theorems for the broader class of equi-balanced forcings.

Nested balanced forcings are those which contain a sequence of balanced conditions for a suitable infinite nested sequence of generic extensions, as in Definition 9.4.1. All compactly balanced forcings are nested balanced—Example 9.4.7. A prominent example of a nested balanced but not compactly balanced forcing is the poset adding an $E_0$-transversal by countable approximations—Example 9.4.14. The extensions of the symmetric Solovay model by nested balanced forcings do not contain any inequalities $|E_1| \leq |E|$ where $E$ is an orbit equivalence relation—Theorem 9.4.4.

Adequate forcings are those for which, to establish the balance of a pair $\langle Q, \tau \rangle$, it is enough to show that the pair decides virtual conditions of certain complexity—Definition 10.1.1. Most balanced forcings are adequate; a prominent example of an inadequate poset is the collapse poset adding an injection of the $E_0$-quotient space to $2^\omega$. In adequate extensions of the symmetric Solovay model, the countable-to-one uniformization (and many other uniformization principles) holds—Theorem 10.3.2. In the extension by the inadequate collapse poset, there is no $E_0$-selector which means that the countable-to-one uniformization must fail.

Equi-balanced forcings are those for which equi-balanced conditions exist: these are virtual conditions $\bar{p}$ such that for any pair of equi-generic extensions $V[H_0], V[H_1]$, for any conditions $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ such that $p_0, p_1$ are both stronger than $\bar{p}$, the conditions $p_0, p_1$ are compatible—Definition 12.1.3. Here, the pair $V[H_0], V[H_1]$ is equi-generic if for every analytic equivalence relation $E$ coded in $V$, every virtual $E$-class represented both in $V[H_0]$ and $V[H_1]$ is represented in $V$. All placid forcings are equi-balanced. A prominent example of an equi-balanced poset which is not placid is a poset adding a transversal to $E$ by countable approximations, where $E$ is some turbulent Borel equivalence relation. The equi-balanced extensions of the symmetric Solovay model exhibit a number of interesting features: there is no non-atomic probability measure on $\omega$ (Example 12.2.5) and no transcendence basis for the reals over the rationals, and OCA holds (Theorem 12.3.2), among other things. We also show that for a certain subclass of equi-balanced posets, an $L(\mathbb{R})$-generic exists as a theorem of $\text{ZFC} + \text{large cardinals}$, independently of the size or structure of the continuum—Theorem 12.4.3.

3, 2-balanced forcings form a class useful for ruling out discontinuous homomorphisms between Polish groups. A poset $P$ is 3, 2-balanced if there are 3, 2-balanced virtual conditions $\bar{p}$ in it; these are virtual conditions such that
for any triple $V[H_0], V[H_1], V[H_2]$ of pairwise mutually generic extensions of $V$ and conditions $p_0 \in V[H_0], p_1 \in V[H_1], \text{ and } p_2 \in V[H_2]$ below $\bar{p}$ there is a common lower bound of $\{p_0, p_1, p_2\}$—Definition 14.3.1. A prominent example is the forcing adding a linear ordering of a given quotient space by countable approximations. 3, 2-balanced extensions of the symmetric Solovay model do not contain any discontinuous homomorphisms between Polish groups; in particular, they contain no nonprincipal ultrafilters on $\omega$—Theorem 14.3.2.

The perfect balanced forcings are the balanced Suslin forcings for which, in every perfect set of conditions, there is either a finite subset without a common lower bound, or a perfect subset with a common lower bound—Definition 14.4.1. Prominent examples include the posets like $\mathcal{P}(\omega)$ modulo finite adding an ultrafilter. The perfect balanced extensions of the symmetric Solovay model satisfy such features as the strong Silver dichotomy (Theorem 14.4.3) and a strong version of OCA (Theorem 14.4.6).

There are a number of preservation properties concerned with locally countable structures. They all consider posets of the form $F_{K,L}$ (Definition 11.3.2) and start with a hypothesis on the simplicial complex $K$. Thus, Theorem 11.3.5 shows that if $K$ is Suslin $\sigma$-centered then uncountable Borel chromatic numbers of finitary hypergraphs are preserved by the poset $F_{K,L}$ and Theorem 11.3.13 proves the same for Suslin $\sigma$-linked complexes. There are a number of options, and many classes of posets concerned are closed under the countable support product. It seems difficult to get preservation theorems for locally countable structures without the technology of iterated very Suslin forcings of Chapter 11.

### 1.6 Open question list

The geometric method opens many new directions of research. In this section, we gather questions which are left unanswered in this book and appear to be strategically important.

**Pinned equivalence relations.** Regarding the possible values of the pinned cardinal $\kappa(E)$ for analytic equivalence relation $E$, Theorem 2.5.6 yields an optimal upper bound in case $E$ is Borel, Theorem 2.5.8 yields an optimal upper bound in case $E$ is an orbit equivalence relation, and Theorem 2.5.10 provides a general upper bound of a measurable cardinal. One question is whether this bound is optimal:

**Question 1.6.1.** Find an optimal upper bound for the (non-$\infty$) values of the pinned cardinal $\kappa(E)$ for analytic equivalence relation $E$.

The following is a variation of the Silver dichotomy on $\aleph_1$. Theorem 2.5.11 provides the conclusion in very specific models of ZFC.

**Question 1.6.2.** Let $E$ be an unpinned Borel equivalence relation. Does ZFC imply that $E$ has at least $2^{\aleph_1}$ many classes of pinned names?
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The extent of the class of pinned equivalence relation changes dramatically when the Axiom of Choice is not present, and even more so in the absence of the Axiom of Dependent Choices. Manipulation of the class in such circumstances may be useful. The most daring question in this direction:

**Question 1.6.3.** Is it consistent with ZF that a Borel equivalence relation is pinned just in case it is Borel reducible to an $F_{\sigma}$-equivalence relation?

**The reals.**

**Question 1.6.4.** Does ZF+DC plus the statement that $\mathbb{R}$ can be decomposed into a sum of two disjoint vector spaces over $\mathbb{Q}$ imply that there is a Hamel basis for $\mathbb{R}$?

**Question 1.6.5.** Classify Banach spaces as to the criterion whether ZF+DC plus the statement that $X_0$ has a basis over $\mathbb{Q}$ implies that $X_1$ has a basis over $\mathbb{Q}$.

**Ultrafilters.** Chapter 7 introduces many balanced ways of adding an ultrafilter to the Solovay model. However, there are many other ways, and it is not clear if there are any limitations provable in ZF+DC.

**Question 1.6.6.** Is there a Borel ideal $I$ on $\omega$ such that in ZF+DC, the existence of an ultrafilter disjoint from $I$ implies the existence of an uncountable sequence of pairwise distinct reals?

**Question 1.6.7.** Is there a Borel ideal $I$ on $\omega$ such that in ZF+DC, the existence of an ultrafilter disjoint from $I$ implies $|E_0| \leq |2^\omega|$?

Nearly all posets for adding an ultrafilter we consider are compactly balanced and therefore preserve the inequality $|E_0| > |2^\omega|$ by Theorem 9.2.2. The only exception is the natural poset for adding a stable ordered union ultrafilter in Section 7.4.

**Question 1.6.8.** Does ZF+DC plus the statement that there is a stable ordered union ultrafilter imply $|E_0| \leq |2^\omega|$?

Ultrafilters on $\omega$ possibly interact with countable Borel equivalence relations. We outline one intriguing possibility. Let $E$ be a countable Borel equivalence relation on a Polish space $X$. A choice of ultrafilters on $E$ is a function which to each $E$-class assigns an ultrafilter on it.

**Question 1.6.9.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Does ZF+DC plus the statement that there is a nonprincipal ultrafilter on $\omega$ imply that there is a choice of ultrafilters on $E$?

The answer is positive for hyperfinite equivalence relations; that is as much as we know.
Let $P$ be the poset of infinite subsets of $\omega$ ordered by inclusion. The poset $P$ adds a Ramsey ultrafilter $U$. The generic extension $W[U]$ of the Solovay model by the ultrafilter has been investigated in numerous papers and the present book simplifies and improves many known proofs. Still, many questions remain unanswered:

**Question 1.6.10.** In the model $W[U]$:

1. Is there a classification of ultrafilters on $\omega$?
2. Are there cardinal inequalities $|E| \leq |F|$ for Borel equivalence relations $E, F$ on Polish spaces other than the ones which already appear in the Solovay model $W$?

**Balanced forcing.** The main question left open in the methodology of balanced forcing is the seemingly simple

**Question 1.6.11.** Let $P$ be a Suslin forcing. Can there be a proper class of pairwise inequivalent balanced conditions for $P$?

The definition of compactly balanced forcing in Definition 9.2.1 is exceptionally useful. However, it is also unsatisfactory in that it contains the word “definable”. We failed to find a uniform description of the possible topologies on the compact Hausdorff spaces that can arise in this fashion.

**Question 1.6.12.** Classify compact Hausdorff spaces that arise as spaces of balanced classes in Definition 9.2.1.

There are many possible ZF+DC consistency results which cannot be achieved by balanced forcing by the results of Chapter 14. Some of them seem to present genuinely difficult challenges. We select the following:

**Question 1.6.13.** Does ZF+DC plus the statement that there is a linearly ordered, unbounded subset of $\omega^\omega$ with the modulo finite domination ordering imply that there is an uncountable sequence of distinct reals?

With nearly all of our posets it is clear that they introduce sets of reals which are not Lebesgue measurable and do not have the property of Baire. There are very few exceptions ??? but they are certainly coincidental as opposed to intentional. This brings up a rather broad methodological question:

**Question 1.6.14.** Develop a methodology for designing balanced forcings which preserve statements like “all sets of reals have the property of Baire” or “all sets of reals are Lebesgue measurable”.

Two of the most interesting instances of this broad program are the following:

**Question 1.6.15.** (Marks, Unger) Let $E$ be a (non-hyperfinite) countable Borel equivalence relation on a Polish space $X$. Does ZF+DC plus the statement that $E$ is an orbit equivalence relation of a discontinuous action of $\mathbb{Z}$ imply that there is a set of reals without the Baire property?
Question 1.6.16. Does ZF+DC plus the statement that there is a finitely additive measure on $\omega$ imply that there is a set of reals which is not Lebesgue measurable?

Compatibility questions. Most independence results in this book are obtained via forcing with posets which come from categories naturally closed under countable support product. Consequently, the independence results form large groups, and the results in each group are mutually compatible. It is not clear to which extent this is a feature of the geometric method and to which extent it actually reflects some provable underlying pattern. We include the following sample questions:

Question 1.6.17. Are there $\Sigma^2_1$ sentences $\phi_0, \phi_1$, consequences of the Axiom of Choice, such that ZF+DC plus the statement that there is no uncountable sequence of pairwise distinct reals is consistent with each $\phi_0$ and $\phi_1$, but not with the conjunction $\phi_0 \land \phi_1$?

Question 1.6.18. Are there $\Sigma^2_1$ sentences $\phi_0, \phi_1$, consequences of the Axiom of Choice, such that ZF+DC plus the statement that there is no $E_0$-selector is consistent with each $\phi_0$ and $\phi_1$, but not with the conjunction $\phi_0 \land \phi_1$?

1.7 Navigation

The structure of the book is quite complicated and some navigation advice will greatly enhance the reader’s experience. Part I contains results about reducibility of Borel and analytic equivalence relations and how amalgamation properties of models of set theory can be used to disprove it. Part II provides all of the choiceless independence results. It is difficult to appreciate the methods of Part II without having basic acquaintance with the concepts introduced in Part I. Still, Part I should be viewed as only the gate to Cantor’s paradise, while Part II is already within the gates.

Chapter 2 defines and explores the notion of the virtual quotient space for an analytic equivalence relation $E$ on a Polish space $X$. On an intuitive level, a virtual equivalence class is one which may only exist in some forcing extension but still we already have a sensible calculus for speaking about it.

Section 2.1 provides the basic definitions. If $P$ is a partial order and $\tau$ is a $P$-name for an element of $X$, the name is called $E$-pinned if its $E$-class does not depend on the generic filter on $P$. There is a natural equivalence $\bar{E}$ on pinned names, and the $\bar{E}$ equivalence classes are referred to as the virtual $E$-classes. Section 2.2 shows that if one has a definable structure on the $E$-quotient space, it is possible to consider the virtual version of that structure on the virtual quotient space. This is a structure $\Pi_1$-elementarily equivalent to the original one. Structures such as virtual versions of partially ordered sets of Borel simplicial complexes will come handy later in the book.
Sections 2.3 and 2.4 deal with the most immediate concern: the classification problem for the virtual quotient spaces. The virtual equivalence classes should correspond to some immediately recognizable combinatorial objects, and in many important cases this hope is fulfilled. In a broad class of equivalence relations (the pinned equivalence relations of [39, Section 17]) the virtual quotient space is simply identical to the quotient space. Many jump-type operations on equivalence relations have a natural translation to operations on virtual quotient spaces; for example, the Friedman–Stanley jump [24] is translated to a powerset operation–Theorem 2.3.4. Virtual classes of isomorphism relations on countable structures naturally correspond to uncountable structures of the same type (Theorem 2.4.5), even though in some cases there may be mysterious virtual classes which are not classified in this way [41]. Still, there are many wide open questions. For example, we do not know how to classify the virtual quotient space for the measure equivalence. The virtual space of homomorphism of second countable compact Hausdorff spaces seems to be naturally classified by compact Hausdorff spaces, but there is no theorem to that effect.

Section 2.5 deal with the next most natural concern: the attempt to size up the virtual quotient space in terms of cardinality. The effort is surprisingly successful, generating cardinal invariants of analytic equivalence relations which can take all kinds of exotic and informative values. Given an analytic equivalence relation $E$, one defines the pinned cardinal $\kappa(E)$ as the minimal cardinal such that all virtual classes of $E$ are represented by names on posets of size $< \kappa$. One also defines $\lambda(E)$ as the cardinality of the virtual quotient space, and $\lambda(E, P)$ as the cardinality of the set of virtual classes which are represented by names on the poset $P$. These cardinals respect the Borel reducibility order and so serve as viable tools for the Borel nonreducibility results. In addition, they interact well with a number of operations on equivalence relations. In a jarring development, these cardinals can connect Borel nonreducibility results to hard questions about singular cardinal arithmetic or similar concerns of transfinite set theory.

Chapter 3 restates and greatly generalizes Hjorth’s notion of turbulence in forcing terms. This development shows that nonturbulent equivalence relations are in fact parallel to pinned equivalence relations in a very precise sense. The forcing relation encapsulates many distracting estimates needed in the traditional treatment of turbulence, resulting in a clean and efficient general calculus.

Section 3.1 provides the motivating insight–namely, that building pairs of models $V[y_0], V[y_1]$ where $y_0 \in Y_0$ and $y_1 \in Y_1$ are separately Cohen generic elements of their respective Polish spaces such that $V[y_0] \cap V[y_1] = V$ depends on a suitable notion of a walk–Theorem 3.1.5. The comparison of generic extensions vis-a-vis their intersection is an important concern in the later parts of this book. Section 3.2 provides two basic examples. In the first, Hjorth’s notion of turbulence of group actions is restated in geometric terms–Theorem 3.2.2. The second example provides a complete characterization of those analytic ideals on $\omega$ such that it is possible to find points $x_0, x_1 \in 2^\omega$ separately generic over $V$.  

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such that they are equal modulo $I$ and $V[x_0] \cap V[x_1] = V$–Theorem 3.2.4.

Section 3.3 defines two classes of equivalence relations that are simple from the turbulent point of view. An equivalence relation $E$ on a Polish space $X$ is placid if any $E$-class represented in generic extensions with trivial intersection is represented in the ground model. $E$ is virtually placid if any virtual $E$-class represented in such generic extensions is represented as a virtual class in the ground model. These definitions are preserved under Borel reducibility and under many natural operations on equivalence relations, as shown in Section 3.4. The main application is the generalization of an ergodicity theorem by Hjorth and Kechris [42, Theorem 12.5] (Theorem 3.3.5): any Borel homomorphism from a turbulent equivalence relation to a virtually placid one stabilizes on a comeager set.

Section 3.5 provides a necessary complement to the forcing development in this chapter: a theorem asserting that the notions introduced are absolute among all generic extensions and evaluating their descriptive complexity–Theorem 3.5.6. Finally, Section 3.6 develops turbulence of group actions for measure. It turns out that the appropriate concept uses the familiar notion of concentration of measure of [60].

Chapter 4 presents one of the entirely novel concepts of geometric set theory: that of a coherent nested sequence of models of ZFC, with emphasis on the intersection model. Thus, let $(V_n: n \in \omega)$ be an inclusion-decreasing sequence of transitive models of ZFC and consider the transitive class $V_\omega = \bigcap_n V_n$. A simple coherence condition on the sequence guarantees that $V_\omega$ is a model of ZF (Theorem 4.2.8); a coherence condition with a suitable well-ordering parameter guarantees that $V_\omega$ is a model of ZFC–Theorem 4.3.2. In Section 4.2 we develop the technology of building coherent sequences of generic extensions (Theorem 4.2.7) and produce a case in which choice fails in the intersection model. In Section 4.3 we develop a machinery of building choice-coherent sequences of generic extensions–Theorem 4.3.6. In principle, there is a whole field of coherent set theory hiding behind these concepts and the intersection models can serve as new vehicles for obtaining independence results. However, in this chapter our emphasis is on the connection with Borel equivalence relations. Our main result shows that the choice-coherent sequences of models detect the difference between $E_1$ and orbit equivalence relations: if $E$ is an orbit equivalence relation and $C$ is a virtual $E$-class represented in each model $V_n$, then it is represented in $V_\omega$–Theorem 4.3.7. This typically fails for $E_1$. The topic is revisited in Section 9.4 below.

Chapter 5 develops the basic general theory of balanced Suslin forcing. The treatment of Suslin forcing is quite different from that in [5] or similar treatments on the structure of the real line. Our partial orders are typically (but not necessarily) $\sigma$-closed, and serve to add combinatorial objects by explicit countable approximations in the context of the symmetric Solovay model. It is important to understand that the study of these partial orders takes place exclusively within set theory with choice.
Section 5.1 defines the notion of a virtual condition in a Suslin forcing $P$. A virtual condition is an element of the completion of the poset $P$ which only exists in some generic extension, but for which one can develop a sensible calculus of comparison already in the ground model, quite in parallel to virtual equivalence classes developed in Chapter 2. There are natural equivalence and ordering relations on virtual conditions in $P$.

Section 5.2 defines the key notion of a balanced pair in a Suslin forcing $P$. This is a pair $(Q, \tau)$ where $Q$ is a poset and $\tau$ is a $Q$-name for an element of the forcing $P$ which satisfies a natural amalgamation property in mutually generic extensions. There is a natural notion of equivalence on balanced pairs in $P$. The motherload feature exploited throughout the rest of the book is the simple Theorem 5.2.6: every balanced class contains exactly one virtual condition. The task of understanding a given Suslin forcing $P$ in the context of geometric set theory is then nearly identical to the classification of balanced virtual conditions in a given Suslin forcing $P$. Balanced virtual conditions in balanced Suslin forcing are quite parallel to master conditions in proper forcing.

Section 5.3 defines weakly balanced Suslin forcing. This is a weakening of the notion of balanced Suslin forcing designed to produce some effects (such as MAD families) which are absent from balanced extensions of the symmetric Solovay model. The whole development is quite parallel to balanced Suslin forcing with suitable quantification adjustments. Some examples of weakly balanced forcings can be found in Section 8.7. This book’s primary focus is on balanced forcing, and the potential of the weakly balanced variations is left wide open.

Chapter 6 analyzes a class of balanced forcings arising from simplicial complexes of various algebraic forms. Namely, let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$. Let $P_\mathcal{K}$ be the poset of countable subsets $p \subseteq X$ such that $|p|^{<\aleph_0} \subseteq \mathcal{K}$, ordered by reverse inclusion. Many useful posets in this book arise in this fashion, and the balance properties of the posets depend on model theoretic properties of $\mathcal{K}$. By coincidence or not, most natural examples are matroids. Section 6.2 introduces the notion of a modular simplicial complex (Definition 6.2.11), parallel to a modular matroid, with a number of examples and related notions. Another result (Theorem 6.2.32) shows that the poset adding a basis for a matroid is balanced as soon as the matroid satisfies a natural and common complexity requirement. Section 6.3 investigates simplicial complexes on quotient spaces, and shows that balanced conditions are naturally found in the related virtual quotient spaces. Particularly interesting cases concern the collapse of one quotient cardinality to another, Definition 6.3.2, and the transversal type posets which to each pair $E \subset F$ of Borel equivalence relations on a Polish space $X$ add a maximal set $A \subset X$ such that $E \restriction A = F \restriction A$—Definition 6.3.5.

Chapter 7 analyzes various ways of adding a nonprincipal ultrafilter on $\omega$ to a choiceless model. The basic method, the poset of all infinite subsets of $\omega$ ordered by inclusion, is investigated in Section 7.1. This poset adds a Ramsey ultrafilter, and its balanced conditions are classified simply by ultrafilters. The
resulting model has been investigated from many directions previously [16, 76]. The next simplest type of ultrafilter is added by a poset with Ramsey sequences of finite structures, discussed in Section 7.2. It turns out that balanced conditions in this type of forcing are classified by $\omega$-sequences of ultrafilters of a certain type. Section 7.3 shows how to add an ultrafilter which is not a P-point in the simplest fashion: using the quotient poset of $P(\omega \times \omega)$ modulo $I$ where $I$ is the Fubini power of the Fréchet ideal. The interesting development in this case is the complexity of balanced conditions—they are classified by ultrafilters on the set of ultrafilters on $\omega$ (Theorem 7.3.2). It is interesting to compare this situation with Section 6.3: there, the balanced conditions were also classified by complex sets, but the complexity seemed to be worked into the definitions of the posets from the beginning. Here, the increased complexity appeared without invitation and as a surprise, at least to the authors. Section 7.4 shows how to add a stable ordered union ultrafilter; the most interesting insight—the balanced conditions are classified by idempotent ultrafilters (Theorem 7.4.7).

Chapter 8 contains the analysis of balanced forcings which are designed to perform a specific task, and from the point of view of the previous chapters may seem somewhat ad hoc. In Section 8.1, we consider the problem of adding a coloring of a Borel graph by a fixed number of colors. It turns out that the task is closely related to the coloring number of graphs introduced by Erdős and Hajnal [19]. If a Borel graph has countable coloring number, then there is a natural balanced poset (Definition 8.1.1) adding a coloring by countably many colors; if the graph has coloring number $\leq n$ then there is a balanced poset adding a coloring with $\leq 3n - 2$ many colors (Theorem 8.1.2). The classical examples of such graphs appear on Euclidean spaces as distance graphs (Example 8.1.7 and 8.1.8) and have been studied among others by the Hungarian combinatorial school [47]. There are many open questions in the area.

Section 8.2 considers the problem of adding a discontinuous homomorphism between given Polish groups. This task cannot have a positive resolution in general since many Polish groups have the automatic continuity property, for example the unitary group [81]. However, we provide a balanced forcing in the case of abelian groups, one of which is torsion free, and the other divisible (Theorem 8.2.2). Section 8.3 shows that it is possible to force a nontrivial automorphism of the Boolean algebra $P(\omega)$ modulo finite with a balanced forcing. The forcing consists simply of countable approximations ordered by reverse inclusion, and balanced conditions correspond exactly to automorphisms of the algebra—Theorem 8.3.3. Section 8.4 considers the problem of adding a cofinal Kurepa family on a Polish space with a balanced poset. Again, the natural countable approximation poset works, and there is a very simple classification theorem for the balanced conditions—Theorem 8.4.3.

Section 8.5 shows how one can add saturated models of first order theories to $E$-quotient spaces for a Borel equivalence relation $E$. An especially interesting case is adding a saturated limit of a Fraissé class with strong amalgamation; i.e. a linear ordering, a tournament, or similar structures (Theorem 8.5.4). In this section, the amalgamation problems familiar from model theory and the
amalgamation questions arising in geometric set theory exhibit a particularly strong affinity.

Finally, Section 8.6 presents several orderings which produce balanced extensions of the symmetric Solovay model in which the Axiom of Dependent Choices fails. There is the seminal example, the finite-countable poset of Definition 8.6.1 which introduces a partition of an uncountable Polish space into two parts, one of them without a perfect subset and the other without countably infinite set. Other examples presented add a countable coloring to hypergraphs in cases where it is difficult to reconcile the amalgamation and \( \sigma \)-closure demands.

Chapter 9 compares cardinalities of quotient spaces in the balanced extensions of the symmetric Solovay model \( W \). On the surface, this is an enterprise very similar to the traditional forcing concerns in the context of the axiom of choice. However, it is important to understand that the non-well-ordered quotient cardinals offer many more opportunities for meaningful independence work and also for surprising ZF results preventing various patterns of cardinal collapses.

Section 9.1 shows that no new inequalities between well-ordered cardinalities and quotient cardinalities are added by balanced forcing. In fact, no new well-ordered sequences of elements of \( W \) are added–Theorem 9.1.1. This result is central for the rest of the book. Section 9.2 provides a very flexible tool for verification of the inequality \( |E_0| > |2^\omega| \) in certain classes of balanced extensions. Namely, it turns out that if the balanced classes of the forcing are naturally organized into a compact Hausdorff space, then the inequality holds in the \( P \)-extension of the symmetric Solovay model–Theorem 9.2.2. The compact Hausdorff space in question may certainly be nonmetrizable; in the case of the poset \( P \) of infinite subsets of \( \omega \) ordered by inclusion, it is simply the Čech–Stone remainder of \( \omega \). Among the applications, it is possible to add linear ordering to any Borel quotient space and preserve \( |E_0| > |2^\omega| \)–Corollary 9.2.12, or to add a discontinuous homomorphism from \( \mathbb{R} \) to \( \mathbb{R}/\mathbb{Z} \) and preserve \( |E_0| > |2^\omega| \) as in Corollary 9.2.23. The inequality \( |E_0| > |2^\omega| \) can be verified in other ways when the balanced classes do not form a compact Hausdorff space, for example using Theorem 11.3.13 to maintain the chromatic number of the \( G_0 \)-graph uncountable which automatically implies \( |E_0| > |2^\omega| \) by Proposition 11.1.6(2).

Section 9.3 shows that the nonreducibility results obtained by Hjorth’s turbulence method turn into cardinal inequalities in certain classes of balanced extensions. The main contribution here is Definition 9.3.1, isolating the relevant class of placid Suslin forcings. In placid extensions of the symmetric Solovay model, no inequalities of the type \( |E| \leq |F| \) appear where \( E \) is the orbit equivalence relation of a turbulent Polish group action, and \( F \) is an equivalence relation classifiable by countable structures. A weakening of placidity is used in Chapter 12 to rule out combinatorial objects from generic extensions which have nothing to do with any group actions, such as nonprincipal ultrafilters.

Section 9.4 provides practical criteria for showing that the classical nonreducibility of \( \mathbb{E}_1 \) to any orbit equivalence relation translates into cardinal inequalities in certain classes of balanced extensions of the Solovay model. This turns
out to require close study of nested sequences \((V_n : n \in \omega)\) of models of ZFC ordered in the reverse \(\omega\) type as in Chapter 4. In particular, we must resolve the question whether, for a given \(\Sigma^2_1\) sentence \(\phi\) which is a consequence of the axiom of choice, one can produce a set \(A\) such that for each \(n \in \omega\), the set \(A \cap V_n\) belongs to \(V_n\) and is a witness for the sentence \(\phi\) there. The resulting tricky combinatorial work can be used to separate for example the existence of a Hamel basis from existence of maximal acyclic subgraphs in Borel graphs.

Section 9.6 shows that the nonreducibility results between analytic equivalence relations obtained by the comparisons of their cardinal invariants \(\kappa\) and \(\lambda\) mostly turn into cardinal inequalities in the Solovay model \(W\), and these inequalities survive unharmed into the balanced extension.

Chapter 10 deals with the question of whether countable-to-one uniformization and similar uniformization principles hold in the balanced forcing extensions. Section 10.1 provides a practical criterion on a Suslin forcing \(P\) which implies that many strong uniformization principles hold in the forcing extension of the Solovay model by \(P\). It turns out that the criterion is satisfied in a great number of interesting cases, and its verification mostly follows from the careful classification of balanced conditions for \(P\). A number of examples appear in Section 10.5. Sections 10.2, 10.3, and 10.4 study various uniformization principles in turn and show that they in fact hold in adequate extensions of the symmetric Solovay model. Lastly, Section 10.5 provides a great number of examples and non-examples.

Chapter 11 considers the independence results which appear when one attempts to select one type of structure to each \(E\)-class, and not select another type; here \(E\) is a given countable Borel equivalence relation on a Polish space. The problems considered are motivated by the present practice of descriptive graph theory. The upshot is a discovery of new and very pertinent fracture lines in the area. The main point of the whole section is that the definable c.c.c. posets have great influence on independence results in ZF+DC, even though heretofore they were used only for ZFC independence results.

The fracture lines and their critical objects are defined in Section 11.1. We define two variations of countable Borel chromatic number of hypergraphs on Polish spaces: the Borel \(\sigma\)-finite chromatic number (Definition 11.1.9) and the Borel \(\sigma\)-finite fractional chromatic number (Definition 11.1.18). The relevant classes of hypergraphs contain minimal objects obtained by the same operation of a skew product (Definition 11.1.1). There are several dichotomies in style very similar to the original \(G_\alpha\)-dichotomy of [44]. A number of examples illustrate the new concepts; among them, there is one obtained from the density version of the van der Waerden theorem (Example 11.1.20).

Section 11.2 contains information on the basic technical tool needed to prove independence results on locally finite structures: very Suslin c.c.c. forcing notions (Definition 11.2.3) and their iterations. The section is written in such a way that specialists on ZFC independence results can use its results for their own purposes. The basic result shows that the finite support iterations of very
Suslin forcing notions are very Suslin again (Theorem 11.2.13). A long-standing concern of Stevo Todorcevic also makes appearance: for a given Borel simplicial complex $K$, viewed as a poset with the reverse inclusion ordering, determining whether the poset is c.c.c. and calibrating its chain condition carefully. The section provides a novel angle though: the regularity properties of the posets must be exemplified in a simply definable way. Thus, Section 11.2b discusses posets which can be covered by countably many analytic linked pieces and their iterations, and proves an iteration and product preservation theorem. Sections 11.2c, 11.2d, and 11.2e provide the same treatment for posets which are definably $\sigma$-centered, have a definable version of a finitely additive measure, and are $\sigma$-liminf-centered, respectively.

Section 11.3 finally produces some independence results in ZF+DC. The common theme: there is a Borel locally countable simplicial complex $K$ on a Polish space, and we study the consequences of the existence of a maximal $K$-set of a certain type. Adding colorings for various locally countable hypergraphs, for example, can be viewed from this angle. In all cases, the definable forcing properties of $K$ as a poset ordered by reverse inclusion play central role. We prove several results resonating with the current descriptive graph theory. Let $G$ be an analytic hypergraph with finite hyperedges, with uncountable Borel chromatic number. Given a countable Borel equivalence relation $E$, we prove the consistency of ZF+DC plus “$E$ is an orbit equivalence of a $\mathbb{Z}$-action” plus “the chromatic number of $G$ is uncountable”—Corollary 11.3.10. Given a locally finite acyclic Borel graph $H$, we prove the consistency of ZF+DC plus “the chromatic number of $H$ is not greater than three” plus “the chromatic number of $G$ is uncountable”—Corollary 11.3.12.

Section 11.4 studies the effects of collapses of various Borel cardinals to $|2^{\omega}|$ or to $\mathbb{E}_0$ on locally finite structures. Perhaps somewhat unexpectedly, collapsing a Borel cardinal such as $\mathbb{E}_0$ to $2^{\omega}$ does not yield an $\mathbb{E}_0$-selector; similarly, for many locally countable hypergraphs $G$, it does not make $G$ have countable chromatic number. The notion of the fractional chromatic number of a hypergraph (Definition 11.1.18) makes a key appearance in the main result—Theorem 11.4.1. A collapse of a Borel cardinal to $\mathbb{E}_0$ has no discernible effect on locally countable structures, as Theorem 11.4.5 shows.

Section 11.5 revisits the class of compactly balanced forcing. We show that for many locally countable hypergraphs $G$, compactly balanced forcings do not make $G$ have countable chromatic number. such forcings do not add countable coloring to a number of locally countable hypergraphs. The notion of Borel $\sigma$-finite chromatic number (Definition 11.1.9) is critical here; the diagonal Hamming graph $\mathbb{H}_{<\omega}$ is the most prominent example. In particular, it is consistent with ZF+DC that there be a nonprincipal ultrafilter on $\omega$, yet the chromatic number of $\mathbb{H}_{<\omega}$ is uncountable.

Chapter 13 provides several preservation theorems showing that in certain extensions of the Solovay model, certain types of structures do not appear on certain quotient spaces. It appears that many consistency results are best stated in this way.
Section 13.1 deals with tournaments. Theorem 13.1.2 shows that in \( \aleph_0 \)-adequate extensions, there is no tournament on the \( \mathbb{F}_2 \)-quotient space. In particular, it is consistent with ZF+DC to have a linear ordering on any given pinned quotient space and not have even a tournament of the \( \mathbb{F}_2 \)-space--Corollary 13.1.6. Theorem 13.1.7 shows that in placid extensions, there is no tournament of the quotient spaces arising from turbulent actions of Polish groups. In particular, it is consistent with ZF+DC to have a linear ordering on any given pinned quotient space and yet not have even a tournament on the summable equivalence quotient space--Corollary 13.1.9. Even tighter results are possible. For a Borel ideal \( I \) on \( \omega \), one can consider the equivalence relation \( E_I \) on \( 2^\omega \) of equality modulo \( I \), and consider the graph \( G_I \) connecting the \( E_I \)-classes of complementary sets. It turns out that these graphs are hard to orient. For the summable ideal \( I \), Theorem 13.1.10 shows that in the \( \mathcal{P}(\omega) \) modulo finite extension of the Solovay model, there is no orientation of the \( G_I \)-graph for \( I \) equal to the summable ideal. As a consequence, there is no ultrafilter disjoint from the summable ideal in that extension.

Section 13.2 deals with linear orderings on quotient spaces and the problem of separating the existence of a linear ordering from an existence of a tournament. Theorem 13.2.2 shows that it is impossible to linearly order the \( E_0 \)-quotient space with a 3,2-centered forcing without actually adding an \( E_0 \)-selector. This leads to the construction of many models of ZF+DC in which tournaments on complicated quotient spaces (or other objects) are present while the \( E_0 \)-quotient space cannot be linearly ordered.

Section ?? finally shows that the standard generic linear ordering added to a given quotient space by countable approximations is rigid in the resulting model.

Chapter 12 isolates a dividing line among applications of the Axiom of Choice which was to date clearly visible, but never precisely quantified. The story begins with the concept of equi-generic pair of extensions of a model of ZFC in Definition 12.1.1: this is a notion weaker than mutual genericity but strong enough to yield many interesting effects. The associated preservation notion is that of an equi-balanced forcing, Definition 12.1.3. The subject requires skill in constructing equi-generic pairs of extensions, which in retrospect is not difficult and hidden in the proofs of the consistency results.

Section 12.2 defines the notion of a thick Borel hypergraph--Definition 12.2.1. Thick Borel hypergraphs abound and are often very simple; the point is that it is very difficult to color them with countably many colors. The central application shows that it is possible to force a basis to a Polish vector space (and many other objects) and not add a nonatomic finitely additive measure on \( \omega \) (Corollary 12.2.9), and not add a transcendence basis to \( \mathbb{R} \) over \( \mathbb{Q} \). To exclude the transcendence basis, we use a particularly elegant combinatorial approach: we show that in the model there is no countable coloring to the graph on \( \mathbb{R}^2 \) connecting points of rational distance and use a result of Schmerl [67] showing that in ZF+DC, the existence of a transcendence basis implies that the chromatic number of this graph is countable.
Section 12.3 deals with certain \( \Pi^2_1 \) features of equi-balanced extensions. One popular subject included is the Open Coloring Axiom. It turns out that in equi-balanced extensions of the symmetric Solovay model, OCA holds—Theorem 12.3.2. It is in fact somewhat challenging to produce a balanced extension of the Solovay model in which OCA fails, Example 12.3.3. Among other results, we show that in equi-balanced extensions every uncountable subset of \( \omega^\omega \) contains an uncountable subset dominated by a single function—Theorem 12.3.4.

Section 12.4 deals with an issue that at first sight has nothing to do with equi-balanced forcing. It is natural to consider balanced generic extensions of the inner model \( L(\mathbb{R}) \) instead of the symmetric Solovay model; in a suitable large cardinal context, these generic extensions of \( L(\mathbb{R}) \) have an undeniable claim to canonicity. Theorem 12.4.3 shows that for many equi-balanced posets \( P \), a filter \( G \subseteq P \) generic over the model \( L(\mathbb{R}) \) exists provably in the theory ZFC plus a suitable large cardinal, quite independently of any cardinal arithmetic or continuum-structure issues that arise if one seeks a generic for an arbitrary balanced poset.

**Chapter 14** contains results that for some reason do not fit into the previous chapters; some of our favorite theorems are here. Certain structures are impossible to add by balanced forcing; MAD families (Theorem 14.1.1) and linearly ordered unbounded subsets of many partial orderings (Theorem 14.2.1) fall into this category.

In Section 14.3, we provide a machinery which excludes discontinuous homomorphisms between Polish groups in many extensions—Theorem 14.3.2. For example, it is possible to force any quotient cardinal inequality within trivial constraints without adding discontinuous homomorphisms between Polish groups (Corollary 14.3.5), to add a linear ordering on any Borel quotient space (Corollary 14.3.17), or to add a spanning acyclic subgraph to any Borel graph without introducing discontinuous homomorphisms—Corollary 14.3.15.

In Section 14.4 we isolate the class of perfect balanced forcings and show that certain dichotomies survive the journey from the Borel context to the symmetric Solovay model to the perfect balanced extensions of the symmetric Solovay model. In particular, we define the full Silver dichotomy (the Silver dichotomy for all sets of Polish spaces) and show that it holds for all Borel equivalence relations in perfect balanced extensions—Theorem 14.4.3. We also show that a strong version of OCA incompatible with choice holds in the perfect balanced extensions—Theorem 14.4.6. There are very few examples of perfect balanced forcings, but we at least include the posets for adding a Ramsey ultrafilter, and the ordered union ultrafilter.

Finally, in Section 14.5 we treat the very soft class of posets which possess definable balanced conditions. While at first sight it may seem that this class contains no particularly interesting posets, this is not the case: for example, the posets adding a discontinuous automorphism of \( P(\omega) \) modulo finite or a countable complete section to a given pinned Borel equivalence relation belong to it. We show that these posets preserve uncountable chromatic numbers of analytic hypergraphs (Theorem 14.5.7) and do not introduce inequalities of the
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form $|E_1| \leq |E|$ where $E$ is a pinned orbit equivalence relation. Many other preservation properties of posets in this class remain unstated.

1.8 Notation and terminology

Analytic equivalence relations. A number of concepts and results in this book are stated in terms of Borel equivalence relations on Polish spaces. The following definition records several benchmark relations which are used throughout the book.

Definition 1.8.1.

1. $E_0$ is the Vitali equivalence on $2^\omega$, connecting $x, y \in 2^\omega$ if they differ at only finite set of entries;

2. $E_1$ is the equivalence relation on $(2^\omega)^\omega$ connecting $x, y$ if they differ at only finite number of entries;

3. $E_2$ is the relation on $2^\omega$ connecting $x, y$ if the sum $\sum_{n \geq 1} \{ x(n) \neq y(n) \}$ is finite;

4. $F_2$ is the equivalence relation on $(2^\omega)^\omega$ connecting $x, y$ if $\operatorname{rng}(x) = \operatorname{rng}(y)$;

5. $\text{HC}$ is the equivalence relation on $\mathcal{P}(\omega \times \omega)$ connecting relations $x, y$ if either both are illfounded or fail the axiom of extensionality or fail to have a maximal element, or they are isomorphic;

6. $E_{\omega_1}$ is the equivalence relation on $\mathcal{P}(\omega \times \omega)$ connecting relations $x, y$ if either both are illfounded or are not linear orders, or they are isomorphic;

7. if $\Gamma$ is a coanalytic class of structures on $\omega$ invariant under isomorphism, $E_\Gamma$ is the equivalence relation on countable structures on $\omega$ connecting two such structures if they are both fail to belong to $\Gamma$ or they are isomorphic;

8. if $I$ is an ideal on $\omega$ then $=_I$ on $2^\omega$ is the equivalence relation connecting $x, y$ if $\{ n \in \omega : x(n) \neq y(n) \} \in I$. There is an identically defined equivalence relation on $(2^\omega)^\omega$.

Borel equivalence relations are naturally ordered by Borel reducibility. In the case of analytic equivalence relations, the Borel reducibility relation exhibits certain pathologies, and it is best replaced by some of its strengthenings. This is the content of the following definition.

Definition 1.8.2. Let $E$ and $F$ be analytic equivalence relations on respective Polish spaces $X$ and $Y$.

1. $E$ is Borel reducible to $F$, in symbols $E \leq F$, if there is a Borel function $h : X \to Y$ such that $\forall x_0, x_1 \in X \ x_0 E x_1$ iff $h(x_0) F h(x_1)$. 

2. \(E\) is almost Borel reducible to \(F\), in symbols \(E \leq_a F\), if there exists a Borel function \(h: X \to Y\) and a set \(Z \subset X\) consisting of countably many \(E\)-classes such that \(\forall x_0, x_1 \in X \setminus Z \ x_0 E x_1 \iff h(x_0) F h(x_1)\).

The most permissive comparison of equivalence relations is the one which compares just the cardinalities of the quotient spaces. The following abuse of notation is used throughout:

**Definition 1.8.3.** If \(E\) is an equivalence relation on a Polish space \(X\) then the \(E\)-quotient space is the set of all \(E\)-equivalence classes. Moreover, \(|E|\) denotes the cardinality of the \(E\)-quotient space.

The Silver dichotomy shows that for a Borel equivalence relation \(E\), \(|E|\) is either countable or \(|2^\omega| \leq |E|\). In the context of the axiom of choice, the latter disjunct implies that \(|E| = |2^\omega|\) since the quotient space \(X/E\) is a surjective image of \(2^\omega\). In choiceless context though, the dichotomies satisfying the latter disjunct may represent many different cardinalities, and this is the subject of study of several sections in this book. It is clear that \(E \leq F\) implies that \(|E| \leq |F|\), and in the context of the axiom of dependent choices, \(E \leq_a F\) and \(|F|\) is uncountable implies that \(|E| \leq |F|\).

**Analytic hypergraphs.** A hypergraph on a set \(X\) is an arbitrary subset \(\Gamma \subset P(X)\) consisting of nonempty sets. As a matter of convention, our hypergraphs do not contain any singleton sets. The elements of \(\Gamma\) will be called hyperedges while the elements of \(X\) will be called vertices. A hypergraph is finitary if all of its hyperedges are finite sets. It is a graph if all its hyperedges have cardinality two. A \(\Gamma\)-anticlique is a set \(A \subset X\) such that \(\Gamma \cap P(A) = \emptyset\). A \(\Gamma\)-coloring is any function with domain \(X\) which is not constant on any hyperedge; the elements of the range of a coloring are referred to as colors. The chromatic number \(\chi(\Gamma)\) of the hypergraph \(\Gamma\) is the smallest cardinal number \(\kappa\) such that there is a \(\Gamma\)-coloring with at most \(\kappa\)-many colors. This definition makes sense only in the context of the Axiom of Choice in which cardinalities are well-ordered and therefore \(\chi(\Gamma)\) must exist. In a choiceless context, only a limited version is available: we discern between countable and uncountable chromatic number, and the various values of the countable chromatic numbers.

We will be interested in analytic finitary hypergraphs. Say that \(\Gamma \subset [X]^{<\aleph_0}\) is an analytic hypergraph if \(X\) is Polish and the set \(\{y \in X^\omega: \text{rng}(y) \in \Gamma\}\) is analytic. The Borel chromatic number of \(\Gamma\) is countable if \(X\) can be decomposed into countably many Borel sets \(X = \bigcup_n X_n\) none of which contains all vertices of a \(\Gamma\)-edge.

**Forcing.** A great part of this book is devoted to forcing. We use the Boolean notation: \(q \leq p\) means that the condition \(q\) is stronger, more informative than \(p\). The formula \(P \models \phi\) denotes the statement that every condition \(p \in P\), \(p \models \phi\). Whenever \(P\) is a partial ordering, \(\tau\) is a \(P\)-name, and \(G \subset P\) is a generic filter, the symbol \(\tau/G\) denotes the valuation of the name \(\tau\) according to the filter \(G\). Every Polish space \(X\) and every analytic set \(A \subset X\) have a canonical interpretation in every generic extension, which commutes with all usual descriptive set theoretic operations on Polish spaces. For the detailed
theory of interpretations, see [86]; we will use it without explicit mention as is customary in the current forcing practice. The interpretations obey two basic absoluteness rules:

**Fact 1.8.4.** (Mostowski absoluteness) If $M \subset N$ are transitive models of ZF+DC and $\phi(p)$ is a $\Sigma^1_1$-formula with parameters in $M$, then $M \models \phi(p)$ if and only if $N \models \phi(i \bar{p})$ where $i$ is the interpretation operation.

**Fact 1.8.5.** (Shoenfield absoluteness) If $M \subset N$ are transitive models of ZF+DC such that $\omega_1 \subset M$ holds, and $\phi(p)$ is a $\Pi^1_2$-formula with parameters in $M$, then $M \models \phi(p)$ if and only if $N \models \phi(i \bar{p})$ where $i$ is the interpretation operation.

In particular, interpretations of analytic equivalence relations are equivalence relations again, and interpretations of Polish groups and continuous Polish group actions on Polish spaces are Polish groups and continuous Polish group actions again. Mutual relationships between forcing extensions of ZFC are captured in the following facts:

**Fact 1.8.6.** If $V$ is a model of ZFC, $V[G]$ is a forcing extension of $V$, and $M$ is a model of ZFC such that $V \subset M \subset V[G]$, then $M$ is a forcing extension of $V$ and $V[G]$ is a forcing extension of $M$.

The book is loaded with product forcing notions. This section provides basic information on products.

**Fact 1.8.7.** [36, Lemma 15.9] Let $P,Q$ be posets and in some generic extension, let $G \subset P, H \subset Q$ be filters separately generic over the ground model. The following are equivalent:

1. $G \times H \subset P \times Q$ is a filter generic over the ground model;
2. $G \subset P$ is generic over the model $V[H]$.

In the affirmative case, $V[G] \cap V[H] = V$.

If $G \times H \subset P \times Q$ is a filter generic over the ground model, we say that the filters $G,H$ (or their generic extensions) are mutually generic.

The following humble observation greatly simplifies the methodology of product forcing. It says that mutual genericity of forcing extensions can be characterized without an appeal to the specific generic filters and posets that were used to obtain the extensions, and indeed without any appeal to forcing at all.

**Proposition 1.8.8.** Let $n \in \omega$ be a number and $\{P_i : i \in n\}$ be posets. Let $\{G_i : i \in n\}$ be filters separately generic over the ground model $V$ over the respective posets. The following are equivalent:

1. $\prod_i G_i \subset \prod_i P_i$ is a filter generic over $V$;
2. whenever $\{a_i : i \in n\}$ are subsets of the ground model in the respective models $V[G_i]$ such that $\bigcap_i a_i = 0$ then there are sets $\{b_i : i \in n\}$ in the ground model $V$ such that for all $i \in n$ $a_i \subset b_i$ and $\bigcap_i b_i = 0$ holds.
 Proposition 1.8.10. Let \( M \) be a countable transitive model of set theory and \( P \in M \) be a poset. Then there is a continuous map \( h: 2^\omega \to \mathcal{P}(P) \) such that for every finite set \( a \subset 2^\omega \), the product \( \prod_{x \in a} h(x) \) is a filter on the poset \( P^{\mid a} \) generic over the model \( M \).

Proof. Let \( \langle D_n : n \in \omega \rangle \) be an enumeration of all open dense subsets of various finite powers of \( P \) which appear in the model \( M \). By induction on \( m \in \omega \) build maps \( g_m: 2^m \to P \) satisfying the following demands:

- \( g_0: \{0\} \to P \) is arbitrary;
- for every \( m \in \omega \), every \( s \in 2^m \) and every \( t \in 2^{m+1} \) such that \( s \subset t \), \( g_{m+1}(t) \leq g_m(s) \) holds;
- for every \( m \in \omega \) and every \( n \leq m \), if \( D_n \subset P^k \) is a set for some \( k \leq 2^{m+1} \) and \( \langle t_i : i \in k \rangle \) is a sequence of distinct elements of \( 2^{m+1} \), then \( \langle g_{m+1}(t_i) : i \in k \rangle \in D \) holds.
This is easy to arrange as the last item requires meeting only finitely many open dense sets. In the end, let \( h: 2^\omega \rightarrow \mathcal{P}(\omega) \) be the continuous function defined by setting \( h(x) \) to be the filter generated by the set \( \{ g_m(x \upharpoonright m) : m \in \omega \} \), and check that the demands of the theorem are satisfied.

There are several standard forcing notions used throughout:

**Definition 1.8.11.**

1. If \( X \) is a set then \( \text{Coll}(\omega, X) \) is the poset of finite partial functions from \( \omega \) to \( X \) ordered by reverse extension;
2. if \( \kappa \) is a cardinal then \( \text{Coll}(\omega, < \kappa) \) is the finite support product of the posets \( \text{Coll}(\omega, \alpha) \) for all \( \alpha \in \kappa \);
3. if \( X \) is a topological space then \( P_X \) is the poset of all nonempty open subsets of \( X \) ordered by inclusion. If \( X \) is in addition Polish, then \( \check{x}_{\text{gen}} \) denotes the \( P_X \)-name for the generic point of \( P_X \)–the unique element of the interpretation of the Polish space \( X \) in the \( P_X \)-extension which belongs to all open sets in the generic filter.

There are a number of basic properties of these partial orders used in the text below.

**Fact 1.8.12.** [36, Corollary 26.11] Let \( \lambda \) be a cardinal and let \( P \) be a poset of size \( < \lambda \). Suppose that \( G \subset \text{Coll}(\omega, \lambda) \) is a generic filter and in \( V[G] \), \( H \subset P \) is a filter generic over \( V \). Then there is a filter \( K \subset \text{Coll}(\omega, \lambda) \) generic over \( V[H] \) such that \( V[G] = V[H][K] \).

**Fact 1.8.13.** [36, Lemma 26.7] Let \( P \) be a forcing of size \( \lambda > \aleph_0 \) such that \( P \models |\lambda| = \aleph_0 \). Then \( P \) is in the forcing sense isomorphic to \( \text{Coll}(\omega, \lambda) \).

Let \( \kappa \) be an inaccessible cardinal. The symmetric Solovay model derived from \( \kappa \) is obtained as follows. Let \( G \subset \text{Coll}(\omega, < \kappa) \) be a generic filter and in \( V[G] \), form the model \( \text{HOD}^V[G] \) of all sets hereditarily definable from reals and elements of the ground model. This is the symmetric Solovay model; we will always denote it by \( W \), neglecting the dependence on \( \kappa \) and the filter \( G \) in the notation. The theory of the Solovay model has been thoroughly investigated throughout the years. We note the following:

**Fact 1.8.14.** [73] In \( W \), \( \text{ZF+DC holds, every set has the Baire property and is Lebesgue measurable, and there is no uncountable well-ordered sequence of distinct Borel sets of bounded Borel rank.} \)

During the investigation of the symmetric Solovay model, the following technical fact and its corollary about the symmetric Solovay model will be used without mention.

**Fact 1.8.15.** [36, Section 26] Let \( \kappa \) be an inaccessible cardinal and let \( W \) be the symmetric Solovay extension of \( V \) associated with \( \kappa \). Then
1. every set in $W$ is definable from parameters in $V \cup 2^\omega$;

2. in $W$, whenever $M$ is a generic extension of $V$ by a partial order of size $< \kappa$ then $W$ is a symmetric Solovay extension of $M$.

**Corollary 1.8.16.** Suppose that $\kappa$ is an inaccessible cardinal, $X$ is a Polish space, $\phi(x, \vec{y})$ is a formula of set theory with all free variables displayed, and $\vec{p}$ is a sequence of sets of the same length as $\vec{y}$. The following are equivalent:

1. $\text{Coll}(\omega, < \kappa) \models \exists x \in X \phi(x, \vec{p})$;

2. there exist a poset $R$ of size $< \kappa$ and an $R$-name $\sigma$ for an element of $X$ such that $R \models \text{Coll}(\omega, < \kappa) \models \phi(\sigma, \vec{p})$.

Many models of set theory we investigate are extensions of the symmetric Solovay model by a suitably definable, typically Suslin, forcing. Since the instrumental properties of the Suslin forcing in question may not be absolute between forcing extensions, and often depend on the status of the Axiom of Choice, we use the following key convention to shorten the statements of the results we obtain.

**Convention 1.8.17.** Let $\psi$ be a property of partial orders, and let $\phi$ be a sentence in the language of set theory. The phrase “In $\psi$-generic extensions of the Solovay model, $\phi$ holds” denotes the following long statement. Let $P$ be a Suslin forcing. Let $\kappa$ be an inaccessible cardinal. Suppose that $V_\kappa \models \psi$ in all forcing extensions, $\psi(P)$ holds. Let $W$ be a symmetric Solovay model derived from $\kappa$. Let $G \subset P$ be a filter generic over $W$. Then $W[G] \models \phi$ holds.

In several cases, we will need the basics of the stationary tower forcing.

**Definition 1.8.18.** [52] Let $\kappa$ be an inaccessible cardinal. The symbol $Q_\kappa$ denotes the (countably based) stationary tower up to $\kappa$. That is, $Q_\kappa$ consists of sets $S$ such that $S \subset [\text{dom}(S)]^{\aleph_0}$ is stationary and $\text{dom}(S) \in V_\kappa$. The ordering is defined by $T \leq S$ if $\text{dom}(S) \subset \text{dom}(T)$ and $\{x \cap \text{dom}(S) : x \in T\} \subset S$. If $G \subset Q_\kappa$ is a generic filter, $j : V \to M$ denotes the generic ultrapower derived from $G$.

**Fact 1.8.19.** [52] Let $\kappa$ be an inaccessible cardinal, $G \subset Q_\kappa$ be a generic filter, and $j : V \to M$ be the generic embedding.

1. If $\kappa$ is a Woodin cardinal in the ground model, then $\kappa = \omega_1^{V[G]}$ and $M^{\omega} \subset M$ in $V[G]$. In particular, $M$ is well-founded.

2. If $\kappa$ is a weakly compact Woodin cardinal in the ground model, then for every $z \in 2^\omega$ in $V[G]$ there is a Woodin cardinal $\lambda < \kappa$ such that $G \cap Q_\lambda$ is generic over $V$ and $z \in V[G \cap Q_\lambda]$. In particular, the model $W = V(\mathbb{R}^{V[G]})$ is a symmetric Solovay extension of $V$ derived from $\kappa$.

As a matter of notation, if the generic ultrapower model $M$ is well-founded, it is always identified with its transitive collapse.
Part I

Equivalence relations
Chapter 2

The virtual realm

There are many quotient structures in mathematics. It turns out that a typical quotient structure allows a useful canonical extension to its virtual version. The purpose of this chapter is to lay the foundations of the theory of virtual structures.

2.1 Virtual equivalence classes

The basis of any quotient structure of interest in the present book is a Polish space $X$. A quotient structure worth its salt will also use an equivalence relation on the underlying Polish space $E$. In this section we indicate how to extend the $E$-quotient space in a canonical way to a potentially much larger set or class.

Definition 2.1.1. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P$ be a poset and $\tau$ a $P$-name for an element of $X$.

1. The name $\tau$ is $E$-pinned if $P \times P \models \tau \text{left} \ E \tau \text{right}$;
2. if $\tau$ is $E$-pinned, then the pair $\langle P, \tau \rangle$ will be called an $E$-pin.

The definition may puzzle a novice reader. Its meaning is best illustrated by the following proposition. An $E$-pinned name is one which in all forcing extensions points at the same $E$-class, even though that $E$-class may not have any representative in the ground model.

Proposition 2.1.2. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P$ be a poset and $\tau$ a $P$-name for an element of $X$. The following are equivalent:

1. $\langle P, \tau \rangle$ is an $E$-pin;
2. in every forcing extension, if $G_0, G_1 \subseteq P$ are filters separately generic over the ground model, then $\tau/G_0 \ E \tau/G_1$ holds.
Proof. (2) immediately implies (1) by considering the $P \times P$ extension. To see how (1) implies (2), suppose that $V[H]$ is a forcing extension and in $V[H]$ there are filters $G_0, G_1 \subset P$ separately generic over $V$. Let $G_2 \subset P$ be a filter generic over $V[H]$. By the product forcing theorem, $G_0, G_2 \subset P$ are mutually generic filters, and so are $G_1, G_2 \subset P$. Applying the assumption (1), we see that $\tau/G_0 E \tau/G_2 E \tau/G_1$, and so $\tau/G_0 E \tau/G_1$ by the transitivity of the equivalence relation $E$. \qed

The following is the archetypal example of a non-trivial pinned name.

**Example 2.1.3.** Consider the poset $P = \text{Coll}(\omega, 2^\omega)$ and its name $\tau$ for the generic surjection from $\omega$ to $(2^\omega)^V$. The name $\tau$ is $\mathbb{P}_2$-pinned since no matter which generic filter $G \subset P$ one selects, the range of $\tau/G$ is the same: it is the set $(2^\omega)^V$. Clearly, it is a nontrivial name since the set $2^\omega$ is uncountable and so there is no ground model element of $(2^\omega)^\omega$ that can enumerate it and be equivalent to $\tau$.

Given an analytic equivalence relation $E$ on a Polish space $X$, the $E$-pinned names form a seemingly inexhaustible and complicated class. However, this class admits a natural equivalence relation which usually greatly clarifies matters:

**Definition 2.1.4.** Let $E$ be an analytic equivalence relation on a Polish space $X$. Suppose that $\langle P, \tau \rangle$ and $\langle Q, \sigma \rangle$ are $E$-pins. Define $\langle P, \tau \rangle \bar{E} \langle R, \sigma \rangle$ if $P \times Q \vDash \tau E \sigma$.

**Proposition 2.1.5.** The relation $\bar{E}$ is an equivalence. If $P, Q$ are posets with $E$-pinned names $\tau, \sigma$ on them, the following are equivalent:

1. $\langle P, \tau \rangle \bar{E} \langle Q, \sigma \rangle$;
2. in every forcing extension, if $G \subset P$ and $H \subset Q$ are filters separately generic over the ground model, then $\tau/G \sigma/H$ holds.

**Proof.** $\bar{E}$ is clearly symmetric and reflexive by its definition. To see the transitivity, suppose that $\langle P_0, \tau_0 \rangle \bar{E} \langle P_1, \tau_1 \rangle \bar{E} \langle P_2, \tau_2 \rangle$. This means that $P_0 \times P_1 \times P_2 \vDash \tau_0 E \tau_1 E \tau_2$, and by the transitivity of the equivalence relation $E$, $P_0 \times P_1 \times P_2 \vDash \tau_0 E \tau_2$. By the Mostowski absoluteness between the $P_0 \times P_1 \times P_2$ extension and $P_0 \times P_1 \times P_2$ extension, it is the case that $P_1 \times P_2 \vDash \tau_1 E \tau_2$, and consequently $\langle P_0, \tau_0 \rangle \bar{E} \langle P_2, \tau_2 \rangle$.

Now, (2) immediately implies (1) by considering the $P \times Q$ extension. To see how (1) implies (2), suppose that $V[K]$ is a forcing extension and in $V[K]$ there are filters $G \subset P, H \subset Q$ separately generic over $V$. Let $H' \subset Q$ be a filter generic over $V[K]$. By the product forcing theorem, $G, H'$ are mutually generic filters, and so are $H, H'$. Applying the assumption (1), we see that $\tau/G \sigma/H' E \sigma/H$, and so $\tau/G \sigma/H$ by the transitivity of the equivalence relation $E$. \qed

**Definition 2.1.6.** Let $E$ be an analytic equivalence relation on a Polish space $X$. 

2.2. VIRTUAL STRUCTURES

1. The $\bar{E}$-classes are referred to as the virtual $E$-classes;

2. if $z$ is a virtual $E$-class and in some generic extension $V[G]$ $y$ is an $E$-class, we say that $y$ is a realization of $z$ if for some (equivalently, all) representatives $\langle P, \tau \rangle \in z$ and $x \in y$, $V[G] \models P \models \tau E \bar{x}$ holds.

The following proposition is used throughout this book. It says that $E$-classes represented in mutually generic extensions must be realizations of a virtual $E$-class in the ground model.

**Proposition 2.1.7.** Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P_0, P_1$ be partial orders and $G_0 \subset P_0$ and $G_1 \subset P_1$ be mutually generic filters. If $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ are $E$-equivalent points then $[x_0]_E$ is the realization of some virtual $E$-class from the ground model.

**Proof.** Suppose that $p_0 \in P_0, p_1 \in P_1, \tau_0$ is a $P_0$-name and $\tau_1$ is a $P_1$-name such that $\langle p_0, p_1 \rangle \models \tau_0 E \tau_1$. It immediately follows that $\tau_0$ must be an $E$-pinned name on the poset $P \upharpoonright p_0$ and so $p_0 \models [\tau_0]_E$ is the realization of the virtual $E$-class represented by the pair $\langle P \upharpoonright p_0, \tau_0 \rangle$.

The main question surrounding the virtual $E$-classes is whether they can be classified in some informative way. Is there a proper class of virtual $E$-classes or just a set? If there is just a set, what is its cardinality? Do virtual $E$-classes correspond to some more tangible combinatorial objects? This chapter contains many good answers to similar questions, even though many problems remain unsolved.

2.2 Virtual structures

It is now possible to define virtual versions of quotient structures on Polish spaces.

**Definition 2.2.1.** An analytic quotient structure is a tuple $\mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle$ where

1. $X$ is a Polish space;
2. $E$ is an analytic equivalence relation on $X$;
3. for every $i \in \omega$, $R_i \subset X^{\omega}$ is an analytic relation which is invariant under $E$;
4. for every $j \in \omega$, $f_j \subset X^{\omega+1}$ is an analytic relation which is invariant under $E$, and in the $E$-quotient space it is a graph of a function.

The quotient structure $\mathcal{M}$ is Borel if all the relations above including $E$ are Borel.
There are many popular examples of analytic quotient structures. If \( \langle G, \cdot \rangle \) is a Polish group and \( H \subset G \) is an analytic normal subgroup, one can form the quotient group \( G/H \). If \( \langle X, \leq \rangle \) is an analytic partial ordering, one can form the separative quotient under the assumption that the quotient equivalence relation is analytic. Embeddability of countable structures forms an ordering on the quotients space of all countable structures modulo the equivalence relation of biembeddability etc.

If \( \mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle \) is an analytic quotient structure, then we write \( \mathcal{M}^* = \langle X^*, R_i^* : i \in \omega, f_j^* : j \in \omega \rangle \) for its associated structure on the actual quotient space \( X/E \) of all \( E \)-classes.

**Definition 2.2.2.** Let \( \mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle \) be an analytic quotient structure. The virtual version of \( \mathcal{M} \) is the tuple \( \mathcal{M}^{**} = \langle X^{**}, R_i^{**} : i \in \omega, f_j^{**} : j \in \omega \rangle \) where

1. \( X^{**} \) is the set or class of all virtual \( E \)-classes;
2. for each \( i \in \omega \), \( R_i^{**} \) is the relation on \( X^{**} \) of arity \( n_i \) given by \( \langle \langle Q_k, \tau_k \rangle : k \in n_i \rangle \in R_i^{**} \) if \( \prod_k Q_k \Vdash \langle \tau_k : k \in n_i \rangle \in R_i \);
3. for each \( j \in \omega \), \( f_j^{**} \) is the relation on \( X^{**} \) of arity \( m_j + 1 \) given by \( \langle \langle Q_k, \tau_k \rangle : k \in m_j + 1 \rangle \in f_j^{**} \) if \( \prod_k Q_k \Vdash \langle \tau_k : k \in m_j + 1 \rangle \in f_j \).

The first proposition says that Definition 2.2.2 is sound and that we receive a structure with the same signature as the original one:

**Proposition 2.2.3.** The definition of \( R_i^{**} \) and \( f_j^{**} \) does not depend on the choice of representatives of the virtual classes. Moreover, \( f_j \) is a graph of a (total) function.

**Proof.** The statements “\( R_i, f_j \) are \( E \)-invariant relations” and “\( f_j \) is a graph of a total function in the quotient” are \( \Pi^1_1 \) and therefore absolute between \( V \) and all forcing extensions. The first sentence of the proposition immediately follows. For the second sentence, suppose that the function \( f_j \) has arity \( m_j \) and \( \langle \langle Q_k, \tau_k \rangle : k \in m_j \rangle \) is a tuple of \( E \)-pins. Let \( Q = \prod_k Q_k \) and let \( \tau \) be a \( Q \)-name for an element of \( X \) such that \( f_j(\tau_k : k \in m_j, \tau) \) is forced to hold. Since in the \( Q \times Q \)-extension, \( f_j \) is a graph of a function on the quotient and the names \( \tau_k \)
for \( k \in m_j \) are \( E \)-pinned, it follows that the name \( \tau \) is \( E \)-pinned as well. Clearly, the tuple \( \langle \langle Q_k, \tau_k \rangle : k \in m_j, \langle Q, \tau \rangle \rangle \) belongs to \( f_j^{**} \) and the second sentence of the proposition follows. \( \square \)

In the case of an analytic quotient structure, it is possible that its virtual version is a proper class. In particular, it is possible to express the whole ordinal axis as an isomorph of a virtual version of an analytic quotient structure—Example 2.4.6. However, if the equivalence relation \( E \) is Borel then the virtual version is a set of size \( < \beth_1 \) by Theorem 2.5.6. In any case, it is possible to stratify \( \mathcal{M}^{**} \) into set sized pieces:
**Definition 2.2.4.** Let $\mathcal{M} = (X, E, R_i : i \in \omega, f_j : j \in \omega)$ be an analytic quotient structure. $\mathcal{M}^{**}_\kappa$ is the substructure of $\mathcal{M}^{**}$ consisting of the virtual $E$-classes represented by names on $\text{Coll}(\omega, \kappa)$.

A proof identical to that of Proposition 2.2.3 shows that $\mathcal{M}^{**}_\kappa$ is closed under all functions of $\mathcal{M}^{**}$ and so it is truly a substructure of $\mathcal{M}^{**}$. An elementary name-counting argument shows that for each infinite $\kappa$, the structure $\mathcal{M}^{**}_\kappa$ has size at most $2^\kappa$. If $\kappa \leq \lambda$ are cardinals then $\text{Coll}(\omega, \kappa)$ is regularly embedded in $\text{Coll}(\omega, \lambda)$ and so $\mathcal{M}^{**}_\kappa \subseteq \mathcal{M}^{**}_\lambda$. Every poset is regularly embedded in $\text{Coll}(\omega, \kappa)$ for some $\kappa$ and so $\mathcal{M}^{**}$ decomposes into a monotone union $\bigcup \kappa \mathcal{M}^{**}_\kappa$.

For each analytic quotient structure $\mathcal{M}$, there is a canonical embedding $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}$ which maps each class $[x]_E$ to the virtual $E$-class of $\langle Q, \tilde{x} \rangle$ for a trivial poset $Q$. The most important fact about the virtual structures is that there is some degree of elementaricity:

**Proposition 2.2.5.** The canonical embedding $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}$ is $\Pi_1$-elementary in $L_{\omega_1 \omega}$ logic.

**Proof.** Let $\kappa$ be a cardinal, and let $G \subseteq \text{Coll}(\omega, \kappa)$ be a generic filter. In the model $V[G]$, let $\chi : (\mathcal{M}^{**}_\kappa)^V \rightarrow (\mathcal{M}^*)^{V[G]}$ be the map sending a virtual $E$-class in $(\mathcal{M}^{**}_\kappa)^V$ to its realization. Thus, we have maps $(\mathcal{M}^*)^{V} \xrightarrow{\pi} (\mathcal{M}^{**}_\kappa)^V \xrightarrow{\chi} (\mathcal{M}^*)^{V[G]}$. The composition $\chi \circ \pi$ sends each equivalence class in $V$ to its interpretation in $V[G]$.

Let $\phi$ be a $\Pi_1$ formula with possible parameters such that $\mathcal{M}^* \models \phi$. The statement $\mathcal{M}^* \models \phi$ is a $\Pi_1^3$ sentence about the structure $\mathcal{M}$, and by Shoenfield absoluteness it transfers from the ground model $V$ to the generic extension $V[G]$. Thus, the map $\chi \circ \pi$ is a $\Pi_1$ elementary embedding from $(\mathcal{M}^*)^{V} \rightarrow (\mathcal{M}^{**}_\kappa)^V$. It follows that the map $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}_\kappa$ in $V$ must be a $\Pi_1$-elementary embedding. Since $\mathcal{M}^{**}$ is an increasing union $\bigcup \kappa \mathcal{M}^{**}_\kappa$, the proposition follows.

In particular, if the original quotient structure was a group, a partial order, or an acyclic graph, its virtual version maintains these properties. However, it is important to understand that the embedding does not have to be $\Sigma_2$-elementary, and so virtual versions of connected graphs may become disconnected, virtual versions of divisible groups may not be divisible anymore, and virtual versions of nonatomic partial orders may have atoms. The study of these apparent pathologies is the fuel and fire of this book.

### 2.3 Classification: general theorems

In this section, we provide a number of general classification theorems for virtual equivalence classes. The theorems are all of the same type: if $F$ is an analytic equivalence relation which is obtained from another equivalence relation $E$ using a certain operation, then all virtual $F$-classes are obtained from virtual $E$-classes using a similar operation. This will provide a suitable background to the investigation of specific cases in Section 2.4.
To start with, great many analytic equivalence relations yield only utterly uninteresting virtual classes: only those already realized in the ground model. This phenomenon is isolated in the following definition.

**Definition 2.3.1.** Let $E$ be an analytic equivalence relation on a Polish space $X$. A virtual $E$-class represented by $\langle P, \tau \rangle$ is trivial if there is $x \in X$ such that $P \vdash \tau E \bar{x}$. The equivalence relation $E$ is pinned if it has only trivial virtual classes.

The class of pinned equivalence relations has been investigated for a number of years. The basic pre-existing knowledge about this class is subsumed in the following fact.

**Fact 2.3.2.** [39, Theorem 17.1.3] The analytic equivalence relations in the following classes are pinned:

1. orbit equivalence relations generated by actions of Polish cli groups;
2. Borel equivalence relations with all classes $\Sigma^0_3$;
3. equivalence relations Borel reducible to pinned ones.

The operation on equivalence relations which has the most informative translation into virtual classes is that of the Friedman–Stanley jump.

**Definition 2.3.3.** Let $E$ be an analytic equivalence relation on a Polish space $X$. The Friedman–Stanley jump of $E$ is the equivalence relation $E^+$ on the space $Y = X^{\omega}$ defined by $y_0 E^+ y_1$ if $[\text{rng}(y_0)]_E = [\text{rng}(y_1)]_E$.

Here, the classification of pinned names is right at hand: a pinned name for the jump is essentially just a set of pinned names for the original equivalence relation. For a nonempty set $S = \{\langle P_i, \tau_i \rangle : i \in I\}$ of representatives of virtual $E$-classes, let $\tau_S$ be the name on the poset $Q_S = \prod_i P_i \times \text{Coll}(\omega, I)$ for an element of $X^{\omega}$ enumerating the set $\{\tau_i : i \in I\}$.

**Theorem 2.3.4.** Let $E$ be an analytic equivalence relation on a Polish space $X$.

1. If $S$ is a set of $E$-pinned names then $\langle Q_S, \tau_S \rangle$ is an $E^+$-pinned name;
2. $\langle Q_S, \tau_S \rangle \sim E^+ \langle Q_T, \tau_T \rangle$ iff the sets $S, T$ represent the same set of virtual $E$-classes;
3. whenever $\langle P, \tau \rangle$ is an $E^+$-pinned name, there is a set $S$ of $E$-pinned names such that $\langle P, \tau \rangle \sim E^+ \langle Q_S, \tau_S \rangle$.

**Proof.** Items (1) and (2) are immediate. To prove (3), suppose that $\tau$ is an $E^+$-pinned name on a poset $P$. For every virtual $E$-class $y$, the statement $\phi(y) = \text{rng}(y)$ contains a realization of the class $y^\tau$ must be decided in the same way by every condition in $P$. Let $S$ be any set of $E$-pinned names which
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collects representatives from all virtual $E$-classes $y$ such that $\forall p \in P \ p \models \phi(y)$. We claim that the set $S$ works.

Indeed, suppose that $G_0, G_1 \subset P$ are mutually generic filters. The sets $[\text{rng}(\tau/G_0)]_E$ and $[\text{rng}(\tau/G_1)]_E$ are equal, and by Proposition 2.1.7 they contain only realizations of ground model virtual $E$-classes. By the choice of the set $S$, these sets contain exactly realizations of virtual $E$-classes represented by names in $S$. Since the generic ultrafilter $G_0$ was arbitrary, $\langle P, \tau \rangle \bar{E}^+ \langle Q_S, \tau_S \rangle$ as desired.

Example 2.3.5. The relation $\mathbb{F}_2$ is the Friedman–Stanley jump of the identity on $X = 2^\omega$. The identity is pinned by Fact 2.3.2 and so its virtual classes can be identified with elements of $X$. The $\mathbb{F}_2$-classes then correspond to subsets of $2^\omega$.

The countable product of equivalence relations translates to the virtual realm without change.

Definition 2.3.6. For each $n \in \omega$, let $E_n$ be an analytic equivalence relation on a Polish space $X_n$. The product $\Pi_n E_n$ is the equivalence relation $E$ on $Y = \prod_n X_n$ defined by $y_0 E y_1$ if for every $n \in \omega$, $y_0(n) E_n y_1(n)$.

For every function $g$ such that $\text{dom}(g) = \omega$ and for all $n \in \omega$ $g(n)$ is some $E_n$-pinned name $\langle Q_n, \tau_n \rangle$, let $\tau_g$ be the name on the poset $Q_g = \prod_n Q_n$ (the support applied in the product is irrelevant for the equivalence class of the resulting $\Pi_n E_n$-pin) for the sequence $\langle \tau_n : n \in \omega \rangle$, which is forced to be an element of $Y$. The following is nearly immediate.

Theorem 2.3.7. For each $n \in \omega$, let $E_n$ be analytic equivalence relations on respective Polish spaces $X_n$ and let $E = \prod_n E_n$.

1. If $g$ is a sequence of $E_n$-pinned names, then $\langle Q_g, \tau_g \rangle$ is an $\prod_n E_n$-pin;
2. $\langle Q_g, \tau_g \rangle \bar{E} \langle Q_h, \tau_h \rangle$ iff for each $n \in \omega$, $g(n) \bar{E}_n h(n)$;
3. whenever $\langle P, \tau \rangle$ is an $E$-pinned name, there is a function $g$ such that $\langle P, \tau \rangle \bar{E} \langle Q_g, \tau_g \rangle$.

The countable increasing union of equivalence relations translates to the virtual realm without change as well.

Theorem 2.3.8. Let $\{ E_n : n \in \omega \}$ be an increasing sequence of analytic equivalence relations on a Polish space $X$, and let $E = \bigcup_n E_n$.

1. Whenever $\langle Q, \sigma \rangle$ is an $E_n$-pinned name for some $n \in \omega$ then it is also $E$-pinned;
2. whenever $\langle P, \tau \rangle$ is an $E$-pinned name, there is a number $n \in \omega$ and an $E_n$-pinned name $\langle Q, \sigma \rangle$ such that $\langle P, \tau \rangle \bar{E} \langle Q, \sigma \rangle$. 
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Proof. (1) is immediate as $E_n \subset E$. For (2), by the forcing theorem there must be conditions $p_0, p_1 \in P$ and a number $n$ such that $\langle p_0, p_1 \rangle \Vdash_{P \times P} \tau_{\text{left}} \ E_n \tau_{\text{right}}$. The transitivity of the relation $E_n$ then shows that $\tau$ on the poset $P \upharpoonright p_0$ is an $E_n$-pinned name. The initial assumptions show that $\langle P, \tau \rangle \not\Vdash \langle P \upharpoonright p_0, \tau \rangle$, as desired.

Example 2.3.9. The Louveau jump survives into the virtual realm without change. The Louveau jump of an analytic equivalence relation $E$ on a Polish space $X$ is the equivalence relation $E^{+L}$ on $Y = X^\omega$ connecting $y_0, y_1 \in Y$ if for all but finitely many $n \in \omega$, $y_0(n) \ E y_1(n)$. The Louveau jump can be written as a countable increasing union of countable products of $E$, which by Theorems 2.3.7 and 2.3.8 yields a complete analysis of its virtual classes in terms of virtual $E$-classes. In particular, if $E$ is pinned then its Louveau jump is pinned.

The virtual realm also correctly reflects the situation in which the equivalence classes of one relation consist of countably many equivalence classes of another one.

Definition 2.3.10. Let $E, F$ be analytic equivalence relations on a Polish space $X$. We say that $F$ is countable over $E$ if $E \subset F$ and every $F$-class consists of countably many $E$-classes.

Theorem 2.3.11. Let $E, F$ be analytic equivalence relations on a Polish space $X$, with $F$ countable over $E$.

1. If $\langle Q, \sigma \rangle$ is an $E$-pinned name then it is $F$-pinned as well;

2. if $\langle P, \tau \rangle$ is an $F$-pinned name then there is an $E$-pinned name $\langle Q, \sigma \rangle$ such that $\langle P, \tau \rangle \not\Vdash \langle Q, \sigma \rangle$.

Proof. (1) is immediate. For (2), it will be enough to show that there is a condition $p \in P$ such that $\tau$ is an $E$-pinned name on $P \upharpoonright p$, for then $\langle P, \tau \rangle \not\Vdash \langle Q, \sigma \rangle$.

Suppose towards contradiction that there is no condition $p \in P$ such that $\tau$ is $E$-pinned on the poset $P \upharpoonright p$. It follows by the forcing theorem that $P \times P \Vdash \neg \tau_{\text{left}} \ E \tau_{\text{right}}$ holds. Let $M$ be a countable elementary submodel of a large structure containing $P, \tau$ and the codes for $E, F$. Use Proposition 1.8.8 to find an uncountable collection $\{g_i : i \in I\}$ of filters on $M \cap P$ pairwise mutually generic over the model $M$. By the Mostowski absoluteness between the models $M[g_i, g_j]$ and $V$ for $i \neq j \in I$, the elements $\tau / g_i \in X$ for $i \in I$ are pairwise $F$-related, but pairwise $E$-unrelated, contradicting the initial assumptions on the relations $E, F$.

Example 2.3.12. The Clemens jump survives into the virtual realm without change. Here the Clemens jump of an analytic equivalence relation $E$ on a Polish space $X$ is the equivalence relation $E^{+C}$ on $Y = X^\omega$ connecting $y_0, y_1 \in Y$ if there is $n \in \mathbb{Z}$ such that for every $m \in \mathbb{Z}$ $y_0(m) \ E y_1(m+n)$. The Clemens jump
is countable over the product of $\mathbb{Z}$-many copies of $E$, which yields a complete analysis of its virtual classes by Theorems 2.3.7 and 2.3.8. In particular, if $E$ is pinned, then so is its Clemens jump.

### 2.4 Classification: specific examples

There are many analytic equivalence relations for which the virtual space can be classified by more tangible combinatorial objects, but which do not fit into the context of the theorems of Section 2.3. The purpose of this section is to investigate these more difficult, but also more informative, possibilities.

The most interesting issues arise in equivalence relations classifiable by countable structures. Among these, the Borel equivalence relations are Borel reducible to an iterated jump of the identity [39, Theorem 12.5.2], and so can be handled by Theorems 2.3.4 and 2.3.8. For example, for all equivalence relations $E$ Borel reducible to $\mathbb{F}_2$, the virtual $E$-classes are classifiable by subsets of $2^\omega$ by Example 2.3.5. More interesting issues arise with analytic equivalence relations. Let $E$ be the equivalence relation of isomorphism of structures on $\omega$. It is natural to attempt to classify virtual classes of $E$ by uncountable structures as in the following definition.

**Definition 2.4.1.** Let $M$ be a (possibly uncountable) structure of a countable signature. $\tau_M$ is a $\text{Coll}(\omega,M)$-name for some structure on $\omega$ isomorphic to $M$.

It is immediate that the pair $\langle \text{Coll}(\omega,M), \tau_M \rangle$ is an $E$-pin, and its $\bar{E}$-equivalence relation does not depend on the choice of the name $\tau_M$. It turns out that the $\bar{E}$-equivalence relation on the $E$-pins obtained in this way coincides with a familiar concept of model theory:

**Definition 2.4.2.** [54, Section 2.4] Let $M,N$ be structures with the same signature. Say that $M,N$ are Ehrenfeucht–Fraïssé-equivalent if Player II has a winning strategy in the Ehrenfeucht–Fraïssé game. In the EF-game, the two players take turns, at round $i \in \omega$ Player I starting with an element $n_i \in N$ or $m_i \in M$ and Player II responding with an element $m_i \in M$ or $n_1 \in N$ respectively. After all rounds indexed by $i \in \omega$ have been completed, Player II wins if the map $n_1 \mapsto m_i$ for $i \in \omega$ preserves all relations and functions of $N,M$ in the signature.

**Theorem 2.4.3.** Let $M,N$ be models with the same countable signature. The following are equivalent:

1. $M,N$ are Ehrenfeucht–Fraïssé equivalent;
2. $\langle \text{Coll}(\omega,M), \tau_M \rangle \bar{E} \langle \text{Coll}(\omega,N), \tau_N \rangle$.

**Proof.** For the (1)$\rightarrow$(2) direction, if $M$ is Ehrenfeucht–Fraïssé equivalent to $N$ as witnessed by a winning strategy $\sigma$ for Player I in the EF-game, then $\text{Coll}(\omega,M) \times \text{Coll}(\omega,N) \models \tau_M \in E \tau_N$, since a generic run of the EF-game in which Player II
follows the strategy $\sigma$ will generate an isomorphism between $M$ and $N$ in the extension. For the $(2)\rightarrow(1)$ direction, suppose that $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models \tau_M E \tau_N$ and let $\pi: M \rightarrow N$ be a product name for the isomorphism. The winning strategy for Player II can be described as follows: as the game develops, Player II also maintains on the side conditions $q_i \in \text{Coll}(\omega, M) \times \text{Coll}(\omega, N)$ such that $q_0 \geq q_1 \geq \ldots$ and $q_i \models \pi(\bar{m}_i) = \bar{n}_i$. It is immediate that this is possible and Player II must win in the end. \hfill \Box

However, in general, there are virtual $E$-classes which are not represented by a straightforward collapse name as in Definition 2.4.1. In order to extract more information, we have to restrict to more specific classes of structures.

**Definition 2.4.4.** Let $\Gamma$ be a coanalytic set of structures on $\omega$, closed under isomorphism. A (possibly uncountable) structure $M$ is a $\Gamma^{**}$-structure if $\text{Coll}(\omega, M) \models \tau_M \in \Gamma$.

We proceed to show that for some interesting coanalytic classes $\Gamma$, every $E_\Gamma$-class is represented by a collapse name of a $\Gamma^{**}$-structure as in Definition 2.4.1. In the classification results, we always ignore the trivial class of structures which do not belong to $\Gamma$.

**Theorem 2.4.5.** Let $\Gamma$ be a coanalytic class of countable structures on $\omega$, invariant under isomorphism, consisting of rigid structures only.

1. For $\Gamma^{**}$-structures, the Ehrenfeucht–Fraissé equivalence and isomorphism coincide;

2. for every $E_\Gamma$-pin $(P, \sigma)$ there is a $\Gamma^{**}$-structure $M$ such that $(P, \sigma) \Bar{E}\text{Coll}(\omega, M), \tau_M)$.

**Proof.** Before we begin the argument, note that the statement that every structure in the set $A$ is rigid is $\Pi^1_2$ and so holds also in all generic extensions by the Shoefield absoluteness.

For (1), it is clear that isomorphic structures are Ehrenfeucht–Fraissé equivalent. For the opposite implication, suppose that $M, N$ are $\Gamma^{**}$ structures which are $EF$-equivalent. By Theorem 2.4.3, $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models \tau_M E \tau_N$ must hold. As $\Gamma$ consists of rigid structures even in the collapse extension, $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models$ there is a unique isomorphism $\pi: M \rightarrow N$. Since $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N)$ is a homogeneous notion of forcing, for each $m \in M$ the value of $\pi(\bar{m})$ is decided by the largest condition to be some $h(m) \in N$. The function $h: M \rightarrow N$ is an isomorphism of $M$ to $N$ present already in $V$.

For (2), let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over $V$. In the model $V[G_0, G_1]$, let $N_0 = \tau/G_0$ and $N_1 = \tau/G_1$. To define the model $M \in V$, let $x_0 = \{ s : s$ is the Scott sentence of the model $\langle N_0, a \rangle$ for some $a \in N_0 \} \in V[G_0]$ and $x_1 = \{ s : s$ is the Scott sentence of the model $\langle N_1, a \rangle$ for some $a \in N_1 \} \in V[G_1]$. Since $N_0$ is isomorphic to $N_1$, it follows from Karp’s theorem [25, Lemma 12.1.6] that $x_0 = x_1$, so $x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. The set $x_0$ will be the universe of the model $M$. Note that since the model $N_0$ is rigid, the elements of $N_0$ are
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in one-to-one correspondence with $x_0$ by Karp’s theorem again and the unique isomorphism between $N_0$ and $N_1$ factors through the identity on the set $x_0 = x_1$. To construct the realizations of relational and functional symbols of the model $M$, for every relational symbol $R$ (say binary) of the language of the models and $s, t \in x_0$, let $s R^M t$ if for the unique $a, b \in N_0$ such that $s$ is the Scott sentence of $a$ and $t$ is a Scott sentence of $b$, $N_0 \models s R t$. The same definition using the model $N_1$ yields the same relation, and so $R^M \in V[G_0] \cap V[G_1] = V$. Define the realizations of all functional and relational symbols of the model $M$ in this way. As a result, $M$ is a model in $V$ and the map $a \mapsto$ the Scott sentence of $\langle M, a \rangle$ is an isomorphism of $N_0$ and $M$ in the model $V[G_0]$. Thus, (2) follows. \qed

Theorem 2.4.5 makes it possible to describe some class-sized virtual spaces explicitly:

**Example 2.4.6.** The virtual $E_{\omega_1}$-classes are precisely classified by ordinals, since $E_{\omega_1} = E_{\Gamma}$, where $\Gamma$ is the class of all well-orderings on $\omega$. Well-orderings are rigid, and up to isomorphism are classified by ordinals.

**Example 2.4.7.** The virtual $HC$-classes are classified by transitive sets with the $\in$-relation.

In the case of non-rigid structures, the classification may become more complicated. We will treat the case of acyclic graphs, which has the virtue of being Borel-complete among the equivalence relations classifiable by countable structures. Note that the Ehrenfeucht–Fraissé equivalence on uncountable acyclic graphs is distinct from isomorphism, as the case of an empty graph on $\aleph_1$ or $\aleph_2$ vertices shows.

**Theorem 2.4.8.** Let $\Gamma$ be the class of all acyclic graphs on $\omega$. For every $E_{\Gamma}$-pin $\langle P, \sigma \rangle$ there is an acyclic graph $H$ such that $\langle P, \sigma \rangle \equiv \langle Coll(\omega, H), \tau_H \rangle$.

**Proof.** The point of the proof is that an acyclic graph can be explicitly built from automorphism orbits of its elements. This procedure is captured in the following observation. Suppose $x$ is a set, $f: x^2 \to \omega + 1$ is a function such that $f(s, t) > 0 \iff f(t, s) > 0$, and $g: x^2 \to \omega + 1$ is a function such that $f(s, t) > 0$ implies $g(s) = g(t)$. Then there is, up to an isomorphism unique, acyclic graph $H(x, f, g)$ together with an onto map $h: y \to x$, where $y$ is the set of vertices of $H(x, f, g)$, such that

- for every $s, t \in x$ and every vertex $v \in y$, if $h(v) = s$ then the set of all neighbors of $v$ mapped to $t$ has size $f(s, t)$;
- for every $s \in x$ there are $g(s)$ many connected components of the graph $H$ containing a vertex $v$ with $h(v) = s$.

The construction of the graph $H(x, f, g)$ is straightforward. Note that whenever $u, v \in y$ are two vertices such that $h(u) = h(v)$ then there is an automorphism of the graph $H(x, f, g)$ sending $u$ to $v$. 
Now, suppose that $\sigma$ is an $E$-pinned name on a poset $P$ and let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over $V$. In the model $V[G_0,G_1]$, let $H_0 = \sigma/G_0$ and $H_1 = \sigma/G_1$. To define the graph $H \in V$, let $x_0 = \{ s : s$ is the Scott sentence of the model $\langle H_0,v \rangle$ for some vertex $v$ of $H_0 \} \in V[G_0]$ and $x_1 = \{ s : s$ is the Scott sentence of the model $\langle H_1,v \rangle$ for some vertex $v$ in $H_1 \} \in V[G_1]$. Since $H_0$ is isomorphic to $H_1$, it follows from Karp's theorem [25, Lemma 12.1.6] that $x_0 = x_1$, so $x = x_0 = x_1 \in V[G_0]\cap V[G_1] \in V$. Let $f : x^2 \rightarrow \omega + 1$ be the function defined by $f(s,t) = i$ if every vertex of $H_0$ of type $s$ has $i$-many neighbors of type $t$ when $i \in \omega$, and $f(s,t) = \omega$ if every vertex of $H_0$ of type $s$ has infinitely many neighbors of type $t$. Let $g : x \rightarrow \omega + 1$ be the function defined by $f(s) = i$ if there are $i$-many connected components of $H_0$ containing a node of type $s$ when $i \in \omega$, and $g(s) = \omega$ if there are infinitely many connected components of $H_0$ containing a node of type $s$. Note that these functions are well-defined and the graph $H_0$ is isomorphic to $H(x,f,g)$ in the model $V[G_0]$. Similar definitions using the graph $H_1$ yield the same functions, and so $f,g \in V[G_0]\cap V[G_1] = V$. Working in $V$, consider the graph $H = H(x,f,g)$. This graph is isomorphic to $H_0$ in $V[G_0]$, and so $\tau_H \cong \sigma$.

In the recent years, model theorists proved several other theorems regarding the classification of virtual isomorphism classes by possibly uncountable structures.

**Fact 2.4.9.** [41, Section 4] Let $E$ be the equivalence relation of isomorphism of structures on $\omega$.

1. If $P$ is a poset forcing $|\mathcal{N}_V^\omega| > \aleph_0$ then every $E$-pin $(P,\sigma)$ is $\bar{E}$-equivalent to $(\text{Coll}(\omega,M),\tau_M)$ for some structure $M$ of size $\leq \aleph_1$;

2. if $P$ forces $|\mathcal{N}_V^\omega| = \aleph_0$, then there is an $E$-pin $(P,\sigma)$ which is not $\bar{E}$-equivalent to $(\text{Coll}(\omega,M),\tau_M)$ for any structure $M$.

Virtual spaces for equivalence relations which are not classifiable by countable structures are often quite difficult to understand. To conclude this section, we state another classification theorem and a couple of open questions.

**Theorem 2.4.10.** Let $E$ be the equivalence relation on $X = (\mathcal{P}(\omega))^\omega$ connecting $x_0, x_1$ if $\text{rng}(x_0)$ and $\text{rng}(x_1)$ generate the same filter on $\omega$. The virtual $E$-classes are classified by filters on $\omega$.

**Proof.** On one hand, if $F$ is a filter on $\omega$, one can consider the $\text{Coll}(\omega,F)$-name $\tau_F$ for a generic enumeration of the filter $F$. It is immediate that the name $\tau_F$ is $E$-pinned, and distinct filters yield inequivalent names.

For the more difficult part, suppose that $P$ is a partial order and $\tau$ is an $E$-pinned name on $P$; we must find a filter $F$ on $\omega$ such that $\tau$ is equivalent to $\tau_F$. To do this, let $F = \{ a \subset \omega : \exists p \ p \vdash \check{a} \$ belongs to the filter generated by $\text{rng}(\tau) \}$. By the pinned property of the name $\tau$, the existential quantifier in the definition of $F$ can be replaced by universal without changing the resulting set $F$. It follows immediately that the set $F$ is a filter; we must show that $\tau$ is equivalent to $\tau_F$. 

Suppose towards contradiction that this fails; then it must be the case that for some condition \( p \in P \) and some number \( n \in \omega \), \( p \rhd \tau(\check{n}) \) has no subset in the filter \( F \). Since the name \( \tau \) is \( E \)-pinned, there must exist conditions \( p_0, p_1 \leq p \) and a finite set \( a \subset \omega \) such that \( \langle p_0, p_1 \rangle \rhd \bigcap_{m \in a} \tau(m) \subset \tau(\check{n}) \). Let
\[
b = \{ k \in \omega : \exists r \leq p_1 \text{ such that } k \in \bigcap_{m \in a} \tau(m) \} \text{ and let } c = \{ k \in \omega : p_0 \rhd \cdots \rhd \check{k} \in \tau(\check{n}) \}.
\]
Observe that \( b \in F \) (since \( p_1 \rhd \bigcap_{m \in a} \tau(m) \subset \check{b} \)) and \( c \notin F \) (since \( p_0 \rhd \cdots \rhd \check{c} \subset \tau \)). Thus, there has to be a number \( k \in b \setminus c \), and for this number \( k \) there are conditions \( r_0 \leq p_0 \) and \( r_1 \leq p_1 \) such that \( r_1 \rhd \check{k} \in \bigcap_{m \in a} \tau(m) \) and \( r_0 \rhd \check{k} \notin \tau(\check{n}) \). This, however, contradicts the choice of the conditions \( p_0, p_1 \). \( \square \)

Consider the equivalence relation \( E \) of homeomorphism of compact metrizable spaces. This is known to be a largest equivalence relation reducible to an orbit equivalence in the sense of Borel reducibility [88]. In an attempt to describe its virtual space, consider any compact Hausdorff space \( X \) with a topology basis of size \( \kappa \), and consider the \( \text{Coll}(\omega, \kappa) \)-name \( \tau_X \) for the interpretation of the space \( X \) in the extension in the sense of [86]. The interpretation of \( X \) will have a countable basis, and so the interpretation will be a compact metrizable space. The basic theory of interpretations shows that \( \langle \text{Coll}(\omega, \kappa), \tau_X \rangle \) is an \( E \)-pin. The most natural question is open:

**Question 2.4.11.** Is every virtual \( E \)-class represented by a compact Hausdorff space?

The measure equivalence \( E \) is one of the hardest equivalence relations to understand. It connects two Borel probability measures \( \mu, \nu \) on the Cantor space if they share the same ideal of null sets. The relation \( F_2 \) Borel reduces to \( E \), and so \( E \) is not pinned.

**Question 2.4.12.** Classify the virtual space for the measure equivalence.

### 2.5 Cardinal invariants

There are several cardinal invariants of equivalence relations which are associated with the concept of virtual equivalence classes. They respect the Borel reducibility order and so they are useful as tools for nonreducibility results. They also come conceptually handy in several places in this book.

#### 2.5a Basic definitions

The following definition records the most natural cardinal invariants associated with the virtual spaces.

**Definition 2.5.1.** Let \( E \) be an analytic equivalence relation on a Polish space \( X \).

1. \( \kappa(E) \), the **pinned cardinal** of \( E \), is the smallest \( \kappa \) (if it exists) such that every virtual \( E \)-class has a representative on a poset of size \( < \kappa \). If \( E \) is pinned, we let \( \kappa(E) = \aleph_1 \). If \( \kappa(E) \) does not exist, we write \( \kappa(E) = \infty \);
2. $\lambda(E)$ is the cardinality of the set of all virtual $E$-classes. If the virtual $E$-classes form a proper class, we let $\lambda(E) = \infty$;

3. $\lambda(E, P)$ is the cardinality of the set of all virtual $E$-classes represented on the poset $P$.

The rather mysterious demand that $\kappa(E) = \aleph_1$ for all pinned equivalence relations $E$ is explained by a reference to Theorem 2.8.2: there are no nontrivial pinned names on countable posets for any analytic equivalence relation. It turns out that the cardinals $\kappa(E)$ and $\lambda(E, P)$ can attain all kinds of exotic and informative values. We will start with the two archetypal and somewhat boring computations.

**Example 2.5.2.** $\kappa(F_2) = c^+$ and $\lambda(F_2) = 2^c$.

*Proof.* This follows immediately from the classification of pinned names for the Friedman–Stanley jump. Every virtual $F_2$ class is represented by a subset of $2^{\omega}$, and distinct subsets of $2^{\omega}$ give rise to distinct virtual $F_2$ classes. $\square$

**Example 2.5.3.** $\kappa(E_{\omega_1}) = \infty$. To see this, let $\Gamma$ be the class of all countable well-orders, so $E_{\omega_1} = E_\Gamma$. Note that all elements of $\Gamma$ are rigid. By Theorem 2.4.5, the virtual $E_{\omega_1}$-classes are classified by ordinals, and so there is a proper class of them.

Theorem 2.7.9 shows that in fact $E_{\omega_1}$ is a minimal example of an equivalence relation $E$ with $\kappa(E) = \infty$.

The most appealing fact about the cardinal invariants $\kappa(E)$ and $\lambda(E, P)$ is that they respect the Borel reducibility order and therefore can be used to prove Borel nonreducibility results. In a good number of instances, the comparison of the cardinal invariants is the fastest and most intuitive way of proving nonreducibility. One of the main features of this style of argumentation is that it automatically survives the transfer to nonreducibility by functions more complicated than Borel.

**Theorem 2.5.4.** Let $E, F$ be analytic equivalence relations on Polish spaces $X, Y$ respectively. If $E \leq_a F$ then $\kappa(E) \leq \kappa(F)$ and $\lambda(E, P) \leq \lambda(F, P)$ holds for every partial order $P$.

*Proof.* Suppose that $h: X \to Y$ is a Borel function witnessing the reduction of $E$ to $F$ everywhere except for a set $Z \subset X$ consisting of countably many $E$-classes. By a Shoenfield absoluteness argument, these properties of the function $h$ transfer to all generic extensions. If $P$ is a partial ordering and $\tau$ is an $E$-pinned name on $P$ which is not a name for one of the classes in the set $Z$, then $\dot{h}(\tau)$ is an $F$-pinned name on $P$, and the map $\tau \mapsto \dot{h}(\tau)$ respects virtual $E$- and $F$-classes, and it is an injection from the former to the latter. It follows that $\lambda(E, P) \leq \lambda(F, P)$.

Now, suppose that $Q$ is another poset and that $h(\tau)$ has a $\bar{F}$-equivalent name $\sigma$ on $Q$. By the Shoenfield absoluteness, $Q \models \exists x \in X \setminus Z \ h(x) = \tau$; any $Q$-name for such an $x$ is $E$-pinned and the name for $h(x)$ is $\bar{F}$-equivalent to $h(\tau)$. A
brief bout of diagram chasing now shows that $\kappa(E) \leq \kappa(F)$ and the theorem follows.

The following neat application has been observed by [82] for the case of Borel equivalence relations classified by countable structures.

**Example 2.5.5.** Friedman and Stanley proved that the Friedman–Stanley jump of a Borel equivalence relation $E$ is not Borel reducible to $E$. The most appealing proof of this fact uses the cardinal invariant $\lambda(E)$. Theorem 2.5.6 below shows that for Borel equivalence relations, the value of $\lambda(E)$ is an actual cardinal as opposed to $\infty$. Theorem 2.3.4 shows that virtual $E^+$-classes are classified by sets of virtual $E$-classes, in other words $\lambda(E^+) = 2^{\lambda(E)}$. Theorem 2.5.4 then completes the argument.

### 2.5b Estimates

The key fact about the pinned cardinal is the following theorem. It shows that there are a priori bounds on the size of the pinned cardinal; in particular, if the equivalence relation $E$ is Borel, then $\kappa(E) \leq \beth_\omega$ and the virtual space of $E$ is a set.

**Theorem 2.5.6.** Let $E$ be a Borel equivalence relation on a Polish space $X$ of rank $\Pi^0_\alpha$. Then $\kappa(E) \leq (\beth_\omega)^+$.

**Proof.** Let $\tau$ be an $E$-pinned name on a poset $P$; we must produce a $\text{Coll}(\omega, \beth_\omega)$-name $\sigma$ which is $\bar{E}$-related to $\tau$. Note that $[\tau]_E$ is a $P$-name for a Borel set of rank $\leq \alpha$. As is the case for every name for a Borel set, [73, Corollary 2.9] shows that in the $\text{Coll}(\omega, \beth_\omega)$ extension $V[G]$ there is a Borel code for a Borel set $B \subset X$ such that in every further forcing extension $V[G][H]$ and every $x \in X \cap V[G][H]$ in that extension, $x \in B$ if and only if $V[x] \models P \models \bar{x} \in [\tau]_E$. Note that if $H \subset P$ is generic over $V[G]$, then the set $B$ is nonempty in $V[G][H]$, containing the point $\tau/H$; this follows from the fact that $\tau$ is $E$-pinned. Thus, the set $B$ is nonempty already in $V[G]$ by the Mostowski absoluteness between $V[G]$ and $V[G][H]$. Back in $V$, let $\sigma$ be any $\text{Coll}(\omega, \beth_\omega)$-name for an element of the set $B$. This clearly works. \qed

**Example 2.5.7.** For any given countable ordinal $\alpha$, let $\Gamma_\alpha$ be the class of binary relations on $\omega$ which are extensional and wellfounded of rank $< \alpha$; let $E_\alpha$ be the isomorphism relation. It is not difficult to check (Theorem 2.4.5) that for each countable ordinal $\alpha$, the relation $E_\alpha$ is Borel, and its pinned names are collapse names for isomorphs of the membership relation on sets in $V_\alpha$. Thus, the cardinals $\kappa(E_\alpha)$ converge to $\beth_\omega$.

**Theorem 2.5.8.** Let $E$ be an analytic equivalence relation almost reducible to an orbit equivalence relation of a continuous Polish group action. If $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first $\omega_1$-Erdős cardinal.
Proof. Let $\kappa$ be the first $\omega_1$-Erdős cardinal, and suppose that $\kappa(E) > \kappa$; we must show that $\kappa(E) = \infty$. Since the cardinal $\kappa$ is the Hanf number for the class of wellfounded models of first order sentences, for every cardinal $\lambda$ there is a wellfounded model $M$ such that $M \models \kappa(E) > \lambda$. Now, since $E$ is almost reducible to an orbit equivalence relation, Corollary 2.6.4 shows that the wellfounded model $M$ is correct about $\kappa(E)$ to the extent that $|\kappa(E)^M| \leq \kappa(E)$. It follows that $\kappa(E) > \lambda$, and since $\lambda$ was arbitrary, $\kappa(E) = \infty$. \hfill $\square$

Example 2.5.9. For every countable ordinal $\alpha$ there is a coanalytic class $\Gamma$ of structures on $\omega$, invariant under isomorphism, such that $\kappa(\mathcal{E}_\Gamma) =$ the first $\alpha$-Erdős cardinal.

Proof. Let $\Gamma$ be the class of all binary relations on $\omega$ which are extensional, wellfounded, and do not admit a sequence of indiscernibles of ordertype $\alpha$; we claim that this class works.

Clearly, $\Gamma$ is a coanalytic set of rigid structures invariant under isomorphism. Write $E = \mathcal{E}_\Gamma$ and $\kappa =$ the first $\alpha$-Erdős cardinal. By Theorem 2.4.5, every virtual $E$-class is represented by a transitive set $A$ without indiscernibles of ordertype $\alpha$. It must be the case that $|A| < \kappa$ and so $\kappa(E) \leq \kappa$. On the other hand, whenever $\lambda < \kappa$ is an ordinal, then the structure $(V_\lambda, \in)$ has no indiscernibles of ordertype $\alpha$, and it remains such in every forcing extension by a wellfoundedness argument. Thus, the $\text{Coll}(\omega, V_\lambda)$-name for the generic isomorph of this structure is $E$-pinned, and it is not equivalent to any $E$-pinned name on a poset of size $|V_\lambda|$ since it entails the collapse of $|V_\lambda|$ to $\aleph_0$. Thus, $\kappa(E) = \kappa$ as desired. \hfill $\square$

Theorem 2.5.10. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $\kappa$ be a measurable cardinal. If $\kappa(E) < \infty$ then $\kappa(E) < \kappa$.

Proof. Suppose that there is a poset $P$ and an $E$-pinned name $\tau$ on $P$ which is not $\bar{E}$-related to any name on a poset of size $< \kappa$. We will produce a proper class of pairwise non-$\bar{E}$-related $E$-pinned names.

First note that the poset $P$ and the name $\tau$ can be selected so that $|P| = \kappa$. Simply take an elementary submodel $M$ of size $\kappa$ of large structure with $V_\kappa \subset M$ and consider $Q = P \cap M$ and $\sigma = \tau \cap M$; so $|Q| = \kappa$. As $M$ is correct about pinned names and the equivalence $\bar{E}$ by a Shoenfield absoluteness argument, $\sigma$ is an $E$-pinned name on $Q$ and it is not $\bar{E}$-equivalent to any pinned names on posets of size $< \kappa$.

Thus, assume that the poset $P$ has size $\kappa$. Let $j: V \to N$ be any elementary embedding into a transitive model with critical point equal to $\kappa$. Note that $H(\kappa) \subset N$ and so both $P, \tau$ are (isomorphic to) elements of $N$. Let $\langle N_\alpha, j_{\beta}\alpha: \beta \in \alpha \rangle$ be the usual system of iteration of the elementary embedding $j$ along the ordinal axis. Let $P_\alpha = j_{0\alpha}(P)$ and $\tau_\alpha = j_{0\alpha}(\tau)$. It will be enough to show that the pairs $\langle P_\alpha, \tau_\alpha \rangle$ for $\alpha \in \text{Ord}$ are pairwise $\bar{E}$-unrelated. To see this, pick ordinals $\alpha \in \beta$. As the original poset had size $\kappa$, it is the case that $P_\alpha, \tau_\alpha, P_\beta, \tau_\beta$ are in the model $N_\beta$. By the elementarity of the embedding $j_{\beta}\alpha$, $N_\beta \models \neg \langle P_\alpha, \tau_\alpha \rangle \bar{E} \langle P_\beta, \tau_\beta \rangle$. The wellfounded model $N_\beta$ is correct about $\bar{E}$ by a
Shoenfield absoluteness argument and so $\langle P_\alpha, \tau_\alpha \rangle \bar{E} \langle P_\beta, \tau_\beta \rangle$ fails also in $V$ as required.

Unlike the previous theorems in this section, we do not have a complementary example showing that the measurable cardinal bound is, at least to some extent, optimal.

The last theorem in this section provides an estimate of the $\lambda$ cardinal for unpinned equivalence relations. The well-known Silver dichotomy says that every Borel equivalence relation with uncountably many classes has in fact $2^{\aleph_0}$ many classes. We would like to show that every unpinned Borel equivalence relation has at least $2^{\aleph_1}$ many virtual classes. However, in ZFC this is still open. The best we can do is the following.

**Theorem 2.5.11.** Let $E$ be an unpinned Borel equivalence relation. Let $\kappa$ be an inaccessible cardinal. Then $\text{Coll}(\omega, \kappa) \models \lambda(E, \text{Coll}(\omega, \omega_1)) = 2^{\aleph_1}$.

**Proof.** Let $A \subset \mathcal{P}(\kappa)$ be a set of size $2^\kappa$ such that any two distinct elements of $A$ have intersection of size $< \kappa$. Recall that the poset $P = \text{Coll}(\omega, \kappa)$ is a finite support product of posets $\text{Coll}(\omega, \alpha)$ for $\alpha \in \kappa$. For every set $a \subset \kappa$ write $P_a = \Pi_{\alpha \in a} \text{Coll}(\omega, \alpha) \subset P$. Let $G \subset \text{Coll}(\omega, \kappa)$ be a generic filter. Use Theorem 2.6.1 to argue that for every $a \in A$, $V[G \cap P_a] \models E$ is unpinned, and let $\langle Q_a, \tau_a \rangle \in V[G \cap P_a]$ be a nontrivial $E$-pinned name with its attendant poset. It will be enough to show that the pairs $\langle Q_a, \tau_a \rangle$ for $a \in A$ represent pairwise distinct virtual classes.

Suppose towards contradiction that for some $a \neq b$ in the set $A$ the $E$-pinned names $\tau_a, \tau_b$ are equivalent. Let $c = a \cap b$ and work in the model $V[G \cap P_c]$. The assumptions imply that the poset $(P_{a \setminus c} * Q_a) \times (P_{b \setminus c} * Q_b)$ forces (at least below some condition) that $\tau_a \bar{E} \tau_b$. It follows that the name $\tau_a$ on the poset $P_{a \setminus c} * Q_a$ is $E$-pinned. Since the equivalence relation $E$ is Borel, it follows that the virtual class of $\langle P_{a \setminus c} * Q_a, \tau_a \rangle$ is also represented by some pair $\langle R, \sigma \rangle$ by some poset of size $< \sum_{\aleph_1} < \kappa$ by Theorem 2.5.6. However, in the model $V[G \cap P_a]$, there is a filter $H \subset R$ generic over the model $V[G \cap P_a]$. Thus, it would have to be the case that in the model $V[G \cap P_a]$, $Q_a \models \tau_a \bar{E} \sigma/H$, contradicting the nontriviality of the name $\tau_a$.

**Example 2.5.12.** Similarly to the Silver dichotomy, the conclusion fails for analytic equivalence relations. Consider the case of the equivalence relation $\mathbb{E}_{\omega_1}$ and an inaccessible cardinal $\kappa$ such that $2^\kappa > \kappa^+$. In the $\text{Coll}(\omega, \kappa)$ extension, $\aleph_2 < 2^{\aleph_1}$ will hold; at the same time, $\mathbb{E}_{\omega_1}$-names realized on $\text{Coll}(\omega, \omega_1)$ are classified by ordinals of cardinality $\aleph_1$ and so there are only $\aleph_2$-many of them.

### 2.5c Cardinal arithmetic examples

The cardinal invariants $\kappa(E)$ and $\lambda(E)$ provide a basis on which equivalence relations can be compared with uncountable cardinals of all sorts. In this section, we will introduce several jump operations which have direct translations into cardinal arithmetic operations. Using this approach, one can formally encode
statements such as the failure of the singular cardinal hypothesis into reducibility results between Borel or analytic equivalence relations. For brevity, given a coanalytic class $\Gamma$ of structures on $\omega$ invariant under isomorphism, we write $\kappa(\Gamma)$ for $\kappa(E_\Gamma)$ in this section. A possibly uncountable structure $M$ is a $\Gamma^{**}$-structure if Coll($\omega, |M|$) forces that there is a structure in $\Gamma$ isomorphic to $M$.

The constructions in this section depend on certain types of jump operators on structures and equivalence relation. They are all provisionally denoted by the $+$ sign, not to be confused with the Friedman–Stanley jump. The first jump operation on equivalence relations we will consider translates into the powerset operation using the pinned cardinal:

**Definition 2.5.13.** Let $\Gamma$ be a coanalytic class of structures on $\omega$, invariant under isomorphism. $\Gamma^+$ is the class of structures on $\omega$ of the following description: there is a partition $\omega = a \cup b$ into two infinite sets, there is a $\Gamma$-structure on $a$, and there is an extra relation $R$ on $b \times a$ such that the vertical sections $R_m$ for $m \in b$ are pairwise distinct subsets of $a$.

**Proposition 2.5.14.** Let $\Gamma$ be a coanalytic class of structures on $\omega$, invariant under isomorphism.

1. $\Gamma^+$ is coanalytic, and if $\Gamma$ is Borel then so is $\Gamma^+$;
2. if $\Gamma$ consists of rigid structures, then so does $\Gamma^+$;
3. if $\Gamma$ consists of rigid structures and $\kappa(\Gamma) = \lambda^+$ then $\kappa(\Gamma^+) = (2^\lambda)^+$.

*Proof.* The first two items are obvious. For (3), suppose that $\kappa(\Gamma) = \lambda^+$. Then there must be a $\Gamma^{**}$ structure $M$ of size $\lambda$ and no $\Gamma^{**}$ structures of larger size. Every $\Gamma^{**}$-structure consists of a $\Gamma^{**}$-structure and some family of its pairwise distinct subsets; the largest such structure then is of size $2^\lambda$ exactly. By Theorem 2.4.5 and (2), every virtual $E_\Gamma$-class is associated with a $\Gamma^{**}$ structure. This completes the proof.

**Example 2.5.15.** For every countable ordinal $\alpha$ there is a Borel class $\Gamma$ of rigid structures such that $\kappa(E_\Gamma) = \beth_{\alpha^+}$.

*Proof.* By transfinite recursion on $\alpha$ define Borel classes $\Gamma_\alpha$ consisting of rigid structures as follows. Let $\Gamma_0$ be the class of structures isomorphic to $\langle \omega, \in \rangle$. Let $\Gamma_{\alpha+1} = (\Gamma_\alpha)^+$. For a limit cardinal $\alpha$ let $\Gamma_\alpha$ be the class of structures which consist of exactly one copy of a structure in class $\Gamma_\beta$ for each $\beta \in \alpha$. It is not difficult to prove by induction on $\alpha$ using the Proposition 2.5.14 at the successor stage and Theorem 2.4.5 at the limit stage that $\kappa(E_\Gamma) = \beth_{\alpha^+}$ as desired.

**Definition 2.5.16.** Let $\Gamma$ be a coanalytic class consisting of structures on $\omega$ invariant under isomorphism. $\Gamma^+$ is the class of structures on $\omega$ of the following description. A structure $M \in \Gamma^+$ has a linear order $\leq$ and on cofinally many initial segments $\{m: m \leq n\}$ it has a $\Gamma$-structure $M_n$, so that for distinct elements $n_0, n_1 \in M$, the structures $\langle M_{n_0}, \leq \rangle$ and $\langle M_{n_1}, \leq \rangle$ are nonisomorphic.
Proposition 2.5.17. Let $\Gamma$ be a coanalytic class of structures on $\omega$, invariant under isomorphism.

1. $\Gamma^+$ is coanalytic;
2. if $\Gamma$ is Borel and consists of rigid structures, then $\Gamma^+$ is Borel and consists of rigid structures;
3. if $\Gamma$ consists of rigid structures, then $\kappa(\Gamma^+) = \kappa(\Gamma)$.

Proof. (1) is clear. For (2), the rigidity conclusion is clear. To see the Borelness of the class $\Gamma^+$ note that the statement “$\langle M, \leq_M \rangle$ is isomorphic to $\langle N, \leq_N \rangle$” for structures $M, N \in \Gamma$ is Borel by the rigidity of the structures in $\Gamma$ and the Lusin–Suslin theorem [45, Theorem 15.1]. For (3), write $\kappa = \kappa(\Gamma)$. To show that $\kappa(\Gamma^+) \leq \kappa^+$, use Theorem 2.4.5 to argue that every virtual $E_{\Gamma^+}$ class is classified by a generic $\Gamma^+$-structure. Such a structure contains a linear ordering, and on cofinally many initial segments of the ordering there is a generic $\Gamma$-structure. It follows that every initial segment of the ordering is of size $< \kappa$, and so the underlying set has size at most $\kappa$.

To show that $\kappa(\Gamma^+) \geq \kappa^+$, treat first the case that $\kappa$ is a limit cardinal. Let $M$ be the structure on $\kappa$ which includes the usual $\in$-ordering on $\kappa$ and on each cardinal $\lambda < \kappa$ on which there is a generic $\Gamma$-structure, $M$ contains one. This is a generic $\Gamma^+$-structure of size $\kappa$ and so $\kappa(\Gamma^+) \geq \kappa^+$.

Suppose now that $\kappa$ is a successor cardinal, $\kappa = \lambda^+$. This means that there is a generic $\Gamma$-structure $N$ of size $\lambda$. Let $M$ be the structure on $\kappa$ which includes the usual $\in$-ordering of $\kappa$ and on each ordinal $\mu < \kappa$ of cardinality $\lambda$, it contains a copy of $N$. This is a generic $\Gamma^+$-structure of size $\kappa$ and so $\kappa(\Gamma^+) \geq \kappa^+$ in this case as well. The proof is complete.

Example 2.5.18. For every countable ordinal $\alpha > 0$ there is a Borel equivalence relation $E$ such that (provably) $\kappa(E) = \aleph_1^\alpha$.

Proof. Now, by recursion on a countable ordinal $\alpha$ construct a Borel class $\Gamma_\alpha$ of rigid structures as follows. $\Gamma_0$ is the set of structures isomorphic to $\langle \omega, \in \rangle$, containing just one isomorphism class. Then let $\Gamma_{\alpha+1} = \Gamma_\alpha^+$ and $\Gamma_{\alpha} = \bigcup_{\beta \in \alpha} \Gamma_\beta$ if $\alpha$ is limit. Proposition 2.5.17 can be used to show by transfinite induction that $\kappa(\Gamma_\alpha) = \aleph_1^\alpha$, as desired.

Note that an equivalence relation $E$ as above for $\alpha \geq 2$ cannot be reducible to $\mathcal{F}_2$ and $\mathcal{F}_2$ cannot be reducible to it. This answers a question of Kechris [39, Question 17.6.1] in the negative as well as some related questions of Simon Thomas. To see that $\mathcal{F}_2$ cannot be Borel reducible to any $E$, suppose for contradiction that $h : \text{dom}(E) \rightarrow \text{dom}(\mathcal{F}_2)$ is a Borel reduction. Pass to a generic extension in which $\kappa > \aleph_\omega$. There, $h$ is still a reduction of $E$ to $F$, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 2.5.4. To see that $E$ cannot be reducible to $\mathcal{F}_2$ for any $\alpha > 2$, pass to a generic extension in which the Continuum Hypothesis holds instead.
The following two examples deal with jump operations designed to mimic cardinal exponentiation. They lead to nonreducibility results which, at least on the face of it, use the failure of singular cardinal hypothesis in various situations. This means that the proofs presented use large cardinal assumptions, as they are needed to get the failure of the singular cardinal hypothesis. We make no claim as to whether the large cardinal assumptions are necessary for the conclusion.

**Definition 2.5.19.** Let $\Gamma, \Delta$ be coanalytic classes of structures on $\omega$, invariant under isomorphism. The symbol $\Gamma^\Delta$ stands for the coanalytic class of structures $M$ on $\omega$ of the following form: $\omega$ is partitioned into infinite sets $\omega = a \cup b \cup c$, on $M \upharpoonright a$ is a structure in class $\Gamma$, $M \upharpoonright b$ is a structure in class $\Delta$, and there is an extra relation $R \subset c \times b \times a$ such that for every $m \in c$, the vertical section $R_m$ is a function from $b$ to $a$, and for $m_0 \neq m_1 \in c$ the vertical sections $R_{m_0}$ and $R_{m_1}$ are distinct.

**Proposition 2.5.20.** Suppose that the classes $\Gamma, \Delta$ consist of rigid structures only. Then

1. $\Gamma^\Delta$ consists of rigid structures only;
2. $\kappa(\Gamma^\Delta) = \sup\{\kappa^\lambda : \kappa < \kappa(\Gamma), \lambda < \kappa(\Delta)\}$.

**Proof.** The first item is nearly trivial. For the second item, use Theorem 2.4.5 to note that a $(\Gamma^\Delta)^{**}$ structure is represented by a $\Gamma^{**}$-structure, a $\Delta^{**}$-structure, and an infinite set of functions from the latter to the former. \qed

**Example 2.5.21.** Let $\Delta$ be the class of structures isomorphic to $\langle \omega, \in \rangle$ (consisting of just one equivalence class). For every countable ordinal $\alpha$, let $\Gamma_\alpha$ be the class derived in Example 2.5.18. Let $E$ be the equivalence relation of isomorphism on the class $(\Gamma_\omega)^\Delta$ and $F$ be the isomorphism equivalence relation on the class $E_{\Gamma_\alpha}$. Then $E$ is not Borel reducible to $F_2 \times F$.

**Proof.** Move to a model of ZFC where $\epsilon = \aleph_1$ and $\aleph_\omega^{\aleph_\omega} > \aleph_\omega$. Proposition 2.5.20 shows that $\kappa(E) = \aleph_\omega^{\aleph_\omega} > \epsilon \cdot \aleph_{\alpha} = \kappa(F_2 \times F)$. The conclusion of the example follows from Theorem 2.5.4. \qed

**Definition 2.5.22.** Let $E$ be an analytic equivalence relation on a Polish space $X$ and $F$ be a Borel equivalence relation on a Polish space $Y$. $E^F$ is the equivalence relation $(F \times E)^+$ on the space $(Y \times X)^\omega$ restricted to the Borel set \{z $\in (Y \times X)^\omega$ : $\text{rng}(z)$ is a partial function from $Y$ to $X$ whose domain consists of pairwise $F$-unrelated elements\}.

It seems to be impossible to formulate this concept in an analytic form without the additional demand that $F$ be Borel.

**Proposition 2.5.23.** Let $E$ be an analytic equivalence relation on a Polish space $X$ and $F$ be a Borel equivalence relation on a Polish space $Y$. Then

1. $\lambda(E^F) = \lambda(E)^{\lambda(F)}$;
2. If \( P \) is a partial order such that \( P \Vdash |\lambda(F)| = |\kappa(F)| = \aleph_0 \) then \( \lambda(E^F, P) = \lambda(E, P)^{\lambda(F)} \).

Note that as \( F \) is assumed to be Borel, the value of \( \lambda(F) \) is not \( \infty \).

**Proof.** Both statements follow from the classification of virtual classes for product and Friedman–Stanley jump in Theorem 2.3.4. An \( E_F \)-virtual class is represented by a function from \( F \)-virtual classes to \( E \)-virtual classes.

**Example 2.5.24.** Let \( F_0 \) be any Borel equivalence relation with countably many classes and let \( F_1 \) be the identity on \( 2^\omega \). For every Borel equivalence relation \( E \), \( E \times F_0^{\omega_1} \) is not Borel reducible to \( E \times F_1^{\omega_1} \).

**Proof.** Move to a model of set theory where \( \text{cf} = \aleph_1 \) and there is a cardinal \( \kappa > \aleph_{\omega_1} \) such that \( (\kappa^+)^{\aleph_1} > (\kappa^+)^{\aleph_0} \). The first such a cardinal must be in violation of the singular cardinal hypothesis at cofinality \( \omega_1 \). Let \( P = \text{Coll}(\omega, \kappa) \). Clearly, \( \lambda(E^{\omega_1}, P) = \kappa^+ \), since the pinned names on \( P \) correspond to ordinals below \( \kappa^+ \). Since \( E \) is Borel, \( \lambda(E), \kappa(E) < \aleph_{\omega_1} < \kappa \) by Theorem 2.5.6 and so Proposition 2.5.23 shows that \( \lambda(E \times E^{\omega_0}_1) = \lambda(E) \cdot (\kappa^+)^{\aleph_0} = (\kappa^+)^{\aleph_0} < (\kappa^+)^{\aleph_1} = \lambda(E^{\omega_1}_0) \). The argument is concluded by a reference to Theorem 2.5.4.

A similar argument with a failure of the singular cardinal hypothesis at larger cofinalities yields

**Example 2.5.25.** Let \( F \) be any Borel equivalence relation. For every Borel equivalence relation \( E \), \( E^{\omega_1}_1 \) is not Borel reducible to \( E \times E^{\omega_1}_1 \).

### 2.5d Hypergraph examples

The previous examples were to some extent artificial in the sense that the values of the pinned cardinal were directly built into them. The following examples, all Borel reducible to \( \mathbb{F}_2 \), are connected with combinatorics of small uncountable cardinals via the pinned cardinal, even though the connection is not at first sight obvious.

**Definition 2.5.26.** A hypergraph on a set \( X \) is a subset of \( X^{\leq \omega} \). If \( G \) is a hypergraph on \( X \), a \( G \)-anticlique is a set \( A \subset X \) such that \( A^{\leq \omega} \cap G = 0 \). If \( G \) is an analytic hypergraph on a Polish space \( X \), write \( E_G \) for the equivalence relation on \( X^\omega \) connecting \( y, z \) if the sets \( \text{rng}(y), \text{rng}(z) \) either both fail to be \( G \)-anticliques or they are equal.

It is clear that the equivalence relations of the form \( E_G \) are all analytic and almost Borel reducible to \( \mathbb{F}_2 \). In the common case when \( G \) is Borel and contains only finite edges, the equivalence relation is in fact Borel. A simple Mostowski absoluteness argument shows that every (even uncountable) \( G \)-anticlique remains such in every forcing extension. By Example 2.3.5, the virtual \( E_G \)-classes are classified by \( G \)-anticliques, and so \( \kappa(E_G) = \sup\{|A|^+ : A \subset X\} \) is
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a $\mathcal{G}$-anticlique}. The values of cardinals defined in this way are subject to forcing manipulations and transfinite combinatorics. The interesting examples are always connected with a universality feature of the hypergraph in question.

Our first two examples use Borel hypergraphs of finite arity.

**Definition 2.5.27.** Let $X$ be a Polish space. A Borel relation $R \subseteq [X]^{<\aleph_0} \times X$ is **combinatorially universal** if it has countable vertical sections and for every cardinal $\kappa$ and every relation $T \subseteq \kappa^{<\aleph_0} \times \kappa$ with countable vertical sections, there is a c.c.c. poset $P$ such that $P \Vdash$ there is an injective function $\pi: \kappa \to X$ which is a homomorphism of $T$ to $R$.

It is not clear whether combinatorial universality of this sort is actually a property absolute among transitive models of ZFC, but all universal examples found in this section are absolutely universal.

**Theorem 2.5.28.** Let $X$ be a Polish space. There is a combinatorially universal Borel relation on $[X]^{<\aleph_0} \times X$ with countable vertical sections.

**Proof.** Let $X = \mathcal{P}(\omega)$. We will show that the relation $R \subseteq [X]^{<\aleph_0} \times X$ defined by $(a, x) \in R$ if $x$ is computable from $a$ is universal. Let $\kappa$ be a cardinal and let $T \subseteq [\kappa]^n \times \kappa$ be a relation with countable vertical sections. Let $(k_m : m \in \omega)$ be a recursive sequence of increasing functions in $\omega^\omega$ with disjoint ranges. For a finite set $b \subseteq \mathcal{P}(\omega)$ let $e_b$ be the increasing enumeration of the set $\bigcap b$, for every $m \in \omega$ let $h_m(b) \subseteq \mathcal{P}(\omega)$ be the set of all $l \in \omega$ such that $e_b \circ k_m(l)$ is an odd number. We will produce a forcing which adds an injection $\pi: \kappa \to X$ such that for every finite set $a \subseteq \kappa$ and every $\alpha \in T_a$, there is a number $m \in \omega$ such that $\pi(\alpha)$ is modulo finite equal to $h_m(\pi'' a)$.

Let $P$ be the poset of all tuples $p = (n_p, \pi_p, \nu_p)$ so that

- $n_p \in \omega$, $\pi_p$ is a partial function from $\kappa$ to $\mathcal{P}(n_p)$ with finite domain $\text{dom}(p)$;
- $\nu_p$ is a finite partial function from $\mathcal{P}(\text{dom}(p)) \times \omega$ to $\text{dom}(p)$ such that $(a, \nu_p(a, m)) \in T$ whenever $(a, m) \in \text{dom}(\nu_p)$.

The ordering on $P$ is defined by $q \leq p$ if $n_q \leq n_p$, $\text{dom}(p) \subseteq \text{dom}(q)$, $\forall \alpha \in \text{dom}(p)$ $\pi_p(\alpha) = \pi_q(\alpha) \cap n_p$, $\nu_p \subseteq \nu_q$, and for every $(a, m) \in \text{dom}(\nu_p)$, whenever $l$ is a number in the domain of $(e_{\pi'' a} \setminus e_{\pi'' b}) \circ k_m$ then $e_{\pi'' a} \circ k_m(l)$ is odd if and only if $l \in \pi_q(\nu_p(a, m))$. It is not difficult to see that $P$ is indeed an ordering.

**Claim 2.5.29.** The poset $P$ is c.c.c.

**Proof.** Let $\langle p_\alpha : \alpha \in \omega_1 \rangle$ be conditions in $P$. The usual $\Delta$-system and counting arguments can be used to thin down the collection if necessary so that the sets $\text{dom}(p_\alpha)$ for $\alpha \in \omega_1$ form a $\Delta$-system with root $b$ and for all $a \in |b|^\omega$ and all $\alpha \in \omega_1$, $T_a \cap \text{dom}(p_\alpha) \subseteq b$. Moreover, we can require that the increasing bijection between $\text{dom}(p_\alpha)$ and $\text{dom}(p_\beta)$ extends to an isomorphism of $p_\alpha$ and $p_\beta$ for every $\alpha, \beta \in \omega_1$. 
We claim that any two conditions in such a collection are compatible. Indeed, whenever \( \alpha, \beta \in \omega_1 \), then the condition \( q \) defined by \( n_q = n_{q_0}, \pi_q = \pi_{p_\alpha} \cup \pi_{p_\beta} \) and \( \nu_q = \nu_{p_\alpha} \cup \nu_{p_\beta} \) is easily checked to be a common lower bound of the conditions \( p_\alpha, p_\beta \).

\[ \text{Claim 2.5.30.} \quad \text{Whenever} \ a \in [\kappa]^n \ \text{and} \ \beta \in T_a, \ \text{the set} \ D_{a, \beta} = \{p \in P: a \cup \{\beta\} \subset \text{dom}(p), \exists m \ \nu_p(a, m) = \beta\} \ \text{is dense in} \ P. \]

\[ \text{Proof.} \ \text{Let} \ p \in P; \ \text{we must find a condition} \ q \leq p \ \text{in the set} \ D_{a, \beta}. \ \text{For definiteness assume that} \ \beta \notin \text{dom}(p). \ \text{Choose} \ m \in \omega \ \text{such that} \ \langle a, m \rangle \notin \text{dom}(\nu_p). \ \text{Consider the condition} \ q \leq p \ \text{defined by} \ n_q = n_p, \ \pi_q = \pi_p \cup \{(\alpha, 0): \alpha \in a \setminus \text{dom}(p), (\beta, 0)\}, \ \nu_q = \nu_p \cup \{(a, m, \beta)\}. \ \text{The condition} \ q \leq p \ \text{is in the set} \ D_{a, \beta} \ \text{as required.} \]

\[ \text{Claim 2.5.31.} \quad \text{For every finite} \ a \subset \kappa \ \text{and every} \ k \in \omega, \ \text{the set} \ D_{a, k} = \{p \in P: a \subset \text{dom}(p) \ \text{and the set} \ \bigcap \pi_p^a \ \text{has at least} \ k \ \text{elements}\} \ \text{is dense in} \ P. \]

\[ \text{Proof.} \ \text{Fix} \ a, k \ \text{and let} \ p \in P \ \text{be an arbitrary condition. We must find a condition} \ q \leq p \ \text{in the set} \ D_{a, k}. \ \text{First of all, the previous claim shows that one can strengthen} \ p \ \text{to include all ordinals in} \ a. \ \text{Increasing} \ n_p, \ \text{if necessary, we may also assume that} \ k < n_p. \]

\text{Consider the set} \ b = \pi_p^a \ \text{and the function} \ e_b; \ \text{write} \ k' = \text{dom}(e_b). \ \text{If} \ k \leq k' \ \text{then} \ q = p \ \text{will work. Otherwise, it is easy to find an increasing sequence} \ d = \langle m_i: k' \leq i < k \rangle \ \text{of numbers larger than} \ n_q, \ \text{such that, writing} \ e = e_b \cup d, \ \text{for every natural number} \ m \ \text{such that} \ \langle a, m \rangle \in \text{dom}(\nu_p) \ \text{and every} \ l \ \text{such that} \ k' \leq k_m(l) < k, \ m_{k_m(l)} \ \text{is odd if and only if} \ l \in p(\nu_r(a, m)). \ \text{The condition} \ q \leq p \ \text{defined by} \ n_q = m_{k-1} + 1, \ \text{dom}(\pi_q) = \text{dom}(\pi_p), \ \forall \beta \in a \ \pi_q(\beta) = \pi_p(a) \cup \{m_i: k' \leq i < k\}, \ \forall \beta \in \text{dom}(\pi_p) \ \text{and} \ a \ \pi_q(\beta) = \pi_p(\beta), \ \text{and} \ \nu_q = \nu_p, \ \text{is in the set} \ D_{a, k} \ \text{as desired.} \]

The last two claims show that the function \( \pi: \kappa \rightarrow X \) defined as \( \pi(\alpha) = \bigcup \{\pi_p: p \text{ is in the generic filter}\} \) is forced by \( P \) to be the desired homomorphism.

Recall that if \( R \subset [X]^n \times X \) is a relation then \( a \subset X \) is \( R \)-free if for every \( x \in a \) \( \langle a \setminus \{x\}, x \rangle \notin R. \)

\[ \text{Example 2.5.32.} \quad \text{Let} \ X \ \text{be a Polish space and let} \ R \subset [X]^<n \times X \ \text{be a combinatorially universal Borel relation with countable vertical sections. Let} \ n \geq 1 \ \text{be a number. Let} \ G_n \ \text{be the hypergraph of all sets} \ a \in [X]^n \ \text{which are} \ R \text{-free. Then} \]

1. \( \kappa(E_{G_n}) \leq \aleph_{n+1}; \)
2. if Martin’s Axiom for \( \aleph_n \) holds then equality is attained.
Proof. Fix the number \( n \in \omega \). The argument depends on an old theorem of Sierpiński [27]: \( \aleph_{n+1} \) is the smallest cardinal \( \kappa \) such that every relation \( T \subseteq [\kappa]^n \times \kappa \) with countable vertical sections has a \( T \)-free \( n+1 \)-tuple. To prove (1), apply the Sierpiński theorem to show that there is no \( G_n \)-anticlique of size \( \geq \aleph_{n+1} \). To prove (2), let \( \kappa = \aleph_n \) and use the Sierpiński theorem again to find a relation \( T \subseteq \kappa^{<\aleph_0} \times \kappa \) with countable vertical sections and no \( T \)-free \( n \)-tuple. Then use the universality assumption to find an injective homomorphism \( \pi: \kappa \to X \) which is a homomorphism of \( T \) to \( R \). Observe that \( \text{rng}(\pi) \) is a \( G_n \)-anticlique of size \( \kappa \), representing a \( \kappa(E_{G_n}) \)-pin which is realized only by posets collapsing \( \kappa \) to \( \aleph_0 \).

Definition 2.5.33. Let \( X \) be a Polish space. A Borel equivalence relation \( R \) on \( [X]^{<\aleph_0} \) with countably many classes is combinatorially universal if for every cardinal \( \kappa \) and every equivalence relation \( T \) on \( [\kappa]^{<\aleph_0} \) with countably many classes, there is a c.c.c. poset \( P \) such that \( P \Vdash \text{there is an injection } \pi: \kappa \to X \) which is a homomorphism of \( \neg T \) to \( \neg R \).

Theorem 2.5.34. Let \( X \) be a Polish space. There is a universal Borel equivalence relation on \( [X]^{<\aleph_0} \) with countably many classes.

Proof. Without loss of generality assume \( X = [\omega]^{\aleph_0} \). Consider the following relation \( R \) on \( [X]^{<\aleph_0} \). Define a Borel function \( g: [X]^{<\aleph_0} \to [\omega]^{<\aleph_0} \) by \( g(a) = \{ \min(x \setminus m + 1): x \in a \} \) if \( a \) is a set of size at least two and consists of pairwise almost disjoint sets and \( m \) is the largest number which appears in at least two of them; \( g(a) = \min(x) \) if \( a = \{ x \} \) is a singleton; and otherwise \( g(a) = 0 \). Let \( R \) be the equivalence relation induced by the function \( g \). We will show that \( R \) is universal.

Let \( \kappa \) be a cardinal, \( T \) an equivalence relation on \( [\kappa]^{<\aleph_0} \) with countably many classes, and let \( f: [\kappa]^{<\aleph_0} \to [\omega]^{<\aleph_0} \) be a map inducing the equivalence relation \( T \). Let \( \nu: [\omega]^{<\aleph_0} \to \omega \) be a sufficiently generic map such that \( \nu(g(0)) = f(0) \). Let \( P \) be the poset of all maps \( p \) such that

- \( \text{dom}(p) \subseteq \kappa \) is a finite set;
- for every \( \alpha \in \text{dom}(p) \) the value \( p(\alpha) \) is a nonempty subset of \( \omega \);
- for every \( \alpha \in \text{dom}(p) \), \( \nu(\min(p(\alpha))) = f(\alpha) \);
- for every set \( a \subseteq \text{dom}(p) \) of size at least 2 there is a number which belongs to at least two sets \( p(\alpha), p(\beta) \) for \( \alpha \neq \beta \in a \), and writing \( m \) for the largest such number, \( p(\alpha) \setminus m + 1 \neq 0 \) holds for every \( \alpha \in a \), and \( \nu(\{ \min(p(\alpha)) \setminus m + 1: \alpha \in a \}) = f(a) \).

The ordering is defined by \( q \leq p \) if for every \( \alpha \in \text{dom}(p) \), \( q(\alpha) \) end-extends \( p(\alpha) \), and the sets \( q(\alpha) \setminus p(\alpha) \) are pairwise disjoint for \( \alpha \in \text{dom}(p) \).

Claim 2.5.35. \( P \) is c.c.c.
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Proof. In fact, $P$ is semi-Cohen in the sense of [4], but we will not need that fact here. By the usual $\Delta$-system arguments, it is enough to show that any two conditions $p, q \in P$ such that $p \upharpoonright \text{dom}(p) \cap \text{dom}(q) = q \upharpoonright \text{dom}(p) \cap \text{dom}(q)$, are compatible. To find the lower bound, enumerate $\text{dom}(p) \cup \text{dom}(q)$ as $\beta_i$ for $i \in k$, enumerate $(\text{dom}(p) \setminus \text{dom}(q)) \times (\text{dom}(q) \setminus \text{dom}(p))$ as $u_j$ for $j \in l$. Use the genericity of the function $\nu$ to find numbers $m_0 < m_1 < \cdots < m_{l-1}$ and pairwise distinct numbers $n^i_j$ for $i \in k$ and $j \in l$ so that

- $m_0 > \max(\bigcup \text{rng}(p) \cup \bigcup \text{rng}(q));$
- $m_j < n^i_j < m_{j-1}$ for every $j \in l$;
- for every set $a \subset \text{dom}(p) \cup \text{dom}(q)$, $\nu(\{n^i_j : \beta_i \in a\}) = f(a)$.

The lower bound is then a function $r$ defined by $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$, for $\alpha \in \text{dom}(p)$, $\alpha = \beta_i$ set $r(\alpha) = p(\alpha) \cup \{n^i_j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. Similarly, for $\alpha \in \text{dom}(q)$, $\alpha = \beta_i$ set $r(\alpha) = q(\alpha) \cup \{n^i_j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. It is not difficult to check that $r \leq p, q$ as required. \hfill $\Box$

Claim 2.5.36. The set $D_\alpha = \{p \in P : \alpha \in \text{dom}(p)\}$ is dense in $P$ for every $\alpha \in \kappa$.

Proof. Let $\alpha \in \kappa$ and $p \in P$; we must produce $q \leq p$ such that $\alpha \in \text{dom}(q)$. Enumerate $\text{dom}(p)$ as $\beta_i$ for $i \in k$ and write $\alpha = \beta_k$. Use the genericity of the function $\nu$ to find numbers $m < m_0 < m_1 < \cdots m_{k-1}$ and pairwise distinct numbers $n^i_j$ for $i \in k, j \in k+1$ so that

- $\nu(m) = f(\alpha)$ and $m_0 > \max(\bigcup \text{rng}(p));$
- $m_j < n^i_j$ for every $j \in k+1$;
- for every nonempty set $a \subset \text{dom}(p) \cup \{\alpha\}$ and every $i \in k$, $\nu(\{n^i_j : \beta_j \in a\}) = f(a)$.

Once this is done, just consider the function $q$ defined by $\text{dom}(q) = \text{dom}(p) \cup \{\alpha\}$, $s(\alpha) = \{m, m_i : i \in k, n^i_j : i \in k\}$, and for every $i \in k$, $q(\beta_i) = p(\beta_i) \cup \{m_i, n^i_j : j \in k\}$. It is not difficult to observe that $q \in P$ and $q \leq p$ as desired. \hfill $\Box$

Now, if $G \subset P$ is a generic filter, then in $V[G]$ let $\pi : \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{p \in G} p(\alpha)$. The claims show that $f = \nu \circ g \circ \pi$, in particular $\pi$ is a homomorphism of $-T$ to $-R$. \hfill $\Box$

Example 2.5.37. Let $X$ be a Polish space and $R$ a universal equivalence relation on $[X]^{|\aleph_0|}$ with countably many classes. Let $n > 0$ be a number. Let $\mathcal{G}_n$ be the Borel hypergraph on $X$ consisting of all finite sets $a \subset X$ which can be written in more than $2^n - 1$ ways as a union of two distinct $R$-equivalent sets. Then

1. $\kappa(E_{\mathcal{G}_n}) \leq \aleph_{n+1}$.
2. If Martin’s Axiom for $\aleph_n$ holds, then the equality is attained.

**Proof.** Fix the number $n$. The computations depend on and are motivated by a theorem of Komjáth and Shelah [48]. Let $\kappa = \aleph_n$. For every equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes, there is a finite set $a \subset \kappa$ which can be written in at least $2^n - 1$ ways as a union of two distinct $T$-related sets. In addition, if Martin’s Axiom for $\kappa$ holds then there is an equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes such that every finite set $a \subset \kappa$ can be written in at most $2^n - 1$ ways as a union of two distinct $T$-related sets.

For (1), the first part of this result shows that there is no $G_n$-anticlique of size $\kappa^+$. For (2), use Martin’s Axiom and the second part of the Komjáth–Shelah result to find the equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes as above, and use the universality of the relation $R$ to find an injective homomorphism $\pi: \kappa \to X$ of $\neg T$ to $\neg R$. It is immediate that $\text{rng}(\pi) \subset X$ is a $G_n$-anticlique of size $\kappa$, and therefore associate with a virtual $E_{\psi_n}$-class realized only on posets which collapse $\kappa$ to $\aleph_0$.

Much more complicated effects can be realized if the hypergraph $G$ is allowed to have infinite edges. We conclude this section with one more example of this type.

**Definition 2.5.38.** Let $X$ be a Polish space. A Borel equivalence relation $R$ on $X^2$ with countably many classes is **combinatorially universal** if for every ordinal $\kappa$ and every equivalence relation $T$ on $\kappa^2$ with countably many classes, there is a c.c.c. poset $P$ such that $P \forces$ there is an injective function $\pi: \kappa \to X$ which is a homomorphism of $\neg T$ to $\neg R$.

**Theorem 2.5.39.** There is a combinatorially universal Borel equivalence relation with countably many classes on $X^2$ for every Polish space $X$.

**Proof.** Without loss of generality, assume $X = \mathcal{P}(\omega)$. Let $\omega = \bigcup_{n,m \in \omega} a_{n,m}$ be a partition of $\omega$ into infinite sets. For almost disjoint sets $b, c \subset \omega$ such that $b$ is lexicographically less than $c$ define $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$, in other cases define $f(b, c) = 0$. Let $R$ be the equivalence relation induced by the function $f$. We will show that $R$ is combinatorially universal.

Fix a cardinal $\kappa$ and a function $g: \kappa^2 \to \omega$ which induces an equivalence relation $T$ with countably many classes. Define the poset $P$ as the collection of all functions $p$ such that

- $\text{dom}(p) \subset \kappa$ is a finite set;
- $\text{rng}(p)$ consists of finite subsets of $\omega$ such that neither of them is an initial segment of another;
- for every $\alpha \neq \beta$ such that $p(\alpha)$ is lexicographically smaller than $p(\beta)$, the set $p(\alpha) \cap p(\beta)$ is nonempty, and its maximum belongs to the set $a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$. 

The ordering on $P$ is defined by $q \leq p$ if $\text{dom}(p) \subseteq \text{dom}(q)$, for every $\alpha \in \text{dom}(p)$ the set $p(\alpha)$ is an initial segment of $q(\alpha)$, and the sets $\{q(\alpha) \setminus p(\alpha): \alpha \in \text{dom}(p)\}$ are pairwise disjoint. The following routine claims complete the proof of the theorem.

**Claim 2.5.40.** The poset $P$ is c.c.c.

**Proof.** By the usual $\Delta$-system arguments it is only necessary to show that any two conditions $p, q \in P$ such that $p \upharpoonright \text{dom}(p) \cap \text{dom}(q) = q \upharpoonright \text{dom}(p) \cap \text{dom}(q)$ are compatible in the poset $P$. Strengthening the conditions $p, q$ on $\text{dom}(p) \setminus \text{dom}(q)$ and $\text{dom}(p) \setminus \text{dom}(q)$ respectively if necessary, we may assume that no set in $\text{rng}(p) \cup \text{rng}(q)$ is an initial segment of another. Enumerate $(\text{dom}(p) \setminus \text{dom}(q)) \times (\text{dom}(q) \setminus \text{dom}(p))$ as $u_i$ for $i \in j$ and find pairwise distinct numbers $m_i$ for $i \in j$ such that

- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically smaller than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$;
- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically greater than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = n$ and $g(\beta, \alpha) = m$;
- all numbers $m_i$ are greater than $\max(\bigcup \text{rng}(p) \cup \bigcup \text{rng}(q))$.

In the end, let $r$ be the function defined by $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$, for all $\alpha \in \text{dom}(p)$ let $r(\alpha) = p(\alpha) \cup \{m_i: \alpha \text{ appears in } u_i\}$, and for all $\beta \in \text{dom}(q)$ let $r(\beta) = q(\beta) \cup \{m_i: \beta \text{ appears in } u_i\}$. It is not difficult to check that $r$ is a common lower bound of the conditions $p, q$ as desired. □

**Claim 2.5.41.** For every $\alpha \in \kappa$ the set $D_\alpha = \{p \in P: \alpha \in \text{dom}(r)\}$ is dense in $P$.

**Proof.** Let $\alpha \in \kappa$ be an ordinal and $p \in P$ be a condition; we must produce a condition $q \leq p$ such that $\alpha \in \text{dom}(q)$. It will be the case that $q(\alpha) \cap \max(\bigcup \text{rng}(p)) + 1 = 0$; this way, $q(\alpha)$ will be lexicographically smaller than all $q(\beta)$ for $\beta \in \text{dom}(p)$. List $\text{dom}(p)$ as $\beta_i$ for $i \in j$, and find pairwise distinct numbers $m_i$ for $i \in j$ so that

- $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\alpha, \beta) = n$;
- each $m_i$ is greater than $\max(\bigcup \text{rng}(p))$.

Then, let $q$ be the function defined by $\text{dom}(q) = \text{dom}(p) \cup \{\alpha\}$ and $q(\beta_i) = p(\beta_i) \cup \{m_i\}$ and $q(\alpha) = \{m_i: i \in j\}$. It is immediate that the condition $q$ works. □

Now it is easy to see that if $G \subseteq P$ is a generic filter, the function $\pi: \kappa \rightarrow \mathcal{P}(\omega)$ defined by $\pi(\alpha) = \bigcup\{p(\alpha): p \in G\}$ induces a homomorphism of $\neg T$ to $\neg R$ as desired. □
Example 2.5.42. Let $X$ be a Polish space and let $R$ be a combinatorially universal Borel equivalence relation on $X^2$ with countably many classes. Let $\mathcal{G}$ be the hypergraph on $X$ consisting of all unions $b_0 \cup b_1$ where $b_0, b_1$ are infinite sets such that $b_0 \times b_1$ is a subset of a single $R$-class.

1. if Chang’s conjecture holds, then $\kappa(E_{\mathcal{G}}) \leq \aleph_2$;
2. if Martin’s Axiom for $\aleph_2$ holds then Chang’s conjecture is equivalent to $\kappa(E_{\mathcal{G}}) \leq \aleph_2$.

Proof. The argument is based on and motivated by two results of Todorcevic [78]. Namely, if Chang’s conjecture holds, then for every partition of $\omega_2^2$ into countably many pieces, one piece of the partition contains a product of infinite sets. In addition, if Martin’s Axiom for $\aleph_2$ holds and Chang’s conjecture fails, then there is a partition of $\omega_2^2$ into countably many pieces such that no piece of the partition contains a product of infinite sets.

Now, for (1) use the first result of Todorcevic to argue that under Chang’s conjecture there is no $\mathcal{G}$-anticlique of size $\geq \aleph_2$. For (2), suppose that Martin’s Axiom holds and Chang’s conjecture fails. By the second result of Todorcevic, find an equivalence relation $T$ on $\omega_2^2$ with countably many classes without any monochromatic infinite rectangles. Use the universality of the equivalence relation $R$ to find an injection $\pi: \omega_2 \to X$ which is a homomorphism of $\neg T$ to $\neg R$. Observe that $\text{rng}(\pi)$ is a $G$-anticlique of size $\aleph_2$ which therefore identifies a virtual $E_G$-class realized only on posets which collapse $\aleph_2$ to $\aleph_0$. 

2.6 Absoluteness

This section compiles the available information regarding the absoluteness of various notions regarding the virtual equivalence classes. The most substantial and useful statement is Corollary 2.6.3: if $E$ is a Borel equivalence relation then its pinned status is absolute among all generic extensions.

Theorem 2.6.1. Suppose that $E$ is a Borel equivalence relation on a Polish space $X$. The following are equivalent:

1. $E$ is pinned;
2. For every $\omega$-model $M$ of ZFC containing the code for $E$, $M \models E$ is pinned.

Proof. For simplicity assume $X = \omega^\omega$. The implication (2)$\rightarrow$(1) is trivial. If (1) fails, then $E$ is not pinned, and the failure of (2) is witnessed by $M = V$.

The implication (1)$\rightarrow$(2) is more difficult. We start with an abstract claim which can be used in several other absoluteness results. Say that an $\omega$-model $N$ of ZFC is correct about stationary sets if for all sets $a, b \in N$, if $N \models a$ is uncountable then in fact $a$ is uncountable, and if $N \models b \subset [a]^{\aleph_0}$ is a stationary system of countable sets, then in fact $b$ is a stationary system of countable sets. Note that as $N$ is an $\omega$-model, for any set $c$, if $N \models c$ is countable then in fact $c$ is countable.
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Claim 2.6.2. Let $M$ be a countable $\omega$-model of ZFC. Then there is an elementary embedding $j: M \to N$ such that $N$ is correct about stationary sets.

Proof. We start with setting up a bookkeeping tool. By a classical result of Solovay, $\omega_1$ can be decomposed into $\aleph_1$ many stationary sets. Thus, there is a surjective function $f: \omega_1 \to \omega_1 \times \omega$ such that preimages of singletons are stationary.

Now, by recursion on $\alpha \in \omega_1$ build countable $\omega$-models $M_\alpha$ of ZFC and maps $j_\beta: M_\gamma \to M_\alpha$ for all $\beta \in \alpha$. With each model $M_\alpha$ we also pick a surjection $\pi_\alpha: \omega \to M_\alpha$ arbitrarily. The following are the recursive demands on the construction:

- $M = M_0$ and $\langle j_\gamma: M_\gamma \to M_\beta: \gamma \in gb \in \alpha \rangle$ is a commuting system of elementary embeddings;
- if $\alpha \in \omega_1$ is limit then $M_\alpha$ is the direct limit of the previous models and embeddings;
- whenever $\alpha \in \omega_1$ is an ordinal such that $f(\alpha) = \langle \beta, n \rangle$ and $\beta \in \alpha$ and $M_\beta \models b = \pi_\beta(n)$ is a stationary system of countable sets on $a = \bigcup b$, then writing $c = j_\alpha^\beta(j_\beta^\alpha)(a)$ we get that $M_{\alpha+1}$ contains $c$ and $M_{\alpha+1} \models c \in j_\beta^\alpha(b)$.

It is only necessary to show how the last item is arranged. Thus, suppose that its assumptions are satisfied at $\alpha \in \omega_1$. Work in the model $M_\alpha$ and consider the poset $P$ of all stationary subsets of $b$, ordered by inclusion. Since the model $M_\alpha$ is countable, there is a filter $G \subset P$ generic over the model $M_\alpha$. Let $j_{\alpha+1}: M_\alpha \to M_{\alpha+1}$ be its associated generic ultrapower as in \cite[Lemmas 22.13 and 14]{36}. Then $M_{\alpha+1}$ will be again an $\omega$-model. Note that the set $c$ belongs to the model $M_{\alpha+1}$ as it is represented by the identity function; it also has the requested properties by the Loś theorem.

In the end, let $j: M \to N$ be the direct limit of the system of elementary embeddings produced during the recursion; we claim that this embedding works as required. Suppose then that $M \models b$ is a stationary system of countable sets on $a = \bigcup b$. To show that $b$ is indeed stationary, let $d$ be a closed unbounded set of countable subsets of $a$, and work to show that $d \cap b \neq \emptyset$ holds. Fix an ordinal $\beta \in \omega_1$ and a number $n \in \omega$ such that $b = j_{\beta\omega_1}(\pi_\beta(n))$. Let $K$ be a countable elementary submodel of a large substructure, containing in particular the set $d$, the ordinal $\beta$, and the system of elementary embeddings constructed above, and such that $f(\alpha) = \langle \beta, n \rangle$ where $\alpha = K \cap \omega_1$. Such a model exists by the initial choice of the function $f$. Clearly $K \cap a \in d$; we will be done once we show that $K \cap a \in b$.

To see this, let $c = j_{\alpha+1}^\beta j_\beta^\alpha(\bigcup \pi_\beta(n))$. By the elementarity of the embedding $j_{\alpha+1}\omega_1$ and the last item above, it is the case that $j_{\alpha+1}\omega_1(c) \in b$. Also, $j_{\alpha+1}\omega_1(c) = j_{\alpha+1}\omega_1 c = j_{\alpha+1}\omega_1 \pi_\beta(n)$. By the elementarity of the model $K$ and the second item above, the latter expression is exactly equal to $K \cap j_{\beta\omega_1}(\bigcup \pi_\beta(n)) = K \cap a$. This completes the proof. \qed
Now, back to the theorem. Suppose that (2) fails. Fix a \( \omega \)-model \( M \) of ZFC containing the code for \( E \) such that \( M \models E \) is unpinned; taking an elementary submodel if necessary we may assume that \( M \) is countable. Let \( j: M \rightarrow N \) be an elementary embedding from the claim. By elementarity, \( N \models E \) is unpinned, and so there is a poset \( P \) and a \( P \)-name \( \tau \) such that \( N \models \tau \) is a nice nontrivial \( E \)-pinned name. We will show that in fact \( \tau \) really is a nontrivial \( E \)-pinned name.

First of all, \( \tau \) indeed is \( E \)-pinned. If \( G_0, G_1 \subset P \) are mutually generic filters then \( N[G_0, G_1] \models \tau/G_0 \models E \tau/G_1 \) holds, and by the Borel absoluteness between \( N[G_0, G_1] \) and \( V[G_0, G_1] \), \( \tau/G_0 \models E \tau/G_1 \) holds in \( V[G_0, G_1] \) as required. To show that \( \tau \) is a nontrivial \( E \)-pinned name, assume towards contradiction that there is a point \( x \in X \) such that \( P \models \tau \mathrel{E} \check{x} \); the difficulty resides in the possibility that \( x \notin N \).

Let \( \kappa \) be some large cardinal in the model \( N \). Let \( K \) be a countable elementary submodel of a large structure containing in particular \( N, P, \tau, \) and \( \kappa \), such that \( c = K \cap N \cap V_\kappa \) belongs to the model \( N \) and is countable there. Such a model \( K \) exists since the model \( N \) is correct about the stationarity of the set it views as \( [V_\kappa]^{\aleph_0} \). Work in the model \( N \). Since \( c \) is an elementary submodel of \( V_\kappa \), it follows that \( \sigma = \tau \cap c \) is an \( E \)-pinned name on the poset \( Q = P \cap c \).

Since the poset \( Q \) is countable, by Theorem 2.8.2 below applied in the model \( N \), there must be a point \( y \in N \) such that \( Q \models \sigma \mathrel{E} \check{y} \). Now, work in the model \( K \) and observe that \( Q \models \sigma \mathrel{E} \check{y} \). It follows that \( x \mathrel{E} y \). Now, back in the model \( N \) it must be the case that \( P \models \tau \mathrel{E} \check{y} \), contradicting the assumption that \( \tau \) is a nontrivial \( E \)-pinned name.

\[ \square \]

**Corollary 2.6.3.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \). The statement “\( E \) is pinned” is absolute between all generic extensions.

**Proof.** The validity of item (2) of Theorem 2.6.1 does not change if one only considers countable models. This follows from an immediate downward Löwenheim-Skolem argument. The countable model version of (2) is a coanalytic statement, and as such it is absolute among all forcing extensions by the Mostowski absoluteness. \[ \square \]

The absoluteness of the pinned status of analytic equivalence relations is a more difficult question. A rather primitive example shows that in the constructible universe it fails in general; on the other hand, in the presence of sufficiently large cardinals the pinned status of every analytic equivalence relation is absolute. We refrain from giving the rather uninformative proofs. Another natural question concerns the computation of the various cardinal invariants of equivalence relations in inner models. In particular, we would like to see that if \( M \subset N \) are transitive models of ZFC containing the code for an analytic equivalence relation \( E \), then \( (\kappa(E))^M \leq (\kappa(E))^N \). Interestingly enough, this is not clear, and we only have a positive answer in the case of orbit equivalence relations and their relatives.
Theorem 2.6.4. Let $E$ be an analytic equivalence relation almost reducible to an orbit equivalence relation. Let $M$ be a transitive model of large portion of set theory containing the codes for $E$, the group action, and the almost reduction. Then $|\kappa(E)^M| \leq \kappa(E)$ holds.

Proof. The proof uses the transitivity of the model $M$ in one fairly tricky abstract point. Let $M \models B$ is a complete Boolean algebra, completely generated by a set $A \subset B$. Then (viewed from $V$!) $RO(B)$ is a complete Boolean algebra completely generated by $A$ again. To see this, observe that every element of $B$ is obtained by a transfinite series of Boolean operations from the set $A$ in the model $M$, which is also a transfinite series of operations in the algebra $RO(B)$ in $V$, and these operations are evaluated in the same way in $B$ as in $RO(B)$ as $B$ is dense in $RO(B)$. It follows that $A$ generates a dense subset of $RO(B)$ in $V$, and therefore it generates $RO(B)$ as well.

Let $\Gamma$ be a Polish group continuously acting on a Polish space $Y$, $F$ its orbit equivalence relation, and $h: X \to Y$ a Borel almost reduction of $E$ to $F$ such that the codes of these objects belong to the model $M$. Suppose that $\lambda < |\kappa(E)|^M$ is a cardinal; we have to find an $E$-pinned name which has no equivalent on a poset of size $< \lambda$.

Work in the model $M$. Let $\tau$ be an $E$-pinned name on a poset $P$ which does not have an equivalent on a poset of size $< \lambda$. It will be enough to show that $\tau$ remains such also in $V$. To this end, let $\sigma = \dot{h}(\tau)$. This is an $F$-pinned name which has no $F$-equivalent on a poset of size $< \lambda$. Consider the poset $P$. It follows by Theorem 2.8.3 that the poset $P$ collapses the cardinal $\lambda$. The initial remark of the proof shows that $P$ as evaluated in $M$ is dense in $P$ as evaluated in $V$.

Now, step out of the model $M$ and argue that $\langle P, \tau \rangle$ does not have an $E$-equivalent on a poset of size $< \lambda$ even in $V$. Suppose for contradiction that $\langle Q, \eta \rangle$ is such an equivalent. Let $\chi = \dot{h}(\eta)$; this is a name $F$-equivalent to $\sigma$. It follows from Claim 2.8.6 that $Q \times P_G$ adds a point which is $P_G$-generic over $V$ and therefore also over the smaller model $M$. Therefore, $\lambda$ is collapsed in the $Q \times P_G$-extension. This contradicts the fact that the density of $Q \times P_G$ is smaller than $\lambda$.

2.7 Dichotomies

This section presents three theorems which characterize various complex features of the virtual realm in terms of dichotomies. It turns out that each of the dichotomies uses a preparatory proposition or two on the descriptive theory of forcing and model theory. These preparatory propositions are perhaps more difficult to state properly than to prove. Nevertheless, they are unavoidably needed and they do not seem to be present in published literature. We gather them in the first subsection, while the second subsection contains the dichotomies themselves.
2.7a Preliminaries

The first two auxiliary propositions concern pure model theory; they show that
the satisfaction relation and the construction of models from indiscernibles are
in a suitable sense Borel affairs. The proofs are straightforward and left to the
reader. For both of them, fix a countable language and let $X$ be the Polish
space of models of that language whose universe is $\omega$.

**Proposition 2.7.1.** Suppose that $f : 2^\omega \to X$ is a Borel function, $\phi$ is a formula
of the language with $n$ free variables, and $g_i : 2^\omega \to \omega$ are Borel functions for
every $i \in n$. The set $\{ x \in X : f(x) \models \phi(g_0(x), g_1(x) \ldots g_{n-1}(x)) \}$ is Borel.

Let $T$ be a Skolemized complete consistent theory containing a linear ordering.
Let $S$ be a theory of an infinite ordered set of indiscernibles in the linear order
of $T$. Let $Y$ be the space of all linear orderings on $\omega$.

**Proposition 2.7.2.** There are Borel functions $f : Y \to X$ and $g : Y \times \omega \to \omega$
such that for each $y \in Y$,

1. $f(y)$ is a model of $T$;
2. the function $g_y : \omega \to \omega$ is an order-preserving map from the ordering $y$ to
   the ordering of $f(y)$ and its range consists of indiscernibles with theory $S$;
3. the model $f(y)$ is a Skolem hull of $\text{rng}(g_y)$.

The rest of the subsection deals with models of set theory, showing that
the construction of their generic extensions is a Borel affair. Thus, let $X$ be
the Polish space of all models for the language with one binary relation whose
universe is $\omega$. Let $Y$ be any uncountable Polish space, serving as an index space.
Let $M : Y \to X$ and $P : Y \to \omega$ be Borel functions such that for each $y \in Y$,
$M(y)$ is a model of (a large fragment of) ZF and $M(y) \models P(y)$ is a partially
ordered set.

**Proposition 2.7.3.** There is a Borel function $G : Y \to P(\omega)$ such that for
every $y \in Y$, $G(y)$ is a filter on $P(y)$ which is generic over $M(y)$.

**Proof.** By induction on $n \in \omega$ define Borel functions $f_n : Y \to \omega$ so that

- for every $y \in Y$, $M(y) \models f_n(y)$ is the largest element of the poset $P(y)$;
- for every $y \in Y$ and $n \in \omega$, if $M(y) \models n$ is an open dense subset of the
  poset $P(y)$, then $f_{n+1}(y)$ is the smallest number $m$ such that $M(y) \models m$
is an element of $n$ and it is smaller that $f_n(y)$ in the poset $P(y)$; otherwise,
  $f_{n+1}(y) = f_n(y)$.

The functions defined in this way are Borel by Proposition 2.7.1. Define the
function $G : Y \to P(\omega)$ as follows: for every $y \in \omega$, $G(y)$ is the set of all $m$
such that $M(y) \models m$ is an element of the poset $P(y)$ and for some $n \in \omega$,
$M(y) \models f_n(y)$ is smaller than $m$ in the poset $P(y)$. It is clear that the function
$G$ works. $\square$
Now, suppose that a Borel function \( G : Y \to P(\omega) \) as in Proposition 2.7.3 is given; we want to produce the generic extensions. To wit, we have to produce the generic extensions as Borel functions on \( Y \) and also the valuation functions for names in the ground models as Borel functions.

**Proposition 2.7.4.** There are Borel functions \( M[G] : Y \to X \) and \( v : Y \times \omega \to \omega \) such that for every \( y \in Y \), \( M[G](y) \) is a generic extension of \( M(y) \) by \( G(y) \) and for every \( n \in \omega \), \( v(y)(n) = n/G(y) \) whenever \( M(y) \models n \) is a \( P(y) \)-name.

**Proof.** Let \( E \) be the Borel equivalence relation on \( Y \times \omega \) connecting pairs \( \langle y_0, n_0 \rangle \) and \( \langle y_1, n_1 \rangle \) if \( y_0 = y_1 \) and (denoting their common value by \( y \)) either \( M(y) \models n_0, n_1 \) is a \( P(y) \)-name or \( M(y) \models n_0, n_1 \) are \( P(y) \)-names and there is some condition \( p \in G(y) \) such that \( M(y) \models p \models n_0 = n_1 \). Let \( v : Y \times \omega \to \omega \) be the Borel function defined in the following way: \( f(y)(n) = 0 \) if \( M(y) \models n \) is not a \( P(y) \)-name, and \( f(y)(n) = m \) if \( M(y) \models n \) is a \( P(y) \)-name and, writing \( n' \) for the smallest name \( E \)-related to \( n \), there are exactly \( m \)-many \( E \)-classes of \( P(y) \)-names represented by numbers smaller than \( n' \). Let \( M[G] \) be the Borel function defined in the following way: \( M[G](y) \) connects the pair \( \langle m_0, m_1 \rangle \) just in case there are numbers \( n_0, n_1 \in \omega \) such that \( M(y) \models n_0, n_1 \) are \( P(y) \)-names, \( f(y)(n_0) = m_0 \), \( f(y)(n_1) = m_1 \), and there is a condition \( p \in G(y) \) such that \( M(y) \models p \models n_0 = n_1 \). The Borelness of the functions \( M[G] \) and \( v \) follows from Proposition 2.7.1. \( \square \)

The construction of the generic filters may be subject to an additional constraint. Suppose that \( E \) is an equivalence relation on a Polish space \( Z \), coded in each model \( M(y) \). Suppose that \( \tau : Y \to \omega \) is a Borel function such that for each \( y \in Y \), \( M(y) \models \tau(y) \) is a \( P(y) \)-name of an element of \( Z \). Suppose that \( z : Y \to Z \) is a Borel function. Can we find a Borel function \( G : Y \to P(\omega) \) such that for each \( y \in Y \), \( G(y) \) is a filter on \( P(y) \) generic over \( M(y) \), such that the valuation of \( \tau(y) \) over \( G(y) \) is \( E \)-related to \( z(y) \)?

In this generality, the answer is clearly negative. To come to a useful conclusion, we assume that the models in the range of the function \( M \) are all well-founded, and the equivalence relation \( E \) is induced as an orbit equivalence relation of a continuous action of a Polish group \( \Gamma \) on the space \( Z \). In such circumstances, we have the following:

**Proposition 2.7.5.**

1. the set \( B = \{ y \in Y : \text{there is a filter } G \subset P(y) \text{ generic over } M(y) \text{ such that } \tau(y)/G \text{ is } E \text{-related to } z(y) \} \) is Borel;

2. there is a Borel function \( G : B \to P(\omega) \) such that for every \( y \in B \), \( G(y) \subset P(y) \) is a filter generic over \( M(y) \) such that \( \tau(y)/G(y) \) is \( E \)-related to \( z(y) \).

**Proof.** Let \( P_\Gamma \) be the poset of all nonempty open subsets of \( \Gamma \) ordered by inclusion, with its name \( \check{\cdot} \) for a generic element of \( \Gamma \). Use Proposition 2.7.1 to find Borel functions \( Q, \sigma : Y \to \omega \) so that for every \( y \in Y \), \( M(y) \models Q(y) \) is the
complete subalgebra of the complete Boolean algebra of \( P(y) \times P\Gamma \) generated by the name \( \sigma(y) = \dot{\gamma} \cdot \tau(y) \).

Let \( D = \{ (y, \delta, G) \in Y \times \Gamma \times \mathcal{P}(\omega) : G \subseteq Q(y) \} \) is a filter generic over the model \( M(y) \) such that \( \nu(y)/G = \delta \cdot z(y) \). This is a Borel set by Proposition 2.7.4. The projection of \( D \) into the \( Y \) coordinate is the set \( B \) by definitions.

**Claim 2.7.6.** For each pair \( \langle y, \delta \rangle \in Y \times \Gamma \), the vertical section \( D_{y\delta} \) is either empty or a singleton.

**Proof.** This uses the wellfoundedness of the model \( M(y) \). Suppose that the section \( D_{y\delta} \) is nonempty, containing some filter \( G \). As \( M(y) \models \text{"}Q(y)\text{"} \) is completely generated by the name \( \sigma(y) \), the filter \( G \) can be recovered by transfinite induction using infinitary Boolean expressions in \( M \) applied to \( \delta \cdot z(y) \), and therefore it is unique. \( \square \)

Write \( C \subset Y \times \Gamma \) for the projection of \( D \) into the first two coordinates. Since one-to-one projections of Borel sets are Borel [45, Theorem 15.1], the set \( C \) is Borel.

**Claim 2.7.7.** For each point \( y \in Y \), the section \( C_y \) either empty or else comeager in \( \Gamma \).

**Proof.** To simplify the notation, fix \( y \in Y \) and omit the argument \( y \) from the expressions like \( M(y), \sigma(y), \ldots \). Suppose that the \( \Gamma \)-section \( C_y \) is nonempty. Thus, there is a filter \( G \subset P \) be a filter generic over \( M \) such that \( \tau/G \) is \( E \)-related to \( z \). Let \( A \subset \Gamma \) be the set of all elements of the group \( \Gamma \) which are \( P\Gamma \)-generic over the model \( M[G] \); this is a co-meager set. Let \( \delta \in \Gamma \) be an element of the group such that \( \tau/G = \delta \cdot z \). Then the vertical section \( C_y \) contains the set \( A \cdot \delta \) which is co-meager as the right multiplication by \( \delta \) is a self-homeomorphism of the group \( \Gamma \). \( \square \)

As the category quantifier yields Borel sets [45, Theorem 16.1], the set \( B \) as the projection of \( C \) into the first coordinate is Borel. Borel sets with nonmeager vertical sections allow Borel uniformizations [45, Theorem 18.6], and so there is a Borel uniformization \( f : B \rightarrow \Gamma \) of \( C \). As the set \( D \) has singleton vertical sections, it is itself its uniformization \( g : C \rightarrow \mathcal{P}(\omega) \). Let \( H : B \rightarrow \mathcal{P}(\omega) \) be the function defined by \( H(y) = g(y, f(y)) \). Thus, for every \( y \in B \), \( H(y) \subseteq Q(y) \) is a filter generic over \( M(y) \) such that \( \sigma(y)/H(y) \) is \( E \)-related to \( z(y) \).

Now, let \( R : Y \rightarrow \omega \) be a Borel function such that for every \( y \in 2^\omega \), \( M(y) \models R(y) \) is a name for the quotient poset \( P(y) \times P\Gamma/Q(y) \). Use Propositions 2.7.3 and 2.7.4 to find a Borel function \( K : B \rightarrow \mathcal{P}(\omega) \) such that \( K(y) \subset R(y) \) is a filter generic over the model \( M(y)[H(y)] \). Let \( G : B \rightarrow \mathcal{P}(\omega) \) be a Borel function indicating a filter on \( P(y) \) which is the \( (P(y))-\text{coordinate} \) the composition of \( H(y) \ast K(y) \). The function \( G \) has the required properties. \( \square \)

**Corollary 2.7.8.** Suppose that \( \Gamma \) is a Polish group continuously acting on a Polish space \( X \), inducing an orbit equivalence relation \( E \). Let \( M \) be a countable transitive model of set theory containing a code for the action, \( P \in M \) and \( \tau \in M \) be a poset and a name for an element of the space \( X \). Then
2.7. DICHOTOMIES

1. the set \( B = \{ x \in X : \exists G \subset P \text{ generic over } M \text{ such that } \tau/G = x \} \) is Borel;

2. the equivalence relation \( E \upharpoonright B \) is Borel.

Proof. (1) is immediate from the proposition. To see (2), let \( Q = P \times P \Gamma \) and let \( \sigma \) be the \( Q \)-name for the pair \( \langle \tau, \gamma_{\text{gen}} \cdot \tau \rangle \). Now, the set \( C = \{ \langle x, y \rangle \in X^2 : \exists G \times H \subset P \times P \Gamma \text{ generic over } M \text{ such that } \langle x, y \rangle = \sigma/G \times H \} \) is Borel by the proposition again. Now, for \( x, z \in B \), \( x E z \) just in case the set \( A_{xz} = \{ \gamma \in \Gamma : \langle x, \gamma \cdot z \rangle \in C \} \) is comeager in \( \Gamma \). To see that, note that if \( x E z \) fails then the set \( A_{xz} \subset \Gamma \) is actually empty, and if \( x E z \) holds with \( \delta \cdot x = z \), then the \( A_{xz} \supset \{ \gamma \in \Gamma : \gamma \cdot \delta^{-1} \text{ is } P_{\Gamma}\text{-generic over } V[x] \} \) and the latter set is comeager since it is a shift of the comeager set of all points in \( \Gamma \) which are \( P_{\Gamma}\text{-generic over } V[x] \).

Finally, observe that the equivalence relation \( E \upharpoonright B \) is obtained by an application of the category quantifier to the Borel set \( \{ \langle x, z, \gamma \rangle \in B \times B \times \Gamma : \gamma \in A_{xz} \} \) and so is Borel. \( \square \)

2.7b Results

The first dichotomy characterizes the existence of the pinned cardinal in analytic equivalence relations.

Theorem 2.7.9. Assume that there is a measurable cardinal. Let \( E \) be an analytic equivalence relation on a Polish space \( X \). The following are equivalent:

1. \( \kappa(E) = \infty \);

2. \( \mathbb{E}_{\omega_1} \leq_\alpha E \).

Proof. (2) implies (1) by Example 2.4.6 and Theorem 2.5.4. The large cardinal assumption is not needed for this direction. For the \( (1) \rightarrow (2) \) implication, suppose that \( \kappa(E) = \infty \). Let \( \kappa \) be a measurable cardinal.

Claim 2.7.10. There is a poset \( P \) of size \( \kappa \) and an \( E \)-pinned \( P \)-name \( \tau \) such that \( \tau \) is not \( E \)-related to any name on a poset of size \( < \kappa \).

Proof. Since \( \kappa(E) = \infty \), there is a poset \( Q \) and an \( E \)-pinned \( P \)-name \( \sigma \) which is not \( E \)-equivalent to any name on a poset of size \( < \kappa \). Choose an elementary submodel \( M \) of a large enough structure such that \( |M| = \kappa, V_\kappa \subset M \) and \( Q, \sigma \in M \), and let \( P = Q \cap M \) and let \( \tau = \sigma \cap M \). We claim that \( P, \tau \) works as required.

Indeed, if \( G \times H \subset P \times P \) is a filter generic over \( V \), then it is also generic over \( M \), by the elementarity of \( M \) and the forcing theorem in \( M \). \( M[G,H] \models \tau/G \in E \tau/H \), and by the Mostowski absoluteness between \( M[G,H] \) and \( V[G,H] \), \( V[G,H] \models \tau/G \in E \tau/H \). This proves that the name \( \tau \) is \( E \)-pinned. If \( R \) is a poset in \( V \) of size \( < \kappa \) and \( \nu \) is an \( E \)-pinned \( R \)-name and \( G \times H \subset P \times R \) is a generic filter over \( V \), then \( R, \nu \in M \), the filter \( G \times H \) also generic over \( M \), by
the elementarity of $M$ and the forcing theorem in $M M[G, H] \models \neg \tau/G E \nu/H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models \neg \tau/G E \sigma/H$. This proves that $\nu \not\equiv \tau$ fails and completes the proof of the claim.

Choose a poset $P$ on $\kappa$ and an $E$-pinned name $\tau$ as in the claim. Choose a large enough cardinal $\theta$ and skolemize the structure $(H_\theta, \in, P, \tau)$ of sets whose transitive closure has cardinality $< \theta$. Let $T$ be the theory of the skolemized structure. Let $j : V = M_0 \to M_1$ be an ultrapower embedding associated with some normal measure on the measurable cardinal $\kappa$, and let $j_n : V \to M_n$ be its iterations of length $n$ for $n \leq \omega$. The ordinals $\{j_n(\kappa) : n \in \omega\}$ form a sequence of indiscernibles in the model $j_\infty(H_\theta)$. Let $S$ be the theory of the indiscernibles in this structure. Let $Y$ be the space of all linear orderings on $\omega$, and let $Z$ be the space of all binary relations on $\omega$. By Propositions 2.7.2 and 2.7.3, there are Borel functions $M : Y \to Z, G : Y \times M \to \mathcal{P}(\omega)_\omega, P : Y \to \omega$ and $\tau : Y \to \omega$ such that whenever $y \in Y$ is a linear ordering then $M(y)$ is a model of $T$ which is a Skolem hull of indiscernibles of ordertype $y$ satisfying the theory $S$, $P(y)$ is its version of the poset $P$, and $\tau(y)$ is its version of the name $\tau$, and $G(y) \subset P(y)$ is a filter generic over the model $M(y)$. Note that all models $M(y)$ are $\omega$-models and so compute the Polish space $X$ correctly. Let $k : Y \to X$ be the function defined by $k(y) = \tau(y)/G(y)$, which is Borel by Proposition 2.7.4. It will be enough to show that $k$ is a Borel reduction of $E_{\omega_1}$ to $E$ on the set of $y \in Y$ which are well-orders.

Suppose first that $y_0, y_1 \in Y$ are well-orders of the same length. Then the models $M(y_0), M(y_1)$ are wellfounded and isomorphic. Write $N$ for their common transitive isomorph and $Q, \sigma$ for its version of the poset $P$ and the name $\tau$. Thus, $N \models \sigma$ is an $E$-pinned $Q$-name. The filters $G(y_0), G(y_1) \subset Q$ are separately generic over $N$. Let $G \subset Q$ be a filter generic over both countable models $N[G(y_0)]$ and $N[G(y_1)]$. By the product forcing theorem, the filters $G(y_0) \times G$ and $G(y_1) \times G$ are both $Q \times Q$-generic over $N$. As $N \models \sigma$ is an $E$-pinned name, it follows that $N[G(y_0), G] \models \sigma/G(y_0) E \sigma/G$ and $N[G(y_1), G] \models \sigma/G(y_1) E \sigma/G$. By the Mostowski absoluteness between these two models and $V$, and the transitivity of the relation $E$, $\sigma/G(y_0) = k(y_0) E k((y_1) = \sigma/G(y_1)$ follows.

Suppose now that $y_0, y_1 \in Y$ are well-orders of different lengths; say that $y_0$ is shorter than $y_1$. Let $N_0$ be the transitive isomorph of $M(y_0)$, let $\kappa_0, P_0, \sigma_0$ be the versions of $\kappa, P,$ and $\tau$ in the model $N_0$, and let $G_0 \subset P_0$ be the filter generic over $N_0$ indicated by $G(y_0)$. Similar notation prevails for subscript $1$. Let $i : N_0 \to N_1$ be the elementary embedding obtained by sending the indiscernibles of $N_0$ to an initial segment of the indiscernibles of $N_1$. Note that the critical point of $i$ is exactly $\kappa_0$ and so $P_0 \in N_1$, as $P_0 = P_1 \upharpoonright \kappa_0$. Also, $N_1 \models \kappa_0 = |P_0| < \kappa_1$. Find a filter $G \subset P_0$ generic over both the countable models $N_0[G_0]$ and $N_1[G_1]$. Since $N_0 \models \sigma_0$ is $E$-pinned, the Mostowski absoluteness between $V$ and $N_0[G, G_0]$ implies that $\sigma_0/G_0 E \sigma_0/G$ holds. Since $N_1 \models \sigma_1$ is not $E$-equivalent to any name on a poset of size $< \kappa_1$, in particular to $\sigma_0$, it follows from the Mostowski absoluteness between $V$ and $N_1[G, G_1]$ that $\sigma_0/G E \sigma_1/G_1$.
In this case, we will conclude that $E \equiv \mathbb{1}/G_0 \leq E \sigma_1/G_1 = k(y_1)$ fails as required.

The status of a Borel equivalence relation as pinned/unpinned may be absolute between models of ZFC by Corollary 2.6.3. However, if one dares to look at choiceless models, much more colorful picture comes to sight. The simplest description of the pinned status occurs in the Solovay model, where it actually obeys a former conjecture of Kechris [39, Question 17.6.1].

**Theorem 2.7.11.** The following holds in the Solovay model derived from a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. The following are equivalent:

1. $E$ is unpinned;
2. $\mathcal{F}_2 \leq E$ or $E_{\omega_1} \leq E$.

*Proof.* Let $\kappa$ be a measurable cardinal and let $W$ be the derived Solovay model. In $W$, (2) certainly implies (1) as the proofs that $\mathcal{F}_2, E_{\omega_1}$ are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the reducibility orderings works in ZF+DC.

For the implication (1) $\rightarrow$ (2), assume that $W \models E$ is unpinned. There must be a poset $P$ and a nontrivial $E$-pinned name $\tau$ on the poset $P$, both in $W$. Both $P$ and $\tau$ must be definable in $W$ from a ground model parameter $z \in 2^\omega$.

For simplicity of the notation assume that $z \in V$. Return to $V$. Let $Q$ be the two-step iteration $\text{Coll}(\omega, \kappa) \ast \dot{P}$, and write $\sigma$ for the $Q$-name obtained from the $\dot{P}$-name $\tau$. There are two cases.

**Case 1.** There is a condition $q \in Q$ such that the $Q \restriction q$-name $\sigma$ is $E$-pinned. In this case, we will conclude that $E_{\omega_1} \leq E$ and use the Shoenfield absoluteness to transfer the almost reducibility to the Solovay model. To simplify the notation assume that $q$ is the largest element of the poset $Q$.

Observe that the name $\sigma$ cannot be $\dot{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that $R$ is a poset of size $< \kappa$ and $\chi$ an $E$-pinned $R$-name such that $(R, \chi) \dot{E} (Q, \sigma)$. In $W$, let $H \subset R$ be a filter generic over $V$. By Proposition 2.1.5 applied over $V$, $W \models P \Vdash \tau E \chi/H$. This contradicts the assumption that $V[G] | P \Vdash \tau$ is not $E$-related to any point in $V[G]$.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 2.5.6(2), since $\kappa$ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 2.7.9, $E_{\omega_1} \leq E$ as desired.

**Case 2.** For every condition $q \in Q$, the $Q \restriction q$-name $\sigma$ is not $E$-pinned. In this case, we will conclude that $\mathcal{F}_2$ is reducible to $E$. Then, a Shoenfield absoluteness argument shows that the Borel reduction of $\mathcal{F}_2$ to $E$ remains a Borel reduction also in the Solovay model.

Fix a countable elementary submodel $M$ of a large structure containing the code for $E$, the posets $P, Q$ and the name $\tau$. Write $R = \text{Coll}(\omega, \kappa) \cap M$. Let $Z = 2^\omega$, and use Proposition 1.8.8 to find a Borel function $G$ from $Z$ to $\mathcal{P}(M)$ such that for every finite set $a \subseteq Z$ the filters $G(z) \subset R$ for $z \in a$ are mutually generic over $M$. For every set $a \subseteq Z$ write $\mathcal{F}_a = \bigcup\{2^\omega \cap M[\prod_{\gamma \in b} G(z)]: b \subseteq a \}$. The following holds in the Solovay model derived from a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. The following are equivalent:

1. $E$ is unpinned;
2. $\mathcal{F}_2 \leq E$ or $E_{\omega_1} \leq E$.

*Proof.* Let $\kappa$ be a measurable cardinal and let $W$ be the derived Solovay model. In $W$, (2) certainly implies (1) as the proofs that $\mathcal{F}_2, E_{\omega_1}$ are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the reducibility orderings works in ZF+DC.

For the implication (1) $\rightarrow$ (2), assume that $W \models E$ is unpinned. There must be a poset $P$ and a nontrivial $E$-pinned name $\tau$ on the poset $P$, both in $W$. Both $P$ and $\tau$ must be definable in $W$ from a ground model parameter $z \in 2^\omega$.

For simplicity of the notation assume that $z \in V$. Return to $V$. Let $Q$ be the two-step iteration $\text{Coll}(\omega, \kappa) \ast \dot{P}$, and write $\sigma$ for the $Q$-name obtained from the $\dot{P}$-name $\tau$. There are two cases.

**Case 1.** There is a condition $q \in Q$ such that the $Q \restriction q$-name $\sigma$ is $E$-pinned. In this case, we will conclude that $E_{\omega_1} \leq E$ and use the Shoenfield absoluteness to transfer the almost reducibility to the Solovay model. To simplify the notation assume that $q$ is the largest element of the poset $Q$.

Observe that the name $\sigma$ cannot be $\dot{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that $R$ is a poset of size $< \kappa$ and $\chi$ an $E$-pinned $R$-name such that $(R, \chi) \dot{E} (Q, \sigma)$. In $W$, let $H \subset R$ be a filter generic over $V$. By Proposition 2.1.5 applied over $V$, $W \models P \Vdash \tau E \chi/H$. This contradicts the assumption that $V[G] | P \Vdash \tau$ is not $E$-related to any point in $V[G]$.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 2.5.6(2), since $\kappa$ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 2.7.9, $E_{\omega_1} \leq E$ as desired.

**Case 2.** For every condition $q \in Q$, the $Q \restriction q$-name $\sigma$ is not $E$-pinned. In this case, we will conclude that $\mathcal{F}_2$ is reducible to $E$. Then, a Shoenfield absoluteness argument shows that the Borel reduction of $\mathcal{F}_2$ to $E$ remains a Borel reduction also in the Solovay model.
Lemma 2.7.12. 1. whenever \( a \subset Z \) is a nonempty set then in there is a filter \( h \subset R \) generic over \( M \) such that \( 2^\omega \cap M[h] = 2^\omega \); 

2. whenever \( a, b, c \subset I \) are pairwise disjoint countable nonempty sets then there is a filter \( h_a \times h_b \times h_c \subset R^3 \) generic over \( M \) such that \( 2^\omega_a = 2^\omega \cap V[h_a] \) and similarly for \( 2^\omega_b \) and \( 2^\omega_c \); 

3. whenever \( a, b \) are distinct countable subsets of \( I \) then \( P_a \times P_b \forces \neg \tau_a \ E \tau_b \).

Proof. For (1), let \( S = \{ k : \exists \alpha \in \kappa \cap M \exists b \subset a \) finite \( k \in \text{Coll}(\omega, < \alpha) \cap M \) is a filter generic over \( M \) and \( k \in M[G(z) : z \in b] \} \) and order \( R \) by inclusion. Let \( K \subset S \) be a sufficiently generic filter; we claim that \( h = \bigcup K \) works as desired. Indeed, a simple density argument shows that \( h \subset R \) is an ultrafilter all of whose proper initial segments are generic over \( M \). By the \( k \)-c.c. of \( \text{Coll}(\omega, < \kappa) \), the filter \( h \) is in fact generic over \( M \) itself. A straightforward genericity argument then shows that \( 2^\omega = 2^\omega \cap M[h] \) as desired.

(2) follows easily from (1). Let \( h_a, h_b, h_c \subset R \) be any filters obtained from (1); we will show that these filters are in fact mutually generic over the model \( V \). Since \( \text{Coll}(\omega, < \kappa)^3 \) has \( k \)-c.c.c., it is enough to show that for every ordinal \( \alpha \in \kappa \), the filters \( h_a^\alpha = h_a \cap \text{Coll}(\omega, < \alpha) \), \( h_b^\alpha = h_b \cap \text{Coll}(\omega, < \alpha) \), and \( h_c^\alpha = h_c \cap \text{Coll}(\omega, < \alpha) \) are mutually generic over \( V \). Since the filters \( h_a^\alpha \), \( h_b^\alpha \) and \( h_c^\alpha \) are coded by reals in the models \( M[h_a] \), \( M[h_b] \), and \( M[h_c] \), there are finite subsets \( a', b', c' \) of \( a, b, c \) respectively such that \( h_a^\alpha \in M[\prod_{\alpha \in \omega} g_i] \) etc. The mutual genericity now follows from the general Corollary 1.8.9 about product forcing.

(3) is proved in several parallel cases depending on the mutual position of the sets \( a, b \) vis-a-vis inclusion. We will treat the case in which all three sets \( a \cap b \), \( a \setminus b \), \( b \setminus a \) are nonempty. Suppose for contradiction that \( P_a \times P_b \forces \tau_a \ E \tau_b \). From (2), it follows that in \( V \), the triple product \( \text{Coll}(\omega, < \kappa)^3 \) forces \( \bar{P}_{(1,1)} \times \bar{P}_{(1,2)} \forces \tau_{(1,1)} \ E \tau_{(1,2)} \). Then, the quadruple product \( \text{Coll}(\omega, < \kappa)^4 \) forces in \( V \) that \( \bar{P}_{(0,1)} \times \bar{P}_{(1,2)} \times \bar{P}_{(2,3)} \forces \tau_{(0,1)} \ E \tau_{(1,2)} \ E \tau_{(2,3)} \), in particular \( \bar{P}_{(0,1)} \times \bar{P}_{(2,3)} \forces \tau_{(0,1)} \times \tau_{(2,3)} \). In view of (2) again, this means that the product \( \text{Coll}(\omega, < \kappa) \times \text{Coll}(\omega, < \kappa) \) forces \( \bar{P}_{(0,1)} \times \bar{P}_{(2,3)} \forces \sigma_{(0,1)} \ E \sigma_{(2,3)} \). In other words, \( (\text{Coll}(\omega, < \kappa) \times \text{Coll}(\omega, < \kappa) \times \bar{P}) \times (\text{Coll}(\omega, < \kappa) \times \bar{P}) \) forces \( \sigma_{(0,1)} \ E \sigma_{(2,3)} \), contradicting the case assumption.

Write \( Y = (2^\omega)^\omega = \text{dom}(\mathbb{F}_2) \). Use the Lusin–Novikov theorem to find a Borel map \( g : Y \to (2^\omega)^\omega \) such that for every \( y \in Y \), \( g(y) \) enumerates the set \( 2^\omega_y \). Use Lemmas 2.7.12(1) and 2.7.5 to find a Borel map \( h : Y \to \mathcal{P}(Q \cap M) \) such that for every \( y \in Y \), \( h(y) \subset Q \) is a filter generic over \( M \) and \( \text{rng}(g(y)) = 2^\omega \cap V[h(y)] \), where \( h_0(y) \subset R \) is the filter generic over \( M \) obtained from \( h(y) \). Let \( k : Y \to X \) be given by \( k(y) = \tau/h(y) \); this is a Borel map by Lemma 2.7.4. We will show that \( k \) is a reduction of \( \mathbb{F}_2 \) to \( E \).
First, assume that $y_0, y_1 \in Y$ are $F_2$-related. Then $\text{rng}(y_0) = \text{rng}(y_1)$, $\text{rng}(g(y_0)) = \text{rng}(g(y_1))$, and so $M_{y_0} = M_{y_1}, P_{y_0} = P_{y_1}$ and $\tau_{y_0} = \tau_{y_1}$. Let $H \subseteq P_{y_0}$ be a filter generic over both countable models $M_{y_0}[k(y_0)]$ and $M_{y_1}[k(y_1)]$ and let $x = \tau_{y_0}/H$. By the forcing theorem applied in the model $M_{y_0} = M_{y_1}$ and the fact that $\tau_{y_0}$ is an $E$-pinned name, conclude that $x E k(y_0)$ and $x E k(y_1)$ and so $k(x_0) E k(x_1)$ as desired.

Second, assume that $y_0, y_1 \in Y$ are not $F_2$-related. Choose a sufficiently generic filter $H_0 \times H_1 \subseteq P_{y_0} \times P_{y_1}$ so that $H_0$ is generic over $M_{y_0}[k(y_0)]$ and $H_1$ is generic over $M_{y_1}[k(y_1)]$. As the names $\tau_{y_0}$ and $\tau_{y_1}$ are $E$-pinned, the forcing theorem in the models $M_{y_0}$ and $M_{y_1}$ implies that $k(y_0) E \tau_{y_0}/H_0$ and $k(y_1) E \tau_{y_1}/H_1$. Now, $\tau_{y_0}/H_0 E \tau_{y_1}/H_1$ fails by Lemma 2.7.12(3), and so $k(y_0) E k(y_1)$ must fail as well. This completes the proof.

**Corollary 2.7.13.** The following holds in the symmetric Solovay model derived from a measurable cardinal. Let $E$ be a Borel equivalence relation on a Polish space $X$. $E$ is unpinned if and only if $F_2 \leq E$.

**Proof.** This follows from Theorem 2.7.11 once we show that the option $E_{\omega_1} \leq_n E$ is not available for any Borel equivalence relation $E$. This in turn follows easily from results on the pinned cardinal $\kappa(E)$ obtained in Section 2.5: $\kappa(E) < \infty$ by Theorem 2.5.6(1), $\kappa(E_{\omega_1}) = \infty$ by Example 2.4.6, and the pinned cardinal is monotone with respect to the reducibility ordering $\leq_n$—Theorem 2.5.4.

Among the equivalence relations Borel reducible to $F_2$, $F_2$ is the only one whose pinned cardinal can be proved in ZFC to be equal to $\mathfrak{c}^+$. This is the contents of the following theorem:

**Theorem 2.7.14.** Let $\kappa > \beth_1$ be a regular cardinal such that $\kappa^{\aleph_0} = \kappa$. Let $P$ be the usual poset for adding $\kappa$ Cohen reals. In the $P$-extension, exactly one of the following holds for every Borel equivalence relation $E$ Borel-reducible to $F_2$:

1. $E$ is bireducible with $F_2$;
2. $\kappa(E) \leq \mathfrak{c}$.

**Proof.** It is clear that (1) implies the negation of (2). If $F_2$ is Borel reducible to $E$ then $\kappa(E) \geq \kappa(F_2) = \mathfrak{c}^+$ holds in every generic extension, and so (2) fails. We must show that the negation of (2) implies that $F_2$ is Borel-reducible to $F_2$.

For simplicity assume that $E$ is a Borel equivalence relation on $X = 2^\omega$ with a code in the ground model, and fix a Borel reduction $h : 2^\omega \rightarrow (2^\omega)^\omega$ of $E$ to $F_2$.

Let $G \subseteq P$ be a filter generic over the ground model and work in $V[G]$. Since $V[G] \models \kappa(E) = \mathfrak{c}^+$, There must be a poset $Q$ and an $E$-pinned name $\sigma$ on the poset $Q$ such that $\sigma$ is not $E$-equivalent to any name on a poset of size $< \mathfrak{c}$. The $Q$-name $\overline{h}(\sigma)$ is $F_2$-pinned and so there is a set $A \subseteq 2^\omega$ such that $Q \models \overline{A} = \text{rng}(\overline{h}(\sigma))$ by Example 2.3.5. By the Mostowski absoluteness, there is a Coll($\omega_1, A$)-name $\chi$ for an element of $X$ such that $\text{rng}(\overline{h}(\chi)) = A$. The name $\chi$ is $E$-related to $\sigma$. Since $\sigma$ was chosen to have no equivalents on posets of size $< \mathfrak{c}$, it follows that $|A| = \mathfrak{c}$.


Back to the ground model. The previous paragraph shows that there must be a P-name \( \dot{A} \) for a subset of \( 2^\omega \) of size \( \mathfrak{c} = \kappa \) such that in some further forcing extension (by Mostowski absoluteness it is enough to take \( \text{Coll}(\alpha, \kappa) \)-extension) there is an element \( x \in X \) such that \( \text{rng}(h(x)) = \dot{A} \).

Since \( \kappa > \beth_\omega \), essentially by Theorem ??? there is a countable model \( M \) elementarily equivalent to ??? such that the integers of \( M \) are isomorphic to \( \omega \) and \( M \) has an infinite collection of ordered indiscernibles below \( \kappa^M \). Let \( N \) be a model obtained as a Skolem hull of indiscernibles of ordertype \( 2^\omega \) with the usual lexicographic ordering from \( M \). More precisely, let \( Y = \{(t, \vec{a}) \mid \text{ for some } n \in \omega, t \text{ is an } n\text{-ary Skolem term and } \vec{a} \in (2^\omega)^n\} \). Let \( F \) be the equivalence relation on \( Y \) defined by \( (t, \vec{a}) \sim (s, \vec{b}) \) if for some sequences \( \vec{a}', \vec{b}' \) of indiscernibles in the model \( N \) ordered in the same way as \( \vec{a}, \vec{b} \), \( M = t(\vec{a}') = t(\vec{b}') \). Build the model \( N \) on the collection of \( F \)-classes in the usual way: ???

The following piece of notation will be useful below. For a pair \( y = (t, \vec{a}) \in Y \), write \( \text{supp}(y) \) for the intersection of all sets \( b \subseteq 2^\omega \) such that for some Skolem term \( s \) and some sequence \( \vec{c} \) with \( \text{rng}(\vec{c}) \subseteq b \), \( (s, \vec{c}) \sim (t, \vec{a}) \). An easy indiscernibility argument shows that \( y \mapsto \text{supp}(y) \) is a Borel function in a natural sense.

**Claim 2.7.15.** \( N \) is (can be presented as) a Borel model. The natural numbers of \( N \) are isomorphic to \( \omega \).

*Proof.* The equivalence relation \( F \) is clearly Borel; we will show that it is smooth. ???

**Claim 2.7.16.** There is a filter \( G \subset P^N \) generic over the model \( N \) which is moreover a Borel subset of \( N \).

Let \( G \subset P^N \) be a filter generic over the model \( N \). For any point \( z \in (2^\omega)^\omega \), let \( N_z \) be the Skolem hull of the \( N \)-indiscernibles in the set \( \text{rng}(z) \), naturally presented as a structure on \( \omega \). Note that \( G_z = G \cap N_z \) is a filter generic over the model \( N_z \); this follows from the fact that \( N_z \) is an elementary submodel of \( N \), \( N \models P^N \) is c.c.c., and all natural numbers of \( N \) belong to \( N_z \). Let \( a_z = \dot{A}^{N_z}/G_z \); this is a subset of \( 2^\omega \).

**Claim 2.7.17.** For \( z_0, z_1 \in (2^\omega)^\omega \), \( z_0 \equiv_F z_1 \leftrightarrow a_{z_0} = a_{z_1} \) holds.

Use Proposition 2.7.3 to find a Borel map \( z \mapsto H_z \) assigning to every point \( z \in (2^\omega)^\omega \) a filter \( H_z \subset \text{Coll}(\omega, \kappa)^{N_z} \) generic over the model \( N_z[G_z] \). Use Proposition 2.7.4 to find a Borel map \( z \mapsto x_z \) such that \( x_z \in X^{N_z[G_z]}[H_z] \) such that \( N_z[G_z][H_z] \models \text{rng}(h(x_z)) = a_z \). By the Borel absoluteness between \( V \) and the model \( N_z[G_z][H_z] \), the statement \( \text{rng}(h(x_z)) = a_z \) holds even in \( V \). Finally, Claim 2.7.17 shows that the map \( z \mapsto x_z \) is the desired Borel reduction of \( \equiv_F \) to \( E \).
2.8 Restrictions on partial orders

Given an analytic equivalence relation $E$ on a Polish space $X$, it may be informative to investigate which posets can carry nontrivial $E$-pinned names. After all, essentially all pinned names discussed in this chapter naturally live on collapse posets, and so one may easily (and wrongly) assume that no forcing sophistication is needed when it comes to the investigation of the virtual realm. This section contains several theorems on this topic.

First of all, there are some partial orders which can never carry a nontrivial pinned name.

**Definition 2.8.1.** [22] A poset $P$ is **reasonable** if for every ordinal $\lambda$ and for every function $f : \lambda^{\omega} \to \lambda$ in the $P$-extension there is a set $a \subseteq \lambda$ which is closed under $f$, belongs to the ground model, and it is countable in the ground model.

In particular, all c.c.c. and all proper forcings are reasonable. Good examples of unreasonable forcings are posets which collapse $\aleph_1$, Namba forcing and Prikry forcing.

**Theorem 2.8.2.** Let $E$ be an analytic equivalence relation on a Polish space $X$. If $P$ is a reasonable forcing and $\tau$ is an $E$-pinned name on $P$, then $\tau$ is trivial.

**Proof.** Suppose that $P$ is a reasonable poset and $\tau$ is an $E$-pinned name on $P$. We will produce a condition $p \in P$ and a point $x \in X$ such that $p \Vdash \tau E \check{x}$. Towards this end, choose a large structure and use the reasonability of $P$ to find a countable elementary submodel $M$ of it containing $P, E$ and $\tau$ and a condition $p \in P$ such that $p \Vdash \check{G} \cap M$ is generic over $M$, where $\check{G}$ is the canonical $P$-name for its generic ultrafilter. As $M$ is countable, there is a filter $H \subset P \cap M$ generic over $M$ in the ground model $V$. Let $x = \tau/H \in X$. Proposition 2.1.5 applied to the model $M$ and the filters $H$ and $\check{G} \cap M$ now says that $p \Vdash \check{x} E \tau$, completing the proof.

The key feature of partial orders from the point of view of existence of pinned names is collapsing $\aleph_1$ as the following theorem shows:

**Theorem 2.8.3.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Exactly one of the following occurs:

1. $E$ is pinned;
2. for every poset $P$ collapsing $\aleph_1$, $P$ carries a nontrivial $E$-pinned name.

**Proof.** Clearly (1) implies the negation of (2). For the difficult direction, suppose that (1) fails and work to confirm (2). Let $\tau$ be a nontrivial $E$-pinned name on some poset $P$. Let $\langle M_\alpha : \alpha \in \omega_1 \rangle$ be a continuous $\in$-tower of countable elementary submodels of a large structure containing $X$ and $E$. Let $M = \bigcup_\alpha M_\alpha$, let $Q = P \cap M$ and let $\sigma = \tau \cap M$. 


Observe that in any generic extension, whenever $G, H \subset Q$ are generic filters over $M$, then $\sigma / G \mathrel{E} \sigma / H$ and their equivalence class contains no ground model elements. The first part of this statement follows from Proposition 2.1.5 applied in the transitive collapse of the model $M$. For the second part of the statement, it is enough to show that in $V$, $Q \models \sigma$ is not $E$-equivalent to any element of the ground model. Suppose for contradiction that there is a point $x \in X$ such that $Q \mathrel{E} \sigma \mathrel{E} \check{x}$. We will show that there then must be $y \in M \cap X$ which is $E$-related to $x$. Then $Q \models \sigma \mathrel{E} \check{y}$, by the Mostowski absoluteness between the $Q$-extensions of $M$ and $V$. $M \models P \models E \mathrel{E} \check{y}$, and this contradicts the elementarity of the model $M$ and the nontriviality of the name $\tau$.

To find the point $y \in M \cap X$, let $N$ be a countable elementary submodel of a large structure containing $\langle M_\alpha : \alpha \in \omega_1 \rangle, Q, x$. Since the tower of models $\langle M_\alpha : \alpha \in \omega_1 \rangle$ is continuous, there is a limit ordinal $\alpha \in \omega_1$ such that $M_\alpha = N \cap M$. Let $Q_\alpha = Q \cap M_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \sigma \cap M_\alpha = \tau \cap M_\alpha$. By elementarity of the model $N$ and analytic absoluteness between the $Q_\alpha$-extension of $N$ and $V$, $Q_\alpha \models \sigma_\alpha \mathrel{E} \check{x}$. Since $Q_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \tau \cap M_\alpha$, both $Q_\alpha, \tau_\alpha$ belong to the model $M_{\alpha+1}$. By the elementarity of the model $M_{\alpha+1}$, there must be a point $y \in X \cap M_{\alpha+1}$ such that $Q_\alpha \models \sigma_\alpha \mathrel{E} \check{y}$ (since the point $x$ is such). By the transitivity of $E$, it follows that $x \mathrel{E} y$. The point $y \in M_{\alpha+1} \subset M$ works.

Now, suppose that $R$ is a poset collapsing $\aleph_1$. Since $|M| = \aleph_1$, in the $R$-extension there is a filter $Q$-generic over $M$. Let $\check{H}$ be an $R$-name for such a filter and let $\nu$ be the $R$-name for $\sigma / \check{H}$. We have just proved that $\nu$ is a nontrivial $E$-pinned name on $R$.

Theorem 2.8.3 does not rule out the possibility that some $\aleph_1$ preserving posets carry nontrivial $E$-pinned names. This does not occur for orbit equivalence relations. In them, every pinned name is inescapably connected with a collapse of a certain cardinal to $\aleph_0$. This is the contents of the following:

**Theorem 2.8.4.** Let $E$ be an analytic equivalence relation on a Polish space $X$, Borel reducible to an orbit equivalence relation. Let $\langle P, \tau \rangle$ be an $E$-pin. There is a cardinal $\kappa$ such that in every forcing extension, the following are equivalent:

1. $|\kappa| = \aleph_0$;

2. the virtual $E$-class of $\langle P, \tau \rangle$ is realized.

**Proof.** We will start with a preliminary general claim.

**Claim 2.8.5.** Suppose that $P$ is a poset such that $P \models \check{P}$ has a countable dense subset. Then $P$ is in the forcing sense equivalent to $\text{Coll}(\omega, \kappa)$ where $\kappa$ is the minimum size of a dense subset of $P$.

**Proof.** Let $\check{x}$ be a $P$-name for an enumeration of a countable dense subset of $P$. Then $D = \{ q \in P : \exists p \in P \exists n \in \omega \ p \models \check{q} = \check{x}(n) \}$ must be a dense subset of $P$ and as such has size at least $\kappa$. In the $P$-extension, the set $D$ is equal to $\{ q \in P : \exists p \in \text{rng}(x) \exists n \in \omega \ p \models \check{q} = \check{x}(n) \}$ which is clearly countable. Thus, $P \models |\kappa| = \aleph_0$. Now, every poset collapsing its own density character to $\aleph_0$ is
in the forcing sense equivalent to the collapse poset by a classical theorem of McAloon–Fact 1.8.13.

Now, some notation. Let $\Gamma$ be a Polish group acting on a Polish space $X$ inducing $E$ as its orbit equivalence relation. Write $P_\Gamma$ for the Cohen poset on the acting group $\Gamma$, adding a generic group element $\dot{\gamma}$. For an $E$-pinned name $\tau$ on a poset $P$, write $\dot{\tau}$ for the $P \times P_\Gamma$-name $\dot{\gamma} \cdot \tau$ and $\dot{P}$ for the complete subalgebra of $RO(P \times P_\Gamma)$ generated by the name $\dot{\tau}$. Note that $\langle \dot{P}, \dot{\tau} \rangle \equiv (P, \tau)$.

**Claim 2.8.6.** Suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \equiv \tau \in E \dot{x}$. In $V[H]$, $P_\Gamma \equiv \dot{y} \cdot \dot{x}$ is $\dot{P}$-generic over $V[H]$. Moreover, for every condition $\dot{p} \in \dot{P}$ there is a condition $q \in P_\Gamma$ forcing the filter generated by $\dot{y} \cdot \dot{x}$ to contain $\dot{p}$.

**Proof.** Let $p \in P$ and $q \in P_\Gamma$ be arbitrary conditions. Let $K \subset P$ be a generic filter over $V[H]$ and write $y = \tau/K$. It will be enough to find points $g, h \in \Gamma \cap P_\Gamma$-generic over the model $V[H][K]$, such that $h \in q$ and $gx = hy$. For then, $g$ is $P_\Gamma$-generic over $V[H]$ and $gx$ is a $\dot{P}$-generic point over $V[H]$ meeting the projection of the condition $(p, q)$ into $\dot{P}$.

By the assumption on the point $x$, it is the case that $V[H][K] \equiv x \in E y$, and there is an element $\beta \in \Gamma \cap V[H][K]$ such that $\dot{\beta} \cdot \dot{x} = \dot{y}$. Let $\delta \in q$ be a $P_\Gamma$-generic point over $V[H][K]$ and let $\gamma = \delta \beta$. As the topology on the group $G$ is invariant under translations, $\gamma$ is also $P_\Gamma$-generic over $V[H][K]$, and $\gamma \cdot \dot{x} = \delta \beta \cdot \dot{x} = \delta \cdot \dot{y}$ as required.

**Claim 2.8.7.** $\dot{P}$ is in the forcing sense equivalent to $\text{Coll}(\omega, \kappa)$ for some cardinal $\kappa$.

**Proof.** Since $\tau$ is an $E$-pinned name, $\dot{P} \equiv P \equiv \dot{\tau} \in E \tau$. Thus, by Claim 2.8.6, in the $\dot{P}$-extension the algebra $RO(\dot{P})$ is completely embedded into $RO(P_\Gamma)$ and therefore $\dot{P}$ contains a countable dense set. The proof is completed by a reference to Claim 2.8.5.

We will now show that $\kappa$ works as desired. On one hand, suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \equiv \tau \in E \dot{x}$. By Claim 2.8.6, in $V[H]$ the poset $P_\Gamma$ adds a $P$-generic and therefore by Claim 2.8.7, it collapses $\kappa$ to $\aleph_0$. Since $P_\Gamma$ is just the Cohen poset, this can happen only if $|\kappa| = \aleph_0$ already in $V[H]$.

On the other hand, suppose that $V[H]$ is a generic extension in which $|\kappa| = \aleph_0$. Back in the ground model, find an elementary submodel $M$ of a large enough structure containing $P, \tau$ such that $|M| = \kappa$ and $M \cap \dot{P}$ is dense in $\dot{P}$. In the model $V[H]$, there is an $\dot{P}$-generic filter $K$ over $M$. Let $x = \dot{\tau}/K$. It follows that $\dot{P} \equiv \dot{\tau} \in E \dot{x}$ and so $P \equiv \tau \in E \dot{x}$.

**Corollary 2.8.8.** Let $E$ be a Borel equivalence relation on a Polish space $X$, Borel reducible to an orbit equivalence relation. If $E$ is not pinned then the following are equivalent:
CHAPTER 2. THE VIRTUAL REALM

1. $P$ carries a nontrivial $E$-pinned name;

2. $P$ collapses $\aleph_1$ to $\aleph_0$.

Example 2.8.9. Consider the equivalence relation $F_2$ on $(2^\omega)^\omega$. This is clearly an orbit equivalence relation. Every $E$-pinned name is represented by a set $A \subset 2^\omega$ by Example 2.3.5. The cardinal $\kappa$ of Theorem 2.8.4 is equal to $|A|$.

Example 2.8.10. Let $E$ be the equivalence relation on $X = (\mathcal{P}(\omega))^\omega$ defined by $x_0 E x_1$ if $\text{rng}(x_0)$ and $\text{rng}(x_1)$ generate the same filter on $\omega$. Suppose that in $V$, there is a modulo finite strictly decreasing sequence $a = \langle a_\alpha : \alpha \in \omega_2 \rangle$ of subsets of $\omega$. Let $P$ be the Namba forcing. It is well-known that $P$ preserves $\aleph_1$ and adds a cofinal sequence $\sigma : \omega \rightarrow \omega_2^\omega$ to $\omega_2$. Let $\tau$ be the $P$-name for $a \circ \sigma$. It is immediate that $\tau$ is a nontrivial $E$-pinned name on $P$. Thus, the assumption that $E$ be reducible to an orbit equivalence in Corollary 2.8.8 cannot be omitted.
Chapter 3

Turbulence

In this chapter, we will investigate pairs of generic extensions $V[H_0]$ and $V[H_1]$ such that $V[H_0] \cap V[H_1] = V$. In a particularly successful application, this will allow us to restate Hjorth’s concept of turbulence in geometric terms and produce many generalizations of the associated ergodicity theorem [42, Theorem 12.5].

3.1 Independent functions

The purpose of this section is to find a practical way of producing many extensions which satisfy the condition $V[H_0] \cap V[H_1] = V$ without being mutually generic. We need a folkloric observation. Recall that for a Polish space $X$, the poset $P_X$ consists of nonempty open subsets of $X$ ordered by inclusion; $\dot{x}_{gen}$ is its name for the unique element of $X$ belonging to all open sets in the generic filter.

**Proposition 3.1.1.** Let $X, Y$ be Polish spaces and $f: X \to Y$ be a continuous open function. Then $P_X \vDash f(\dot{x}_{gen})$ is a $P_Y$-generic element of $Y$.

**Proof.** It is just necessary to show that $P_X \vDash f(\dot{x}_{gen}) \in D$ for every open dense set $D \subset Y$. To this end, let $O \in P_X$ be a condition. The set $f^nO \subset Y$ is open, and therefore it has nonempty intersection with $D$. Consider the nonempty open set $O' = (f^{-1}(f^nO \cap D)) \cap O \subset O$. For every point $x \in O'$, $f(x) \in D$ holds, and so $O' \vDash f(\dot{x}_{gen}) \in D$ as required. \qed

Now suppose that $X, Y_0, Y_1$ are Polish spaces and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ are continuous open maps. We want to find a criterion implying that $P_X \vDash V[f_0(\dot{x}_{gen})] \cap V[f_1(\dot{x}_{gen})] = V$. The following turbulence-like definition is central.

**Definition 3.1.2.** Let $X, Y_0, Y_1$ be Polish spaces and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ be continuous open maps. If $O \subset X$ is an open set, a walk in $O$ is a finite
sequence of points in $O$ such that for any two successive points $x, x'$ on the sequence, either $f_0(x) = f_0(x')$ or $f_1(x) = f_1(x')$ holds.

First, a simple complexity observation.

**Proposition 3.1.3.** Let $X, Y_0, Y_1$ be Polish spaces and $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ be continuous open maps. Let $O \subset X$ and $D \subset O$ be nonempty open sets and $k \in \omega$ be a number. The set $C(k, D) \subset O$ of all points $x \in O$ such that there is a walk in $O$ starting at $x$ and ending at $D$ is open.

**Proof.** By induction on $k$ simultaneously for all open sets $D \subset O$. The case $k = 0$ is trivially satisfied, since $C(k, O) = D$. The case $k = 1$ is satisfied with $f_0^{-1}f_0'(f_0^{-1}(f_1''D) \cap O) \cap O$. Now suppose that the proposition is verified for all nonempty open sets $D \subset O$. To check it for $k + 1$ and a fixed $D$, apply the induction hypothesis to conclude that the set $C(k, D) \subset O$ is open. Apply the case $k = 1$ to show that the set $C(1, C(k, D)) \subset O$ is open. The definitions show that $C(1, C(k, D)) = C(k + 1, D)$, completing the induction step. □

**Definition 3.1.4.** Let $X, Y_0, Y_1$ be Polish spaces and $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ be continuous open maps. The functions $f_0, f_1$ are independent if for every nonempty open set $O \subset X$ there is a nonempty open set $A \subset X$ such that for all nonempty open subsets $B_0, B_1 \subset A$ there is a walk in $O$ starting in $f_0^{-1}B_0$ and ending in $f_0^{-1}B_1$.

It may appear that the definition is not symmetric with respect to the maps $f_0, f_1$. In fact, the definition is symmetric, and this follows as a small corollary from the following central theorem.

**Theorem 3.1.5.** Suppose that $X, Y_0, Y_1$ are Polish spaces and $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ are continuous open maps. The following are equivalent:

1. $f_0, f_1$ are independent;
2. $P_X \models V[f_0(\check{x}_{gen})] \cap V[f_1(\check{x}_{gen})] = V$.

**Proof.** For (1) $\implies$ (2) implication, since the models in question are models of set theory with choice, it is enough to show that every set of ordinals in the intersection is actually in $V$. To this end, suppose that $\tau_0, \tau_1$ are $P_{Y_0}$ and $P_{Y_1}$ names for sets of ordinals respectively and $O \subset X$ is a nonempty open set forcing $\tau_0/f_0(\check{x}_{gen}) = \tau_1/f_1(\check{x}_{gen})$. We need to find an open set $O' \subset O$ deciding the statement $\check{\alpha} \in \tau_0/f_0(\check{x}_{gen})$ for every ordinal $\alpha$.

Let $A \subset Y_0$ be a nonempty open set standing witness to the independence of the functions $f_0, f_1$. We claim that for every ordinal $\alpha$, in the poset $P_{Y_0}$, $A \forces \check{\alpha} \in \tau_0$ or $A \forces \check{\alpha} \notin \tau_0$. Once this is proved, let $O' = O \cap f_0^{-1}A$ and note that for every ordinal $\alpha$, in the poset $P_X$, $O' \forces \check{\alpha} \in \tau_0/f_0(\check{x}_{gen})$ or $O' \forces \check{\alpha} \notin \tau_0/f_0(\check{x}_{gen})$. This will complete the proof.

Suppose towards contradiction that there is an ordinal $\alpha$ and nonempty open sets $B_0, B_1 \subset A$ such that $B_0 \forces \check{\alpha} \in \tau_0$ and $B_1 \forces \check{\alpha} \notin \tau_0$. Let $k \in \omega$ be the minimal number such that there is a walk $(x_i : i \leq 2k)$ in $O$ with $f_0(x_0) \in B_0$ and $f_0(x_{2k}) \in B_1$. The following is the central walk claim:
Claim 3.1.6. There are nonempty open sets $O_i \subset O$ for $i \leq 2k$ such that

1. $f''_0 O_0 \subset B_0$ and $f''_0 O_{2k} \subset B_1$;
2. for each $i \leq k$, $f''_0 O_{2i}$ in $P_{Y_0}$ decides the statement $\bar{\alpha} \in \tau_0$;
3. for each $i < k$, $f''_1 O_{2i+1}$ in $P_{Y_1}$ decides the statement $\bar{\alpha} \in \tau_1$;
4. for each $i < k$, the sets $f''_0 O_{2i+1} \subset f''_0 O_{2i}$ and $f''_0 O_{2i+2} \subset f''_0 O_{2i+1}$.

Proof. For every number $i \leq k$ let $P_i = \{x \in O : \text{there is a walk } \langle x_j : j \in 2k + 1 - 2i \rangle \in O \text{ such that } x = x_2i, \text{ and for all } j \ x_{2j} \in O_{2j} \text{ and } f_0(x_{2k}) \in B_1 \}$. The sets $P_i$ are open by Proposition 3.1.3. The construction of the sets $O_{2i}, O_{2i+1}$ proceeds by recursion on $i$ with the additional hypothesis that $O_{2i} \subset P_i$. To start the recursion, just let $O_0 = O \cap f_0^{-1} B_0 \cap P_0$; this is a nonempty set by the initial assumptions. Now suppose that the set $O_{2i} \subset X$ has been found.

Let $Q = f_1^{-1}(f''_1 f_0^{-1}(f''_0 O_{2i}) \cap O) \cap O$. The set $Q \subset O$ is open, and must have nonempty intersection with $P_{i+1}$. Find an open set $O_{2i+2} \subset Q \cap P_{i+1}$ such that $f''_0 O_{2i+2}$ in $P_{Y_0}$ decides the statement $\bar{\alpha} \in \tau_0$ and $f''_1 O_{2i+2}$ in $P_{Y_1}$ decides the statement $\bar{\alpha} \in \tau_1$, and let $O_{2i+1} = f_1^{-1} f''_1 O_{2i+2} \cap f_0^{-1} f''_0 O_{2i} \cap O$. A review of the demands of the claim shows that the sets $O_{2i+1}$ and $O_{2i+2}$ work as required. \(\square\)

Now, note that by the initial assumptions on the sets $B_0, B_1$ there has to be $i \leq k$ such that in the poset $O_{2i} \models \bar{\alpha} \notin \tau_0/f_0(x_{\text{gen}})$ and $O_{2i+2} \models \bar{\alpha} \notin \tau_0/f_0(x_{\text{gen}})$. For such $i \leq k$, no matter how $O_{2i+1}$ decides the statement $\bar{\alpha} \in \tau_1$, it will be in disagreement with the decision made by either $O_{2i}$ or $O_{2i+2}$. Appropriately, one of the conditions $O_{2i+1}$ or $O_{2i+2}$ in the poset $P_X$ will force $\bar{\alpha} \in \tau_0/f_0(x_{\text{gen}}) \Delta \tau_1/f_1(x_{\text{gen}})$, contradicting the initial choice of the open set $O \subset X$.

To see why the negation of (1) implies the negation of (2), let $O \subset X$ be a nonempty open set witnessing the failure of (1). For each $y \in Y_0$, define the orbit of $y$ to be the set of all elements $z \in Y_0$ such that there is a walk $\langle x_i : i \leq 2k \rangle$ in $O$ such that $f_0(x_0) = y$ and $f_0(x_{2k}) = z$. Let $x \in O$ be a $P_X$-generic and work in the model $V[x]$.

Claim 3.1.7. The orbit of $f_0(x)$ is nowhere dense in $Y_0$.

Proof. If the orbit were dense in some nonempty open set $B \subset Y_0$, then in the ground model, the set $B$ would show that $O$ is not a witness to the failure of (1). To see this, suppose that $B_0, B_1 \subset B$ are nonempty open subsets of $B$. Working in $V[x]$, find walks $w_0, w_1$ in $O$ which lead from $f_0(x)$ to $B_0$ and $B_1$, invert $w_0$ and concatenate with $w_1$ and get a walk from $B_0$ to $B_1$. By the Mostowski absoluteness between $V[x]$ and $V$, there must be such a walk in $V$ as well. \(\square\)

Now look at the set $A = \{B \subset Y_0 : B \text{ is a basic open subset of } Y_0 \text{ and } B \text{ contains no element of the orbit of } f_0(x)\}$.

Claim 3.1.8. $A \in V[f_0(x)] \cap V[f_1(x)]$. 
Proof. In $V[x]$, the set $A$ is defined as the set of all basic open sets $B \subset Y_0$ such that there is no walk $\langle x_i : i \leq 2k \rangle$ in $O$ such that $f_0(x_0) = f_0(x)$ and $f_0(x_{2k}) \in B$. This is a $\Pi^1_1$ definition with parameter $f_0(x)$ which therefore yields the same set in the model $V[f_0(x)]$.

In $V[x]$, the same set $A$ also has an alternative definition: it is the set of all basic open sets $B \subset Y_0$ such that there is no walk $\langle x_i : i \leq 2k \rangle$ in $O$ such that $f_1(x_0) = f_1(x)$ and $f_0(x_{2k}) \in B$. To see this, note that a walk with $f_0(x_0) = f_0(x)$ can be transformed into a walk with $f_1(x_0) = f_1(x)$ (and also vice versa) by simply adding the point $x$ twice to the beginning of the walk. This alternative definition of the set $A$ is again $\Pi^1_1$ with parameter $f_1(x)$, and so by the Mostowski absoluteness yields the same set in the model $V[f_1(x)]$.

To show that (2) fails, it is enough to argue that $A \notin V$. However, if $A \in V$ then $\bigcup A \in V$ is an open dense subset of $Y_0$ by Claim 3.1.7, and $f_0(x) \notin \bigcup(A)$. This contradicts Proposition 3.1.1: the point $f_0(x) \in Y$ is forced to be $P_f$-generic over the ground model and therefore to belong to every open dense set in the ground model.

3.2 Examples

Several groups of results in this book depend on an identification of suitable pair of independent maps. The most important example comes from Hjorth’s notion of turbulence of group actions. Recall the standard definition:

**Definition 3.2.1.** [39, Section 13.1] Let $\Gamma$ be a Polish group continuously acting on a Polish space $Y$.

1. If $U \subset \Gamma$, $O \subset Y$ are sets then a $U,O$-walk is a sequence $\langle y_i : i \leq k, \gamma_i : i < k \rangle$ such that for each $i \leq k$ $\gamma_i \in O$ holds, and for each $i < k$ $\gamma_i \in U$ and $y_{i+1} = \gamma_i \cdot \gamma_i$ both hold;

2. if $y \in O$ then $U,O$-orbit of $y$ is the set of all $z \in O$ such that there is a $U,O$-walk starting at $y$ and ending at $z$;

3. The action is turbulent at $y \in Y$ if for all open sets $U \subset \Gamma$ and $O \subset X$ with $1 \in U$ and $y \in O$ the $U,O$-orbit of $y$ is somewhere dense.

4. The action is generically turbulent if its orbits are meager and dense and the set of points of turbulence is comeager.

Now, suppose that $\Gamma$ is a Polish group acting continuously on a Polish space $Y$. Let $X = \{ (\gamma, y_0, y_1) \in \Gamma \times Y \times Y : \gamma \cdot y_0 = y_1 \}$; this is a closed subset of $\Gamma \times Y \times Y$ and therefore Polish in the inherited topology. Let $f_0 : X \to Y$ be the projection into the second coordinate, $f_1 : X \to Y$ be the projection into the third coordinate, and let $f_2 : X \to \Gamma \times Y$ be the projection into the first two coordinates. It is not difficult to check that these mappings are continuous and open.
3.2. EXAMPLES

**Theorem 3.2.2.** Let \( \Gamma \) be a Polish group continuously acting on a Polish space \( Y \) such that all orbits are meager and dense. The following are equivalent:

1. the action is generically turbulent;
2. \( f_0, f_1 : X \to Y \) is an independent pair of functions;
3. \( P_\Gamma \times P_Y \) forces \( V[\hat{x}_{\text{gen}}] \cap V[\gamma_{\text{gen}} \cdot \hat{x}_{\text{gen}}] = V \).

**Proof.** For the implication (1) \( \to \) (2), suppose that the action is generically turbulent. Let \( O \subset X \) be an open set. Find a point \( \delta \in \Gamma \) and open sets \( U \subset \Gamma \) and \( O_0, O_1 \subset Y \) such that \( 1 \in U, U = U^{-1}, \) and \( \delta \cdot U \times O_0 \times O_1 \cap X \subset O \). Find a point \( y \in O_0 \) such that the \( U, O_0 \)-orbit of \( y \) is somewhere dense; let \( A \subset Y \) be a nonempty open set such that the \( U, O_0 \)-orbit of \( y \) is dense in \( A \). We claim that \( A \subset Y \) is an open set witnessing the independence of functions \( f_0 \) and \( f_1 \).

To see this, let \( B_0, B_1 \subset A \) be nonempty open sets. Concatenating \( U, O_0 \)-walks from \( y \) to \( B_0 \) and \( B_1 \), we can find a \( U, O \)-walk \( \langle y_i : i \leq k, \gamma_i : i < k \rangle \) which starts in \( B_0 \) and ends in \( B_1 \). Consider the sequence \( \langle x_i : i \leq 2k \rangle \) of points in the open set \( O \subset X \) given by \( x_{2i} = \langle \delta, y_i, \delta \cdot y_i \rangle \) and \( x_{2i+1} = \langle \delta \gamma_i, y_i, \delta y_{i+1} \rangle \) if \( i < k \). It is immediate that this is an \( O \)-walk in the sense of Definition 3.1.2, confirming the independence of functions \( f_0, f_1 \).

For the implication (2) \( \to \) (1), suppose that the functions \( f_0, f_1 \) are independent. Suppose that \( U \subset \Gamma \) and \( O_0 \subset Y \) are nonempty open sets and let \( O = \{ (\gamma, y_0, y_1) \in X : \gamma, \gamma^{-1} \in U, y_0, y_1 \in O_0 \} \). Suppose that \( \langle x_i : i \leq 2k \rangle \) is an \( O \)-walk in the sense of Definition 3.1.2 and \( x_i = \langle \delta_i, z_i, z_i' \rangle \); let \( y_0 = z_0 \) and for \( i \leq 2k \) even greater than \( 0 \) let \( y_i = z_i, y_{i-1} = z_i' \) and \( \gamma_i = \delta_i^{-1} \) and \( \gamma_{i-1} = \delta_i \). It is not difficult to see that \( \langle y_i : i \leq 2k, \gamma_i : i < 2k \rangle \) is a \( U, O_0 \)-walk in the sense of Definition 3.2.1 beginning at \( z_0 \) and ending at \( z_{2k} \).

Now, if \( A \subset O_0 \) is an open set witnessing the independence of \( f_0, f_1 \) for \( O \) and \( B \subset A \) is a nonempty open set, then the set \( \{ y \in A : \text{there is an } O \text{-walk from } y \text{ to } B \} \) is open dense in \( O \), and by the previous paragraph it is a subset of the set \( \{ y \in A : \text{there is an } U, O_0 \text{-walk from } y \text{ to } B \} \). Thus, the set \( C_{U, O_0} = \{ y \in Y : \text{either } y \notin A \text{ or the } U, O_0 \text{-orbit of } y \text{ is dense in } A \} \) is comeager. The set of turbulent points is then the intersection \( \bigcap_{U, O_0} C_{U, O_0} \) and is comeager.

The restatement (3) is immediate from the fact that the function \( f_2 \) is continuous and open and so the first two coordinates of the \( P_X \)-generic point are generic for \( P_\Gamma \times P_Y \) by Proposition 3.1.1.

A somewhat similar characterization theorem can be proved for analytic ideals on \( \omega \).

**Definition 3.2.3.** [10] A set \( I \subset P(\omega) \) is \( \omega \)-hitting if for every countable collection of infinite subsets of \( \omega \) there is a set \( b \in I \) which has nonempty intersection with each set in the collection.

A typical analytic \( \omega \)-hitting ideal is \( I_p \), where \( p \) is a partition of \( \omega \) into finite sets and \( I_p \) is the collection of all sets \( a \subset \omega \) such that for some number \( n \in \omega \), for every set \( b \in p, |a \cap b| \leq n \). In fact, [34] shows that every Borel \( \omega \)-hitting ideal
CHAPTER 3. TURBULENCE

contains one of the ideals \( I_p \) as a subset in a suitable sense. \( \omega \)-hitting ideals give rise to pairs of independent functions in the proof of the following theorem:

**Theorem 3.2.4.** Let \( I \) be an analytic ideal on \( \omega \). The following are equivalent:

1. \( I \) is \( \omega \)-hitting;

2. in some forcing extension, there are points \( y_0, y_1 \in 2^\omega \) separately Cohen-generic over \( V \) such that \( y_0 = y_1 \) and \( V[y_0] \cap V[y_1] = V \).

**Proof.** For the direction (1)\( \to \) (2), assume that the ideal \( I \) is \( \omega \)-hitting. Let \( h: \omega^\omega \to P(\omega) \) be a continuous map such that \( I = \text{rng}(h) \). Let \( T = \{ t \in \omega^{<\omega}: h''[t] \text{ is } \omega \text{-hitting} \} \). The set \( T \subset \omega^{<\omega} \) is clearly closed under initial segment.

**Claim 3.2.5.** For all \( t \in T \) there is \( n \in \omega \) such that for all \( m > n \) there is \( z \in [T \upharpoonright t] \) such that \( m \in h(z) \).

**Proof.** Suppose towards contradiction that the conclusion fails for some node \( t \in T \). Let \( a = \omega \setminus \bigcup h''[T \upharpoonright t] \); the set \( a \subset \omega \) is infinite. For each \( s \in \omega^{<\omega} \setminus T \) let \( c_s \) be a countable collection of infinite subsets of \( \omega \) such that for all \( z \in [s] \), \( h(z) \) has finite intersection with one element of \( c_s \). Let \( c = \bigcup_{s \in T} c_s \cup \{ a \} \); it is not difficult to see that for every \( z \in [t] \), \( h(z) \) has finite intersection with an element of \( c \), contradicting the assumption that \( t \in T \). \( \square \)

Now, let \( X \) be the closed set of all triples \( \langle y_0, y_1, z \rangle \subset 2^\omega \times 2^\omega \times [T] \) such that \( \{ n \in \omega: y_0(n) \neq y_1(n) \} \subset h(z) \). Let \( f_0, f_1: X \to 2^\omega \) be the projections into the first and second coordinate. It is not difficult to see that both of these maps are continuous and open. In view of Theorem 3.1.5, the following claim concludes the proof of (2).

**Claim 3.2.6.** The maps \( f_0, f_1 \) are independent.

**Proof.** Let \( O \subset X \) be an open set. Find \( u_0, u_1 \in 2^{<\omega} \) and \( t \in T \) such that \( (\{ u_0 \} \times \{ u_1 \} \times [t]) \cap X \) is a nonempty subset of \( O \). Thinning out further, we may assume that the binary strings \( u_0, u_1 \) have the same length and for all \( z \in [T \upharpoonright t] \) and every \( t \) such that \( u_0(t) \neq u_1(t) \), \( n \in h(z) \) holds. Let \( n \in \omega \) be such that for all \( m > n \) there is \( z \in [T \upharpoonright t] \) such that \( m \in h(z) \). Let \( v \in 2^{<\omega} \) be any binary string extending \( u_0 \) of length \( > n \) and \( A = [v] \). We claim that the open set \( A \) witnesses the definition of independence for \( f_0, f_1 \) and \( O \).

Indeed, suppose that \( w_0, w_1 \) are two binary strings of the same length extending \( v \). To produce a walk from \( [w_0] \) to \( [w_1] \), list all entries on which the strings \( w_0, w_1 \) differ by \( \{ m_j: j \in i \} \). Let \( y \in [w_0] \) be any point, and let \( y_j \in 2^\omega \) be the point obtained from \( y \) by flipping the outputs at all \( m_k \) for \( k \in j \). Thus, \( y_j \) differs from \( y_{j+1} \) only at entry \( m_j \) and \( y_i \in [w_1] \). Let \( y_j' \in 2^\omega \) be the point obtained by replacing the initial segment \( u_0 \subset y_j \) with \( u_1 \). Also, let \( z_j \in [T \upharpoonright t] \) be any point such that \( m_j \in h(z_j) \). Now, define points \( x_{2j+1}, x_{2j+2} \in X \) for \( j \in i \) as follows:

- \( x_0 = \langle y_0, y_0', z \rangle \) for any \( z \in [T \upharpoonright t] \);
3.3. PLACID EQUIVALENCE RELATIONS

- $x_{2j+1} = (y_j, y'_{j+1}, z_j)$;
- $x_{2j+2} = (y_{j+1}, y'_{j+1}, z)$ for any $z \in [T \upharpoonright t]$.

Clearly, the sequence $\langle x_k : k \leq 2i \rangle$ is a walk from $[w_0]$ to $[w_1]$ as required. □

To prove that (2) implies (1), assume that (1) fails, as witnessed by a countable collection $\{a_n : n \in \omega\}$ of infinite subsets of $\omega$. Suppose that in some generic extension, $y_0, y_1 \in 2^\omega$ are $=_I$-related points which are separately Cohen-generic over $V$. The set $b = \{n \in \omega : y_0(n) \neq y_1(n)\}$ belongs to the ideal $I$, and therefore there is a number $n \in \omega$ such that $b \cap a_n$ is finite. It follows that the functions $y_0 \upharpoonright a_n$ and $y_1 \upharpoonright a_n$ are modulo finite equal and so they belong to $V[y_0] \cap V[y_1]$.

By a genericity argument, $y_0 \upharpoonright a_n \notin V$ holds, and so (2) fails. □

3.3 Placid equivalence relations

The characterization of turbulence in Theorem 3.2.2 leads to many Borel nonreducibility results and cardinal preservation results in the generic extensions of the Solovay model. To state these results succinctly, the following definitions are helpful.

**Definition 3.3.1.** Let $E$ be an analytic equivalence relation on a Polish space $X$. We say that $E$ is **placid** if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of $V$ (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ are $E$-related points in the space $X$, then they are $E$-related to some point in $V$.

**Definition 3.3.2.** Let $E$ be an analytic equivalence relation on a Polish space $X$. We say that $E$ is **virtually placid** if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of $V$ (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $(Q_0, \tau_0) \in V[H_0]$ and $(Q_1, \tau_1) \in V[H_1]$ are $\bar{E}$-related $E$-pins, then they are related to some $E$-pin in $V$.

In other words, an equivalence relation $E$ is placid if disjoint generic extensions do not share any $E$-classes which are not represented in $V$. An equivalence relation $E$ is virtually placid if disjoint generic extensions do not share any virtual $E$-classes which are not represented in $V$. The following propositions encapsulate the basic properties of the two classes of equivalence relations.

**Proposition 3.3.3.** Let $E$ be an analytic equivalence relation on a Polish space $X$. $E$ is virtually placid if and only if for any separately generic extensions $V[H_0], V[H_1]$ such that $V[H_0] \cap V[H_1] = V$ and $E$-related points $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$, $x_0, x_1$ are realizations of some virtual $E$-class in $V$.

**Proof.** The left-to-right implication is immediate from the definitions. For the right-to-left implication, suppose that $V[H_0]$ and $V[H_1]$ are separately generic extensions such that $V[H_0] \cap V[H_1] = V$ and $\langle Q, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are $\bar{E}$-related pins. Let $K_0 \subseteq Q_0$ and $K_1 \subseteq Q_1$ be filters mutually generic.
over the model \( V[H_0, H_1] \) and let \( x_0 = \tau_0 / K_0 \) and \( x_1 = \tau_1 / K_1 \). By the mutual
genericty, \( V[H_0][K_0] \cap V[H_1][K_1] = V \), and the assumption on \( E \) gives that
\( x_0, x_1 \) are realizations of some virtual \( E \)-class from the ground model. It follows
that both \( E \)-pins are \( \bar{E} \)-related to some \( E \)-pin from \( V \).

**Proposition 3.3.4.** An analytic equivalence relation \( E \) on a Polish space \( X \) is
placid if and only if it is pinned and virtually placid.

**Proof.** The right-to-left implication is immediate. If \( E \) is a pinned and virtually
placid equivalence relation and \( V[H_0], V[H_1] \) are separately generic extensions
such that \( V[H_0] \cap V[H_1] = V \), then any \( E \)-class represented in both must be represented
as a virtual class in \( V \) by the virtual placid assumption, and this virtual
class must be trivial by the pinned assumption. This means that \( E \) is placid.
For the left-to-right implication, if \( E \) is placid then it must be pinned because
if \( V[H_0], V[H_1] \) are mutually generic extensions then \( V[H_0] \cap V[H_1] = V \) by
the product forcing theorem. To show that \( E \) must be virtually placid, suppose
that \( V[H_0], V[H_1] \) are separately generic extensions. By Proposition 3.3.3, it is
enough to verify that every \( E \)-class represented in both extensions is a realization
of some virtual \( E \)-class in \( V \). However, the placid assumption even implies
that it is a realization of a trivial virtual \( E \)-class in \( V \).

A good example of an equivalence relation which is virtually placid but not
placid is \( F_2 \). It is not placid since it is not pinned.

The following ergodicity theorem is a great generalization of the ergodicity
theorems of Hjorth and Kechris [42, Theorem 12.5], and it is the main motiva-
tion behind the definition of placid and virtually placid classes of equivalence
relations.

**Theorem 3.3.5.** Suppose that a Polish group \( \Gamma \) acts continuously and turbu-
ently on a Polish space \( X \) such that the resulting orbit equivalence relation \( E \) is
analytic, with all orbits dense. Suppose that \( F \) is a virtually placid equivalence
relation on a Polish space \( Y \). Suppose that \( h : X \to Y \) is a homomorphism of
\( E \) to \( F \). Then there is a comeager set \( B \subset X \) which is mapped into a single
\( F \)-class.

**Proof.** Let \( \gamma \in \Gamma, x \in X \) be a mutually generic pair of points for the posets \( P_\gamma \)
and \( P_X \). Theorem 3.2.2 implies the instrumental equality: \( V[x] \cap V[\gamma \cdot x] = V \).
Since \( x \in E \gamma \cdot x \) holds, it must be the case that \( h(x) F h(\gamma \cdot x) \) must hold. By
the virtual placidity of the equivalence relation \( F \), it must be the case that \( h(x) \)
is a realization of a ground model virtual \( F \)-class. However, the poset \( P_X \) is
countable, so all virtual classes realized in its extension are already represented
in the ground model by Theorem 2.8.2. Thus, there must be a point \( y \in Y \cap V \)
such that \( h(x) F y \). Since the generic point \( x \in X \) avoids all ground model coded
meager sets, it must be the case that the set \( B = h^{-1}(\{y\}) \subset X \) is nonmeager.
Since this is an analytic set which is invariant under the continuous group action
all of whose orbits are dense, it follows that the set \( B \) is comeager.
3.4 Examples and operations

Theorem 3.4.1. The class of placid equivalence relations is closed under the following operations:

1. Borel almost reduction;
2. countable product;
3. countable increasing union;
4. countable factor;

The class of virtually placid equivalence relations is closed under the same operations and the Friedman–Stanley jump.

Proof. In view of Proposition 3.3.4, it is enough to show the closure of virtual placidity under these operations, since the class of pinned equivalence relations is closed under (1–4) by the work of Chapter 2. For virtual placidity, we will prove (1) and the closure under the Friedman–Stanley jump.

For (1), suppose that $E, F$ are analytic equivalence relations on Polish spaces $X, Y$ and $h : X \to Y$ is a Borel function which is a reduction of $E$ to $Y$ everywhere except for a set $Z \subseteq X$ consisting of countably many $E$-classes. Suppose that $F$ is virtually placid and work towards the conclusion that $E$ is virtually placid. Let $V[\mathcal{H}_0], V[\mathcal{H}_1]$ be generic extensions such that $V[\mathcal{H}_0] \cap V[\mathcal{H}_1] = V$ and let $x_0 \in V[\mathcal{H}_0]$ and $x_1 \in V[\mathcal{H}_1]$ be $E$-related points in $X$; we need to show that they are realizations of some virtual $E$-class. If $x_0 \in Z$ then this certainly occurs as $V$ contains a countable set of points whose $E$-classes cover $Z$, and this feature of the countable set persists to $V[\mathcal{H}_0]$ by a Shoenfield absoluteness argument. If $x_0 /\in Z$ then $x_1 /\in Z$; look at the points $h(x_0) \in V[\mathcal{H}_0]$ and $h(x_1) \in V[\mathcal{H}_1]$. These are $F$-related points; since $F$ is placid they realize some virtual $F$-class represented by some $F$-pin $\langle Q, \sigma \rangle$. By a Shoenfield absoluteness argument, $Q \vdash \exists x \in X \setminus Z h(x) E \sigma$; let $\tau$ be a $Q$-name for such a point $x \in X$. It is not difficult to see that $\langle Q, \tau \rangle$ is an $E$-pin and $x_0, x_1$ realize the virtual class of $\langle Q, \sigma \rangle$.

For (5), suppose that $E$ is a virtually placid equivalence relation on a Polish space $X$, $V[\mathcal{H}_0]$ and $V[\mathcal{H}_1]$ are separately generic extensions of $V$ such that $V[\mathcal{H}_0] \cap V[\mathcal{H}_1] = V$ and $y_0, y_1 \in X^\omega$ are elements in the respective models such that $\text{rng}(y_0)\_E = \text{rng}(y_1)\_E$. Since $E$ is virtually placid, every element of $\text{rng}(y_0)$ is a realization of a virtual $E$-class from $V$. Let $B$ be the set of all virtual $E$-classes in $V$ which have realizations in $\text{rng}(y_0)$. Since $B$ is also the set of all virtual $E$-classes in $V$ which have realizations in $\text{rng}(y_1)$, it is clear that $B \in V[\mathcal{H}_0] \cap V[\mathcal{H}_1] = V$. Thus, the points $y_0, y_1 \in X^\omega$ are realizations of the virtual $E^+$-class represented by the set $B$. By Proposition 3.3.3, we conclude that $E^+$ is virtually placid.

Theorem 3.4.2. Suppose that $\Gamma$ is a Polish group continuously acting on a Polish space $X$ and $E$ is the resulting orbit equivalence relation. The following are equivalent:
1. $E$ is virtually placid;

2. for every Borel set $B \subseteq X$ such that $E \upharpoonright B$ is Borel, $E \upharpoonright B$ is virtually placid.

Proof. (1) immediately implies (2) since $E \upharpoonright B$ is reducible to $E$ by the identity map on $B$. Now suppose that (1) fails, and let $P$ be a poset and $\tau$ be a name for an element of $X$ such that $P \times P \forces \tau_{\text{left}} \ E \tau_{\text{right}}$ fails, and let $Q$ be a poset with names $\dot{H}_0$, $\dot{H}_1$ for filters generic over the ground model such that $Q \forces V[\dot{H}_0] \cap V[\dot{H}_1] = V$ and $\tau/\dot{H}_0 = \tau/\dot{H}_1$.

Let $M$ be a countable elementary submodel of a large structure containing all objects mentioned so far. Let $\bar{M}$ be the transitive collapse of $M$, and $\bar{P}, \bar{Q}$ etc. be the images of $P, Q$ etc. under the transitive collapse. By Proposition 3.5.5, $\bar{Q} \forces V[\bar{H}_0] \cap V[\bar{H}_1] = V$ holds even in $\bar{V}$ (as opposed to $\bar{M}$). Let $B = \{x \in X : \exists H \subseteq \bar{P} \text{ generic over } \bar{M} \text{ such that } x = \dot{\tau}/H\}$. We have just proved that $E \upharpoonright B$ is not virtually placid. The proof is completed by a reference to Corollary 2.7.8—the set $B \subseteq X$ is Borel and the equivalence relation $E \upharpoonright B$ is Borel as well. ☐

The theorem allows the transfer of the virtual placid property from Borel fragments of a given orbit equivalence relation to the whole equivalence relation. Consider the following attractive corollary:

Corollary 3.4.3. Every equivalence relation classifiable by countable structures is virtually placid.

Proof. In view of Theorem 3.4.1(1), it is only necessary to show that every orbit equivalence relation $E$ obtained from an action of $S_\infty$ is virtually placid. By Theorem 3.4.2, it is only necessary to show that all Borel fragments of $E$ are virtually placid. To this end, consider the transfinite sequence $\langle F_\alpha : \alpha \in \omega_1 \rangle$ obtained from $F_1 =$identity on $2^\omega$ by repeated application of Friedman–Stanley jump, at limit stages taking disjoint unions. It is well-known cite[Theorem 12.5.2]kanovei:book that every Borel equivalence relation classifiable by countable structures is Borel reducible to $F_\alpha$ for some countable ordinal $\alpha$. Theorem 3.4.1 iterated transfinitely shows that each $F_\alpha$ is virtually placid. Thus, every Borel fragment of $E$ is virtually placid, and so is $E$. ☐

There are many Borel placid equivalence relations which cannot be obtained from the identity by repeated application of the operations indicated in Theorem 3.4.1. The following examples deal with equivalence relations associated with ideals on countable sets. If $a$ is a countable set and $J$ is an ideal on $a$, write $=_J$ for the equivalence relation on the space $X = (2^a)^d$ connecting points $x_0, x_1$ if the set $\{n \in a : x_0(n) \neq x_1(n)\}$ belongs to $J$.

Example 3.4.4. Let $a = 2^{<\omega}$ and let $J$ be the branch ideal: the ideal generated by the subsets of $a$ linearly ordered by inclusion. The equivalence relation $=_J$ is placid.
3.4. EXAMPLES AND OPERATIONS

Proof. Write \( X = (2^\omega)^{2^{<\omega}} \). For every node \( t \in \omega^{<\omega} \) write \( [t] \) for the set of all nodes in \( 2^{\omega^{<\omega}} \) extending \( t \). Let \( V[G_0], V[G_1] \) be two generic extensions containing respective \( =_J \)-related points \( x_0, x_1 \in X \). Assume that \( V[G_0] \cap V[G_1] = V \) and work to find a ground model point \( x \in X \) which is \( =_J \)-related to both \( x_0, x_1 \).

Let \( T = \{ t \in 2^{<\omega} : x_0 \upharpoonright |t| \text{ is not } =_J \text{-equivalent to any point in the ground model} \} \). This is a subtree of \( \omega^{<\omega} \). If \( 0 \notin T \) then the proof is complete; thus, it is only necessary to derive a contradiction from the assumption \( 0 \in T \). First, observe that the tree \( T \) cannot have any terminal nodes: if \( t \) was a terminal node of \( T \) then one could combine the ground model witnesses for \( t \uparrow 0 \notin T \) and \( t \uparrow 1 \notin T \) to find a witness for \( t \notin T \). Second, observe that the definition of the tree \( T \) depends only on the \( =_J \)-class of \( x_0 \) and so \( T \in V[G_0] \cap V[G_1] = V \). Since \( T \) is a nonempty ground model tree without endnodes, it must have an infinite branch \( y \in 2^\omega \) in the ground model. Since \( x_0 =_J x_1 \), there is a number \( n \in \omega \) such that for every \( t \in 2^{<\omega} \) such that \( x_0(t) \neq x_1(t) \), either \( t \) is incompatible with \( y \downarrow n \) or else \( t \) is an initial segment of \( y \). Let \( e = [y \upharpoonright n] \setminus y \upharpoonright m : m \geq n \) and observe that \( e \in V \) and \( x_0 \) and \( x_1 \) coincide on \( e \), therefore \( x_0 \upharpoonright e \in V \). If \( z \in V \) is any function in \( 2^{[y \upharpoonright n]} \) extending \( x_0 \upharpoonright e \), then \( z =_J x_0 \upharpoonright |t| \), contradicting the assumption that \( y \upharpoonright n \notin T \).

A whole class of ideals with placid equivalence relations is isolated in the following definition.

Definition 3.4.5. An ideal \( I \) on \( \omega \) is countably separated if there is a countable collection \( A \) of subsets of \( \omega \) such that for every \( b \in I \) and \( c \notin I \) there is \( a \in A \) such that \( b \cap a = 0 \) and \( c \cap a \notin I \).

Countably separated ideals include for example the ideal of nowhere dense subsets of the rationals; a characterization theorem is proved in [50]. Note that the instrumental property of the set \( A \) is coanalytic, and by Mostowski absoluteness it will hold in all forcing extensions.

Theorem 3.4.6. The equivalence relation \( =_I \) is placid for every countably separated Borel ideal \( I \) on \( \omega \).

Proof. Let \( \{ a_n : n \in \omega \} \) be a countable collection of subsets of \( \omega \) witnessing the countable separation of the ideal \( I \). Write \( X = (2^\omega)^{2^{<\omega}} \). Let \( V[G_0], V[G_1] \) be generic extensions containing respective \( =_I \)-related points \( x_0, x_1 \in X \). Assume \( V[G_0] \cap V[G_1] = V \) and work to find a ground model point \( x \in X \) which is \( =_J \)-related to both \( x_0, x_1 \).

Consider the set \( b = \{ n \in \omega : x_0 \upharpoonright a_n \text{ is } I\text{-equivalent to some element of the ground model} \} \). The definition of the set \( b \) depends only on the \( =_I \)-equivalence class of \( x_0 \), therefore the set \( b \) belongs to both \( V[G_0] \) and \( V[G_1] \), and by the initial assumptions, to \( V \). Let \( f \) be the map with domain \( b \) which to each \( n \in b \) identifies the \( =_I \)-class in \( V \) which contains \( x_0 \upharpoonright a_n \). Again, the definition of \( f \) depends only on the \( =_J \)-class of \( x_0 \) and so \( f \in V[G_0] \) and \( f \in V[G_1] \), therefore \( f \in V \). By the Mostowski absoluteness between \( V \) and \( V[G_0] \), there is a point \( x \in X \) such that for all \( n \in b \), \( x \upharpoonright a_n \in f(n) \). We will show that \( x =_J x_0 \) holds.
Suppose towards contradiction that this fails. Consider the set \( c = \{ i \in \omega: x(i) \neq x_0(i) \} \notin I \) and the set \( d = \{ i \in \omega: x_0(i) \neq x_1(i) \} \in I \). By the countable separation of the ideal \( I \), there must be a number \( n \in \omega \) such that \( c \cap a_n \notin I \) and \( d \cap a_n = 0 \). However, the latter equality shows that \( x_0 \upharpoonright a_n = x_1 \upharpoonright a_n \) and so \( x_0 \upharpoonright a_n \in V \) and therefore \( n \in b \). But then \( \{ i \in a_n: x(i) \neq x_0(i) \} \in I \) by the choice of the point \( x \). This contradicts the fact that \( c \cap a_n \notin I \). The proof of the theorem is complete. \( \square \)

There is a natural operation on analytic ideals which seeks a countably separated closure. For the discussion below, let \( \{ a_n: n \in \omega \} \) be a fixed countable collection of subsets of \( \omega \).

**Definition 3.4.7.** Let \( I \) be an ideal on \( \omega \). \( I^+ \) is the ideal on \( \omega \) consisting of all sets \( b \subseteq \omega \) such that there is \( c \in I \) such that \( \forall n \in \omega \) \( b \cap a_n \notin I \rightarrow c \cap a_n \neq 0 \).

We will start with some simple observations. It is clear that \( I^+ \) is an ideal containing \( I \) as a subset. If \( I \) is analytic, then so is \( I^+ \). Moreover, \( I \) is separated by the sequence \( \{ a_n: n \in \omega \} \) just in case \( I = I^+ \). We will show that under suitable additional assumption, the jump preserves placticity of the equivalence relation \( =_I \).

**Definition 3.4.8.** Let \( I \) be an ideal on \( \omega \). \( I \) is skew if for every set \( c \subseteq \omega \) there is a subset \( c' \subseteq c \) such that for every finite set \( d \subseteq \omega \), if \( c \cap \bigcap_{n \in d} a_n \notin I \) then \( c' \cap \bigcap_{n \in d} a_n \notin I \) and if \( c \cap \bigcap_{n \in d} a_n \in I \) then \( c' \cap \bigcap_{n \in d} a_n \) is finite.

It is not difficult to see that every \( F_\sigma \) ideal is skew and every analytic \( P \)-ideal is skew.

**Theorem 3.4.9.** If \( I \) is a skew analytic ideal on \( \omega \) then \( I^+ \) is a skew analytic ideal as well. If in addition the equivalence relation \( =_I \) is placid then the equivalence relation \( =_{I^+} \) is placid as well.

**Proof.** For the first sentence, if \( c \subseteq \omega \) is a set and \( c' \subseteq c \) witnesses the skew property for \( I \) and \( c \), then \( c' \) also witnesses the skew property for \( I^+ \) and \( c \). The proof of the second sentence is more involved. Write \( X = (2^\omega)^\omega \). Let \( V[G_0], V[G_1] \) be generic extensions containing respective \( =_{I^+} \)-related points \( x_0, x_1 \in X \). Assume \( V[G_0] \cap V[G_1] = V \) and work to find a ground model point \( x \in X \) which is \( =_{I^+} \)-related to both \( x_0, x_1 \).

Consider the set \( b = \{ n \in \omega: x_0 \upharpoonright a_n \text{ is } I^+-\text{equivalent to some element of the ground model} \} \). The definition of the set \( b \) depends only on the \( =_{I^+} \)-equivalence class of \( x_0 \), therefore the set \( b \) belongs to both \( V[G_0] \) and \( V[G_1] \), and by the initial assumptions, to \( V \). Let \( f \) be the map with domain \( b \) which to each \( n \in b \) identifies the \( =_{I^+} \)-class in \( V \) which contains \( x_0 \upharpoonright a_n \). Again, the definition of \( f \) depends only on the \( =_{I^+} \)-class of \( x_0 \) and so \( f \in V[G_0] \) and \( f \in V[G_1] \), therefore \( f \in V \). By the Mostowski absoluteness between \( V \) and \( V[G_0] \), there is a point \( x \in X \) such that for all \( n \in b \), \( x \upharpoonright a_n \in f(n) \). We will show that \( x =_{I^+} x_0 \) holds.

Suppose towards contradiction that this fails. Consider the set \( c = \{ i \in \omega: x(i) \neq x_0(i) \} \notin I^+ \) and the set \( d = \{ i \in \omega: x_0(i) \neq x_1(i) \} \in I^+ \). It is now
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Time to use the skew assumption on the ideal $I$. Find a set $c' \subset c$ witnessing the skew property of $I^+$. Use the definition of the jump to find a set $e \in I$ such that for all $n \in \omega$, $d \cap a_n \notin I$ implies $e \cap a_n \neq 0$. Since the set $c'$ is positive in the jump, there must be a number $n \in \omega$ such that $c' \cap a_n \notin I$ and $e \cap a_n = 0$. The choice of the set $c'$ shows that $c' \cap a_n$ is in fact $I^+$-positive. The choice of the set $e$ shows that $d \cap a_n \in I$. This means that $x_0 \upharpoonright a_n = x_1 \upharpoonright a_n$, and by the placid assumption on the equivalence relation $=_I$, $n \in b$ must hold. This stands in contradiction with the fact that $c' \cap a_n$ and so $c \cap a_n$ is $I^+$-positive. The proof of the theorem is complete.

Example 3.4.10. Let $I$ be the branch ideal on $2^{<\omega}$, and let \{${a_n : n \in \omega}$\} be an enumeration of the basic open subsets of $2^{<\omega}$. The ideal $I$ is skew. One can start iterating the jump for countable ordinals $\alpha$, at limit stages taking unions. The resulting ideals consist of subsets of $\omega^{<\omega}$ whose closure in $2^{\omega}$ is countable with Cantor–Bendixson rank $\leq \alpha$. All the resulting equivalence relations are placid.

3.5 Absoluteeness

The placid and virtually placid classes of equivalence relations are defined in such a way that it is not clear whether the membership in them is absolute, and what the actual complexity is. This section provides a satisfactory resolution to these questions.

To reach the absoluteness result, one has to perform a computation of intersections of forcing extensions of independent interest. The computation starts with several definitions:

Definition 3.5.1. Let $B$ be a Boolean algebra. A subalgebra $A \subset B$ is projective if the projection function $\pi : B \to A$ assigning to each $b \in B$ the smallest element of $A$ which is $\geq b$, is defined for every $b$.

A good example of a projection is any complete subalgebra of a complete Boolean algebra. The point of the current definition is that the property of being projective is first order in $A, B$, and so (unlike completeness) transfers from model to model without damage. Note that if $A \subset B$ is a projective subalgebra, then every maximal antichain of $A$ is also a maximal antichain of $B$, and therefore the intersection of a generic filter on $B$ will be a generic filter on $A$. To see this, let $D \subset A$ be a maximal antichain of $A$ and $b \in B$ be a condition. Let $d \in D$ be an element of $A$ compatible with $\pi(b)$. Then $\pi(b) - d \in A$ is strictly smaller element of $A$ than $\pi(b)$, and so it is not greater than $b$ anymore. This means that $d$ and $b$ must be compatible.

Definition 3.5.2. Let $B$ be a Boolean algebra and $A_0, A_1 \subset B$ be projective subalgebras with projection functions $\pi_0, \pi_1$.

1. The projection sequence starting at $b \in B$ is the sequence $\langle b_n : n \in \omega\rangle$ defined by $b_0 = b$, $b_{2n+1} = \pi_0(b_{2n})$ and $b_{2n+2} = \pi_1(b_{2n+1})$;
2. the pair \( \{A_0, A_1\} \) is projective if for each \( b \in B \), the supremum of the projection sequence starting at \( b \) exists in \( B \) and belongs to \( A_0 \cap A_1 \).

Again, a good example of a projective pair is a pair of complete subalgebras of a complete Boolean algebra. The notion of a projective pair is no longer a first order statement about \( A_0, A_1, B \), but it still transfers between \( \omega \)-models of ZFC without damage.

**Proposition 3.5.3.** Let \( B \) be a Boolean algebra and \( \{A_0, A_1\} \) be a projective pair of subalgebras. Then

1. \( A_0 \cap A_1 \subset B \) is a projective subalgebra of \( B \);
2. if \( \tau_0, \tau_1 \) and \( A_0 \)- and \( A_1 \)-names for sets of ordinals and \( B \models \tau_0 = \tau_1 \), then there is an \( A_0 \cap A_1 \)-name \( \tau_2 \) such that \( B \models \tau_0 = \tau_1 = \tau_2 \).

**Proof.** Let \( \pi_0, \pi_1 \) be the projections of \( B \) to \( A_0, A_1 \). Let \( \pi : B \to A_0 \cap A_1 \) be the function which assigns to each \( b \in B \) the supremum of the projection sequence starting from \( b \). It is clear from the definitions that \( \pi(b) \) is the smallest element of \( A_0 \cap A_1 \) above \( b \) and (1) follows.

For (2), first note that for every \( b \in B \) and every ordinal \( \alpha, b \models \bar{\alpha} \in \tau_0 \) just in case \( \pi(b) \models \bar{\alpha} \in \tau \). To see this, by induction on \( n \in \omega \) argue that \( b_n \models \bar{\alpha} \in \tau_0 \). If this holds for \( n = 2k \), then use the fact that \( \tau_0 \) is an \( A_0 \)-name and so \( b_{n+1} = \tau_0(b_n) \) must force \( \bar{\alpha} \in \tau \); if this holds for \( n = 2k + 1 \), then use the fact that \( \tau_1 \) is an \( A_1 \)-name forced to be equal to \( \tau_0 \) and so \( b_{n+1} = \tau_1(b_n) \) must force \( \bar{\alpha} \in \tau \). It follows that \( b \models \bar{\alpha} \in \tau \) just in case \( \pi(b) \models \bar{\alpha} \in \tau \), and so one can let \( \langle \pi(b), \bar{\alpha} \rangle \in \tau_2 \) if \( b \models \bar{\alpha} \in \tau_0 \).

The above notions have the slightly unpleasant feature that they do not necessarily survive localization well, so an additional local definition is needed.

**Definition 3.5.4.** Let \( B \) be a Boolean algebra and \( \{A_0, A_1\} \) be a pair of subalgebras. The pair is locally projective if for every nonzero element \( b \in B \), in the algebra \( B \upharpoonright b \) the subalgebras \( A_0 \upharpoonright b = \{a \wedge b : a \in A_0\} \) and \( A_1 \upharpoonright b = \{a \wedge b : a \in A_1\} \) form a projective pair.

The following is the central motivation of the notion of a projective pair of subalgebras:

**Proposition 3.5.5.** Let \( B \) be a Boolean algebra and \( A_0, A_1 \) be a locally projective pair of subalgebras. The following are equivalent:

1. \( B \models \forall b \in \dot{G} \cap A_0 \cap V[\dot{G} \cap A_1] = V \);
2. for every \( b \in B \), the intersection algebra \( A_0 \upharpoonright b \cap A_1 \upharpoonright b \) has an atom.

**Proof.** Suppose that (2) fails for some \( b \in B \). Write \( C = A_0 \upharpoonright b \cap A_1 \upharpoonright b \) and use Proposition 3.5.3 to argue that \( C \) is a projective subalgebra of \( B \upharpoonright b \), and so \( b \models \dot{G} \cap C \) is a \( C \)-generic filter, and since \( C \) is nonatomic, \( \dot{G} \cap C \notin V \). At the same time, \( \dot{G} \cap C \) belongs to \( V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] \), and so (1) fails.
3.6. A VARIATION FOR MEASURE

Suppose now that (2) holds. Suppose that \( \tau_0, \tau_1 \) are \( A_0 \)- and \( A_1 \)-names for sets of ordinals and some condition \( b \in B \) forces \( \tau_0 = \tau_1 \). Write \( C = A_0 \upharpoonright b \cap A_1 \upharpoonright b \) and let \( c \) be an atom of \( C \). To verify (1), it is enough to argue that \( c \) decides the membership of any given ordinal in \( \tau_0 \). However, this follows immediately from Proposition 3.5.3 (2) applied below \( b \).

Finally, we are in a position to give a succinct and principled proof of the main absoluteness result of this section.

**Theorem 3.5.6.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \). The statement “\( E \) is virtually placid” is absolute among transitive models of set theory containing the code for \( E \).

**Proof.** We will show that the statement “\( E \) is not virtually placid” is in ZFC equivalent to the statement “there is a countable \( \omega \)-model \( M \) of ZFC- containing the code for \( E \) such that \( M \models E \) is not virtually placid”. This is an analytic statement and therefore absolute among transitive models of set theory.

Now, the left-to-right implication is immediate: one just needs to take a countable elementary submodel of a large enough structure to get the requisite \( M \). The right-to-left direction is more interesting. Suppose that \( M \) is a countable \( \omega \)-model containing the code for \( E \) which satisfies that \( E \) is not virtually placid. Working in the model \( M \), there must be complete algebra \( B \), complete subalgebras \( A_0, A_1 \subset B \), and respective \( A_0, A_1 \)-names \( \tau_0, \tau_1 \) for elements of the underlying space \( X \) such that \( B \models V[\hat{G} \cap A_0] \cap V[\hat{G} \cap A_1] = V \) and \( \tau_0 E \tau_1 \), and \( B \times B \vdash \neg(\tau_0)_{\text{left}} E (\tau_0)_{\text{right}} \). The subalgebras \( A_0, A_1 \subset B \) form a locally projective pair in \( M \) since they are complete. By Proposition 3.5.5 applied in \( M \) in the (1) \( \rightarrow \) (2) direction, for every \( b \in B \), the algebra \( A_0 \upharpoonright b \cap A_1 \upharpoonright b \) has an atom. Stepping out of the model \( M \), we see that the pair \( A_0, A_1 \subset B \) is a projective pair and for every \( b \in B \), the algebra \( A_0 \upharpoonright b \cap A_1 \upharpoonright b \) has an atom. By Proposition 3.5.5 applied in the (2) \( \rightarrow \) (1) direction, \( B \models V[\hat{G} \cap A_0] \cap V[\hat{G} \cap A_1] = V \) holds. Moreover, \( B \models \tau_0 \in E (\tau_1) \), and \( B \times B \vdash \neg(\tau_0)_{\text{left}} E (\tau_0)_{\text{right}} \), since \( E \) is absolute between generic extensions of \( V \) and generic extensions of \( M \) by Borel absoluteness. In conclusion, \( E \) is not virtually placid in \( V \).

**Corollary 3.5.7.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \). The statement “\( E \) is placid” is absolute among transitive models of set theory containing the code for \( E \).

**Proof.** The placidity of \( E \) is a conjunction of virtual placidity and the pinned property of \( E \) by Proposition 3.3.4. The conjuncts are absolute by Theorem 3.5.6 and 2.6.1, and so is the conjunction.

3.6 A variation for measure

The purpose of this section is to introduce a parallel for turbulence for measure, with attendant ergodicity results for measure. Curiously enough, the measure theoretic parallel for turbulence is intimately connected to the concentration of
measure phenomenon. The following definition is close to the whirly actions of [26]:

**Definition 3.6.1.** Let $X$ be a Polish space with a Borel probability measure $\mu$ and a metric $d$. Let $\Gamma$ be a Polish group acting on $X$ in a measure preserving and distance preserving fashion. We say that the action has **concentration of measure** if for every open neighborhood $U \subset \Gamma$ containing the unit and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $d$-ball $B$ of radius $< \delta$ and every Borel set $C \subset B$ of relative $\mu$-mass $> \varepsilon$, the set $(U \cdot C) \cap B$ has relative $\mu$-mass $> 1/2$.

To formulate the main results of this section, let $P_\Gamma$ be the Cohen forcing on $\Gamma$ and $P_\mu$ be the random forcing with $\mu$, i.e. the forcing with $\mu$-positive Borel subset of $X$, ordered by inclusion. The poset $P_\Gamma$ adds a generic point $\dot{\gamma}_{\text{gen}} \in \Gamma$ while the poset $P_\mu$ adds a random point $\dot{x}_{\text{gen}} \in X$.

**Theorem 3.6.2.** Suppose that $\Gamma$ is a Polish group acting on a Polish space $X$ with a Borel probability measure $\mu$ and an ultrametric $d$ in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Then $P_\Gamma \times P_\mu \Vdash V[\dot{\gamma}_{\text{gen}}] \cap V[\dot{x}_{\text{gen}} \cdot \dot{\gamma}_{\text{gen}}] = V$.

The proof of Theorem 3.6.2 hinges on a new walk concept and a proposition.

**Definition 3.6.3.** Let $U \subset \Gamma$ be an open neighborhood of the unit and $\delta > 0$. A **$U,\delta$-walk** is a sequence $\langle x_i : i \leq j \rangle$ of points in $X$ such that for every $i \leq j$, either $d(x_i, x_{i+1}) < \delta$ or there is $\gamma \in U$ such that $\gamma \cdot x_i = x_{i+1}$.

**Definition 3.6.4.** Let $U \subset \Gamma$ be an open neighborhood of the unit and $D \subset X$ be a closed set. We say that the set $D$ is **$U$-connected** if for any two points $x_0, x_1 \in D$ and any $\delta > 0$ there is a $U,\delta$-walk from $x_0$ to $x_1$ using only points from $D$.

The main import of the concentration of measure assumption is that for any open neighborhood $U$ of the unit, closed $U$-connected $\mu$-positive sets can be found under every stone.

**Proposition 3.6.5.** Suppose that the action of $\Gamma$ has concentration of measure and $d$ is an ultrametric. Then for every open neighborhood $U \subset \Gamma$ of the unit and every $\mu$-positive Borel set $C \subset X$ there is a $\mu$-positive $U$-connected compact set $D \subset C$.

*Proof.* We will first fix useful terminology. For a symmetric neighborhood $V \subset \Gamma$ containing the unit and $d$-balls $B_0, B_1 \subset X$ of the same radius we say that $B_0$ is **$V$-related to** $B_1$ if there is an element $\gamma \in V$ such that $\gamma \cdot B_0 = B_1$. Since $d$ is an ultrametric, this is equivalent to the statement that there is $\gamma \in V$ such that $\gamma \cdot B_0 \cap B_1 \neq \emptyset$. A set $\mathcal{B}$ consisting of $d$-balls of the same radius will be called **$V$-connected** if for any two balls $B_0, B_1 \in \mathcal{B}$ one can find a sequence of balls in $\mathcal{B}$ which starts with $B_0$, ends in $B_1$, and successive balls in it are $V$-related. Lastly, for Borel sets $B, D \subset X$ write $\mu_B(D)$ for the relative measure of $D$ in $B$, i.e. the ratio $\frac{\mu(B \cdot D)}{\mu(B)}$. 
Let \( U \subset \Gamma \) be an open neighborhood of the unit, and let \( C \subset X \) be a Borel \( \mu \)-positive set. Find a symmetric open neighborhood \( V \subset \Gamma \) of the unit such that \( V^4 \subset U \), and thin down \( C \) if necessary to a compact \( \mu \)-positive set. Let \( \varepsilon_0 = 1/2 \) and for each \( n \in \omega \) let \( \varepsilon_{n+1} = \frac{\varepsilon_n}{2n+2} \). Let \( \delta_n > 0 \) be the numbers witnessing the concentration of measure of the action for \( V \) and \( \delta_n \). Thinning down \( C \) if necessary, use the Lebesgue density theorem to find a \( d \)-ball \( B_{ini} \) of radius \( \varepsilon_0 \) such that \( C \subset B_{ini} \) and \( \mu_B(C) > \varepsilon_0 \).

Let \( B_0 = \{B_{ini}\} \) and by induction on \( n \in \omega \) find finite families \( B_n \) of pairwise disjoint \( d \)-balls of the same radius \( \varepsilon_n \) such that for every number \( n \in \omega \), \( B_{n+1} \) refines \( B_n \) and \( C \subset \bigcup B_n \). We will find families \( B'_{n+1} \subset B_n \) such that \( B'_{n+1} \subset C \cup B_n \). We will find families \( B'_{n+1} \subset B_n \) such that \( B'_{n+1} \subset C \cup B_n \). Let \( B'_n \) be an open neighborhood of the unit, and let \( B'_n \) be a \( B \)-ball \( \Gamma \) such that for every number \( n \in \omega \), \( B_{n+1} \) refines \( B_n \) and \( C \subset \bigcup B_n \). We will find families \( B'_{n+1} \subset B_n \) such that \( B'_{n+1} \subset C \cup B_n \). We will find families \( B'_{n+1} \subset B_n \) such that \( B'_{n+1} \subset C \cup B_n \). Let \( Y_0 = \{A \in B_{n+1} : A \subset B \} \). Let \( Y_1 = \{A \in Y_0 : \mu(A) > (2^n + 2)\varepsilon_{n+1}\} \). Thus, \( \mu_B(C \cap \bigcup (Y_0 \setminus Y_1)) < 2^{n+1}\varepsilon_{n+1} < \varepsilon_n \).

**Claim 3.6.6.** There is a \( V^2 \)-connected component \( Y_2 \subset Y_1 \) such that \( \mu_B(C \cap \bigcup Y_2) > (1 - 2^{-n})\mu_B(C) \).

**Proof.** We first show that there is a \( V^2 \)-connected component \( Y_2 \subset Y_1 \) such that \( \mu_B(C \cap \bigcup Y_2) > \varepsilon_n \). If this were not the case, it would be possible to divide \( Y_1 \) into sets \( Y' \) and \( Y'' \) which are both invariant under \( V^2 \)-connections, and \( \mu_B(C \cap \bigcup Y') \) and \( \mu_B(C \cap \bigcup Y'') \) are both greater than \( \varepsilon_n \). By the concentration of measure assumption, the sets \( V \cdot (C \cap \bigcup Y') \) and \( V \cdot (C \cap \bigcup Y'') \) are both of \( \mu_B \)-mass greater than \( 1/2 \) and therefore intersect. It follows that \( V^2 \cdot (C \cap \bigcup Y') \cap (C \cap \bigcup Y'') \neq \emptyset \), contradicting the invariance of the sets \( Y' \) and \( Y'' \) under \( V^2 \)-connections.

Now, the \( V^2 \)-connected component \( Y_2 \) found in the first paragraph must in fact be such that \( \mu_B(C \cap \bigcup (Y_1 \setminus Y_2)) < \varepsilon_n \), by an argument identical to the one in the previous paragraph with \( Y' = Y_2 \) and \( Y'' = Y_1 \setminus Y_2 \). This completes the proof of the claim.

Letting \( Y(B) = Y_2 \) completes the construction. The \( V^2 \)-connectedness is clear from the choice of \( Y_2 \), the first item is clear from the choice of \( Y_1(B) \), and the second item follows from some arithmetic: \( \mu_B(C \cap \bigcup (Y \setminus Y_2)) < 2\varepsilon_n \) which is smaller than \( 2^{-n}(2^n + 2)\varepsilon_n \).

Now, let \( D = \bigcap B_n' \). This is a closed subset of \( C \) of positive mass. Its mass distribution is governed by the following claim:

**Claim 3.6.7.** For every ball \( B \in B_n' \), \( \mu_B(D) > \varepsilon_n \).
Proof. To get the set $D \cap B$, we subtract from $C \cap B$ a set of size at most $\Sigma_{m>n}2^{-m}\mu_B(C)$ by the second item above, and so $\mu_B(D) > 1/2\mu_B(C) > \varepsilon_n$ by the first item above.

Towards the connectedness of the set $D$, consider the following claim:

**Claim 3.6.8.** Let $m \leq n$ be natural numbers, $B \in B'_m$ a ball, and $x_0, x_1 \in B \cap D$ be any points. Then there is a $U, \delta_n$-walk from $x_0$ to $x_1$ using only points in $B \cap D$.

**Proof.** This is proved by induction on $n - m$, which is the reason for the convoluted statement of the claim. The case $n - m = 0$ is trivial since then $d(x_0, x_1) < \delta_n$. Now suppose that the statement is known for $m + 1 \leq n$ and proceed to show that it is also true for $m \leq n$. Let $B_0, B_1 \in B'_{m+1}$ be balls such that $x_0 \in B_0$ and $x_1 \in B_1$. If $B_0 = B_1$ then the induction hypothesis immediately applies. Otherwise, by the third item above, the set $Y(B)$ is $V^2$-connected, and so there must be a sequence of balls in $Y(B)$ starting at $B_0$ and ending at $B_1$ such that successive balls are $V^2$-connected. Suppose for simplicity that this sequence has length 2—i.e. $B_0$ and $B_1$ are $V^2$-connected. Fix an element $\gamma \in V^2$ such that $\gamma \cdot B_0 = B_1$.

Since $\mu_{B_0}(D)$ and $\mu_{B_1}(D)$ are both greater than $\varepsilon_{m+1}$, the concentration assumption yields that the numbers $\mu_{B_0}(V \cdot (D \cap B_0)) = \mu_{B_1}(V \cdot (D \cap B_1))$ are both greater than $1/2$. It follows that there is a point $x \in (V \cdot (D \cap B_0)) \cap B_0$ such that $\gamma \cdot x \in (V \cdot (D \cap B_1))$. Find points $x'_0 \in D \cap B_0$ and $x'_1 \in D \cap B_1$ and group elements $\beta_0, \beta_1 \in V$ such that $\beta_0 \cdot x'_0 = x$ and $\beta_1 \cdot f(x) = x'_1$. Use the induction hypothesis to find a $U, \delta_n$-walk from $x_0$ to $x'_0$. Follow it by acting by $\beta_0 \gamma \beta_1 \in U$ to get to the point $x'_1$, and then use the induction hypothesis again to extend the walk from $x'_1$ to $x_1$. The claim follows.

Since the numbers $\delta_n$ tend to 0 as $n$ tends to infinity and all points in $D$ belong to the unique ball in $B'_0$, the claim shows that the set $D$ is $U$-connected and completes the proof of the proposition.

**Proof of Theorem 3.6.2.** Suppose towards contradiction that $(U, C)$ is a condition in the product $P_T \times P_\mu$ which forces $V[x_{\text{gen}}] \cap V[\hat{\gamma}_{\text{gen}} \cdot \hat{x}_{\text{gen}}] \neq V$. The following claim follows from abstract forcing considerations.

**Claim 3.6.9.** $(U, C) \Vdash 2^\omega \cap V[x_{\text{gen}}] \cap V[\hat{\gamma}_{\text{gen}} \cdot \hat{x}_{\text{gen}}] \neq 2^\omega \cap V$.

**Proof.** It will be enough to show that $P_\mu$ forces that for every ordinal $\alpha$ and every function $f: \alpha \to 2$ in the extension, if $f \notin V$ then there is a ground model countable set $b \subset \alpha$ such that $f \upharpoonright b \notin V$. It will follow immediately that $P_T \times P_\mu \Vdash$ if $f \in V[x_{\text{gen}}] \cap V[\hat{\gamma}_{\text{gen}} \cdot \hat{x}_{\text{gen}}] \cap V$ then $f \upharpoonright b \in V[x_{\text{gen}}] \cap V[\hat{\gamma}_{\text{gen}} \cdot \hat{x}_{\text{gen}}] \cap V$ for some ground model countable set $b$, and therefore $2^\omega \cap V[x_{\text{gen}}] \cap V[\hat{\gamma}_{\text{gen}} \cdot \hat{x}_{\text{gen}}] \neq 2^\omega \cap V$ as desired.

The forcing property of $P_\mu$ in question is well-known; we include a complete proof. Suppose towards contradiction that $B \in P_\mu$ and $B \Vdash \tau: \alpha \to 2$ is a function which is not in the ground model, and for every countable set $b \in V$,
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τ \mid b \in V. Let (M_3, \beta \in \omega_1) be an \varepsilon\text{-tower of countable elementary submodels of a large structure containing } B, \tau \text{ as elements. For each } \beta \in \omega_1 \text{ use the contradictory assumption to find a function } g_\beta : M_3 \cap \alpha \to 2 \text{ in the model } M_{\beta+1} \text{ such that some condition below } B \text{ forces } \tau \mid M_3 = g_\beta. \text{ Let } B_\beta \subset B \text{ be the Borel set representing the nonzero Boolean value of the latter statement; } B_\beta \in M_{\beta+1} \text{ holds by elementarity of the model } M_{\beta+1}, \text{ but also } B_\beta \notin M_\beta \text{ since } \tau \text{ is forced not to belong to the ground model. Use a counting argument and the Lebesgue density theorem to find a basic open set } O \subset X \text{ such that the set } C = \{ \beta \in \omega_1 : \mu(B_\beta \cap O) > \frac{1}{2} \mu(O) \} \text{ is uncountable. Since the conditions } \{ B_\beta : \beta \in C \} \text{ are pairwise compatible, the functions } \{ g_\beta : \beta \in C \} \text{ must form an increasing chain and so in fact the conditions } \{ B_\beta : \beta \in C \} \text{ form a strictly decreasing chain in } P_\mu. \text{ This contradicts the countable chain condition of } P_\mu. \square

Strengthening the condition \langle U, C \rangle \text{ if necessary, we may find a continuous function } f : C \to 2^\omega \text{ and a name } \tau \text{ in the complete Boolean algebra generated by the name for } \hat{\gamma} \text{ such that the fibers of } f \text{ are } \mu\text{-null and } \langle U, C \rangle \models f(\hat{\gamma}) = \tau/\hat{\gamma}. \text{ Let } M \text{ be a countable elementary submodel of a large structure containing } f, \tau \text{ in particular, and let } C' \subset C \text{ be a set of points } P_\mu\text{-generic over the model } M. \text{ The set } C' \text{ is Borel and } \mu\text{-positive. Find open subsets } V, U' \text{ of } \Gamma \text{ such that } 1 \in V, U' \subset U, \text{ and } U' \cdot V \subset U. \text{ Use Proposition 3.6.5 to find a } V\text{-connected compact } \mu\text{-positive subset } D \subset C'.

Since the fibers of } f \text{ are } \mu\text{-null, there must be points } x_0, x_1 \in D \text{ such that } f(x_0) \neq f(x_1). \text{ Find a number } n \in \omega \text{ such that } f(x_0)(n) \neq f(x_1)(n). \text{ Let } O_0 = \{ x \in D : f(x)(n) = f(x_0)(n) \} \text{ and } O_1 = \{ x \in D : f(x)(n) = f(x_1)(n) \}. \text{ These are complementary relatively open subsets of the compact set } D \text{ and so they are separated by some distance } \delta > 0. \text{ Use the connectedness of the set } D \text{ to produce a } V, \delta\text{-walk from } x_0 \text{ to } x_1. \text{ There must be successive points } x'_0, x'_1 \text{ on the walk such that } f(x'_0)(n) \neq f(x'_1)(n). \text{ The two points are at a distance } \geq \delta \text{ by the choice of } \delta, \text{ and so there must be a group element } \beta \in V \text{ such that } \beta \cdot x'_0 = x'_1. \text{ By the Baire category theorem, there must be an element } \gamma \in U' \text{ which belongs to no meager subset of } \Gamma \text{ coded in the model } V[x'_1] \text{ and also to no right } \beta^{-1}\text{-shift of any meager subset of } \Gamma \text{ coded in the model } V[x'_0]. \text{ As a result, the point } \gamma \text{ is } P_\gamma\text{-generic over } V[x'_1] \text{ and the point } \gamma \cdot \beta \text{ is } P_\gamma\text{-generic over the model } V[x'_0]. \text{ Both of these points belong to the set } U. \text{ By the product forcing theorem, the pairs } \langle \gamma \beta, x'_0 \rangle \text{ and } \langle \gamma, x'_1 \rangle \text{ are both } P_Y \times P_\mu\text{-generic over the model } M, \text{ meeting the condition } (U, C). \text{ However, } f(x'_0) \neq f(x'_1) \text{ while } \tau/\gamma \beta x'_0 = \tau/\gamma x'_1, \text{ violating the forcing theorem in view of the initial contradictory assumption. Theorem 3.6.2 follows!} \square

The main corollaries are encapsulated in the following ergodicity result.

**Corollary 3.6.10.** Suppose that } \Gamma \text{ is a Polish group acting on a Polish space } X \text{ with a Borel probability measure } \mu \text{ and an ultrametric } d \text{ in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Suppose that } E \text{ is the orbit equivalence relation, } Y \text{ is a Polish space, } F \text{ on } Y \text{ is an analytic virtually placid equivalence relation, and } h : X \to Y \text{ is a
Borel homomorphism from $X$ to $Y$. Then there is an $F$-equivalence class with \( \mu \)-positive $h$-preimage.

**Proof.** Let $\gamma \in \Gamma$ and $x \in X$ be mutually $P_\Gamma$-generic and $P_\mu$-generic points, and look at the models $V[x]$ and $V[\gamma \cdot x]$. Since $h$ is a homomorphism of $E$ to $F$ and $x E \gamma \cdot x$, $h(x) F h(\gamma \cdot x)$ must hold. Since $F$ is virtually placid and $V[x] \cap V[\gamma \cdot x] = V$ holds per Theorem 3.6.2, there must be a virtual $F$-class in the ground model such that $h(x)$ and $h(\gamma \cdot x)$ realize it. Since the poset $P_\Gamma \times P_\mu$ is c.c.c., Theorem 2.8.2 shows that all virtual classes realized in its extension are in fact represented in the ground model. Thus, there is a ground model element $y \in Y$ such that $h(x) F y$ holds. Since $x$ is a $P_\mu$-generic point, it belongs to no analytic ground model coded $\mu$-small sets. Thus, $\mu(h^{-1}[y]_F) > 0$ as desired. \( \square \)

Examples of actions with concentration of measure are not easy to identify. The following examples use $F_\sigma$ $P$-ideals $I$ on $\omega$ (which are Polish groups with the symmetric difference operation and a suitable topology by a result of Solecki [72]) and their standard action on $2^\omega$ ($a \cdot x = y$ just in case $\{n \in \omega : x(n) \neq y(n)\} = a$), inducing the equivalence relation $\equiv_I$. The action preserves the usual Borel probability measure $\mu$ on $2^\omega$ and also the usual minimum difference metric $d$ on $2^\omega$.

**Example 3.6.11.** Let $a_n \colon n \in \omega$ be positive real numbers such that $\Sigma_n a_n$ is infinite while $\Sigma_n a_n^2$ is finite. Let $I$ be the ideal of all sets $b \subset \omega$ such that $\Sigma_n \in b a_n$ is finite. The standard action of $I$ on $2^\omega$ exhibits the concentration of measure. To see this, let $U$ be a neighborhood of the unit in $\Gamma$, and $\varepsilon > 0$ be a real number. Find a real number $\eta > 0$ such that the $d, \eta$-ball around the unit is a subset of $U$, and find a number $m \in \omega$ such that $2^\exp(-\eta^2/8\Sigma_n=m a_n) < \varepsilon$. The concentration of measure formula in [60, Theorem 4.3.19] then shows that $\delta = 2^{-m}$ works as required in Definition 3.6.1.

**Corollary 3.6.12.** Let $I$ be the usual summable ideal on $\omega$. Let $h \colon 2^\omega \to X$ be a Borel homomorphism of $\equiv_I$ to a virtually placid analytic equivalence relation $F$ on a Polish space $X$. Then there is an $F$-class whose $h$-preimage has full $\mu$-mass.

**Proof.** By Corollary 3.6.10, there is an $F$-class whose $h$-preimage has positive $\mu$-mass. However, the $h$-preimage is an $=I$-invariant set, the equivalence relation $=I$ includes $E_0$ as a subset, and by the usual $E_0$-ergodicity considerations, the $h$-preimage must in fact have full $\mu$-mass. \( \square \)

The concentration of measure for actions fails in many cases. Typically, there is a homomorphism of the orbit equivalence relation which violates the conclusion of Corollary 3.6.10 and therefore witnesses the failure of the concentration of measure in a strong sense.

**Example 3.6.13.** There is a summable-type ideal $I$ on $\omega$ and a Borel homomorphism $h \colon B \to 2^\omega$ of $=I$ to $E_0$ such that preimages of $E_0$-equivalence classes are $\mu$-null.
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Proof. The key tool is the following:

**Claim 3.6.14.** For every \( i \in \omega \) and every \( \varepsilon > 0 \) there is a number \( n \in \omega \) and sets \( a, b \subset 2^n \) of the same relative size \( > \frac{1-\varepsilon}{2} \) each, such that for every \( x \in a \) and \( y \in b \) the set \( \{ m \in n : x(m) \neq y(m) \} \) contains at least \( i \) many elements.

**Proof.** Fix \( i \) and \( \varepsilon \). Elementary computation shows that there is \( n \in \omega \) such that the size of the set \( \{ a \subset n : ||a| - \frac{n}{2}^i | < i + 1 \} \) is less than \( \varepsilon 2^n \). Let \( a = \{ x \in 2^n : \text{the set } \{ m \in n : x(m) = 1 \} \text{ contains at least } \frac{n}{2} - i \text{ many elements} \} \) and \( b = \{ x \in 2^n : \text{the set } \{ m \in n : x(m) = 1 \} \text{ contains at least } \frac{n}{2} + 1 \text{ many elements} \} \). This works.

Towards the proof of the example, find a partition \( \omega = \bigcup_n I_n \) into finite sets such that for every \( n \in \omega \), the set \( 2^{I_n} \) contains subsets \( a_n, b_n \) of the same size such that their relative size in \( 2^{I_n} \) is greater than \( 1/2 - 2^{-n} \), and if \( x \in a_n \) and \( y \in b_n \) are arbitrary elements, then the set \( \{ i \in I_n : x(n) \neq y(n) \} \) has size at least \( n \).

Now, define \( w(m) = 1/n + 1 \) if \( m \in I_n \) and \( I = \{ a \subset \omega : \Sigma_{n \in a} w(n) < \infty \} \). Define \( B = \{ x \in 2^{\omega} : \forall \alpha \in \omega \text{ } x \upharpoonright \alpha \in a_\alpha \cup b_\alpha \} \); this is a Borel set of full mass. For \( x \in B \), define \( h_0(x) \in 2^{\omega} \) by \( h_0(x)(n) = 0 \leftrightarrow x \upharpoonright I_n \in a_n \). It is not difficult to check that \( h_0 : B \to 2^{\omega} \) is a continuous homomorphism from \( B \) to \( E_\omega \) such that preimages of \( E_\omega \)-classes are of zero mass. The rest of the proof only adjusts \( h_0 \) to a Borel homomorphism \( h \) defined on the whole space.

To this end, let \( C_n \) for \( n \in \omega \) be inclusion increasing compact subsets of \( B \) whose mass converges to 1. The set \( \bigcup_n C_n = K_\sigma \) and the equivalence relation \( =_I \) is \( K_\sigma \), and so the saturation \( D = [\bigcup_n C_n]_{=_I} = K_\sigma \) as well. Let \( F \subset D \times 2^\omega \) be the Borel set of all pairs \( (x, y) \) such that for some (equivalently, for all) \( x' \in B \) such that \( x' =_I x \), \( h_0(x') \in E_\omega \)-related to \( y \). \( F \) is a Borel set with nonempty countable sections, and by the Lusin–Novikov theorem, it has a Borel uniformization \( h \). Extend \( h \) to all of \( 2^\omega \) by defining \( h(x) \) for \( x \notin D \) to be an arbitrary fixed element of \( 2^\omega \). It is not difficult to check that \( h \) has the required properties.

**Example 3.6.15.** There is a Tsirelson-type ideal \( I \) on \( \omega \) and a Borel homomorphism \( h : B \to 2^\omega \) of \( =_I \) to \( E_\omega \) such that preimages of \( E_\omega \)-equivalence classes are \( \mu \)-null.

**Proof.** We will deal with a certain special kind of Tsirelson submeasures. Let \( \alpha > 0 \) be a real number and \( f : \omega \to \mathbb{R}^+ \) be a function. In a typical case, the function \( f \) will converge to 0 and never increase. By induction on \( n \in \omega \) define submeasures \( \nu_n \) on \( \omega \) by setting \( \nu_0(a) = \sup_{i \in a} f(i) \), and \( \nu_{n+1}(a) = \sup\{ \nu_n(a), \alpha \sum_{b \in F} \nu_n(b) \} \) where the variable \( b \) ranges over all finite sequences \( \langle b_0, b_1, \ldots, b_j \rangle \) of finite subsets of \( a \) such that \( j < \min(b_0) \leq \max(b_0) < \min(b_1) \leq \max(b_1) < \ldots \). In the end, let the submeasure \( \nu \) be the supremum of \( \nu_n \) for \( n \in \omega \). Some computations are necessary to verify that \( \nu \) is really a lower semicontinuous submeasure on \( \omega \). The Tsirelson ideal \( I = \{ a \subset \omega : \lim m, \nu(x \upharpoonright m) = 0 \} \) turns out to be an \( F_\sigma \) P-ideal [21].

By induction on \( i \in \omega \) choose intervals \( I_i \subset \omega \) such that \( \max(I_i) > \min(I_{i+1}) \) and such that \( \min(I_i) > i/\alpha \) there are sets \( a_i, b_i \subset 2^{I_i} \) of the same relative size.
such that for any elements \( x \in a_i, y \in b_i \) the set \( \{ m \in I_i : x(m) \neq y(m) \} \) has size at least \( \alpha i \). This is easily possible by Claim 3.6.14. Now, consider the function \( f \) defined by \( f(t) = 1/i \) for \( t \in (\max(I_{i-1}), \max(I_i]) \) and let \( \nu \) be the derived submeasure and \( I \) the derived Tsirelson ideal. Observe that with this choice of the function \( f \), for any \( i \in \omega \) and elements \( x \in a_i, y \in b_i \) the set \( \{ m \in I_i : x(m) \neq y(m) \} \) has \( \nu \)-mass at least 1, since it has \( \nu_1 \)-mass at least 1.

The rest of the proof follows the lines of Example 3.6.13.

Example 3.6.16. Let \( J \) be the Rado graph ideal on \( \omega \). There is a Borel homomorphism \( h : 2^\omega \to 2^\omega \) from \( =_J \) to \( E_0 \) such that preimages of \( E_0 \)-classes are \( \mu \)-null.

Proof. Let \( G \) be the Rado graph, interpreted so that \( \omega \) is the set of its vertices; then \( J \) is the ideal on \( \omega \) generated by \( J \)-cliques and \( J \)-anticliques. To construct \( h \), by induction on \( n \in \omega \) find pairwise disjoint finite sets \( I_n \subset \omega \) and sets \( a_n, b_n \subset 2^{I_n} \) such that

- each \( I_n \) is a \( G \)-anticlique;
- if \( n \neq m \) then \( I_n \times I_m \subset G \);
- for every \( x \in a_n \) and every \( y \in b_n \) the set \( \{ t \in I_n : x(t) \neq y(t) \} \) has size at least \( n \);
- the sets \( a_n, b_n \subset 2^{I_n} \) have the same relative size larger than \( 1/2 - 2^{-n} \).

This is easy to do using the universality properties of the Rado graph and Claim 3.6.14 repeatedly. Let \( B = \{ x \in 2^\omega : \forall n x \mid I_n \in a_n \cup b_n \} \) and let \( h_0 : B \to 2^\omega \) be the Borel map defined by \( h_0(x) = 0 \leftrightarrow x \mid I_n \in a_n \). It is immediate that the set \( B \) has full \( \mu \)-mass and the function \( h \) is a homomorphism from \( =_J \) to \( E_0 \). The rest of the proof follows the lines of Example 3.6.13.

There are numerous questions left open by the development of this section, of which we quote two.

Question 3.6.17. Can the assumption that \( d \) be an ultrametric be eliminated from the assumptions of Theorem 3.6.2?

Question 3.6.18. Is there a Tsirelson ideal whose natural action on \( 2^\omega \) exhibits concentration of measure?
Chapter 4

Nested sequences of models

4.1 Prologue

The purpose of this chapter is to set up a calculus for infinite nested sequences of models of ZFC, which turn out to be critical for the treatment of the $E_1$ cardinal. As a motivation, we include a simple proof of the fact that $E_1$ is not Borel reducible to any orbit equivalence relation. It is quite different from the standard one [39, Theorem 11.8.1], and it has the advantage that it can be easily adapted to the context of inequalities between cardinalities of quotient spaces.

**Theorem 4.1.1.** $E_1$ is not Borel reducible to any orbit equivalence relation.

*Proof.* Let $Y$ be the Polish space $(2^\omega)^\omega$ and $\Gamma$ be a Polish group continuously acting on a Polish space $Z$, inducing the orbit equivalence relation $F$. Suppose towards contradiction that there is a Borel reduction $h: Y \to Z$ of $E_1$ to $F$.

Let $\langle x_n : n \in \omega \rangle$ be a sequence of mutually Cohen generic points in $2^\omega$. Let $y_n$ denote the element of $Y$ for which $y(n) =$ the zero sequence in $2^\omega$ if $m < n$, and $y(m) = x_m$ if $m \geq n$, and let $V_n = V[y_n]$.

Work in the model $V_0$. The reinterpretation of the Borel map $h$ in $V_0$ is still a reduction of $E_1$ to $F$. For each number $n > 0$ fix a group element $\gamma_n \in \Gamma$ such that $\gamma_n \cdot h(y_n) = h(y_0)$. Let $\gamma \in \Gamma$ be a point Cohen-generic over $V[y_0]$ and look into the model $V[\gamma \cdot h(y_0)]$. By a Mostowski absoluteness argument, there must be a point $y \in V[\gamma \cdot h(y_0)]$ such that $h(y) F \gamma \cdot h(y_0)$. Since the function $h$ is a reduction of $E_1$ to $F$ even in the model $V_0[\gamma]$, this point $y \in Y$ must be $E_1$-related to $y_0$ and so for all but finitely many numbers $n$ it must be the case that $y(n) = x_n$. Choose a number $n \in \omega$ such that $y(n) = x_n$ and look at the model $V_{n+1}$. The point $\gamma \gamma_{n+1} \in \Gamma$ is Cohen generic over $V_0$ by the invariance of the meager ideal under right shift, and by the product forcing theorem it follows that the models $V_0$ and $V_{n+1}[\gamma \gamma_{n+1}]$ are mutually generic over the model $V_{n+1}$. Now, the points $y_{n+1}$ and $\gamma \gamma_{n+1} \cdot h(y_{n+1}) = \gamma \cdot h(y_0)$ belong to the model $V_{n+1}[\gamma \gamma_{n+1}]$ and so does $y$. Thus, even the point $y(n) = x_n \in 2^\omega$ belongs to this model; however, it is a point of $V_0$ Cohen generic over $V_{n+1}$. This contradicts the product forcing theorem. \( \square \)
4.2 Coherent sequences of models

It is clear from the argument that its generalizations will require codification of decreasing $\omega$-sequences of generic extensions. In addition to the approach from Theorem 4.1.1, we pay close attention to the intersection model. This is the contents of the following definitions and theorems.

**Definition 4.2.1.** Let $\langle V_n : n \in \omega \rangle$ be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is **coherent** if for every ordinal $\lambda$ and every natural number $n$, the intersection of the relation $R_\lambda = \{ \langle m, x \rangle : m \in \omega, \text{rank of } X \text{ is smaller than } \lambda, x \in V_m \}$ with $V_n$ belongs to $V_n$. Given a coherent sequence of models $\langle V_n : n \in \omega \rangle$, a sequence $\langle v_n : n \in \omega \rangle$ is **coherent** if for every number $m \in \omega$, $\langle v_n : n \geq m \rangle \in V_m$ holds.

**Example 4.2.2.** Let $P$ be the countable support product of countably many copies of the Sacks forcing, adding a sequence $\langle x_m : m \in \omega \rangle$. Let $V_n = V[x_m : m \geq n]$. Then $\langle V_n : n \in \omega \rangle$ is a coherent sequence of models.

**Example 4.2.3.** Let $x$ be a set and let $M$ be a class containing all ordinals. By induction on $n \in \omega$, define models $V_n$ by $V_0 = V$ and $V_{n+1}$ by the collection of all sets hereditarily definable from parameters in $M$ and the parameter $x \cap V_n$ in the model $V_n$. The sequence $\langle V_n : n \in \omega \rangle$ is coherent.

**Example 4.2.4.** Let $\kappa$ be a measurable cardinal, $U$ a measure on it, and $j : V \rightarrow M$ the $U$-ultrapower, with iterands denoted by $j_\alpha$ for every ordinal $\alpha$. For each $n \in \omega$ let $V_n = j_\alpha(V)$. The sequence $\langle V_n : n \in \omega \rangle$ is coherent.

The coherent sequences of models are most often formed as generic extensions of the constant sequence $\langle V_n = V : n \in \omega \rangle$ using the following definition and theorem.

**Definition 4.2.5.** Let $P, Q$ be posets. A **projection** of $Q$ to $P$ is a pair of order-preserving functions $\pi : Q \rightarrow P$ and $\xi : P \rightarrow Q$ such that

1. $\pi \circ \xi$ is the identity on $P$;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

**Definition 4.2.6.** Let $\langle V_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC. A **coherent sequence of posets** is a sequence $\langle P_n, \pi_{nm} : P_n \rightarrow P_m, \xi_{mn} : P_m \rightarrow P_n : n \leq m \in \omega \rangle$ such that

1. the maps $\pi_{nm}, \xi_{mn}$ form a projection of $P_n$ to $P_m$;
2. the functions $\pi_{nm}$ form a commutative system, same for $\xi_{mn}$, and $\pi_{nn} = \xi_{nn} =$ the identity on $P_n$;
3. for every number $k \in \omega$, the sequence $\langle P_n, \pi_{nm} : P_n \rightarrow P_m, \xi_{mn} : P_m \rightarrow P_n : k \leq n \leq m \in \omega \rangle$ belongs to the model $V_k$. 

4.2. COHERENT SEQUENCES OF MODELS

In particular, every commutative sequence of projections is coherent over the constant coherent sequence \( \langle V_n : n \in \omega \rangle \).

**Theorem 4.2.7.** Let \( \langle V_n : n \in \omega \rangle \) be a coherent sequence of models of ZFC and \( \langle P_n, \pi_{nm} : P_n \to P_m, \xi_{nm} : P_m \to P_n : n \leq m \in \omega \rangle \) be a coherent sequence of posets. Let \( G \subset P_0 \) be a filter generic over \( V_0 \), and let \( G_n = \xi_{0n}^{-1}G \). The sequence \( \langle V_n[G_n] : n \in \omega \rangle \) is a coherent sequence of models of ZFC again.

**Proof.** Observe that for every number \( k \in \omega \), in the model \( V_k[G_k] \), one can form the sequence \( \langle G_n : n \geq k \rangle \) as \( \langle \xi_{nk}^{-1}G_k : n \geq k \rangle \) by the commutativity of the \( \xi \)-maps. Thus, if \( \lambda \) is a limit ordinal larger than the rank of all the posets on the coherent sequence, one can form the relation \( \{ (n, x) : n \geq k, x \in V_n[G_n] \} \) has rank \( < \lambda \) in the model \( V_k[G_k] \) as the set \( \{ (n, \tau/G_n) : n \geq k \) and \( \tau \in V_n \) is a \( P_n \)-name of rank \( < \lambda \) \) by the coherence of the original sequence \( \langle V_n : n \in \omega \rangle \). \( \square \)

The critical object for understanding a coherent sequence of models is the intersection model \( V_\omega = \bigcap_n V_n \). In Example 4.2.2, the intersection model is a model of ZFC, and it has been discussed in [40, Theorem 9.3.4]; we will come back to it below—Theorem 4.3.6. In the context of general coherent sequences, the intersection model is a transitive model of ZF, and the Axiom of Choice may fail in it. This is the contents of the following theorem and example.

**Theorem 4.2.8.** If \( \langle V_n : n \in \omega \rangle \) is a coherent decreasing sequence of generic extensions of \( V \), then \( V_\omega = \bigcap_n V_n \) is a class in all models \( V_n \), and it is a model of ZF.

**Proof.** We will only show that \( V_\omega \) is a class in \( V_0 \); it then follows by the same argument that \( V_\omega \) is a class in each \( V_n \) for \( n \in \omega \). To show that \( V_\omega \) is a model of ZF, we need to show that \( V_\omega \) is closed under the Gödel operations and it is universal. The closure under the Gödel operations follows from the fact that these operations are evaluated in the same way in each model \( V_n \). For the universality, suppose that \( a \subseteq V_\omega \) is a set, choose an ordinal \( \alpha \) such that all sets in \( a \) have rank \( < \alpha \), and form the set \( b = \{ x \in V_\omega : \text{the rank of } x \text{ is smaller than } \alpha \} \). Since \( V_\omega \) is a class in each model \( V_n \), the set \( b \) is in all models \( V_n \) and so in \( V_\omega \). Clearly, \( a \subseteq b \), concluding the proof of universality.

To show that \( V_\omega \) is a class in \( V_0 \), it is enough to show that \( V_\omega \) is closed under the Gödel operations and it is universal. The closure under the Gödel operations follows from the fact that these operations are evaluated in the same way in each model \( V_n \). For the universality, suppose that \( a \subseteq V_\omega \) is a set, choose an ordinal \( \alpha \) such that all sets in \( a \) have rank \( < \alpha \), and form the set \( b = \{ x \in V_\omega : \text{the rank of } x \text{ is smaller than } \alpha \} \). Since \( V_\omega \) is a class in each model \( V_n \), the set \( b \) is in all models \( V_n \), and so in \( V_\omega \). Clearly, \( a \subseteq b \), concluding the proof of universality.
every pair \( \langle P, G \rangle \in f(n), x \in V[G] \). Since \( V \) is a class in \( V_0 \), this shows that \( V_\omega \) is a class in \( V_0 \) as well. \( \square \)

**Example 4.2.9.** Let \( \kappa \) be a measurable cardinal, with a normal measure \( U \) on \( \kappa \) and the associated ultrapower \( j: V \rightarrow M \). Let \( j_{nm}: M_n \rightarrow M_m \) be the iterands of \( j \) for \( n \leq m \leq \omega \). Then \( \bigcap_{n \in \omega} M_n = M_\omega[c] \) where \( c = (j_{0n}(\kappa): n \in \omega) \).

It is well-known and follows from the geometric description of Prikry genericity by Mathias [56] that the set \( c \) is generic over the model \( M_\omega \) for the Prikry forcing associated with the measure \( j_0, U \).

**Proof.** It is immediate that \( M_\omega \subseteq M_n \) and \( c \in M_n \) for all \( n \in \omega \), proving the right-to-left inclusion. For the left-to-right inclusion, suppose that \( a \) is a set of ordinals which belongs to all models \( M_n \), and work to show that \( a \in M_\omega[c] \).

For each \( n \in \omega \), the set \( b_n = j_{0n}^{-1}a \) belongs to the model \( M_n \), and as such it must be equal to \( (j_{0n}f_n)(\kappa_i: i \in n) \) for some function \( f \). This follows from the fact that the embedding \( j_0 \) is an iterated \( U \)-ultrapower. The elementarity of the embedding \( j_{0n} \) shows that for every ordinal \( \alpha \in M_n \), \( \alpha \in b_n \iff j_{0n}(\alpha) \in j_{0n}(f_n)(\kappa_i: i \in n) \). Now, let \( F \) be the function assigning each number \( n \in \omega \) the function \( f_n \), and use the fact that \( M_\omega \) is a direct limit of the models \( M_n \) for \( n \in \omega \) to conclude that \( a = \text{lim sup}_n j_{0n}(F)(\kappa_i: i \in n) \) and therefore \( a \in M_\omega[c] \). \( \square \)

**Example 4.2.10.** Let \( c_0: \omega_1 \times \omega \rightarrow 2 \) be a Cohen-generic map, and let \( c_n = c_0 \upharpoonright \omega_1 \times (\omega \setminus n) \). Let \( V_n = V[c_n] \). In the model \( V_\omega = \bigcap_n V_n \), the chromatic number of \( G_0 \) is greater than 2; thus, AC must fail in \( V_\omega \).

**Proof.** Work in \( V \). For each number \( n \in \omega \), let \( P_n \) be the poset of all finite functions from \( \omega_1 \times (\omega \setminus n) \) to 2 ordered by extension. Note that \( c_0 \) is \( P_0 \)-generic and the model \( V_\omega \) is equal to \( V[\mathcal{G} \cap P_n] \). For each ordinal \( \alpha \in \omega_1 \) and a number \( n \in \omega \), let \( d_{\alpha n} \) be a name for the function defined by \( d_{\alpha n}(m) = 0 \) if \( m \in n \) and \( d_{\alpha n}(m) \) the unique value of \( p(\alpha, m) \) for all conditions \( p \) in the generic filter with the pair \((\alpha, m)\) in their domain if \( m \geq n \). Note that \( d_{\alpha n} \) is really a \( P_n \)-name and it is forced to belong to the intersection model \( V_\omega \).

Now, let \( p \in P \) be a condition and \( \sigma \) be a name such that \( p \Vdash \sigma: 2^\omega \rightarrow 2 \) is a function in \( V_\omega \); we will find an ordinal \( \alpha \in \omega_1 \), a number \( n \in \omega \) and a condition \( p \) which forces \( d_{\alpha n} \) to differ in an even number of entries if and only if \( \sigma(d_{\alpha 0}) \neq \sigma(d_{\alpha n}) \). This cannot occur for a coloring of \( G_0 \).

By a standard \( \Delta \)-system argument, strengthening \( p \) if necessary, we may find a stationary set \( S \subseteq \omega_1 \), conditions \( p_\alpha \in P \) for \( \alpha \in S \) and a number \( n \in \omega \) such that the conditions \( p_\alpha \) for \( \alpha \in S \) form a \( \Delta \)-system with root \( p \), \( \text{dom}(p_\alpha) \subseteq \omega_1 \times n - 1 \), and each \( p_\alpha \) decides the value of \( \sigma(d_{\alpha 0}) \) to be some bit \( b_\alpha \in 2 \). Find a condition \( q \leq p \) and a \( P_n \)-name \( \tau \) such that \( q \Vdash \sigma = \tau \); this is possible as \( \sigma \) is forced to belong to \( V_\omega \). Find an ordinal \( \alpha \in S \) such that \( p_\alpha \) is compatible with \( q \). Find a condition \( r \in P_n \) such that \( r \leq q \upharpoonright \omega_1 \times (\omega \setminus n) \) and \( r \) decides the value of \( \tau(d_{\alpha n}) \) to be some specific bit \( b \in 2 \). Note that \( p_\alpha \) and \( r \) are compatible in \( P \), and the pair \( \langle \alpha, n - 1 \rangle \) does not belong to \( \text{dom}(p_\alpha \cup r) \).
4.3 Choice-coherent sequences of models

In most of our examples, we will want to look at sequences of models which have a greater degree of coherence. Certain constructions arising from the axiom of choice will have to be performed in a coherent way. The following definition records the demands:

**Definition 4.3.1.** Let \( \langle V_n : n \in \omega \rangle \) be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is **choice-coherent** if it is coherent and for every ordinal \( \lambda \) there is a well-ordering \( \leq_{\lambda} \) of sets of rank \( < \lambda \) (in \( V_0 \)) such that its intersection with each model \( V_n \) belongs to \( V_n \).

**Theorem 4.3.2.** If \( \langle V_n : n \in \omega \rangle \) is a choice-coherent decreasing sequence of generic extensions of \( V \), then \( V_{\omega} = \bigcap_n V_n \) is a model of ZFC.

**Proof.** Let \( \lambda \) be any ordinal. In view of Theorem 4.2.8, we only need to produce a well-ordering \( \leq \) of the collection of sets of rank \( < \lambda \) in the model \( V_{\omega} \) such that \( \leq \in V_{\omega} \). Fix a wellordering \( \leq_{\lambda} \) witnessing the fact that \( \langle V_n : n \in \omega \rangle \) is a choice-admissible decreasing sequence, and note that \( \leq = \leq_{\lambda} \cap V_{\omega} \) works as desired.

The theorem makes the following definition possible:

**Definition 4.3.3.** The sequence \( \langle V_n : n \in \omega \rangle \) is **generic over** \( V \) if \( V \) is a model of ZFC contained in all \( V_n \) for \( n \in \omega \) and \( V_0 \) is a generic extension of \( V \).

The usual abstract forcing arguments (Fact 1.8.6) show that if the sequence of models is generic over \( V \) then all models \( V_n \) and also \( V_{\omega} \) are generic extensions of \( V \) and if \( n < m \leq \omega \) then \( V_n \) is a generic extension of \( V_m \). Most examples of choice-coherent sequences are generic and obtained from the trivial one \( \langle V_n = V : n \in \omega \rangle \) by a coherent forcing which satisfies a certain degree of completeness.

**Definition 4.3.4.** Let \( \langle P_n, \pi_{nm} : P_n \to P_m, \xi_{mn} : P_m \to P_n : n \leq m \in \omega \rangle \) be a commutative system of projections from posets \( P_n \) to \( P_m \) for \( n \leq m \).

1. The **diagonal game** is the following infinite game between Players I and II, at round \( n \) Player I plays \( p_n \in P_n \) and Player II responds by \( q_n \leq p_n \). Additionally, \( p_{n+1} \leq \pi_{nm+1}(q_n) \). In the end, Player II wins if there is a condition \( r \in P_0 \) such that \( \pi_{nm}(p) \leq q_n \).

2. The sequence is **diagonally distributive** if Player I has no winning strategy in the diagonal game.
Example 4.3.5. Suppose that $\langle Q_m : m \in \omega \rangle$ are arbitrary posets, and let $P_n = \prod_{m \geq n} Q_m$ be the countable support product with the natural projection maps from $P_n$ to $P_m$ for $n \leq m$. Player II has a simple winning strategy in the diagonal game in this setup: set $q_n = p_n$.

Theorem 4.3.6. Let $\langle V_n : n \in \omega \rangle$ be a choice-coherent sequence of models of ZFC. Let $\langle P_n, \pi_mm : P_n \to P_m, \xi_mm : P_m \to P_n : n \leq m \in \omega \rangle$ be a coherent sequence of posets which is diagonally distributive in $V_0$. Let $G \subseteq P_0$ be a filter generic over $V_0$, and let $G_n = \xi_m^{-1}G$. Then

1. the sequence $\langle V_n[G_n] : n \in \omega \rangle$ is choice-coherent;

2. the models $\bigcap_n V_n$ and $\bigcap_n V_n[G]$ contain the same $\omega$-sequences of ordinals.

Proof. Let $V_\omega = \bigcup_n V_n$. We start with (1). The main task is to find a poset $P_\omega \in V_\omega$ and a filter $G_\omega \subset P_\omega$ generic over $V_\omega$ such that $\bigcap_n V_n[G_n] = V_\omega[G_\omega]$. This is an uneventful case of diagram chasing.

Using the choice coherence of the original sequence $\langle V_n : n \in \omega \rangle$, we may assume that there is a sequence $\langle \alpha_n : n \in \omega \rangle$ such that the underlying set of the poset $P_n$ is exactly $\alpha_n$. For each condition $p \in P_0$, the ordinal $\omega$-sequence $\langle \pi_0(p) : n \in \omega \rangle$ belongs to $V_\omega$, since for each number $k \in \omega$, the tail $\langle \pi_0(p) : n \geq k \rangle$ is reconstructed as $\langle \pi_k(\pi_0(p)) : n \geq k \rangle$ in the model $V_k$. Similarly, the set $P_\omega = \{ q \in \prod_n P_n : \exists k \in \omega \forall n \geq k \ q(n) = \pi_k(q(k)) \}$ belongs to the model $V_\omega$. For elements $q_0, q_1 \in P_\omega$, let $q_1 \leq q_0$ if for all but finitely many numbers $n \in \omega$, $q_1(n) \leq q_0(n)$ in the poset $P_n$, and conclude that the poset $\langle P_\omega, \leq \rangle$ belongs to the model $V_\omega$.

Define a function $\pi_0 : P_0 \to P_\omega$ by $\pi_0(p) = q$ where $q(n) = \pi_0(p)$, and a function $\xi_0 : P_\omega \to P_0$ by $\xi_0(q) = \xi_k(q(k))$ where $k \in \omega$ is such that for all $n \geq k$, $\pi_k(q(k)) = q(n)$. One can also similarly define maps $\pi_m : P_n \to P_\omega$ and $\xi_m : P_\omega \to P_n$. It is a matter of trivial diagram chasing to show that the maps form a commuting system of projections from $P_n$ to $P_\omega$ and moreover $\pi_m, \xi_m \in V_\omega$. Thus, letting $G_\omega = \xi_0^{-1}G_0$, one can conclude that $G_\omega \subset P_\omega$ is a filter generic over $V_0$ and therefore over $V_\omega$. Also $G_\omega \in \bigcap_n V_n[G_n]$ since $G_\omega$ can be reconstructed in $V_n[G_n]$ as $G_\omega = \xi_\omega^{-1}G_\omega$. In conclusion, $G_\omega \in \bigcap_n V_n[G_n]$.

Finally, we have to prove that every element of the intersection $\bigcap_n V_n[G_n]$ belongs to $V_\omega[G_\omega]$. This is where the diagonal distributivity of the original poset sequence is used. Suppose that $\tau \in V_0$ is a $P_0$-name for a set of ordinals and $p \in P_0$ is a condition forcing $\tau \in \bigcap_n V_n[G_n]$; we must produce a condition $p' \leq p$ and a $P_\omega$-name $\tau_\omega \in V_\omega$ such that $p' \Vdash \tau = \tau_\omega / G_\omega$. Consider a strategy by Player I in the diagonalization game in which he plays $p_n$ so that $p_n \leq p$, $\tau_0 = \tau$, and there is a $P_n$-name $\tau_{n+1} \in V_{n+1}$ such that $p_n \Vdash \tau_{n+1} = \tau_{n+1}/G_{n+1}$. This is possible by the assumption on the name $\tau$. By the diagonalization assumption, Player II has a counterplay with conditions $q_n \leq p_n$ such that there is a condition $p' \leq p$ for which $\pi_0(p) \leq q_n$ for all $n \in \omega$. Let $\tau_\omega$ be the $P_\omega \Vdash \pi_0(p')$-name defined by $q \Vdash \alpha \in \tau_\omega$ just in case $\xi_\omega(q) \Vdash p_\omega \alpha \in \tau$. The name $\tau_\omega$ can be reconstructed in every model $V_n$ by the definition $q \Vdash \alpha \in \tau_\omega$ just in case $\xi_\omega(q) \Vdash p_\omega \alpha \in \tau$ by the choice of the strategy for Player I in the
diagonalization game. As a result, \( \tau_\omega \in V_\omega \). It is immediate from the definition of \( \tau_\omega \) that \( p' \Vdash \tau = \tau_\omega / G_\omega \) as desired.

Now we are ready to construct the requisite well-orderings verifying the choice-coherence of the models \( \langle V_n[G_n] : n \in \omega \rangle \). Let \( \lambda \) be an ordinal larger than the ranks of all the posets \( P_n \) for \( n \in \omega \). Let \( \leq \) be a coherent well-ordering of sets of rank \( < \lambda \) in \( V_0 \). We will now describe a coherent well-ordering \( \leq' \) of sets of rank \( < \lambda \) in the model \( V_0[G_0] \). In this well-ordering, the sets in \( V_\omega[G_\omega] \) come first, ordered by some well-ordering in the model \( V_\omega[G_\omega] \). The sets in \( V_0[G_0] \setminus V_1[G_1] \) come next, well-ordered by their \( \leq' \)-first \( P_0 \)-name in the model \( V_0 \) representing them. The sets in \( V_1[G_1] \setminus V_2[G_2] \) come next with a similar well-order, and so on. The coherence of the resulting well-ordering \( \leq' \) is due to the fact that for each \( k \in \omega \), the sequence \( \langle G_n : n \geq k \rangle \) belongs to the model \( V_k[G_k] \).

(2) is much easier. Suppose that \( \tau \in V_0 \) is a \( P_0 \)-name for an \( \omega \)-sequence of ordinals in the model \( V_\omega[G_\omega] \) and \( p \in P_0 \) is a condition; we must find a condition \( r \leq p \) and an \( \omega \)-sequence \( z \in V_\omega \) such that \( r \Vdash \tau = z \). Consider a strategy for Player I in the diagonal game in which he plays conditions \( p_n \in P_n \) and on the side produces \( P_n \)-names \( \tau_n \in V_n \) so that \( p_0 \leq p, \tau_0 = \tau \) and \( p_n \Vdash p_{n+1} \tau_n = \tau_{n+1} \) evaluated by the \( \chi_{n+1} \)-image of the generic filter on \( P_n \), and also \( p_n \) decides the value \( \tau_n(n) \) to be some ordinal \( z(n) \). The assumptions on the name \( \tau \) shows that this is a valid strategy. The initial assumptions on the coherent sequence of posets show that this is not a winning strategy and so there must be a play against it such that in the end there is a condition \( r \leq p \) with \( \chi_0(n)(r) \leq p_n \) for all \( n \in \omega \). Let \( \tau_n \in V_n \) be the names produced during that counterplay, and let \( z \) be the \( \omega \)-sequence of ordinals obtained. The definitions show that for all \( n \in \omega \), \( \chi_0(n)(r) \Vdash p_{n+1} \tau_n = z \). It follows that \( z \in V_n \) for all \( n \in \omega \), and therefore \( r, z \) are as required in (2).

The main feature of choice-coherent sequences of models we use later is the following theorem connecting them with orbit equivalence relations:

**Theorem 4.3.7.** Let \( \langle V_n : n \in \omega \rangle \) be a choice-coherent, generic sequence of models. Let \( E \) be an orbit equivalence relation on a Polish space \( X \) with code in \( V_\omega = \bigcap_n V_n \). If a virtual \( E \)-class is represented in \( V_n \) for every \( n \in \omega \), then it is represented in \( V_\omega \).

Note that a virtual \( E \)-class is an equivalence class of \( E \)-pins. Thus, the theorem says that if there are pairwise equivalent \( E \)-pins \( \langle P_n, \tau_n \rangle \in V_n \) for all \( n \in \omega \), then there is an \( E \)-pin equivalent to them in the intersection model.

**Proof.** Let \( \Gamma \) be a Polish group continuously acting on the space \( X \), inducing the equivalence relation \( E \). Let \( d \) be a compatible right-invariant metric on \( \Gamma \). Let \( \langle P_0, \tau_0 \rangle \in V_0 \) be an \( E \)-pin which has an equivalent in the model \( V_n \) for every \( n \in \omega \). Let \( \lambda \) be a cardinal so large that for each \( n \in \omega \), \( V_0 \) is a generic extension of \( V_n \) by a poset of size \( < \lambda \), and \( V_n \) contains an \( E \)-pin on a poset of size \( < \lambda \) equivalent to the pin \( \langle P_0, \tau_0 \rangle \).
Let $P_1$ be the Cohen forcing on the Polish group $\Gamma$, with its name $\dot{\gamma}_{\text{gen}}$ for a generic point. Let $\gamma \in \Gamma$ be a Cohen-generic point, $H \subset P_0$ be a generic filter and $K \subset \text{Coll}(\omega, \lambda)$ be a generic filter, mutually generic over $V_0$; let $x_0 = \sigma_0/H$. In the model $V_0[\gamma, H, K]$, form the model $M$ as the class of all sets hereditarily definable from $\gamma \cdot x_0$ and parameters in $V_\omega$. The model $M$ is an intermediate model of ZFC between $V_\omega$ and $V_0[\gamma, H]$, and so by Fact 1.8.6, the model $M$ is a forcing extension of $V_\omega$. We will argue that $M, V_0[H]$ are mutually generic extensions of $V_\omega$.

First note that this will prove the theorem. Let $Q, \tau \in V_\omega$ be a poset and a name and $L \subset Q$ be a filter generic over the model $V_0[H]$ such that $V_\omega[L] = M$ and $\tau/L = \gamma \cdot x_0$. By the forcing theorem in the model $V_0$, there have to be conditions $p \in H$ and $q \in L$ such that $(p, q) \Vdash \tau_0 \in E \cdot \tau$. It is immediate that $\tau$ as a name on $Q \restriction q$ is $E$-pinned, and the $E$-pin $(Q \restriction q, \tau)$ is equivalent to $(P_1, \tau_0)$. This confirms the conclusion of the theorem.

To argue that $M, V_0[H]$ are mutually generic extensions of $V_\omega$, we use the criterion of Proposition 1.8.8. In other words, if $a \in V_0[H]$ and $b \in M$ are disjoint subsets of some ordinal $\alpha$, we must find a set $c \in V_\omega$ of ordinals such that $a \subset c$ and $b \cap c = 0$. Towards this end, move back to the model $V_0$. Suppose that $O \subset \Gamma$ is a nonempty open set, $p \in P_0$ is a condition, $\dot{a}$ is a $P_0$-name for a set of ordinals, and $\phi$ is a formula with parameters in $V_\omega$ such that in the poset $P_1 \times P_0, (O, p) \Vdash \text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \dot{a} \phi(\beta, \dot{\gamma}_{\text{gen}}, \tau_0)$ holds. Due to the definition of the model $M$, it will be enough to find a set $c \in V_\omega$ and a condition $\langle O', p' \rangle \leq (O, p)$ which forces $\dot{a} \subset \dot{c}$ and $\text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \dot{c} \phi(\beta, \dot{\gamma}_{\text{gen}}, \tau_0)$ holds.

Finally, we are in a position to use some coherence arguments. Let $\prec$ be a coherent well-ordering of the collection of sets of rank $< \lambda$; i.e. the restriction of $\prec$ to each $V_\alpha$ belongs to $V_\omega$. We will use the ordering to perform some coherent constructions. A typical construction of a coherent sequence of elements of $V_\alpha$ proceeds by induction on $\alpha \in \omega$. If $\langle v_\alpha \colon \alpha \in \omega \rangle$ is coherent, $w_0 \in V_\omega$, and $\phi$ is some formula with parameters in $V_\omega$, one can select the $\prec$-least $w_{\alpha+1} \in V_{\alpha+1}$ such that $V_\omega \models \phi(v_\alpha, w_\alpha, w_{\alpha+1})$ if it exists; then, the sequence $\langle w_\alpha \colon \alpha \in \omega \rangle$ is coherent. The routine details of these constructions will be suppressed below.

Find a coherent sequence $\langle P_\alpha, \tau_\alpha \colon \alpha \in \omega \rangle$ of pairwise equivalent $E$-pins starting with $P_0, \tau_0$; i.e. for every number $\alpha \in \omega$ it is the case that $P_\alpha \times P_{\alpha+1} \Vdash \sigma_\alpha \in E \cdot \tau_{\alpha+1}$. Find a coherent sequence $\langle \dot{\gamma}_n \colon n \in \omega \rangle$ such that for each $n \in \omega$, $\dot{\gamma}_n$ is a $P_\alpha \times P_{\alpha+1}$-name for an element of the group $\Gamma$ such that $\tau_n = \dot{\gamma}_n \cdot \tau_{n+1}$. Let $D \subset \Gamma$ be a fixed countable dense set in the model $V_\omega$, and let $\delta_0 \in D$ and $\varepsilon > 0$ be such that the open $d$-ball $B(\delta_0, \varepsilon) \subset \Gamma$ is a subset of the open set $O$. Find a coherent sequence $\langle p_\alpha, \delta_\alpha \colon n \in \omega \rangle$ such that $p_0 \leq p, p_n \in P_n, \delta_\alpha \in D$, and in the poset $P_\alpha \times P_{\alpha+1}, \langle p_\alpha, p_{\alpha+1} \rangle \Vdash d(\gamma_n, \gamma_\alpha, \tau_{n+1}) < \varepsilon \cdot 2^{-n-3}$. Let $O_n = B(\delta_\alpha, \varepsilon/2)$. The point in these definitions is the following claim:

\textbf{Claim 4.3.8.} Let $n > 0$. The condition $\langle p_i \colon i \leq n \rangle$ forces in the product $\prod_{i \leq n} P_i$ the following:

1. $O_n \subset B(\delta_0, \varepsilon) \cdot \gamma_0 \gamma_1 \ldots \gamma_{n-1}$;
2. $B(\delta_0, \varepsilon/4) \cdot \dot{\gamma}_0 \dot{\gamma}_1 \ldots \dot{\gamma}_{n-1} \subset O_n$. 
Proof. Use the right invariance of the metric $d$ to argue by induction on $i \in n$ that $d(\delta_{i+1}, \delta_i \gamma_0 \gamma_1 \ldots \gamma_i)$ is forced to be smaller than $\varepsilon \cdot \Sigma_{j \leq i} 2^{-j-3}$. In conclusion, $d(\delta_n, \delta_0 \gamma_0 \gamma_1 \ldots \gamma_{n-1})$ is forced to be smaller than $\varepsilon/4$. The two items then follow immediately by the right invariance of the metric $d$ again.

Now, for every number $n \in \omega$, in the model $V_n$ form the set $c_n = \{ \beta \in \kappa :$ in the poset $P_i \times P_n, \langle O_n, p_n \rangle \Vdash \text{Coll}(\omega, \lambda) \Vdash \phi(\tilde{\beta}, \tilde{\gamma}_\text{gen} \cdot \tau_n) \}$. Finally, let $c = \limsup_n c_n = \{ \beta \in \kappa : \exists \gamma n \beta \in c_n \}$. It is immediate that the sequence $\langle c_n : n \in \omega \rangle$ is coherent and therefore the set $c$ belongs to the model $V_\omega$. Let $O' = B(\delta_0, \varepsilon/4)$ and $p' = p_0$. The following two claims stated in the model $V_0$ complete the proof of the theorem.

Claim 4.3.9. In the poset $P_0$, $p' \Vdash \dot{a} \subset \dot{c}$.

Proof. Let $p'' \leq p'$ be a condition and $\beta \in \kappa$ an ordinal such that $p'' \Vdash \tilde{\beta} \in \dot{a}$. It will be enough to show that for all $n > 0$, $\beta \in c_n$. To this end, fix a number $n > 0$ and let $H_i : i \leq n$ be filters on the respective posets $P_i$ mutually generic over the model $V_0$ such that $p_i \in H_i$ and moreover $p'' \in H_0$. Write $x_i = \tau_i/H_i$ and $\gamma_i = \tilde{\gamma}_i/H_i; H_{i+1}$; so $x_0 = \gamma_0 \gamma_1 \ldots \gamma_{n-1} x_n$.

Let $\gamma \in O_0$ be a point $P_1$-generic over the model $V_0[H_i : i \leq n]$. Let $\gamma' = \gamma_0 \gamma_1^{-1} \gamma_2^{-1} \ldots \gamma_{n-1}^{-1}$. By the invariance of the meager ideal on $\Gamma$ under right translations, $\gamma' \in \Gamma$ is a point Cohen generic over the model $V_0[H_i : i \leq n]$. By Claim 4.3.8(1), $\gamma' \in B(\delta_0, \varepsilon) \subset O$; moreover, $\gamma \cdot x_n = \gamma' \cdot x_0$.

Let $K \subset \text{Coll}(\omega, \lambda)$ be a filter generic over the model $V_0[H_i : i \leq n][\gamma]$. The model $N = V_0[H_i : i \leq n][\gamma][K]$ is a $\text{Coll}(\omega, \lambda)$-extension of both $V_0[\gamma', H_0]$ and $V_n[\gamma, H_n]$ by the choice of $\lambda$ and Fact 1.8.12. By the forcing theorem in the model $V_0$ and the initial assumptions on the name $\dot{a}$ and the formula $\phi$, $N \Vdash \phi(\beta, \gamma' \cdot x_0)$. By the forcing theorem in the model $V_n$, the filter on $P_i \times P_n$ given by $\gamma, H_n$ must contain a condition forcing $\text{Coll}(\omega, \lambda) \Vdash \phi(\tilde{\beta}, \gamma \cdot \tau_n)$. However, $\gamma, H_n$ were arbitrary generics meeting the condition $\langle O_n, p_n \rangle$, so it must be the case that this condition forces $\text{Coll}(\omega, \lambda) \Vdash \phi(\tilde{\beta}, \tilde{\gamma}_\text{gen} \cdot \tau_n)$. This means that $\beta \in c_n$ as required.

Claim 4.3.10. In the poset $P_1 \times P_0$, for every ordinal $\beta \in c$, $\langle O', p' \rangle \Vdash \text{Coll}(\omega, \lambda) \Vdash \phi(\tilde{\beta}, \gamma \cdot \tau_0)$.

Proof. Find a number $n > 0$ such that $\beta \in c_n$. Let $H_i \subset P_i$ for $i \leq n$ be filters mutually generic over $V_0$ containing the conditions $p_i$ respectively. Write $x_i = \tau_i/H_i$ and $\gamma_i = \tilde{\gamma}_i/H_i, H_{i+1}$; so $x_0 = \gamma_0 \gamma_1 \ldots \gamma_{n-1} x_n$.

Let $\gamma' \in O'$ be a point $P_1$-generic over the model $V_0[H_i : i \leq n]$. Let $\gamma = \gamma_0 \gamma_1^{-1} \gamma_2^{-1} \ldots \gamma_{n-1}^{-1}$; by the invariance of the meager ideal on $\Gamma$ under right translations, $\gamma \in \Gamma$ is a point Cohen generic over the model $V_0[H_i : i \leq n]$. By Claim 4.3.8(2), $\gamma \in O_n$; moreover, $\gamma \cdot x_n = \gamma' \cdot x_0$.

Let $K \subset \text{Coll}(\omega, \lambda)$ be a filter generic over the model $V_0[H_i : i \leq n][\gamma]$. The model $N = V_0[H_i : i \leq n][\gamma'[K]$ is a $\text{Coll}(\omega, \lambda)$-extension of both $V_0[\gamma', H_0]$ and $V_n[\gamma, H_n]$ by the basic properties of collapse posets–Fact 1.8.12. By the forcing theorem in the model $V_n$ and the initial assumptions on the name $\dot{a}$ and the
formula $\phi$. $N \models \phi(\gamma \cdot x_n)$. By the forcing theorem in the model $V_0$, the filter on $P_1 \times P_0$ given by $\gamma', H_n$ must contain a condition forcing $\text{Coll}(\omega, \lambda) \vDash \phi(\beta, \gamma \cdot \tau)$.

However, $\gamma', H_0$ were arbitrary meeting the condition $\langle \gamma', p' \rangle$, so it must be the case that this condition forces $\text{Coll}(\omega, \lambda) \vDash \phi(\beta, \gamma \cdot \tau)$ as required.

\[ \square \]

**Example 4.3.11.** Consider the equivalence relation $E_1$ on $X = (2^\omega)\omega$; it is well-known not to be reducible to any orbit equivalence relation [39, Theorem 11.8.1]. The conclusion of Theorem 4.3.7 fails for $E_1$. To see that, choose any partial order $Q$ which adds a new point $\dot{y} \in 2^\omega$. Let $P$ be the full support product of $\omega$-many copies of $Q$, and let $P_n$ be the product of the copies of the copies of $Q$ indexed by natural numbers $\geq n$. The poset $P_{n+1}$ is naturally regularly embedded in $P_n$. Let $G \subset P$ be a generic filter, and for each $n \in \omega$ let $G_n \subset P_n$ be the restriction of $G$ to conditions in $P_n$. Theorem 4.3.6 shows that $\langle V[G_n] : n \in \omega \rangle$ is a choice-coherent sequence of models, and $\bigcap_n V[G_n]$ contains no new reals compared to $V$. In $V[G_n]$, let $x_n \in X$ be the sequence defined by $x_n(m) =$ the zero sequence if $m \in n$ and $x_n(m) = \dot{y}$ evaluated by the $n$-th coordinate of the generic filter $G$. It is clear that the points $x_n$ all represent the same $E_1$-class, which is not represented in $V$ and therefore in $\bigcap_n V[G_n]$.

As a final remark, in general it is necessary to consider virtual $E$-classes as opposed to just $E$-classes in the wording of Theorem 4.3.7. To see this, start with the trivial coherent sequence $V_n = V$ for all $n \in \omega$ and let $P_n$ be the countable support product of copies of the poset $\text{Coll}(\omega, 2^\omega)$ indexed by natural numbers $\geq n$, with the natural projections from $P_n$ to $P_m$ added. This is a diagonally complete sequence of posets, and by Theorem 4.3.7 it induces a choice-coherent sequence of models $\langle V_n[G_n] : n \in \omega \rangle$ such that the model $\bigcap_n V_n[G_n]$ contains only ground model $\omega$-sequences of ordinals. Now, every model $V_n[G_n]$ contains an enumeration of the set $(2^\omega \cap V)$ by natural numbers, and all of these enumerations are $F_2$-related. Clearly, there is no $F_2$-equivalent of them in the intersection model. Yet, there is a virtual $F_2$-class related to these enumerations in the intersection model, and even in the ground model $V$. 


Part II

Balanced extensions of the Solovay model
Chapter 5

Balanced Suslin forcing

5.1 Virtual conditions

We look at the class of Suslin posets from an angle quite distinct from the standard treatment in [5]; in particular, the center of attention is on \( \sigma \)-closed Suslin forcings as opposed to c.c.c. or proper forcings adding reals. Recall:

**Definition 5.1.1.** A poset \( \langle P, \leq \rangle \) is Suslin if there is a Polish space \( X \) such that

1. \( P \) is an analytic subset of \( X \);
2. the ordering \( \leq \) is an analytic subset of \( X^2 \);
3. the incompatibility relation is an analytic subset of \( X^2 \).

It is important to understand that Suslin forcings may fail to be separative. Recall that conditions \( p_0, p_1 \) are called *inseparable* if for every condition \( q \leq p_0 \), \( q, p_1 \) have a lower bound, and vice versa, for every condition \( q \leq p_1 \), \( q, p_0 \) have a lower bound. The inseparability relation \( E_P \) is an equivalence; the partial order \( P \) is separative if this equivalence relation is the identity. A simple yet informative example of a nonseparable Suslin forcing is in order. Consider the poset of countable partial functions from \( 2^\omega \) to \( 2 \) ordered by reverse extension. As stated, it is not a Suslin forcing, since it is not an analytic subset of a Polish space. One has to make an innocuous adjustment: \( P \) is in fact the set of all functions from \( \omega \) to \( 2^\omega \times 2 \) whose range is a function, and order \( P \) by setting \( q \leq p \) if \( \text{rng}(p) \subseteq \text{rng}(q) \). This adjustment sacrifices the separability for Suslinness. Note that the equivalence relation \( E_P \) is unpinned in this case.

Throughout the rest of the book, we will make use of *virtual conditions* in Suslin posets. Similar to virtual equivalence classes, these are conditions which may not exist in the present model of set theory and appear only in some generic extension, and yet we have a sensible calculus for dealing with them in the ground model. We want the space of virtual conditions not to depend on a
particular presentation of a given Suslin poset, and to be rich enough to harvest certain critical features. This leads to the following definition:

**Definition 5.1.2.** Let \( \langle P, \leq \rangle \) be a Suslin forcing.

1. Let \( Q \) be a poset and \( \tau \) a \( Q \)-name for an analytic subset of \( P \). We say that the name \( \tau \) is \( P \)-pinned if \( Q \times Q \models \Sigma \tau_{\text{left}} = \Sigma \tau_{\text{right}} \) in the completion of the separative quotient of the poset \( P \). If \( \tau \) is \( P \)-pinned, then the pair \( \langle P, \tau \rangle \) is called a \( P \)-pin.

2. Let \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) be \( P \)-pins. Define \( \langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle \) if \( Q_0 \times Q_1 \models \Sigma \tau_0 \leq \Sigma \tau_1 \) in the separative quotient of the poset \( P \), and \( \langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle \) if \( \langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle \) and \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \).

3. The *virtual conditions* of \( P \) are the equivalence classes of \( \equiv \).

For the last item of the above definition, we must verify that \( \equiv \) is in fact an equivalence relation. This is the contents of the following propositions. As the first technical remark, note that for analytic sets \( A_0, A_1 \subset P \), the statement \( \Sigma A_1 \leq \Sigma A_0 \) is equivalent to \( \forall p \in P \exists p_1 \in A_1 \; p \leq p_1 \rightarrow \exists q \leq p \exists p_0 \in A_0 \; q \leq p_0 \). Since the ordering \( \leq \) on \( P \) is analytic, this is a \( \Pi^1_2 \) statement and therefore absolute among all generic extensions by the Shoenfield absoluteness.

**Proposition 5.1.3.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle \) be posets and names for analytic subsets of \( P \). The following statements are absolute among all forcing extensions:

1. \( \langle Q_0, \tau_0 \rangle \) is a \( P \)-pin;
2. \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \);
3. \( \langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle \).

*Proof.* We will prove (2), the other items are parallel. Suppose first that \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \) holds, and \( V[H] \) is a generic extension of \( V \); we must verify that the \( \leq \)-relation transfers to \( V[H] \). To see this, suppose that \( G_0 \subset Q_0, G_1 \subset Q_1 \) are filters mutually generic over \( V[H] \), and let \( A_0 = \tau_0/G_0 \) and \( A_1 = \tau_1/G_1 \). Thus, \( A_0, A_1 \subset P \) are analytic sets with codes in \( V[G_0] \) and \( V[G_1] \), respectively. By the assumption, \( V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0 \) in the completion of the separative quotient of the poset \( P \). This \( \Pi^1_2 \) statement transfers to \( V[H][G_0, G_1] \) by the Shoenfield absoluteness. In conclusion, \( V[H][G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0 \) and therefore \( V[H] \models \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \).

For the other direction, if \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \) fails in \( V \) and \( V[H] \) is a generic extension of \( V \), we must show that the failure of the \( \leq \) relation transfers to \( V[H] \). Fix conditions \( q_0 \in Q_0, q_1 \in Q_1 \) such that in \( V \), \( \langle q_0, q_1 \rangle \models \Sigma \tau_1 \not\leq \Sigma \tau_0 \); we will show that this property of \( q_0, q_1 \) transfers to the model \( V[H] \). This follows from the same Shoenfield absoluteness argument as the one used in the previous paragraph. \( \square \)
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Proposition 5.1.4. \(\equiv\) is an equivalence relation on \(P\)-pins and \(\leq\) is an ordering on \(\equiv\)-equivalence classes.

Proof. We will show that \(\leq\) is transitive, the other statements are similar. Suppose that \(\langle Q_2, \tau_2 \rangle \leq \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle\); we need to show that \(\langle Q_2, \tau_2 \rangle \leq \langle Q_0, \tau_0 \rangle\) holds. Let \(G_0 \subset Q_0, G_2 \subset Q_2\) are mutually generic filters and \(A_2 = \tau_2/G_2\) and \(\tau_0/G_0\); we must show that \(V[G_0, G_2] \models \Sigma A_2 \leq \Sigma A_0\). Let \(G_1 \subset Q_1\) be a filter generic over \(V[G_0, G_2]\) and let \(A_1 = \tau_1/G_1\). By the assumption, \(V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0\) and \(V[G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1\). By the Shoenfield absoluteness, \(V[G_0, G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1 \leq \Sigma A_0\), in particular \(\Sigma A_2 \leq \Sigma A_0\). By another application of the Shoenfield absoluteness, the inequality \(\Sigma A_2 \leq \Sigma A_0\) transfers from \(V[G_0, G_1, G_2]\) to \(V[G_0, G_2]\). \(\square\)

It is now time to exhibit some virtual conditions for familiar partial orders.

Example 5.1.5. Let \(P\) be the poset of infinite subsets of \(\omega\), ordered by inclusion. Let \(F\) be a nonprincipal filter on \(\omega\). Let \(\tau\) be a Coll(\(\omega, F\))-name for the set of all conditions \(p \in P\) which diagonalize the filter \(F\), i.e. \(\forall a \in F\ p \setminus a\) is finite. The name \(\tau\) is \(P\)-pinned, since its valuation does not depend on the choice of the generic filter on Coll(\(\omega, F\)). It is not difficult to see that distinct nonprincipal filters generate distinct virtual conditions.

Example 5.1.6. Let \(P\) be the poset of all countable functions from \(2^\omega\) to 2, ordered by reverse inclusion. Let \(f\) be any function from \(2^\omega\) to 2, perhaps uncountable. Let \(\tau\) be a Coll(\(\omega, f\))-name for the set of all conditions \(p \in P\) such that \(f \subset p\). Again, the name \(\tau\) is \(P\)-pinned. It is clear that distinct functions \(f\) generate distinct virtual conditions in \(P\).

In the preceding examples, the virtual conditions were generated by natural combinatorial objects. However, the space of virtual conditions usually contains very many objects of no earthly interest, as is apparent from the following proposition.

Proposition 5.1.7. Suppose that \(\langle P, \leq \rangle\) is a \(\sigma\)-closed Suslin poset such that below any element \(p \in P\) there are two incompatible ones. Then the equivalence \(\equiv\) has proper class many equivalence classes.

Proof. Let \(h: 2^{<\omega} \to P\) be a function such that for every \(t \in \omega^{<\omega}\), the conditions \(h(t^0)\) and \(h(t^1)\) are incompatible and stronger than \(h(t)\). Let \(g: \omega \to \omega^2\) be a bijection. For every ordinal \(\alpha\), consider the Coll(\(\omega, \alpha\)) name \(\tau_{\alpha}\) for the set of all conditions \(p \in P\) such that for every \(n \in \omega\) there is (exactly one) string \(t(p, n) \in 2^n\) such that \(p \leq h(t)\), and the binary relation \(g''\{n \in \omega: t(p, n + 1)(n) = 1\}\) is isomorphic to \(\alpha\). It is not difficult to see that the pair \(\langle\text{Coll}(\omega, \alpha), \tau_{\alpha}\rangle\) is a \(P\)-pin; in fact the evaluation of \(\tau_{\alpha}\) yields the same analytic set no matter what the generic filter on Coll(\(\omega, \alpha\)) is. The \(P\)-pins obtained in this way are \(\equiv\)-inequivalent; in fact, for distinct ordinals \(\alpha \neq \beta\), Coll(\(\omega, \alpha\)) \times Coll(\(\omega, \beta\)) \(\equiv\) \(\Sigma\tau_{\alpha}\) and \(\Sigma\tau_{\beta}\) are incompatible elements of the completion of the separative quotient of \(P\). \(\square\)
However, certain type of virtual conditions is entirely central to the technology developed in this book, and normally allows a neat classification by natural combinatorial objects. These are the balanced virtual conditions of the next section.

5.2 Balanced conditions

This section isolates the notion of a balanced condition in a given Suslin poset and a natural equivalence of balanced conditions. It turns out that every balanced class is represented by a virtual condition.

Definition 5.2.1. Let \langle P, \leq \rangle be a Suslin forcing. Let \( Q \) be a poset and \( \tau \) a \( Q \)-name for an analytic subset of \( P \). We say that \( \langle Q, \tau \rangle \) is a balanced pair in \( P \) if for all posets \( R_0, R_1 \) and all \( R_0 \times Q \)- and \( R_1 \times Q \)-names \( \sigma_0, \sigma_1 \) for elements of \( P \) such that \( R_0 \times Q \vDash \sigma_0 \leq \Sigma \tau \) and \( R_1 \times Q \vDash \sigma_1 \leq \Sigma \tau \), it is the case that \( (R_0 \times Q) \times (R_1 \times Q) \vDash \sigma_0, \sigma_1 \) are compatible conditions in the poset \( P \).

Note that the definition of a balanced pair depends on the Suslin poset \( P \). The Suslin poset is not mentioned as it will be always understood from the context and no opportunity for confusion arises. The expressions of the type \( \sigma \leq \Sigma \tau \) will be shortened to \( \sigma \leq \tau \) below as there can be no confusion. Balanced pairs and virtual conditions are quite different things; the main topic of interest in this book are pairs \( \langle Q, \tau \rangle \) which are simultaneously balanced and virtual conditions.

The following proposition restates the notion of a balanced pair in terms of the generic extension as opposed to the forcing relation.

Proposition 5.2.2. Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( Q \) be a poset and \( \tau \) a \( Q \)-name for an analytic subset of \( P \). The following are equivalent:

1. \( \langle Q, \tau \rangle \) is a balanced pair;
2. whenever \( V[H_0] \) and \( V[H_1] \) are mutually generic extensions, \( G_0, G_1 \subset Q \) are filters generic over the ground model in the respective extensions and \( p_0 \leq \tau/G_0, p_1 \leq \tau/G_1 \) are conditions in \( P \) in the respective extensions \( V[H_0], V[H_1] \), then \( p_0, p_1 \) are compatible in \( P \).

The class of balanced conditions is greatly simplified by introducing the following equivalence relation on it:

Definition 5.2.3. Let \( \langle P, \leq \rangle \) be a Suslin forcing. If \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are balanced pairs, we say that \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \) if for all posets \( R_0, R_1 \) and all \( R_0 \times Q_0 \)- and \( R_1 \times Q_1 \)-names \( \sigma_0, \sigma_1 \) for elements of \( P \) such that \( R_0 \times Q_0 \vDash \sigma_0 \leq \tau \) and \( R_1 \times Q_1 \vDash \sigma_1 \leq \tau \), it is the case that \( (R_0 \times Q_0) \times (R_1 \times Q_1) \vDash \sigma_0, \sigma_1 \) are compatible conditions in the poset \( P \).

Proposition 5.2.4. The relation \( \equiv_b \) is an equivalence on balanced pairs. Moreover, if \( \langle Q, \tau \rangle \) is a balanced pair, then
5.2. BALANCED CONDITIONS

1. if \( Q \vDash \sigma \leq \tau \) then \( \langle Q, \sigma \rangle \) is a balanced pair \( \equiv_b \)-related to \( \langle Q, \tau \rangle \);

2. if \( Q \) is a regular subposet of \( R \) then \( \langle R, \tau \rangle \) is a balanced pair \( \equiv_b \)-related to \( \langle Q, \tau \rangle \).

**Proof.** We argue for the first sentence; items (1) and (2) are immediate consequences of the definition. The relation \( \equiv_b \) is clearly symmetric and contains the identity; we will show that it is transitive. Let \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \equiv \langle Q_2, \tau_2 \rangle \); it must be shown that \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_2, \tau_2 \rangle \) follows. To this end, let \( R_0, R_2 \) be posets and \( \sigma_0, \sigma_2 \) be \( R_0 \times Q_0 \)- and \( R_2 \times Q_2 \)-names for conditions in \( P \) stronger than \( \tau_0, \tau_1 \) respectively. By the assumption about \( Q_0, Q_1 \), there is a \((R_0 \times Q_0) \times (R_1 \times Q_1)\)-name \( \eta \) which is a lower bound of \( \sigma_0 \) and \( \sigma_2 \). Thus, in the long product extension \((R_0 \times Q_0) \times (R_1 \times Q_1) \times (R_2 \times Q_2)\), the conditions \( \sigma_0, \sigma_2 \) are compatible, and by Mostowski absoluteness this statement transfers to the \((R_0 \times Q_0) \times (R_2 \times Q_2)\)-extension. This completes the proof of the transitivity of \( \equiv_b \). \( \square \)

**Proposition 5.2.5.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle \) be balanced pairs. The following are equivalent:

1. \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \);

2. whenever \( V[H_0] \) and \( V[H_1] \) are mutually generic extensions, \( G_0 \subset Q_0, G_1 \subset Q_1 \) are filters generic over the ground model in the respective extensions and \( p_0 \leq \tau_0/G_0, p_1 \leq \tau_1/G_1 \) are conditions in \( P \) in the respective extensions \( V[H_0], V[H_1] \), then \( p_0, p_1 \) are compatible in \( P \).

It is now time for a simple and informative example. Let \( P \) be the poset of countable functions from \( 2^\omega \) to \( 2 \), ordered by reverse inclusion. On one hand, if \( f : 2^\omega \to 2 \) is a total function, then \( \langle \text{Coll}(\omega, 2^\omega), f \rangle \) is a balanced pair. On the other hand, if \( \langle Q, \tau \rangle \) is a balanced pair, then for every point \( x \in 2^\omega \) in the ground model, it must be the case that \( Q \) forces \( f(x) \in \text{dom}(\tau) \) and in fact \( Q \) has to decide the value of \( \tau(\dot{x}) \) as well–otherwise it would be easy to violate the balance of the pair. Let \( f : 2^\omega \to 2 \) be the total function given by \( \forall x \in 2^\omega \ Q \vDash \tau(\dot{x}) = f(\dot{x}) \) and note that \( \langle Q, \tau \rangle \equiv_b \langle \text{Coll}(\omega, 2^\omega), f \rangle \). Thus, the balanced classes for \( P \) are exactly classified by total functions from \( 2^\omega \) to \( 2 \).

One of the main concerns of this book is the classification of balanced classes for various Suslin posets \( \langle P, \leq \rangle \). The following theorem is the basic contribution in this direction.

**Theorem 5.2.6.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Every \( \equiv_b \)-class contains a virtual condition. The condition is unique up to the virtual condition equivalence.

**Proof.** Let \( \langle Q, \tau \rangle \) be a balanced pair; by Proposition 5.2.4 we may assume that \( \tau \) is a name for a single element of \( P \). Consider the poset \( Q' = \text{Coll}(\omega, P(Q)) \) and the name \( \tau' \) for the analytic set \( \{ p \in P : \text{for some filter } G \subset Q \text{ meeting all open dense sets enumerated by the } Q'-\text{generic, } p = \tau/G \} \). We will first show that the pair \( \langle Q', \tau' \rangle \) is balanced via Proposition 5.2.2.
To this end, suppose that \( V[H_0], V[H_1] \) are mutually generic extensions, \( G_0', G_1' \subseteq Q' \) are filters in the respective extensions generic over \( V \), and \( r_0 \leq \tau'/G_0' \) and \( r_1 \leq \tau'/G_1' \) are conditions in the poset \( P \); we must show that \( r_0, r_1 \in P \) are compatible conditions. By the definition of the name \( \tau' \), in \( V[H_0] \) it is possible to find a filter \( G_0 \subseteq Q \) generic over \( V \) such that \( p_0 = \tau/G_0 \) is compatible with \( r_0 \), and similarly in the model \( V[G_1] \). Let \( s_0, s_1 \in P \) be lower bounds of \( r_0 \) and \( p_0 \) in \( V[G_0] \) and \( r_1 \) and \( p_1 \) in \( V[G_1] \) respectively. Since \( \langle Q, \tau \rangle \) was a balanced condition in \( P \), the conditions \( s_0, s_1 \) must be compatible in \( P \), and their lower bound is a lower bound of \( r_0 \) and \( r_1 \) as well.

To show that \( \langle Q, \tau \rangle \equiv_b \langle Q', \tau' \rangle \), use Proposition 5.2.4. To show that the pair \( \langle Q', \tau' \rangle \) is a virtual condition, suppose that \( G_0', G_1' \subseteq Q' \) are mutually generic filters. Looking at the model \( V[G_0, G_1] \), it in fact turns out that \( \tau'/G_0' \) and \( \tau'/G_1' \) are in fact equal as analytic subsets of \( P \).

Finally, suppose that \( \langle Q'', \tau'' \rangle \) is another virtual condition which is an element of the \( \equiv \)-class of the balanced class of \( \langle Q, \tau \rangle \); we must show that \( Q' \times Q'' \models \Sigma \tau' = \Sigma \tau'' \). For the \( \leq \) direction, suppose that \( G' \subseteq Q' \) and \( G'' \subseteq Q'' \) are mutually generic filters and \( r \in P \) is a condition in \( V[G', G''] \) which is below \( \Sigma \tau'/G' \); we must show that it is compatible with some element of \( \tau''/G'' \). Let \( H \subseteq Q'' \) be a filter generic over the model \( V[G', G''] \). Since the pairs \( \langle Q', \tau' \rangle \) and \( \langle Q'', \tau'' \rangle \) come from the same balanced class, it must be the case that the condition \( r \) must be compatible with every element of \( \tau''/H \); let \( s \) be such a lower bound. Since the pair \( \langle Q'', \tau'' \rangle \) is a virtual condition, the condition \( s \leq \Sigma \tau''/H \) must be compatible with some condition of \( \tau''/G'' \). By the Mostowski absoluteness between the models \( V[G', G''] \) and \( V[G', G'', H] \), \( r \) is compatible with some element of \( \tau''/G'' \) in the model \( V[G', G''] \) as required.

The following Suslin posets do not have any balanced virtual conditions and therefore are not balanced:

1. nonatomic c.c.c. posets;
2. nonatomic tree posets;
3. the posets \( \mathcal{P}(\omega)/I \) where \( I \) is a countably separated Borel ideal on \( \omega \).
Figure 5.1: A balanced virtual condition $\bar{p} \leq p$. 
Here, a tree poset on a Polish space $X$ is an analytic family of closed subsets of $X$ closed under nonempty intersections with closures of basic open sets, ordered by inclusion.

**Proof.** For (1), suppose towards contradiction that $\langle P, \leq \rangle$ is a Suslin c.c.c. poset and $\langle Q, \tau \rangle$ is a balanced pair. For every maximal antichain $A \subset P$, its maximality is a coanalytic statement and therefore absolute to the $Q$-extension. Thus, $Q \models \tau$ is compatible with some element of $A$. The balance of $\tau$ immediately shows that there can be only one element of $A$ with which $\tau$ is compatible, and the largest condition in $Q$ identifies this element. It follows that the set $\{ p \in P : Q \models \tau \leq \check{p} \}$ in the separative quotient of $P$ is a filter on $P$ which meets all maximal antichains, an impossibility in nonatomic posets.

For (2), suppose towards contradiction that $\langle P, \leq \rangle$ is a tree poset on a Polish space $X$ and $\langle Q, \tau \rangle$ is a balanced pair. For every basic open set $O \subset X$, the statement $\tau \cap \bar{O} \neq 0$ must be decided by the largest condition in $Q$ by the balance of $\tau$. This decision cannot be positive for two disjoint basic open subsets of $X$ by the balance of $\tau$ again. Thus, $\tau$ would have to be forced by $Q$ to be a singleton (even a specific singleton in the ground model), an impossibility in nonatomic tree posets.

For (3), suppose towards contradiction that $I$ is a countably separated ideal on $\omega$ as witnessed by a countable separating set $A \subset \mathcal{P}(\omega)$ and $\langle Q, \tau \rangle$ is a balanced pair in the poset $\mathcal{P}(\omega)/I$; so $\tau$ is a name for an $I$-positive subset of $\omega$. By the balance of $\tau$, for every set $a \in A$ it must be decided by the largest condition in $Q$ whether $\tau \cap \check{a} \in I$ or not. Let $B = \{ a \in A : Q \models \tau \cap \check{a} \notin I \}$. Use the balance of the name $\tau$ to argue that any intersection of finitely many elements of $B$ is an infinite set. Let $b \subset \omega$ be an infinite set such that for every $a \in B$, $b \setminus a$ is finite. Use the density of the ideal $I$ to argue that thinning out the set $b$ if necessary, we may assume that $b \in I$. It is now immediate that no set $a \in A$ can separate the $I$-small set $b$ from the $I$-positive set $\tau$ in the $Q$-extension: if $a \in A \setminus B$ then $\tau \cap a \in I$ is forced, and if $a \in B$ then $b \cap a \neq 0$. This is a contradiction. \qed

Most balanced Suslin posets used in this book are $\sigma$-closed. There are $\sigma$-closed posets which are not balanced, such as the one which adds a maximal almost disjoint family in $\mathcal{P}(\omega)$ by countable approximations, cf. Theorem 14.1.1. There are some posets which are balanced and in ZFC even collapse $\aleph_1$, cf. Theorem 8.6.9. Even such posets are valuable; remember that they prove their worth in the choiceless symmetric Solovay extension. The balanced status of certain posets is nonabsolute, see Corollary 6.3.4. Thus, even though typically the balanced conditions correspond to traditional objects of combinatorial set theory, the balanced status is a complicated matter. There is only one general preservation theorem, which is nevertheless extremely useful for obtaining consistency results:

**Theorem 5.2.10.** Let $P = \prod_n P_n$ be a countable support product of Suslin forcing notions. Balanced conditions in $P$ are exactly classified by sequences $\langle p_n : n \in \omega \rangle$ where for every $n \in \omega$, $p_n$ is a balanced condition in $P_n$. 

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Proof. On one hand, if \( \langle Q_n, \tau_n \rangle \) are balanced pairs for each \( n \in \omega \), the name for the sequence \( \langle \tau_n : n \in \omega \rangle \) in the product \( Q = \prod_n Q_n \) is balanced for the poset \( P \) essentially by the definitions. The choice of the support in the product \( Q \) is immaterial. On the other hand, if \( Q \) is a poset and \( \tau = \langle \tau_n : n \in \omega \rangle \) is a balanced \( Q \)-name for the poset \( P \), it must be the case that each of the names \( \tau_n \) for \( n \in \omega \) is balanced for the poset \( P_n \). It equally easy to see that equivalent balanced names for \( P \) give equivalent balanced names on each coordinate, and sequences of balanced names on the posets \( P_n \) which are coordinatewise equivalent yield equivalent balanced names for the product forcing.

Corollary 5.2.11. The countable support product of balanced Suslin forcings is balanced.

One issue that is constantly present in this book is the lack of absoluteness of the notions surrounding balance. As long as Question 5.2.7 remains open, it will also be necessary to relativize the definition of a balanced poset to \( V_\kappa \) where \( \kappa \) is an inaccessible cardinal. One may think that with suitable large cardinal hypothesis on \( \kappa \), one could use reflection to show that relativization is unnecessary. The best result we have in this direction is

Proposition 5.2.12. Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \kappa \) be a strong cardinal. The following are equivalent:

1. \( P \) is balanced;
2. \( V_\kappa \models P \) is balanced.

Proof. The argument uses two simple absoluteness claims of independent interest:

Claim 5.2.13. If \( M \) is a transitive model of ZFC, \( \langle Q, \tau \rangle \) is a pair in \( M \) and the pair is balanced in \( V \), then it is balanced in \( M \).

Proof. Immediate by Mostowski absoluteness between generic extensions of \( M \) and \( V \).

Claim 5.2.14. If \( M \) is a transitive model of ZFC, \( \langle Q, \tau \rangle \) is a pair in \( M, M \models \langle Q, \tau \rangle \) is balanced, and \( \mathcal{P}(Q) \subset M \), then the pair \( \langle Q, \tau \rangle \) is balanced in \( V \).

Proof. Work in \( V \). Suppose that the conclusion fails, as witnessed by posets \( R_0, R_1 \) and names \( \sigma_0, \sigma_1 \) on \( R_0 \times Q \) and \( R_1 \times Q \) respectively. Take an elementary submodel \( N \) of a large enough structure such that \( Q \subset N \) and \( |N| = |Q| \). The posets \( R_0 \cap N, R_1 \cap N \) and names \( \sigma_0 \cap N \) and \( \sigma_1 \cap N \) still witness the failure of the balance of the pair \( \langle Q, \tau \rangle \): if \( G_0 \subset (R_0 \cap N) \times Q \) and \( G_1 \subset (R_1 \cap N) \times Q \) are mutually generic filters, \( p_0 = \sigma_0/G_0 \) and \( p_1 = \sigma_1/G_1 \in P \), then \( N[G_0, G_1] \models p_0, p_1 \in P \) are incompatible conditions by the forcing theorem applied in the model \( N \), and by the Mostowski absoluteness between \( N[G_0, G_1] \) and \( V[G_0, G_1] \), this is still true in \( V[G_0, G_1] \). Now, the assumption \( \mathcal{P}(Q) \subset M \) shows that the posets \( R_0 \cap N, R_1 \cap N \) and names \( \sigma_0 \cap N \) and \( \sigma_1 \cap N \) have isomorphic copies in the model \( M \), obtaining the failure of balance of the pair \( \langle Q, \tau \rangle \) in \( M \).
Claim 5.2.13 again with \( j \) embedding \( M \) shows that balanced pairs. Say that the pairs are balanced pairs. Let Definition 5.3.2.

Now, Claim 5.2.14 applied to \( M = V_\kappa \) immediately yields the implication (2) \( \rightarrow \) (1). For the converse, suppose that \( P \) is balanced, \( p \in P \), and find a balanced pair \( \langle Q, \tau \rangle \) below \( p \). We have to deal with the unseemly possibility that \(|Q| > \kappa \) holds. Use the large cardinal hypothesis to find an elementary embedding \( j: V \rightarrow M \) with critical point \( \kappa \) such that \( \langle Q, \tau \rangle \in M \cap V_j(\kappa) \). Claim 5.2.13 shows that \( M \models \langle Q, \tau \rangle \) is a balanced pair below \( p \). By elementarity of the embedding \( j \), there must be a balanced pair \( \langle Q', \tau' \rangle \in V_\kappa \) below \( p \). Applying Claim 5.2.13 again with \( M = V_\kappa \), we see that \( V_\kappa \models \langle Q', \tau' \rangle \) is a balanced pair below \( p \). Since the condition \( p \in P \) was arbitrary, (2) follows.

\[ \square \]

5.3 Weakly balanced Suslin forcing

There is an interesting generalization of balanced Suslin forcing which can realize additional effects in extensions of the symmetric Solovay model. The basic definitions can be stated as a minor variation of the work done in the previous sections.

**Definition 5.3.1.** Let \( P \) be a Suslin forcing. A pair \( \langle Q, \tau \rangle \) is weakly balanced if \( Q \Vdash \tau \subset P \) is an analytic set and whenever \( R_0, R_1 \) are posets, \( \sigma_0, \sigma_1 \) are \( R_0 \times Q \) and \( R_1 \times Q \)-names for elements of \( P \) below \( \Sigma \tau \) and \( \langle r_0, q_0 \rangle \in R_0 \times Q \) and \( \langle r_1, q_1 \rangle \in R_1 \times Q \) are conditions then in some forcing extension there are filters \( H_0 \subset R_0 \times Q \) and \( H_1 \subset R_1 \times Q \) which are separately generic over \( V \), \( \langle r_0, q_0 \rangle \in H_0, \langle r_1, q_1 \rangle \in H_1 \), and \( \sigma_0/H_0, \sigma_1/H_1 \) are compatible elements of \( P \).

**Definition 5.3.2.** Let \( P \) be a Suslin forcing and \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are weakly balanced pairs. Say that the pairs are equivalent and write \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \) if whenever \( R_0, R_1 \) are posets, \( \sigma_0, \sigma_1 \) are \( R_0 \times Q_0 \) and \( R_1 \times Q_1 \)-names for elements of \( P \) below \( \tau_0 \) and \( \tau_1 \) respectively and \( \langle r_0, q_0 \rangle \in R_0 \times Q_0 \) and \( \langle r_1, q_1 \rangle \in R_1 \times Q_1 \) are conditions then in some forcing extension there are filters \( H_0 \subset R_0 \times Q_0 \) and \( H_1 \subset R_1 \times Q_1 \) which are separately generic over \( V \), \( \langle r_0, q_0 \rangle \in H_0, \langle r_1, q_1 \rangle \in H_1 \), and \( \sigma_0/H_0, \sigma_1/H_1 \) are compatible elements of \( P \).

**Proposition 5.3.3.** The relation \( \equiv_{wb} \) is an equivalence relation on weakly balanced pairs.

**Proof.** The relation is clearly symmetric and contains the identity as a subset. For the transitivity, let \( \langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle \) and \( \langle Q_2, \tau_2 \rangle \) be weakly balanced pairs such that \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \) and \( \langle Q_1, \tau_1 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \). To show that \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \) holds, let \( R_0, R_2 \) be posets and \( \sigma_0, \sigma_2 \) be \( R_0 \times Q_0 \)- and \( R_2 \times Q_2 \)-names for elements of \( P \) which are forced to be smaller than some element of \( \tau_0 \) and \( \tau_2 \) respectively, and let \( \langle r_0, q_0 \rangle \) and \( \langle r_2, q_2 \rangle \) be conditions in the two product posets. Let \( \kappa \) be a cardinal bigger than \( |P(R_0 \times Q_0)| \).

Since \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \), holds, an absoluteness argument shows that in the \( \text{Coll}(\omega, \kappa) \times Q_0 \)-extension, there is a filter \( H_0 \subset R_0 \times Q_0 \) generic over the ground model containing \( \langle r_0, q_0 \rangle \) such that \( \sigma_0/H_0 \) and \( \tau_1 \) as conditions in \( P \) have a lower bound; call the \( \text{Coll}(\omega, \kappa) \times Q_0 \)-name for the bound \( \sigma_1 \).

Since \( \langle Q_1, \tau_1 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \), holds, in some forcing extension there are filters
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Let \( P \) be a Suslin forcing. Every \( \equiv_{wb} \)-class contains a virtual condition, which is unique up to \( \equiv \)-equivalence.

**Proof.** Let \( \langle Q, \tau \rangle \) be a weakly balanced condition; strengthening \( \tau \) if necessary, we may assume that \( \tau \) is in fact a name for an element of \( P \). Let \( \kappa \) be an ordinal such that \( P(\langle Q, \tau \rangle) \subset V_\kappa \) and let \( \sigma \) be a \( \text{Coll}(\omega, V_\kappa) \)-name for the set \( \{ p \in P : \exists G \subset Q \, G \text{ is a filter meeting all the dense subsets of } Q \} V_\kappa^G \) such that \( p = \tau / G \).

It is clear that the pair \( \langle \text{Coll}(\omega, V_\kappa), \sigma \rangle \) is a \( P \)-pin. We will show that the pair \( \langle \text{Coll}(\omega, V_\kappa), \sigma \rangle \) is \( \equiv_{wb} \)-related to \( \langle Q, \tau \rangle \).
Suppose that $R_0, R_1$ are posets and $\sigma_0, \sigma_1$ are $R_0 \times \text{Coll}(\omega, V_\kappa)$- and $R_1 \times Q$-names for elements of $P$ stronger than $\sigma$ and $\tau$ respectively, and $\langle r_0, q_0 \rangle, \langle r_1, q_1 \rangle$ are conditions in the respective products. Without loss of generality we may assume that there is a cardinal $\lambda > |V_\kappa|$ such that $R_0 = \text{Coll}(\omega, \lambda)$. Let $K$ be the $R_0 \times \text{Coll}(\omega, V_\kappa)$-name for a filter on $Q$ which is generic over $V$ and such that $\sigma_0 \leq \tau/K$ is forced. By abstract forcing theory, strengthening the condition $\langle r_0, q_0 \rangle$ if necessary, we may cast the product $R_0 \times \text{Coll}(\omega, V_\kappa)$ below $\langle r_0, q_0 \rangle$ as the poset $\text{Coll}(\omega, \lambda) \times Q$ below some condition $\langle r'_0, q'_0 \rangle$ so that the name $K$ is a name for the generic filter on the second coordinate of the product. By the weak balance of the pair $\langle \sigma, \tau \rangle$, in some generic extension there are filters $H'_0 \subset \text{Coll}(\omega, \lambda) \times Q$ and $H_1 \subset R_1 \times Q$ containing the conditions $\langle r'_0, q'_0 \rangle$ and $\langle r_1, q_1 \rangle$ such that the condition $\sigma/H'_0, \sigma_1/H_1$ are compatible in $P$. The filter $H'_0$ can be cast as a filter $H_0 \subset R_0 \times \text{Coll}(\omega, V_\kappa)$ meeting the condition $\langle r_0, q_0 \rangle$. This completes the proof of the $\equiv_{\text{wb}}$-equivalence.

For the uniqueness part, suppose that $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ are $P$-pins which are both $\equiv_{\text{wb}}$-related to $\langle Q, \tau \rangle$; we must show that they are $\equiv$-related. Suppose towards contradiction that this fails. Then, in the $Q_1 \times Q_0$-extension, there must be an element $p \in P$ which is below some element of $\tau_0$ and incompatible with any element of $\tau_1$ (or vice versa). By a Mostowski absoluteness argument, this element $p$ will maintain its incompatibility property in every further forcing extension. Let $\sigma_0$ be a $Q_1 \times Q_0$-name for this element, let $\sigma_1$ be a $Q_1$-name for any element of $\tau_1$, and observe that $\langle Q_1 \times Q_0, \sigma_0 \rangle$ and $\langle Q_1, \sigma_1 \rangle$ witness that the the pairs $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ are $\equiv_{\text{wb}}$-unrelated, contradicting the initial assumptions.

Unlike the balanced conditions, the weakly balanced virtual conditions can be actually recognized in the ordering of virtual conditions by a natural first order property.

**Proposition 5.3.6.** Let $P$ be a Suslin forcing. Let $\langle Q, \tau \rangle$ be a $P$-pin. The following are equivalent:

1. $\langle Q, \tau \rangle$ is weakly balanced;

2. $\langle Q, \tau \rangle$ is an atom in the ordering of virtual conditions.

Recall that an element $\bar{p}$ of a partial order is an atom if every element compatible with $\bar{p}$ is in fact above $\bar{p}$.

**Proof.** For (1)$\Rightarrow$(2) direction, let $\langle R, \sigma \rangle$ be a $P$-pin which is not above $\langle Q, \tau \rangle$: we must show that it is incompatible with $\langle Q, \tau \rangle$. Suppose towards contradiction that it is compatible, with a lower bound $\langle S, \chi \rangle$. In the $R \times Q$ extension, the inequality $\Sigma \tau \leq \Sigma \sigma$ must fail, and so there is a condition $\bar{p}_0$ for an element of $P$ which is below $\Sigma \tau$ but incompatible with $\Sigma \sigma$. In the $S \times R \times Q$-extension, there is a condition $\bar{p}_1$ of $P$ which is below $\Sigma \chi$, and therefore also below $\Sigma \sigma$ and $\Sigma \tau$. Use the weak balance of the pair $\langle Q, \tau \rangle$ to find, in some generic extension, filters $H_0 \times G_0 \subset R \times Q$ and $K_1 \times H_1 \times G_1 \subset S \times R \times Q$ separately generic over $V$ such that the conditions $\bar{p}_0/H_0 \times G_0$ and $\bar{p}_1/K_1 \times H_1 \times G_1$ are compatible
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in $P$, with a lower bound $p$. Note that the sums $\Sigma\sigma/H_0$ and $\Sigma\sigma/H_1$ in the completion of the poset $P$ must coincide, as $(R,\sigma)$ is a $P$-pin. However, the condition $p$ should be incompatible with the former and below the latter by the forcing theorem. This is a contradiction.

For the (2)$\rightarrow$(1) direction, suppose that $(Q,\tau)$ is a $P$-pin which is an atom in the ordering of virtual conditions. To prove the weak balance, suppose that $R_0, R_1$ are posets, $\sigma_0, \sigma_1$ are $R_0 \times Q$ and $R_1 \times Q$-names for elements of $P$ stronger than $\tau$, and let $(r_0, q_0)$ and $(r_1, q_1)$ be conditions in the products. To find the instrumental generic filters, let $\kappa$ be an ordinal such that $\mathcal{P}(R_0 \times Q)$ and $\mathcal{P}(R_1 \times Q)$ are both subsets of $V_\kappa$, and consider the Coll($\omega, V_\kappa$)-names $\chi_0, \chi_1$ for analytic subsets of $P$ defined by $\chi_0 = \{p \in P : \exists G \subseteq R_0 \times Q \text{ such that } G \text{ is a filter meeting all open dense subsets of } R_0 \times Q \text{ in } V_\kappa \}$ such that $(r_0, q_0) \in G$ and $p = \sigma/G$. The name $\chi_1$ is defined in the same way.

It is not difficult to see that $(\text{Coll}(\omega, V_\kappa), \chi_0)$ and $(\text{Coll}(\omega, V_\kappa), \chi_1)$ are both $P$-pins. They are also both $\leq (Q, \tau)$ by their definition. By the assumption on the $P$-pin $(Q, \tau)$, they must both be $\equiv$-equivalent to $(Q, \tau)$. This means that in the Coll($\omega, V_\kappa$)-extension, there must be conditions $p_0 \in \chi_0$ and $p_1 \in \chi_1$ which are compatible in $P$. Reviewing the definition of the names $\chi_0$ and $\chi_1$, we get the filters $G_0 \subseteq H_0 \times K_0$ and $G_1 \subseteq H_1 \times K_1$ separately generic over $V$ such that $(r_0, q_0) \in G_0$, $(r_1, q_1) \in G_1$, and $\sigma_0/G_0$ and $\sigma_1/G_1$ are compatible conditions in $P$ as desired.

Finally, we record the central definition of this section.

**Definition 5.3.7.** A Suslin poset $P$ is **weakly balanced** if below every condition $p \in P$ there is a weakly balanced virtual condition stronger than $p$. 
Chapter 6

Simplicial complex forcings

6.1 Basic concepts

Many examples of $\sigma$-closed Suslin partially ordered sets in this book are presented in the same way:

Definition 6.1.1. Let $X$ be a set.

1. A set $\mathcal{K}$ of finite subsets of $X$ is a simplicial complex if it is closed under subset;

2. a set $A \subseteq X$ is a $\mathcal{K}$-set if $[A]^{<\aleph_0} \subseteq \mathcal{K}$. It is maximal if it is not a proper subset of another $\mathcal{K}$-set;

3. the poset $P_{\mathcal{K}} \subset X^\omega$ consists of countable $\mathcal{K}$-sets ordered by reverse inclusion;

4. $\check{A}_{gen}$ is the $P_{\mathcal{K}}$-name for the union of all sets in the generic filter.

Unless specifically stated otherwise, we will tacitly assume that every singleton belongs to $\mathcal{K}$. When the simplicial context $\mathcal{K}$ is understood from the context, we put $P = P_{\mathcal{K}}$ in this section. The poset $P$ is obviously $\sigma$-closed. By an elementary density argument, the set $\check{A}_{gen}$ is forced to be a maximal $\mathcal{K}$-set. The poset $P$ can be naturally presented as a Suslin forcing by replacing the countable $\mathcal{K}$-sets with their enumerations by natural numbers, and replacing the reverse inclusion ordering by reverse inclusion of ranges of the enumerations. We will neglect this innocuous step in this section as it merely complicates the notation.

Nearly every poset considered in this book can be presented as a poset of the form $P_{\mathcal{K}}$ for a Borel simplicial complex $\mathcal{K}$ on a Polish space $X$. Namely, for a poset $Q$ let $\mathcal{K}$ be the simplicial complex of the finite subsets of $Q$ which have a common lower bound. Under suitable assumptions on definability and existence of lower bounds (which are invariably satisfied), the posets $Q$ and $P_{\mathcal{K}}$ are naturally forcing equivalent. However, this point of view rarely brings any new insight. In this chapter, we deal with simplicial complexes that are
in some way algebraically natural, and their algebraic structure leads to the classification of the balanced conditions. We discovered a number of possibilities. Many of them are categorized by the properties of the following central concepts associated with any simplicial complex whatever.

**Definition 6.1.2.** Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$. $\Gamma_{\mathcal{K}}$ is the Borel graph on $\mathcal{K}$ consisting of all pairs $\langle a, b \rangle$ where $a, b \in \mathcal{K}$, $a \cup b \notin \mathcal{K}$, and $a, b$ are inclusion minimal such: if $a' \subseteq a$ and $b' \subseteq b$ then $a' \cup b' \notin \mathcal{K}$ iff $a' = a$ and $b' = b$.

In particular, if $\langle a, b \rangle \in \Gamma_{\mathcal{K}}$ then any other disjoint partition of the set $a \cup b$ into nonempty pieces $c, d \in \mathcal{K}$ and $\langle c, d \rangle \notin \Gamma_{\mathcal{K}}$.

**Definition 6.1.3.** Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$.

1. A $\mathcal{K}$-walk is a finite sequence $\langle a_i : i \in j \rangle$ of elements of $\mathcal{K}$ such that for every $k \in j \setminus \{0\}$ there is a set $b \subset \bigcup_{i \in k} a_i$ such that $\langle a_k, b \rangle \in \Gamma_{\mathcal{K}}$;
2. A $\mathcal{K}$-walk $\langle a_i : i \in j \rangle$ is said to start at a point $x_0 \in X$ if $a_0 = \{x_0\}$ and reach a point $x_1 \in X$ if for some $i \in j$ $x_1 \in a_i$;
3. $E_{\mathcal{K}}$ is the set of all pairs $\langle x_0, x_1 \rangle$ such that there is a $\mathcal{K}$-walk starting from $x_0$ and reaching $x_1$ and a $\mathcal{K}$-walk starting from $x_1$ and reaching $x_0$.

**Proposition 6.1.4.** Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$. $E_{\mathcal{K}}$ is an analytic equivalence relation. It is the smallest equivalence relation $E$ on $X$ such that for every finite set $a \subset X$, $a \in \mathcal{K}$ if and only if for every $E$-class $c$, $a \cap c \in \mathcal{K}$.

**Proof.** Write $\Gamma = \Gamma_{\mathcal{K}}$ and $E = E_{\mathcal{K}}$. It is immediate that the relation $E$ is an equivalence. The complexity assertion is also immediate from the definitions. For the optimality assertion, for an equivalence relation $F$ on $X$ write $\phi(F)$ for the statement that for every finite set $a \subset X$, $a \in \mathcal{K}$ if and only if for every $F$-class $c$, $a \cap c \in \mathcal{K}$.

We first need to verify that $\phi(E)$ holds. Suppose towards contradiction that it does not, and let $a \subset X$ be an inclusion-minimal counterexample. Note that for every $E$-class $c$ such that $a \cap c \neq 0$ it must be the case that $\langle a \cap c, a \setminus c \rangle \in \Gamma$. Now let $x \in a$ be any point and let $c$ be the $E$-class of $x$. Produce a $\mathcal{K}$-walk from $x$ which reaches $\Gamma_{\mathcal{K}}$ all the finitely many points in $c \cap a$. The walk can be extended by the set $a \setminus c$, reaching an arbitrary element of $a$. This means that all points of $a$ belong to the same $E$-class, a contradiction.

Now suppose that $F$ be an equivalence relation on $X$ for which $\phi(F)$ holds; we must argue that $E \subseteq F$. Suppose towards contradiction that this fails and let $x_0, x_1 \in X$ be $E$-related points which are not $F$-related. Let $\langle a_i : i \in j \rangle$ be a $\mathcal{K}$-walk starting from $x_0$ and reaching $x_1$. Let $k \in j$ be the smallest number such that $a_k \not\subseteq \{x_0\}_F$. There must be a set $b \subset \bigcup_{i \in k} a_i$ such that $\langle b, a_k \rangle \in \Gamma$. Applying $\phi(F)$ to the union $a_k \cup b$, we see that it must be the case that $\langle a_k \cup b \cap \{x_0\}_F \notin \mathcal{K}$.

Thus, considering the proper subset $a_k' = a_k \cap \{x_0\}_F$, we get $b \cup a_k' \notin \mathcal{K}$, contradicting the assumption that $\langle b, a_k \rangle \in \Gamma$. \qed
To conclude this section, we mention two operations on simplicial complexes. Both of them may seem trivial from the simplicial complex view, but they generate quite unpredictable operations on the associated posets \( P_K \). Several classes of complexes introduced below are nearly trivially closed under these operations.

**Definition 6.1.5.** Let \( K \) be a Borel simplicial complex on a Polish space \( X \), and let \( B \subset X \) be a Borel set. The restriction \( K \upharpoonright B \) is just the set \( K \cap [B]^{<\aleph_0} \).

**Definition 6.1.6.** Let \( X \) be a Polish space and for each \( n \in \omega \), let \( K_n \) be a Borel complex on a Borel set \( B_n \subset X \). The product \( K = \bigsqcup_n K_n \) is the simplicial complex of all finite sets \( a \subset X \) such that \( \forall n a \cap B_n \in K_n \).

One way to use the latter operation appears is when the Borel sets \( B_n \) for \( n \in \omega \) are pairwise disjoint. In this case, the poset \( P_K \) corresponds to the countable support product of the posets \( P_{K_n} \) for \( n \in \omega \). However, if the Borel sets \( B_n \) overlap, the forcing effects of this operation become much more difficult to analyze.

### 6.2 Smooth simplicial complexes

In this section we consider simplicial complexes \( K \) which are smooth in a rather vague sense: there is no obvious equivalence relation \( E \) (apart from the identity) such that the membership of a finite set \( a \) in \( K \) depends only on \([a]_E\). There are several natural classes of simplicial complexes of this type for which we can provide informative classification of balanced virtual conditions.

#### 6.2a Locally countable complexes

**Definition 6.2.1.** Let \( K \) be a Borel simplicial complex on a Polish space \( X \). We say that \( K \) is **locally countable** if the graph \( \Gamma_K \) is locally countable.

It follows from a routine application of the Lusin–Novikov theorem that if a Borel simplicial complex \( K \) on a Polish space \( X \) is locally countable, then \( E_K \) is a countable Borel equivalence relation. In particular, a Borel simplicial complex \( K \) is locally countable if and only if there is a countable Borel equivalence relation \( E \) on \( X \) such that for every finite set \( a \subset X \), \( a \in K \) if and only if \( a \cap c \in K \) for every \( E \)-class \( c \). The classification of balanced pairs in the case of locally countable simplicial complexes is particularly simple: they correspond to maximal \( K \)-sets. This is the content of the following theorem:

**Theorem 6.2.2.** Let \( K \) be a locally countable Borel simplicial complex on a Polish space \( X \).

1. For every maximal \( K \)-set \( A \subset X \), the pair \( (\text{Coll}(\omega, X), \tilde{A}) \) is balanced in the poset \( P = P_K \);
2. for every balanced pair \( \langle Q, \tau \rangle \), there is a maximal \( K \)-set \( A \subset X \) such that the balanced pairs \( \langle Q, \tau \rangle \) and \( \langle \text{Coll}(\omega, X), A \rangle \) are equivalent;

3. distinct maximal \( K \)-sets yield inequivalent balanced pairs.

In particular, the poset \( P_K \) is balanced as every condition (=a countable \( K \)-set) can be extended to a virtual balanced condition (=maximal \( K \)-set).

**Proof.** For (1), suppose that \( V[H_0], V[H_1] \) are mutually generic extensions of \( V \) and \( p_0 \in V[H_0], p_1 \in V[H_1] \) are conditions extending \( A \); we must show that \( p_0 \cup p_1 \) is a \( K \)-set. Suppose towards contradiction that \( a_0 \not\in p_0 \) and \( a_1 \not\in p_1 \) are finite sets such that \( a_0 \cup a_1 \notin K \). Choosing \( a_0, a_1 \) inclusion-minimal, we conclude that \( \{a_0, a_1\} \in \Gamma_K \). Since the graph \( \Gamma_K \) is locally countable, by the Mostowski absoluteness between \( V[H_0] \) and \( V[H_0, H_1] \) it must be the case that as \( a_0 \in V[H_0], a_1 \in V[H_0] \) must hold as well. Thus, \( a_1 \in V[H_0] \cap V[H_1] = V \). Since \( A \) was a maximal \( K \)-set and \( A \cup a_1 \subset p_1 \) is a \( K \)-set, it follows that \( a_1 \subset A \) and so \( a_0 \cup a_1 \subset p_0 \), contradicting the assumption that \( a_0 \cup a_1 \notin K \).

For (2), let \( \langle Q, \tau \rangle \) be a balanced pair; strengthening \( \tau \) if necessary we may assume that it is a \( Q \)-name for a single \( K \)-set. Replacing \( Q \) with \( Q \times \text{Coll}(\omega, X) \) and strengthening the condition \( \tau \) if necessary, we may assume that \( Q \models \forall x \in X \forall \tau \) and there is \( a \subset \tau \) such that \( \{x\} \cup a \notin K \). For each point \( x \in X \), either \( Q \models x \in \tau \) or \( Q \models \exists a \subset \tau \in \Gamma_K \) and \( a \subset \tau \) by a balance argument. Let \( A \subset X \) be the set of all points \( x \in X \) for which the first option prevails. Then \( Q \models A = \tau \cap V \) and so \( A \) is a \( K \)-set; in view of (1) and Proposition 5.2.4, it is enough to show that \( A \) is a maximal \( K \)-set. Let \( x \in X \setminus A \). Then \( Q \models \exists a \subset \tau \{x\} \cup a \notin K \); choosing \( a \) inclusion minimal, we conclude that \( Q \models \{x\}, a \in \Gamma_K \). By the local countability of the graph \( \Gamma_K \) and a Mostowski absoluteness between the ground model and its \( Q \)-extension, it must be that \( Q \models a \subset V \) and so \( a \subset A \). It follows that \( A \cup \{x\} \) is not a \( K \)-set as desired.

Finally, (3) is obvious. \( \square \)

**Example 6.2.3.** Let \( E \) be a countable Borel equivalence relation on a Polish space \( X \). Let \( K \) be the set of all finite sets consisting of pairwise \( E \)-unrelated elements. Then \( \Gamma_K \) consists of all edges \( \{x_0, x_1\} \) where \( x_0, x_1 \in X \) are distinct \( E \)-related points and so \( K \) is locally countable. The balanced pairs are classified by maximal \( K \)-sets, which are precisely the \( E \)-transversals.

For the following example, recall the notion of a perfect matching of a graph; this is just a set of edges such that each vertex gets exactly one edge in the set adjacent to it. For a locally finite bipartite graph \( \Gamma \), the existence of a perfect matching is equivalent to Hall’s marriage condition [30]: for every finite set \( a \) of vertices on one side of the bipartition, the set of neighbors of \( a \) has cardinality at least that of \( a \).

**Example 6.2.4.** Let \( \Gamma \) be a locally finite bipartite Borel graph on a Polish space \( X \) satisfying the Hall’s marriage condition; view \( \Gamma \) as a subset of \( |X|^2 \). Let \( K \) be the simplicial complex of all finite subsets of \( \Gamma \) which can be completed to a perfect matching. Then \( K \) is a Borel locally countable simplicial complex:
given a finite set \( a \subset \Gamma \), the membership of \( a \) in \( K \) can be detected by checking all fragments of \( a \) in the components of \( \Gamma \) which are countable. The balanced pairs are classified by maximal \( K \)-sets which are precisely the perfect matchings of \( \Gamma \).

For the following example, recall the notion of an end of an infinite connected graph [28]. If the graph in question is in addition acyclic, an end can be identified with an orientation of the graph in which every vertex gets exactly one edge flowing out of it.

**Example 6.2.5.** Let \( \Gamma \) be a locally finite acyclic Borel graph on a Polish space \( X \) with all components infinite; view \( \Gamma \) as a symmetric subset of \( X^2 \). Let \( K \) be the simplicial complex of all finite subsets \( a \subset \Gamma \) consisting of finite sets \( a \) such that for each \( \Gamma \)-path-connectedness class \( c \subset X \) the set \( a \cap c^2 \) can be extended to an end. It is immediate that \( K \) is a Borel locally countable simplicial complex. The balanced conditions are classified by maximal \( K \)-sets, which are orientations of the graph \( \Gamma \) in which every vertex has exactly one point in its outflow.

**Example 6.2.6.** Let \( \Gamma \) be a locally countable Borel graph on a Polish space \( X \) of chromatic number \( \leq n \) for some finite \( n \in \omega \). Let \( K \) be the simplicial complex of all finite partial \( \Gamma \)-colorings by at most \( n \) colors which can be completed to a total coloring by at most \( n \) colors. Then \( K \) is a locally countable simplicial complex: given a finite coloring \( a \), the membership of \( a \) in \( K \) can be detected by checking all fragments of \( a \) in the components of \( \Gamma \) which are countable. The balanced pairs are classified by maximal \( K \)-sets which are precisely the total \( \Gamma \)-colorings.

### 6.2b Complexes of Borel coloring number \( \aleph_1 \)

The next class of examples is obtained as a generalization of the locally countable simplicial complex using a definable version of a concept due to Erdős and Hajnal.

**Definition 6.2.7.**

1. Let \( \Gamma \) be a Borel graph on a Polish space \( X \). We say that the graph has **Borel coloring number** \( \aleph_1 \) if there is a Borel directed graph \( \vec{\Gamma} \) (an **orientation** of \( \Gamma \)) such that for each edge \( \{x_0, x_1\} \in \Gamma \), either \( \langle x_0, x_1 \rangle \in \vec{\Gamma} \) or \( \langle x_1, x_0 \rangle \in \vec{\Gamma} \), and the \( \vec{\Gamma} \)-outflow of each point is countable.

2. Let \( K \) be a Borel simplicial complex on a Polish space \( X \). We say that \( K \) has **Borel coloring number** \( \aleph_1 \) if the graph \( \Gamma_K \) has Borel coloring number \( \aleph_1 \).

In particular, every locally countable Borel simplicial complex has Borel coloring number \( \aleph_1 \). A Mostowski absoluteness argument shows that if a Borel graph has Borel coloring number \( \aleph_1 \) then this feature will persist to all generic extensions. The Borel coloring number \( \aleph_0 \) is a related but different notion, explored in Chapter 11.
To prove the balance for the posets associated with Borel simplicial complexes of Borel coloring number \( \aleph_1 \) and to classify the balanced conditions, we need a technical weakening of maximality.

**Definition 6.2.8.** Let \( K \) be a Borel simplicial complex on a Polish space \( X \). A \( K \)-set \( A \subset X \) is **weakly maximal** if for every point \( x \in X \) either \( x \in A \) or for every countable set \( b \subset X \) there is \( c \in K \) disjoint from \( b \) such that \( (A \cap b) \cup c \) is a \( K \)-set, but \( (A \cap b) \cup c \cup \{x\} \) is not.

In particular, every maximal \( K \)-set is weakly maximal. For a weakly maximal \( K \)-set \( A \subset X \), let \( \tau_A \) be the \( \Coll(\omega, X) \)-name for the analytic set of all \( p \in P \) such that for all \( x \in X \setminus V \), if \( x \in A \) then \( x \in p \) and if \( x \notin A \) then \( p \cup \{x\} \) is not a \( K \)-set. In particular, if \( A \) is a maximal \( K \)-set, then \( \Coll(\omega, X) \models \Sigma\tau_A = A \).

**Theorem 6.2.9.** Let \( K \) be a Borel simplicial complex on a Polish space \( X \) of Borel coloring number \( \aleph_1 \).

1. For every weakly maximal \( K \)-set \( A \subset X \), the pair \( \langle \Coll(\omega, X), \tau_A \rangle \) is balanced in the poset \( P = P_K \);
2. for every balanced pair \( \langle Q, \tau \rangle \), there is a weakly maximal \( K \)-set \( A \subset X \) such that the balanced pairs \( \langle Q, \tau \rangle \) and \( \langle \Coll(\omega, X), \tau_A \rangle \) are equivalent;
3. distinct weakly maximal \( K \)-sets yield inequivalent balanced pairs.

**Proof.** For (1), we must first show that \( \Coll(\omega, X) \models \tau_A \neq 0 \). By a Mostowski absoluteness argument, it is just enough to produce some generic extension in which there is a \( K \)-set \( p \subset X \) such that for all \( x \in X \setminus V \), if \( x \in A \) then \( x \in p \) and if \( x \notin A \) then \( p \cup \{x\} \) is not a \( K \)-set. To do this, let \( Q \) be the poset of nonstationary subsets of \( [X]^{\aleph_0} \) and let \( j: V \to M \) be the \( Q \)-name for the associated generic ultrapower. In particular, \( M \) is an \( \omega \)-model of ZFC containing \( j''X \) as an element, represented by the identity function; this set is forced to be countable in \( M \). Now, consider a product of copies \( Q_x \) of \( Q \) indexed by elements \( x \in X \setminus A \), with mutually generic filters \( G_x \subset Q_x \), and in the model \( V[G_x : x \in X \setminus A] \) consider the generic ultrapower embeddings \( j_x: V \to M_x \). By the weak maximality of \( A \) and elementarity, there is a set \( c_x \in M_x \) disjoint from \( V \) such that \( A \cup c_x \) is a \( K \)-set, and \( A \cup c_x \cup \{x\} \) is not. We claim that \( p = A \cup \bigcup_x c_x \) is a \( K \)-set; this will complete the proof. Suppose towards contradiction that this is not the case, and find finite sets \( a \subset A \) and \( a_x \subset c_x \) for \( x \) in some finite index set \( I \) such that \( a \cup \bigcup_x a_x \notin K \) and \( a_x \) are inclusion-minimal possible. Pick \( x_0 \) be any element of \( I \) such that \( a_{x_0} \neq 0 \), let \( b = a_{x_0} \), and let \( c = a \cup \bigcup_{x \neq x_0} a_x \). It is clear that \( \{c, d\} \in \Gamma \). Let \( \bar{I} \) be a Borel orientation of \( \Gamma \) with countable outflows, and assume for definiteness that \( \langle c, b \rangle \in \bar{I} \). By a Mostowski absoluteness argument between \( V[G_x : x \neq x_0] \) and \( V[G_x : x \in X \setminus A] \), it must be the case that \( b \), as much as all other points of the countable \( \bar{I} \)-outflow of \( c \), belongs to \( V[G_x : x \neq x_0] \), and so \( b \in V[G_x : x \neq x_0] \cap V[G_{x_0}] = V \). This contradicts the initial choice of the sets \( a_x \subset c_x \).
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Now, for the balance part, suppose that \( V[H_0] \) and \( V[H_1] \) are mutually generic extensions of \( V \), and \( p_0 \in V[H_0] \) and \( p_1 \in V[H_1] \) are conditions such that for all \( x \in X \cap V \), if \( x \in A \) then \( x \in p_0 \) and if \( x \notin A \) then \( p_0 \cup \{x\} \) is not a \( K \)-set, and similarly for \( p_1 \). We must show that \( p_0 \cup p_1 \) is a \( K \)-set. Suppose towards contradiction that this fails, and let \( a_0 \subset p_0 \) and \( a_1 \subset p_1 \) be finite sets such that \( a_0 \cup a_1 \notin K \). Passing to an inclusion-minimal example of this form if necessary, we may assume that \( \{a_0, a_1\} \in \Gamma \). Let \( \bar{\Gamma} \) be a Borel orientation of \( \Gamma \) with countable outflows, and assume for definiteness that \( \langle a_0, a_1 \rangle \in \bar{\Gamma} \). By a Mostowski absoluteness argument between \( V[H_0] \) and \( V[H_0, H_1] \), it must be the case that \( a_1 \), as much as all other points of the countable \( \bar{\Gamma} \)-outflow of \( a_0 \), belongs to \( V[H_0] \), and so \( a_1 \in V[H_1] \cap V[H_0] = V \). It follows that \( a_1 \subset A \subset p_0 \), and this contradicts the assumption that \( a_0 \cup a_1 \notin K \).

For (2), let \( \langle Q, \tau \rangle \) be a balanced pair; strengthening \( \tau \) if necessary we may assume that it is a \( Q \)-name for a single \( K \)-set. Replacing \( Q \) with \( Q \times \text{Coll}(\omega, X) \) and strengthening the condition \( \tau \) if necessary, we may assume that \( Q \models \forall x \in X \forall \tau \exists a \subset \tau \) such that \( \{x\} \cup a \notin K \). For each point \( x \in X \), either \( Q \models \exists \tau \exists a \subset \tau \in \Gamma_K \) and \( a \subset \tau \) by a balance argument. Let \( A \subset X \) be the set of all points \( x \in X \) for which the first option prevails. Then \( Q \models A = \tau \cap V \) and so \( A \) is a \( K \)-set; in view of (1) and Proposition 5.2.4, it is enough to show that \( A \) is a weakly maximal \( K \)-set. Let \( x \in X \setminus A \) and \( b \subset X \) be countable. Observe that \( Q \models \exists \tau \in \tau \{x\} \cup c \notin K \); note that for such set \( c \), \( (A \cup b) \cup c \) is still a \( K \)-set. Use the Mostowski absoluteness argument to find a set \( c \) with these properties in the ground model; this confirms that \( A \) is a weakly maximal \( K \)-set.

Finally, (3) is obvious. \( \square \)

Example 6.2.10. Let \( g: 2^\omega \to (2^\omega)^{\aleph_0} \) be any Borel function in which every point in \( (2^\omega)^{\aleph_0} \) has uncountable preimage. Let \( K \) be the simplicial complex of \( f \)-free sets on \( X = 2^\omega \). It is immediate that \( K \) has Borel coloring number \( \aleph_1 \): the graph \( \Gamma_K \) consists of all pairs \( \{x_0, x_1\} \) such that either \( x_0 \in f(x_1) \) or \( x_1 \in f(x_0) \). The Borel orientation of \( \Gamma_K \) with countable outflows can be defined by considering which of the two cases occurs. Any \( f \)-independent set \( A \subset X \) is weakly maximal, since in any extension making \( 2^\omega \cap V \) countable, there is a point \( y \notin V \) such that \( f(y) = 2^\omega \cap V \setminus A \).

6.2c Modular complexes

Another class of simplicial complexes in this section is not as closely tied to the graph \( \Gamma_K \). The definition is motivated by concerns of geometric model theory.

Definition 6.2.11. A Borel simplicial complex \( K \) on a Polish space \( X \) is modular if there is a Borel modular function \( f: K \to |Y|^{\aleph_0} \) for some Polish space \( Y \) such that

1. (monotonicity) \( f(0) = 0 \) and \( a \subset b \) implies \( f(a) \subset f(b) \);
2. (modularity) for all \( a, b \in K \), \( a \cup b \in K \) if and only if \( f(a) \cap f(b) = f(a \cap b) \).
For a $\mathcal{K}$ set $A \subset X$ we set $f(A) = \bigcup \{ f(a) : a \in [A]^{<\aleph_0} \}$.

Note that the properties of the function $f$ are coanalytic, and therefore transfer to any transitive model of set theory by the Mostowski absoluteness, in particular to all generic extensions. It is not easy to see which Borel simplicial complexes are modular and which are not. The following examples provide several interesting classes of modular complexes.

**Example 6.2.12.** Every locally countable Borel simplicial complex is modular.

*Proof.* Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space $X$. Consider the function $f : \mathcal{K} \to (\Gamma_{\mathcal{K}})^{\aleph_0}$ defined by $\{c, d\} \in f(a)$ just in case either $c \subseteq a$ or $d \subseteq a$. The values of the function $f$ are countable sets by the locally countable assumption on the simplicial complex; it is not difficult to see that the function $f$ is Borel in a suitable sense. The monotonicity of the function $f$ is immediate; this leaves us with verifying the modularity of the function $f$.

Suppose that $a, b \in \mathcal{K}$ are finite sets. Suppose first that $a \cup b \in \mathcal{K}$ and $\{c, d\} \in f(a) \cap f(b)$; we need to show that $\{c, d\} \in f(a \cup b)$.

By the definition of the function $f$, either $c \subseteq a$ or $d \subseteq a$, and $c \subseteq b$ or $d \subseteq b$ must hold. Now, $c \subseteq a$ and $d \subseteq b$ is impossible since $c \cup d \notin \mathcal{K}$ while $a \cup b \in \mathcal{K}$. For the same reason, the conjunction $d \subseteq a$ and $c \subseteq b$ is impossible as well. This means that one of $c, d$ must be a subset of both $a$ and $b$ and so of $a \cap b$. The definition of the function $f$ then shows that $\{c, d\} \in f(a \cap b)$ as required.

Now, suppose that $a \cup b \notin \mathcal{K}$; we must find an edge $\{c, d\} \in f(a) \cap f(b)$ which does not belong to $f(a \cap b)$. To this end, find an inclusion-minimal set $e \subset a \cup b$ such that $e \notin \mathcal{K}$, and let $c = e \cap a$ and $d = e \setminus a$. The minimal choice of $e$ shows that $\{c, d\} \in \Gamma_{\mathcal{K}}$. Since $c \subset a$ and $d \subset b$, it is clear that $\{c, d\} \in f(a) \cap f(b)$. Since $e \not\subset a$ and $e \not\subset b$, it follows that neither $c$ nor $d$ is a subset of $a \cap b$ and so $\{c, d\} \notin f(a \cap b)$ as required. \[\square\]

Many simplicial complexes that we consider, and in particular modular simplicial complexes, are in fact matroids. For completeness, we include the definition of matroids and several basic facts.

**Definition 6.2.13.** [2, Chapters VI and VII] A simplicial complex $\mathcal{K}$ on a set $X$ is a matroid if for any sets $a, b \in \mathcal{K}$, if $|b| > |a|$ then there is a point $x \in b \setminus a$ such that $a \cup \{x\} \in \mathcal{K}$.

It is immediate that for a Borel simplicial complex $\mathcal{K}$ on a Polish space $X$, the statement that $\mathcal{K}$ is a matroid is coanalytic and therefore absolute among transitive models of set theory. Matroid theory introduces several useful basic notions that apply to every matroid. For a set $a \subset X$ let $\text{rk}(a)$ be the largest cardinality of a set $b \subset a$ such that $b \in \mathcal{K}$, if such exists; otherwise $\text{rk}(a) = \infty$. For a set $a \subset X$ let $\bar{a} = \{x \in X : \text{rk}(a) = \text{rk}(a \cup \{x\})\}$. This is a closure operation satisfying the properties from the following fact, which may serve as an alternative definition of a matroid [2, 6.9].

**Fact 6.2.14.** Let $\mathcal{K}$ be a matroid on a set $X$ and let $a, b \subset X$ be finite sets. Then
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1. \( a \subseteq \bar{a} \);

2. (idempotence) \( b \subseteq \bar{a} \) implies \( \bar{b} \subseteq \bar{a} \);

3. (exchange principle) if \( x \in \bar{a} \cup \{y\} \setminus \bar{a} \) then \( y \in \bar{a} \cup \{x\} \).

The set \( X \) with the closure operation satisfying the above properties is a pre-geometry. For a finite set \( a \subset X \), \( a \in K \) just in case for any set \( b \subset a \) and any point \( x \in a \setminus b \), \( x \notin \bar{b} \) holds-\( K \) is the complex of free sets of the pre-geometry. The closed sets (the sets \( a \subset X \) such that \( \bar{a} = a \)) form a lattice under inclusion; the pre-geometry is modular if the lattice of its closed sets is modular. This can be equivalently characterized by \( x \in \bar{a} \cup \bar{b} \) iff there are points \( x_a \in \bar{a} \) and \( x_b \in \bar{b} \) such that \( x \in \bar{a} \cap \bar{b} \). The pre-geometry is locally countable if the closure of any finite set is countable. The following example justifies our choice of terminology.

**Example 6.2.15.** Let \( K \) be a Borel matroid on a Polish space \( X \) whose pre-geometry is locally countable and modular. Then \( K \) is a modular simplicial complex and the closure operation is a modular function on \( K \).

**Proof.** Let \( f : K \to [X]^{\aleph_0} \) be the closure operation; it is immediate from its definition that it is a Borel function. To verify the modularity of \( f \), suppose that \( a, b \in K \) are finite sets. Suppose first that \( f(a) \cap f(b) \neq f(a \cap b) \) and work to show that \( a \cup b \) is free. Let \( x \in X \) be a point in \( f(a) \cap f(b) \) which is not in \( f(a \cap b) \). Let \( a' \subset a \) be inclusion minimal superset of \( a \cap b \) such that \( x \in f(a') \).

Note that \( a' \not\subset a \cap b \), choose a point \( y \in a' \setminus (a \cap b) \), and let \( a'' = a' \setminus \{y\} \). By the exchange property of the pre-geometry, it follows that \( y \in f(a'' \cup \{x\}) \). By the monotonicity and idempotence of the pre-geometry, \( y \in f(a'' \cup b) \) and since \( y \in a \cup b \) and \( y \notin a'' \cup b \), this means that \( a \cup b \notin K \). This direction does not use the modularity assumption on the pre-geometry.

Now suppose that \( a \cup b \notin K \) and work to show that \( f(a) \cap f(b) \neq f(a \cap b) \).

Let \( c \) be an inclusion minimal subset of \( a \cup b \) such that \( a \cap b \subset c \), \( c \) is free, and for some \( x \in a \cup b \), \( c \cup \{x\} \) is not free; pick the witness \( x \). First argue that \( x \in f(c) \).

To this end, let \( d \subset c \cup \{x\} \) be minimal such that there is \( y \in (c \cup \{x\}) \setminus d \) which is in \( f(d) \). If \( x \notin d \) then \( y \) must be equal to \( x \) by the freeness of the set \( c \) and if \( x \in d \), then by the exchange property \( x \in f(d \cup \{y\} \setminus \{x\}) \) and so \( x \in f(c) \) again.

Now, assume for definiteness that \( x \in b \). Let \( a' = c \setminus b \) and \( b' = c \cap b \). Since \( x \in f(a' \cup b') \), the modularity assumption on the pre-geometry yields points \( x_0 \in f(a') \) and \( x_1 \in f(b') \) such that \( x \in f(x_0, x_1) \). Note that \( \{x_0, x_1\} \in K \); if for example \( x_1 \in f(\{x_0\}) \), then by idempotence \( x_1 \in f(a') \), by the exchange property there would have to be some element \( z \in a' \) such that \( z \in f(\{x_1\} \cup (a' \setminus \{z\})) \), and by idempotence again \( z \in f(c \setminus \{z\}) \), contradicting the freeness of the set \( c \). By the freeness of the set \( b \), \( x \notin f(\{x_1\}) \). By the exchange property applied again, \( x_0 \in f(\{x_1, x\}) \subset f(b) \). It follows that \( x_0 \in f(a) \cap f(b) \). At the same time, \( x_0 \notin f(a \cap b) \), since then (as \( a \cap b \subset c \) holds) \( x_0 \in f(b') \) and so \( x \in f(b') \), violating the freeness assumption on \( b \).
Example 6.2.16. (A linear matroid) Let $X$ be a Polish vector space over a countable field $F$, and let $\mathcal{K}$ be the matroid of finite linearly independent subsets of $X$. The pre-geometry of $\mathcal{K}$ is well-known to be modular and so the simplicial complex $\mathcal{K}$ is modular. The poset $P_\mathcal{K}$ is designed to add a basis for the space $X$.

One important point of the present section is that there are interesting matroids modular in the sense of Definition 6.2.11 even though their pre-geometries are not modular.

Example 6.2.17. (A graphic matroid) Let $G$ be a Borel graph on a Polish space $X$. Let $\mathcal{K}$ be the simplicial complex on $G$ consisting of finite acyclic subsets of $G$. Then $\mathcal{K}$ is modular. The poset $P_\mathcal{K}$ is designed a maximal acyclic subgraph to $G$ by countable approximations.

Proof. To describe the modular function $f$, let $a \subset G$ be a finite acyclic set of edges and write $E_a$ for its its path connectivity equivalence relation. The functional value $f(a)$ is the union $f_0(a) \cup f_1(a)$. Here, $f_0(a)$ consists of all finite sequences $(x_i : i \in 2n+1)$ for $n > 0$ of vertices in $X$ mentioned in the edges in $a$, which are pairwise distinct except for $x_{2n} = x_0$ and $\forall j \in n x_{2j} E_a x_{2j+1}$. The set $f_1(a)$ consists of all finite sequences $(x_i : i \in 2n+1)$ for $n > 0$ of vertices in $X$ mentioned in the edges in $a$, which are pairwise distinct except for $x_{2n} = x_0$ and $\forall j \in n x_{2j+1} E_a x_{2j+2}$. We will show that $f$ is modular. The function is clearly monotone with respect to inclusion. To verify the modularity demand, suppose that $a, b$ are acyclic sets.

Case 1. $a \cup b$ is acyclic; we must check that $f(a) \cap f(b) = f(a \cap b)$. Well, in the first place $E_a \cap E_b = E_{a \cap b}$: if vertices $x_0, x_1$ are connected by injective paths $p_a, p_b$ in $a$ and $b$ but by no path in $a \cap b$, then $p_a \cup p_b$ forms a cycle containing at least two edges which are not in $a \cap b$, and so the set $\{e \in G : e$ belongs to exactly one of $p_a, p_b\}$ forms a nonempty injective cycle in $a \cup b$, contradicting the case assumption. Now, suppose that $\vec{x} = \langle x_i : i \in 2n+1 \rangle$ is a sequence in both $f(a)$ and $f(b)$. If $\vec{x} \in f_0(a) \cap f_0(b)$, then $\vec{x} \in f_0(a \cap b)$ since $E_a \cap E_b = E_{a \cap b}$. If $\vec{x} \in f_1(a) \cap f_1(b)$ then $\vec{x} \in f_1(a \cap b)$ for the same reason. The remaining case, when $\vec{x} \in f_0(a)$ and $\vec{x} \in f_1(b)$ (or vice versa) leads to a cycle in $a \cup b$ which is impossible by the case assumption.

Case 2. $a \cup b$ contains a cycle; we must verify that $f(a) \cap f(b) \neq f(a \cap b)$. Let $c$ be an injective cycle in $a \cap b$. There must be a sequence $\vec{c} = \langle c_i : i \in 2n+1 \rangle$ of vertices on that cycle such that $n > 0$, $x_{2n} = x_0$, and the edges in $c$ between $x_{2i}$ and $x_{2i+1}$ are all in $a$ and the edges on $c$ between $x_{2i+1}$ and $x_{2i+2}$ are all in $b$. The sequence $\vec{c}$ belongs to $f(a) \cap f(b)$ since $\vec{c} \in f_0(a)$ and $\vec{c} \in f_1(b)$. The proof in this case will be complete once we prove that $\vec{x} \notin f(a \cap b)$. An indeed, if $\vec{x}$ belonged to $f(a \cap b)$, then it would yield a cycle in $a \cup b$ depending on whether $\vec{x} \in f_0(a \cap b)$ or $\vec{x} \in f_1(a \cap b)$.

Example 6.2.18. (A transversal matroid) Let $X$ be a Polish space partitioned into Borel sets $B_0, B_1$, and let $\Gamma$ be a Borel bipartite graph between $B_0$ and $B_1$. Let $\mathcal{K}$ be the matroid on $\Gamma$ consisting of finite sets of edges in which no two
distinct edges share a vertex. Then \( K \) is modular. The poset \( P_K \) is designed to add a maximal (perhaps perfect) matching to \( G \). To exhibit the modular function, let \( f(a) = \{ x \in X : \text{there is an edge } e \in a \text{ such that } x \text{ is one of the two vertices of } e \} \).

Not every modular simplicial complex needs to be a matroid.

**Example 6.2.19.** Let \( X \) be a Polish space and let \( \Gamma \) be a Borel collection of finite subsets of \( X \). Let \( K \) be the simplicial complex on \( \Gamma \) consisting of finite subsets of \( \Gamma \) which consist of pairwise disjoint sets. It is immediate that the simplicial complex \( K \) is modular, with the modular function \( f(a) = \bigcup a \).

The simplicial complex \( K \) is typically not a matroid: for example, if there are points \( x_0, x_1 \) which are together contained in a single element of \( \Gamma \) and are also contained in respective disjoint elements of \( \Gamma \), then the definitory property of a matroid immediately fails.

Finally, we need to present a couple of examples of simplicial complexes which are not modular. To rule out the existence of a function with the modular property, we use the following simple criterion:

\[
(*) \text{ there are pairwise disjoint sets } \{a, b_\alpha : \alpha \in \omega_1\} \text{ in } K \text{ such that for any } \\
\alpha, \beta \in \omega_1 \text{, } b_\alpha \cup b_\beta \in K \text{ holds and for every } \alpha \in \omega_1, a \cup b_\alpha \notin K \text{ holds.}
\]

It is clear that if \((*)\) holds, then no function \( f \) can have the modularity property: the values of \( f(b_\alpha) \) for \( \alpha \in \omega_1 \) would have to be pairwise disjoint in view of the first demand in \((*)\), and then there would have to be an ordinal \( \alpha \in \omega_1 \) such that \( f(a) \cap f(b_\alpha) = 0 \), which violates the modularity property in view of the second demand in \((*)\).

**Example 6.2.20.** (An algebraic matroid) Let \( X \) be an uncountable Polish field, and let \( K \) be the simplicial complex on \( X \) consisting of sets algebraically free over some countable subfield \( F \subset X \). The simplicial complex \( K \) satisfies \((*)\) and therefore is not modular. To see this, let \( B \subset X \) be an uncountable algebraically free set, let \( a \subset B \) be an arbitrary set of size 2, \( a = \{s, t\} \), and for each \( x \in B \setminus a \) let \( b_x = \{x+s, xt\} \). We will show that \( \{a, b_x : x \in X \setminus a\} \) exemplify \((*)\). Clearly, for each \( x \in B \setminus a \), \( a \cup b_x \) is not free. Also, for distinct elements \( x, y \in B \setminus a \), the set \( b_x \cup b_y \) is free. To see that, it is enough to show that the field \( Q(b_x, b_y) \) contains \( s, t, x, y \) and therefore has transcendence degree 4. To recover \( s, t, x, y \) from \( b_x \cup b_y \), write \( w = x - y \) and \( z = xy^{-1} \); both clearly belong to \( Q(b_x, b_y) \). Then the recovery follows from the string of equalities \( y = w(z - 1)^{-1}, x = zy, t = (x^2)z^{-1} \) and \( s = (x^2) - x \). (This elegant argument was pointed out to us by Peter Sin.)

**Example 6.2.21.** (A gammoid) Let \( X \) be a Polish space. Let \( K \) be the matroid on finite subsets of \( X \) containing those finite sets \( a \subset [X]^{<\omega_0} \) which allow the existence of an injective function \( q : a \to X \) such that \( \forall u \in a \ f(u) \in u \). To verify that \((*)\) is satisfied, select pairwise distinct points \( y_0, y_1 \) and \( x_\alpha \) for \( \alpha \in \omega_1 \) of the underlying space \( X \), let \( a = \{\{y_0\}, \{y_1\}\} \) and \( b_\alpha = \{\{y_0, x_\alpha\}, \{y_1, x_\alpha\}\} \).
CHAPTER 6. SIMPLICIAL COMPLEX FORCINGS

Example 6.2.22. Let $X$ be an uncountable Polish space and let $K$ be the simplicial complex on $[X]^2$ consisting of those finite sets $a \subset [X]^2$ such that there is no set $b \in [X]^3$ such that $|b|^2 \subset a$. The associated poset $P_K$ adds a maximal triangle-free graph on $X$. The simplicial complex $K$ satisfies (*) and so is not modular. To see this, let $x_0, x_1 \in X$ be any distinct points and let $a = \{\{x_0, x_1\}\}$. For each $z \in X$ distinct from $x_0, x_1$ let $b_z = \{\{x_0, z\}, \{x_1, z\}\}$. It is easy to check that $\{a, b_z: z \in X\}$ exemplify (*).

Example 6.2.23. Let $X = \mathcal{P}(\omega)$ and let $K$ be the simplicial complex of all finite sets $a \subset X$ such that $\bigcap a$ is infinite. The associated poset $P_K$ adds a Ramsey ultrafilter. The simplicial complex $K$ satisfies (*) and so is not modular. To see this, let $F$ be any filter on $\omega$ which is not generated by countably many sets. Let $a \subset \omega$ be an infinite set such that the complement of $a$ is in $F$, and let $b_\alpha$ for $\alpha \in \omega_1$ be pairwise distinct elements of $F$ disjoint from $a$. It is not difficult to verify that the sets $\{a\}, \{b_\alpha\}: \alpha \in \omega_1$ exemplify (*).

The examples above are selected so that the associated poset $P_K$ is always balanced. Since the criterion (*) rules out an arbitrary modular function as opposed to a Borel modular function, and in general the modular functions seem to be very difficult to construct by transfinite recursion procedures, the following question suggests itself:

Question 6.2.24. Is there a Borel simplicial complex which possesses a modular function, but no Borel modular function?

The balanced pairs in the poset $P = P_K$ for modular complexes $K$ are again classified by certain $K$-sets, as recorded in the following definition:

Definition 6.2.25. Let $K$ be a simplicial complex on a Polish space $X$. A $K$-set $A \subset X$ is strongly maximal if for every Borel collection $B \subset K$ at least one of the following occurs:

1. there are $a_0 \neq a_1$ in $B$ such that $a_0 \cup a_1 \in K$;
2. there is $b \in B$ such that $b \subset A$;
3. there is a countable set $p \subset A$ such that for no $b \in B$, $p \cup b$ is a $K$-set.

Fittingly, a strongly maximal set $A \subset X$ must be a maximal $K$-set. For this, given $x \in X$ consider the set $B = \{\{x\}\}$ and consult the three options of Definition 6.2.26. The first option is impossible, the second option yields $x \in A$, and the third option shows gives that $A \cup \{x\}$ is not a $K$-set. The maximality of the set $A$ follows.

Theorem 6.2.26. Suppose that $K$ is a modular Borel simplicial complex on a Polish space $X$.

1. Whenever $A \subset X$ is a strongly maximal set then the pair $\langle \text{Coll}(\omega, X), A\rangle$ is balanced;
2. every balanced pair is equivalent to \langle \text{Coll}(\omega, X), A \rangle for some strongly maximal set \( A \subset X \);

3. distinct strongly maximal sets give rise to inequivalent balanced pairs.

**Proof.** Let \( f : K \to [Y]^{\aleph_0} \) be a Borel modular function. For (1), suppose that \( V[H_0] \) and \( V[H_1] \) are mutually generic extensions of the ground model, and \( a_0 \in V[H_0] \) and \( a_1 \in V[H_1] \) are elements of \( K \) such that \( A \cup a_0, A \cup a_1 \) are \( K \)-sets; we must show that \( a_0 \cup a_1 \in K \). Suppose towards contradiction that this fails. Using the maximality of the set \( A \), enlarge the sets \( a_0, a_1 \) so that \( a_0 \cap a_1 = a_0 \cap V = a_1 \cap V \). The modularity of the function \( f \) implies that the set \( f(a_0) \cap f(a_1) \backslash (a_0 \cap a_1) \) is nonempty, containing some element \( y \in Y \). Since \( y \in V[H_0] \cap V[H_1] \), the product forcing theorem implies that \( y \in V \).

Back in \( V \), consider the set \( B \subset K \), \( B = \{ b \in K : y \in f(b) \) and for no proper subset \( c \subset b \), \( y \in f(c) \} \). Apply Definition 6.2.25 to \( A, B \). The first option is impossible by the modularity of the function \( f \). The second option gives a set \( b \in K \) such that \( b \subset A \) and \( y \notin f(b) \). Increasing the set \( b \) if necessary, we may arrange that \( a_0 \cap V \subset b \). But then, \( y \notin f(a_0) \cap f(b) \) while \( y \notin f(a_0 \cap b) = f(a_0 \cap V) = f(a_0 \cap a_1) \). The modularity of the function \( f \) shows that \( b \cup a_0 \notin K \) and so \( A \cup a_0 \) is not a \( K \)-set, contradicting the initial assumptions. In the third option of the trichotomy, there is (in \( V \)) a countable set \( p \subset A \) such that no element of \( B \) can be added to it. Let \( b \subset a_0 \) be an inclusion-minimal subset of \( a_0 \) such that \( y \in f(b) \). Then \( b \in B \) and \( p \cup b \) is a \( K \)-set, contradicting the choice of \( p \). (1) has been proved.

For (2), assume that \( \langle Q, \tau \rangle \) is a balanced pair. Replacing \( Q \) with \( Q \times \text{Coll}(\omega, X) \) and strengthening \( \tau \) if necessary, we may assume that \( \tau \) is in fact a name for an element of \( P_K \), and \( Q \models \forall x \in X \cap V \ x \in \tau \) or \( \{ x \} \cup \tau \) is not a \( K \)-set. By the balance, for each \( x \in X \) it has to be the case that \( Q \models \exists \dot{x} \in \tau \) or \( Q \models \{ \dot{x} \} \cup \tau \) is not a \( K \)-set; let \( A \subset X \) be the set of all points \( x \in X \) for which the former alternative occurs. We will show that \( A \subset X \) is a strongly maximal \( K \)-set. Then, \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X), A \rangle \). Since \( Q \models \tau \leq A \) is forced, in view of (1) and Proposition 5.2.4, \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X), A \rangle \) as desired.

Now, it is clear that \( A \) is a \( K \)-set, since it is forced to be a subset of the \( K \)-set \( \tau \). Suppose towards contradiction that \( A \subset X \) is not strongly maximal, as witnessed by some Borel set \( B \subset K \).

**Claim 6.2.27.** There is a poset \( R \) and an \( R \)-name \( \dot{b} \) such that \( R \models \exists \dot{b} \in K \) and \( A \cup b \) is a \( K \)-set.

**Proof.** Let \( R \) be the poset of all stationary subsets of \( [X]^{\aleph_0} \) ordered by inclusion. Now, if \( G \subset R \) is a generic filter and \( j : V \to M \) is the associated generic ultrapower, the set \( X \cap V \) belongs to \( M \) and it is countable in \( M \), as it is represented by the identity function. Thus, \( A = j(A) \cap (X \cap M) \) is a countable subset of \( j(A) \) in the model \( M \), and by the failure of the third option in Definition 6.2.25 in \( V \) and elementarity of \( j \), \( M \models \exists b \in B \) and \( A \cup b \) is a \( K \)-set. Since \( M \) is an
\(\omega\)-model of set theory, these two statements about \(b\) transfer without change to the generic extension \(V[G]\).

The treatment now divides into two cases.

**Case 1.** For some condition \(q \leq Q\) and a \(Q\)-name \(\dot{b}\) for an element of \(\dot{B}\), \(q \forces \tau \cup \dot{b}\) is a \(K\) set. In this case, note that since the second option in Definition 6.2.25 fails for \(A, B\), it must be the case that \(\dot{b}\) is forced not to be a subset of the ground model. Let \(H_0, H_1 \subseteq Q\) be mutually generic filters meeting the condition \(q\). The sets \(\dot{b}/H_0, \dot{b}/H_1 \in B\) must be distinct by the product forcing theorem. Since the first option of Definition 6.2.25 fails, it must be that \(\dot{b}/H_0 \cup \dot{b}/H_1 \notin K\). Thus, \(\tau/H_0 \cup \dot{b}/H_0\) and \(\tau/H_1 \cup \dot{b}/H_1\) are incompatible conditions in the poset \(P\) in the respective models \(V[H_0]\) and \(V[H_1]\) contradicting the balance of the pair \(\langle Q, \tau \rangle\).

**Case 2.** \(Q \forces \forall b \in B \tau \cup b\) is not a \(K\)-set. Let \(R\) and \(\dot{b}\) be a poset and a name as in the claim. Note that \(Q \times R \forces \tau \cup b\) is not a \(K\)-set by the Mostowski absoluteness between the \(Q\)- and \(Q \times R\)-extension. By passing to a condition in \(Q\) and \(R\) if necessary, find a \(Q\)-name \(\dot{c}\) and a finite set \(d \subseteq A\) such that \(Q \forces \dot{c} \subseteq \tau, d = \dot{c} \cap V\), and \(Q \times R \forces \dot{c} \cup \dot{b} \notin K\). Note that \(R \forces \dot{d} \cup \dot{b} \in K\). By the modularity of the function \(f\), \(Q \times R \forces f(\dot{d} \cup \dot{b}) \cap f(\dot{c} \setminus V) \neq 0\) holds, and by the product forcing theorem, all elements in the intersection must be in the ground model. In particular, there must be a point \(y \in Y\) and a condition \(q \in Q\) forcing \(\dot{y} \in f(\dot{c} \setminus V)\). Let \(H_0, H_1 \subseteq Q\) be filters mutually generic over the ground model, containing the condition \(q \in Q\). Then \(y \in f(\dot{c}/H_0 \setminus V) \cap f(\dot{c}/H_1 \setminus V)\) while \(y \notin f(\dot{c}/H_0 \cap H_1 \setminus V) = f(0) = 0\). The modularity of the function \(f\) shows that \(\dot{c}/H_0 \cup \dot{c}/H_1 \setminus V \notin K\), in particular \(\tau/H_0, \tau/H_1\) are incompatible conditions in the poset \(P\). This contradicts the balance of the pair \(\langle Q, \tau \rangle\).

(2) has just been proved. (3) is obvious.

While the definition of a strongly maximal set may seem to be difficult to check in specific cases, in fact we nearly always achieve a complete classification of strongly maximal sets.

**Theorem 6.2.28.** Let \(K\) be a modular Borel complex.

1. If \(K\) is the collection of finite free sets in a modular Borel pre-geometry, then every maximal \(K\)-set is strongly maximal;
2. if \(K\) is an acyclic complex of a Borel graph, then every maximal \(K\)-set is strongly maximal;
3. if \(K\) is the matching complex of a Borel graph in which every vertex has uncountable degree, a set is strongly maximal if and only if it is a perfect matching;
4. the \(P = P_K\)-generic maximal \(K\)-set is strongly maximal.

**Proof.** For (1), let \(X\) be the Polish domain of the complex \(K\), and let \(f: [X]^{<\aleph_0} \to [X]^{\aleph_0}\) be defined as \(f(a) = \text{the closure of the set } a \text{ in the pregeometry; as in
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Theorem 6.2.15, \( f \mid K \) is a modular function. Let \( A \subset X \) be a maximal \( K \)-set. Suppose that \( B \subset K \) is a set. Let \( M \) be a countable elementary submodel of a large structure containing \( B \) and \( A \). If the third option of Definition 6.2.25 fails for \( B \), there must be a set \( b \in B \) such that \( (M \cap A) \cup b \) is a \( K \)-set. If \( b \in M \) then \( b \subset A \) by the maximality of \( A \) and the elementarity of the model \( M \) and the second option of Definition 6.2.25 holds. Suppose then that \( b \notin M \). By the elementarity of the model \( M \), there must be a set \( a \in B \cap M \) such that \( b \cap M \subset a \).

We claim that \( a \cup b \in K \), confirming the first option of Definition 6.2.25. Suppose towards contradiction that this fails; by the exchange property, there have to be sets \( a' \subset a \), \( M \cap b \subset b' \subset b \) and a point \( x \in b \setminus b' \) such that \( x \in f(a' \cup b') \). By the modularity, there have to be points \( y, z \) such that \( y \in f(a') \), \( z \in f(b') \) and \( x \in f(y, z) \). By the exchange property, \( y \in f(z, x) \subset f(b) \). At the same time, \( y \in M \) and by the modularity of the set \( A \) and the elementarity of the model \( M \), there has to be \( d \subset A \) in the model \( M \) such that \( y \in f(d) \). In total, \( y \in f(b) \cap f(d) \), by the initial assumption on the set \( b \) \( b \cup d \subset K \) holds, and by the modularity of the function \( f \), \( y \in f(b \cap d) \subset f(b') \). Then \( x \in f(y, z) \subset f(b') \), contradicting the freeness of the set \( b \).

For (2), let \( G \) be a Borel graph on a Polish space \( X \) such that \( K \) is the simplicial complex of acyclic subsets of \( G \). Suppose that \( B \subset K \) is a set. Let \( M \) be a countable elementary submodel of a large structure containing \( B \) and \( A \). If the third option of Definition 6.2.25 fails for \( B \), there must be a set \( b \in B \) such that \( (M \cap A) \cup b \) is a \( K \)-set. If \( b \in M \) then \( b \subset A \) by the maximality of \( A \) and the elementarity of the model \( M \) and the second option of Definition 6.2.25 holds. Suppose then that \( b \notin M \). By the elementarity of the model \( M \), there must be a set \( a \in B \cap M \) such that \( b \cap M \subset a \). We claim that \( a \cup b \in K \), confirming the first option of Definition 6.2.25. Suppose towards contradiction that this fails, and let \( c \subset a \cup b \) be a cycle. The cycle has to contain some edges in \( b \setminus M \); let \( c' \subset c \) be a maximal contiguous part of \( c \) containing only edges in \( b \setminus M \). The beginning and ending vertex of \( c' \) (denoted by \( v_0, v_1 \) respectively) must belong to \( M \). Thus, \( v_0, v_1 \) are connected by a \( G \)-path, they are also connected by an \( A \)-path by the maximality of \( A \), and such a path \( d \subset A \) must be found in the model \( M \) by the elementarity of \( M \). Then \( d \cup c' \) is a cycle in \( (M \cap A) \cup b \), contradicting the choice of the set \( b \).

For (3), let \( G \) be a Borel graph on a Polish space \( X \) in which every vertex has uncountable degree. Every strongly maximal set \( A \subset G \) is a \( K \)-set, and therefore a matching. To see that \( A \) has to be a perfect matching, for every vertex \( x \in X \) consider the set \( B_x = \{ \{ e \} : x \text{ is one of the vertices in } e \} \). Consider the alternatives of Definition 6.2.25 for \( B_x \). (1) fails and (3) is impossible as the vertex \( x \) has uncountable degree in \( G \); therefore, (2) has to hold, meaning that the set \( A \) contains an edge with the vertex \( x \) on it and so \( A \) is a perfect matching. For the other implication, suppose that \( A \) is a perfect matching and \( B \subset K \) is a Borel set. To verify Definition 6.2.25 for \( A, B \), let \( M \) be a countable elementary submodel of a large structure containing \( G, A, B \). If the third option of Definition 6.2.25 fails, there must be \( b \in B \) such that \( (M \cap A) \cup b \) is a matching. Let \( b' = b \cap M \) and use the elementarity of the model \( M \) to find \( a \in B \cap M \) such that \( b' \subset a \). There are several possibilities now. If \( a = b = b' \) then \( a \cup A \) is
a matching by the elementarity of $M$, and by the maximality of $A$, $a \subset A$ and the option (2) has been verified. If $a \neq b$ and $a \cup b$ is a matching, then option (1) follows. Otherwise, there has to be an edge $e_0 \in b$ connecting some vertex $x$ mentioned in $a$ with a vertex outside of $M$. Since $A$ is a perfect matching and $M$ is elementary, there is an edge $e_1 \in A \cap M$ containing $x$; since the other vertex of $e_1$ is in $M$, it must be the case that $e_0 \neq e_1$ and so $(M \cap A) \cup b$ is not a matching, contradicting the choice of the set $b$.

For the last item, let $B \subset K$ be a Borel set in the $P$-extension. The poset $P$ is $\sigma$-closed, so the Borel set $B$ is in fact in the ground model. The disjunction of the second and third option of Definition 6.2.25 for the set $B$ is an immediate consequence of the genericity. \[\square\]

An attentive reader noticed that unlike the maximal $K$-sets, strongly maximal $K$-sets cannot be obtained by a straightforward application of Kuratowski–Zorn lemma in ZFC. This brings up the following question.

**Question 6.2.29.** Let $K$ be a Borel modular simplicial complex on a Polish space $X$. Can every countable $K$-set be extended to a strongly maximal $K$-set? In other words, is the poset $P_K$ balanced?

We conjecture that the answer is positive in ZFC. Certainly, Theorem 6.2.28 provides positive answer in ZFC for many specific modular complexes discussed in this book. The following theorem is the best partial general result we can prove.

**Theorem 6.2.30.** Let $K$ be a Borel modular simplicial complex on a Polish space $X$ such that the modular function takes only finite values. Then every countable $K$-set can be extended to a strongly maximal $K$-set.

**Proof.** Let $C \subset X$ be a countable $K$-set. Let $f : K \to [Y]^{<\kappa}$ be a Borel modular function. Let $\kappa$ be the cardinality of the continuum, and let $\{B_\alpha : \alpha \in \kappa\}$ be an enumeration of all Borel subsets of $K$. By recursion on $\alpha \in \kappa$ construct $K$-sets $C_\alpha$ so that

- $C = C_0 \subset C_1 \subset \ldots$, $|C_\alpha| = |\alpha| + \aleph_0$;
- for every ordinal $\alpha$, either there is a set $b \in B_\alpha$ such that $b \subset C_{\alpha+1}$ or there is a countable set $c \subset C_\alpha$ such that for no $b \in B_\alpha$, $c \cup b$ is a $K$-set.

To perform the recursion, for a limit ordinal $\alpha$ let $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$. Now suppose that the set $C_\alpha$ has been constructed and work to find $C_{\alpha+1}$. Let $M$ be a countable elementary submodel of a large structure containing $C_\alpha$. If there is no $b \in B_\alpha$ such that $(C_\alpha \cap M) \cup b$ is a $K$-set, then let $C_{\alpha+1} = C_\alpha$ and proceed to the next ordinal. If, on the other hand, there is a set $b \in B_\alpha$ such that $(C_\alpha \cap M) \cup b$ is a $K$-set, we will work to produce a set $a \in B_\alpha$ such that $C_\alpha \cup a$ is a $K$-set. Setting $C_{\alpha+1} = C_\alpha \cup a$ will complete the recursion step in this case as well.

Use Proposition 1.8.10 to find a perfect collection $\{H_z : z \in 2^\omega\}$ of filters on Coll$(\omega, < \kappa)$ pairwise mutually generic over the model $M$. The product forcing
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Theorem guarantees that for distinct points \( z_0, z_1 \in 2^\omega \), \( M[H_{z_0}] \cap M[H_{z_1}] = M \). Since the set \( C_\alpha \) has size smaller than continuum, a counting argument shows that there is a point \( z \in 2^\omega \) such that \( C_\alpha \cap M[H_z] \) and \( f(C_\alpha) \cap M[H_z] \) are both subsets of \( M \). Note that \( b \cap M, f(b) \cap M, \) and \( X \cap M \) are all sets in \( M[H_z] \) and use a Mostowski absoluteness argument for the model \( M[H_z] \) to find a set \( a \in B_\alpha \) in the model \( M[H_z] \) such that \( a \cap M = b \cap M \) and \( f(a) \cap M = f(b) \cap M \).

We claim that \( C_\alpha \cup a \) is a \( K \)-set as desired.

To see this, suppose that \( d \subset C_\alpha \) is a finite set. Adding some points of \( C_\alpha \cap M \) to \( d \) if necessary, we may assume that \( f(d) \cap f(a) \subset f(d \cap M) \cap f(a) \). The choice of the point \( z \) then implies that \( f(d) \cap f(a) = f(d) \cap f(a) \cap M = f(d \cap M) \cap f(a) \).

The latter set is equal to \( f(d \cap M \cup b) \) since \( (d \cap M) \cup b \in K \) by the initial choice of \( b \) and the fact that \( f \) is a modular function. Finally, \( d \cap M \cup b = d \cup a \) by the choice of the point \( z \) again, and so \( f(d) \cap f(a) = f(d \cap a) \) and \( a \cup d \in K \) by the modularity of the function \( f \).

Once the recursion has been performed, let \( A = \bigcup_\alpha C_\alpha \) and use the second item of the recursion demands to conclude that \( A \subset X \) is a strongly maximal \( K \)-set extending \( C \). \qed

6.2d Co-\( F_\sigma \) matroids

The final class of simplicial complexes investigated in this section deals with complexes associated with pre-geometries of a form particularly simple from the descriptive set theoretic view.

Definition 6.2.31. A simplicial complex \( K \) on a Polish space \( X \) is co-\( F_\sigma \) if the set \( \{ a \in [X]^{<\aleph_0} : a \notin K \} \) is an \( F_\sigma \)-subset of the hyperspace \( K(X) \) of compact subsets of \( X \) equipped with the Vietoris topology.

Theorem 6.2.32. Let \( K \) be a co-\( F_\sigma \) matroid on a Polish space \( X \). Write \( P = P_K \).

1. If \( A \subset X \) is a maximal \( K \)-set, then the pair \( \langle \text{Coll}(\omega, A), \hat{A} \rangle \) is balanced in the poset \( P \);

2. whenever \( \langle Q, \tau \rangle \) is a balanced pair in \( P \) then there is a maximal \( K \)-set such that the balanced pairs \( \langle Q, \tau \rangle \) and \( \langle \text{Coll}(\omega, A), \hat{A} \rangle \) are equivalent;

3. distinct maximal \( K \)-sets yield inequivalent balanced pairs.

In particular, the poset \( P \) is balanced.

Proof. For (1), suppose that \( R_0 \triangleright \sigma_0 \subset X \) is a finite set such that \( A \cup \sigma_0 \) is a \( K \)-set, and similarly for \( R_1, \sigma_1 \); we must show that \( R_0 \times R_1 \triangleright \sigma_0 \cup \sigma_1 \in K \). Suppose towards contradiction that this fails; rearranging the names \( \sigma_0, \sigma_1 \) and passing to a condition in \( R_0, R_1 \) if necessary, we may assume that there is a finite set \( a \subset A \) and \( R_0 \triangleright \sigma_0 \cap V = 0 \) and \( R_1 \triangleright \sigma_1 \cap V = 0 \) and \( R_0 \times R_1 \triangleright \hat{a} \cup \sigma_0 \cup \sigma_1 \notin K \).

Let \( M \) be a countable elementary submodel of a large enough structure, containing all objects named so far. Let \( R_0' = M \cap R_0, \sigma_0' = M \cap \sigma_0, R_1' = M \cap R_1 \).
and $\sigma'_1 = M \cap \sigma_1$. Perhaps more precisely, $\sigma'_0$ is the name on $R'_0$ such that for every basic open set $O \subset K(X)$ and every condition $r \in R'_0$, if $r \Vdash_{R'_0} \sigma_0 \in O$ then $r \Vdash_{R'_0} \sigma'_0 \in O$; similarly for $\sigma'_1$. It is not difficult to see that $\sigma'_0$ is an $R'_0$-name for a finite subset of $X$ disjoint from the ground model. The following claim is key.

**Claim 6.2.33.** $R'_0 \Vdash A \cup \sigma'_0$ is a $K$-set.

**Proof.** Suppose towards contradiction that this fails. Then there must be a finite set $b \subset A$ and a condition $r \in R'_0$ such that $r \Vdash_{R'_0} b \cup \sigma'_0 \notin K$. By the complexity assumption on $K$, strengthening the condition $r$ if necessary we can find a closed set $C \subset K(X)$ in the model $M$ such that $C \cap K = 0$ and $r \Vdash b \cup \sigma'_0 \in C$. Now, step out of the model $M$. The poset $R_0$ forces $b \cup \sigma_0 \notin C$; thus, there is a basic open set $O \subset K(X)$ in the Vietoris topology disjoint from $C$ and a condition $s \leq r$ such that $s \Vdash_{R_0} b \cup \sigma_0 \in O$. The set $O$ is of the form $O = \{ K \in K(X) : K \subset U \text{ and } K \cap V_i \neq 0 \text{ for all } i \in n \}$ for some choice of basic open sets $U, V_i \subset X$ and some number $n \in \omega$. Let $a = \{ i \in n : b \cap V_i = 0 \}$; then $s \Vdash \sigma_0 \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$. By the elementarity of the model $M$, there is a condition $t \leq r$ in $M$ such that $t \Vdash_{R_0} \sigma_0 \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$; then $t \Vdash_{R'_0} \sigma'_0 \subset U$ and $\sigma_0 \cap V_i \neq 0$ for all $i \in a$, in other words $t \Vdash_{R'_0} b \cup \sigma'_0 \in O$. This contradicts the assumption that $r \Vdash_{R'_0} b \cup \sigma'_0 \in C$. \qed

Now, let $H_1 \subset R'_1$ be a filter generic over the model $M$, in $V$. Let $H_0 \subset R'_0$ be a filter generic over $V$. By the product forcing theorem, the filters $H_0 \subset R'_0$ and $H_1 \subset R'_1$ are mutually generic over the model $M$. Write $b_0 = \sigma'_0/H_0$ and $b_1 = \sigma'_1/H_1$; so $b_1 \in V$. By the elementarity of the model $M$ and the forcing theorem applied in it to the poset $R_0 \times R_1$, $M[H_0, H_1] \models a \cup b_1 \in K$ and $a \cup b_1 \cup b_0 \notin K$. By the Mostowski absoluteness $V[H_0] \models a \cup b_1 \in K$ and $a \cup b_1 \cup b_0 \notin K$.

By the maximality of the initial set $A \subset X$, find a finite set $c \subset A$ such that $a \cup b_1$ is a subset of the algebraic closure of $c$. By the claim, $c \cup b_0 \in K$. Find an inclusion maximal set $d \subset a \cup b_1 \cup b_0$ containing $a \cup b_1$ as a subset and such that $d \in K$. Let $x \in (a \cup b_1 \cup b_0) \setminus d$ be an arbitrary point; clearly $x \in b_0$. Consider the pre-geometry associated with the matroid $K$. Denote its closure operation by algebraic closure to eliminate possible confusion with topological closure in the space $X$. By the exchange property of the pre-geometry, $x$ belongs to the algebraic closure of $d$. By the idempotence of the pre-geometry, $x$ belongs to the algebraic closure of $c \cup (b_0 \cap d)$. However, this contradicts the fact that $c \cup b_0 \in K$ holds. (1) has just been proved.

For (2), let $\langle Q, \tau \rangle$ be a balanced pair. Replacing $Q$ with $Q \times \text{Coll}(\omega, X)$ and strengthening $\tau$ if necessary, we may assume that $Q \models \forall x \in X \cap V \, x \in \tau$ or $\{ x \} \cup \tau$ is not a $K$-set. By a balance argument, for each $x \in X$ it must be the case that either $Q \models \dot{x} \in \tau$ or $Q \models \{ \dot{x} \} \cup \tau$ is not a $K$-set. Let $A \subset X$ be the set of those points $x \in X$ for which the former case prevails. Since $Q \models \tau \leq \dot{A}$, in view of (1) and Proposition 5.2.4 it will be enough to show that $A \subset X$ is a maximal $K$-set.
Certainly, \( A \subset X \) is a \( K \)-set since it is forced to be a subset of a \( K \)-set \( \tau \). Now suppose that \( A \subset X \) is not maximal and pick a point \( x \notin A \) such that \( A \cup \{ x \} \) is still a \( K \)-set. Passing to a condition of \( Q \) if necessary, we may find a finite set \( a \subset A \) and a \( Q \)-name \( \sigma \) for a finite subset of \( \tau \) such that \( Q \models \check{x} \) is in the algebraic closure of \( \check{a} \cup \sigma \) and not in the closure of any of its proper subsets. Let \( \eta \) be a \( Q \)-name for an inclusion-maximal subset of \( a \cup \{ x \} \cup \sigma \) which is in \( K \) and contains \( a \cup \{ x \} \). It is not equal to \( a \cup \{ x \} \cup \sigma \), and it contains every element of \( \sigma \setminus \eta \) in its algebraic closure by the exchange property of the pre-geometry. Let \( H_0, H_1 \subset Q \) be filters mutually generic over \( V \) and write \( p_0 = \tau / H_0, b_0 = \sigma / H_0, \) and \( c_0 = \eta / H_0, \) and similarly for \( p_1, b_1, c_1 \). It will be enough to show that \( a \cup b_0 \cup b_1 \notin K \), since this will contradict the initial assumption of balance of the name \( \tau \) and the resulting compatibility of conditions \( p_0, p_1 \in P \).

Note that \( x \) belongs to the algebraic closure of \( a \cup b_0 \). By the idempotence of the pre-geometry, it follows that every element of the nonempty set \( b_1 \setminus c_1 \) belongs to the algebraic closure of \( a \cup b_0 \cup c_1 \). Since \( a \cup b_0 \cup c_1 \subset a \cup b_0 \cup b_1 \), this shows that the latter set is not free and completes the proof of (2).

Finally, (3) is obvious. The last sentence of the theorem follows from a straightforward application of Kuratowski–Zorn lemma—every \( K \)-set can be extended to a maximal \( K \)-set.

**Example 6.2.34.** [33] (An algebraic matroid) Let \( X \) be a Polish field over a countable subfield \( F \). Consider the Borel simplicial complex \( K \) of finite subsets of \( X \) which are algebraically free over \( F \). \( K \) is well-known to be generated by the Borel pre-geometry of algebraic closure. Also, a finite set \( a \subset X \) is not in \( K \) just in case it is a solution to a nontrivial polynomial with coefficients in \( F \). There are only countably many such polynomials and for each of them, the set of solutions is \( F_\sigma \). Thus, the simplicial complex \( K \) is algebraic and the poset \( P_K \) is balanced by Theorem 6.2.32. The poset adds a transcendence basis to the field \( X \) as the union of the generic filter.

### 6.3 Quotient variations

Many simplicial complex forcings are naturally connected with a Borel equivalence relation.

**Definition 6.3.1.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \). A Borel simplicial complex \( K \) on \( X \) is an \( E \)-quotient complex if the membership of any finite set \( a \subset X \) in \( K \) depends only on \( [a]_E \). The letter \( E \) is left out if \( E \) is understood from the context or will be specified later.

In this section we identify two general schemas for defining useful quotient simplicial complexes and their associated posets: collapse posets for quotient cardinals and uniformization posets for quotient spaces.

**Definition 6.3.2.** Let \( E, F \) be Borel equivalence relations on Polish spaces \( X, Y \) with uncountably many classes each. Let \( K_{E,F} = K \) be the simplicial complex on
the set \((X \times Y) \cup Y\) consisting of finite sets \(a\) such that for all \(\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in a\), 
\(x_1 E x_0\) if and only if \(y_1 F y_0\) holds, and for all \(\langle x_0, y_0 \rangle, y_2 \in a\), \(y_0 F y_2\) fails. 
Clearly, \(K\) is an \((E \times F) \cup F\)-quotient complex. The associated poset \(P_K\) is referred to as the \(E, F\)-collapse poset.

If \(A \subseteq (X \times Y) \cup Y\) is a generic set, the part \(A \cap (X \times Y)\) is an injection from \(X/E\) to \(Y/F\), while the part \(A \cap Y\) is the complement of the range of \(A \cap (X \times Y)\). The balanced conditions are neatly classified by injections from the virtual \(E\)-quotient space \(X^*\) to the virtual \(F\)-quotient space \(Y^*\). If \(g: X^* \to Y^*\) is an injection, let \(\tau_g\) be the \(\text{Coll}(\omega, \underline{\tau}_1)\)-name for the set of those elements \(p \in P_K\) such that for each pair \(\langle c, d \rangle \in g\), there is a pair \(\langle x, y \rangle \in p\) such that \(x \in X\) is a realization of the virtual \(E\)-class \(c\) and \(y \in Y\) is a realization of the virtual \(F\)-class \(d\). Moreover, if \(d\) is a virtual \(F\)-class not in the range of \(g\), then \(p\) is required to contain an element \(y \in Y\) which is a realization of the virtual \(F\)-class \(d\).

**Theorem 6.3.3.** Let \(E, F\) be Borel equivalence relations on respective Polish spaces \(X, Y\). Let \(P\) be the \(E, F\)-collapse poset.

1. For every total injection \(g: X^* \to Y^*\), the pair \((\text{Coll}(\omega, \underline{\tau}_1), \tau_g)\) is a balanced \(P\)-pin;

2. for every balanced pair \((Q, \tau)\) there is a total injection \(g: X^* \to Y^*\) such the pair \((\text{Coll}(\omega, \underline{\tau}_1), \tau_g)\) is equivalent to \((Q, \tau)\);

3. distinct injections as in (1) yield inequivalent balanced conditions.

**Proof.** Write \(R = \text{Coll}(\omega, \underline{\tau}_1)\). For (1), the pair \((\text{Coll}(\omega, \underline{\tau}_1), \tau_g)\) is clearly a \(P\)-pin. For the balance, let \(V[H_0], V[H_1]\) be mutually generic extensions of the ground model and \(p_0 \in V[H_0]\) and \(p_1 \in V[H_1]\) be conditions in the analytic set \(\tau_g^* / H_0 = \tau_g^* / H_1\); we must show that \(p_0, p_1 \in P\) are compatible.

To see this, suppose for example that \(\langle x_0, y_0 \rangle \in p_0, \langle x_1, y_1 \rangle \in p_1\), and \(x_0 E x_1\); we must show that \(y_0 E y_1\). By a mutual genericity argument, the points \(x_0, x_1\) must be realizations of the same virtual \(E\)-class \(c\) in the ground model. Since both \(p_0, p_1\) are \(K\)-sets, the definition of the name \(\tau_g\) shows that both \(y_0, y_1\) must be realizations of the virtual \(F\)-class \(g(c)\) and therefore \(y_0 F y_1\) as desired.

Similarly, if \(\langle x_0, y_0 \rangle \in p_0\) and \(y_2 \in p_1\), we need to show that \(y_0 F y_2\) fails. Suppose towards contradiction that \(y_0 F y_2\) holds. By a mutual genericity argument, the points \(y_0, y_2\) are representations of the same virtual \(F\)-class \(d\) in the ground model. Now, if \(d \in \text{rng}(g)\) then \(p_1\) cannot be a \(K\)-set, and if \(d \notin \text{rng}(g)\) then \(p_0\) cannot be a \(K\)-set. There are no other options, and this contradiction completes the proof of (1).

For (2), let \(g: X^* \to Y^*\) be the collection of all pairs \(\langle c, d \rangle \in X^* \times Y^*\) so that \(R \times Q \models \langle x, y \rangle \in \tau\) for some realizations \(x, y\) of the virtual classes \(c, d\). It will be enough to show that \(g\) is a total injection from \(X^*\) to \(Y^*\) and the pair \((Q, \tau)\) is equivalent to \((R, \tau_g)\).

To see that \(\text{dom}(g) = X^*\) and \(g\) is an injection, let \(c \in X^*\) be an arbitrary virtual \(E\)-class. Use the balance of the pair \((Q, \tau)\) to show that there must be
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a unique virtual $F$-class $d \in Y^{**}$ such that $R \times Q \vDash \langle x, y \rangle \in \hat{A}_{\text{gen}}$ for some realizations $x, y$ of the virtual classes $c, d$. Use the balance of the pair $(Q, \tau)$ again to show that for every virtual $F$-class $d \in Y^{**}$, there either must be a virtual $E$-class $c \in X^{**}$ such that $R \times Q \vDash \langle x, y \rangle \in \hat{A}_{\text{gen}}$ for some realizations $x, y$ of the virtual classes $c, d$, or it must be the case that $R \times Q \vDash y \in \tau$ for some realization $y$ of the virtual class $d$.

To see that $(R, \tau_g)$ is equivalent to $(Q, \tau)$, strengthen $\tau$ if necessary so it is a name for an actual element of $P$ and notice that in the $R \times Q$ extension, $\tau \in \tau_g$ holds. The equivalence then follows from Proposition 5.2.4.

Finally, (3) is obvious.

Corollary 6.3.4. Let $E, F$ be Borel equivalence relations on respective Polish spaces $X, Y$ with uncountably many classes. The $E, F$-collapse poset is balanced if and only if $\lambda(E) \leq \lambda(F)$.

The collapse posets exemplify an important phenomenon: a $\sigma$-closed Suslin forcing whose balanced status cannot be decided in ZFC. There are Borel equivalence relations $E, F$ for which the status of the inequality $\lambda(E) \leq \lambda(F)$ cannot be decided in ZFC (see ???) and for them the balanced status of the corresponding collapse forcing is undecidable as well.

Definition 6.3.5. Let $E \subset F$ be Borel equivalence relations on a Polish space $X$. Let $K = K_{E, F}$ be the simplicial complex of all finite sets $a \subset X$ such that for $x_0, x_1 \in X$, $x_0 E x_1$ is equivalent to $x_0 F x_1$. We will refer to the associated poset $P_K$ as the $E, F$-transversal poset.

It is immediate that the simplicial complex $K_{E, F}$ is a Borel $E$-quotient complex. The terminology comes from the examples below; the posets of this type are used to uniformize various relations in products of quotient spaces. The first order of business is to classify the balanced conditions. Let $X^{**}$ be the virtual $E$-quotient space. For a function $f : X/F \rightarrow X^{**}$ such that for each $F$-class $c$ the functional value $f(c)$ is a virtual $E$-class which is (forced to be) a subset of $c$, let $\tau_f$ be the Coll$(\omega, \beth_1)$-name for the set of all conditions $p$ in the uniformization poset such that for each class $c$ represented in the ground model, $p$ contains some point $x \in X$ which is a realization of the virtual $E$-class $f(c)$. It is not difficult to see that the pair $(\text{Coll}(\omega, \beth_1), \tau_f)$ is a $P$-pin.

Theorem 6.3.6. Let $E \subset F$ be Borel equivalence relations on a Polish space $X$ and suppose that $F$ is pinned. Let $P$ be the $E, F$-transversal poset. Then

1. for every total function $f : X/F \rightarrow X^{**}$, the pair $(\text{Coll}(\omega, \beth_1), \tau_f)$ is balanced;

2. for every balanced pair $(Q, \tau)$ there is a total function $f$ as in (1) such that the pairs $(\text{Coll}(\omega, \beth_1), \tau_f)$ and $(Q, \tau)$ are equivalent;

3. distinct functions yield nonequivalent balanced pairs.
Proof. To see (1), let \( V[H_0] \) and \( V[H_1] \) be mutually generic extensions of the ground model and \( p_0 \in V[H_0] \) and \( p_1 \in V[H_1] \) be conditions in the set \( \tau_f/H_0 = \tau_f/H_1 \); we have to show that \( p_0, p_1 \in P \) are compatible. To this end, suppose that \( x_0 \in p_0 \) and \( x_1 \in p_1 \) are points; we must show that if \( x_0 F x_1 \) then \( x_0 E x_1 \) holds. A mutual genericity argument shows that \( x_0, x_1 \) are realizations of the same virtual \( F \)-class \( c \). By the definition of the name \( \tau_f \), it must be the case that \( x_0, x_1 \) are both realizations of the virtual \( E \)-class \( f(c) \) and therefore \( E \)-related.

For (2), suppose that \( \langle Q, \tau \rangle \) is a balanced pair. To find the total function \( f \), let \( c \) be an \( F \)-class. There must be a virtual \( E \)-class \( f(c) \) such that \( Q \models f \) for some point \( x \in \tau \cap \dot{c} \) \( x \) is a realization of \( f(c) \); otherwise, one could find conditions \( q_0, q_1 \in Q \) and names for strengthenings \( \tau_0, \tau_1 \) of \( \tau \) and names \( x_0, x_1 \) for elements of \( \dot{c} \) such that \( \langle q_0, q_1 \rangle \models Q \models \neg(\dot{x}_0) \models F (\dot{x}_1) \). This would violate the balance of the pair \( \langle Q, \tau \rangle \) as \( \langle q_0, q_1 \rangle \models \tau_0, \tau_1 \in P \) are incompatible conditions. Let \( f : c \mapsto d_c \) be the resulting function.

To see that \( \langle R, \tau_f \rangle \) is a balanced pair equivalent to \( \langle Q, \tau \rangle \), strengthen \( \tau \) if necessary to ensure that \( \tau \) is a name for an actual element of \( P \), and observe that \( Q \times \text{Coll}(\omega_1, \omega_1) \models \tau \models \tau_f \). The equivalence of the two balanced pairs then follows from Proposition 5.2.4.

Finally, (3) is obvious. \( \square \)

Corollary 6.3.7. Let \( E \subset F \) be Borel equivalence relations on a Polish space \( X \) such that \( F \) is pinned. Let \( P \) be the \( E, F \)-uniformization poset. Then \( P \) is balanced.

Example 6.3.8. A poset introducing a transversal to a pinned Borel equivalence relation \( F \) on a Polish space \( X \). Just let \( E \) be the identity on the space \( X \) and consider the \( E, F \)-transversal poset. The generic set is an \( F \)-transversal. By Theorem 6.3.6, the balanced conditions are classified by \( F \)-transversals.

Example 6.3.9. A poset introducing a complete countable section to a pinned Borel equivalence relation \( G \) on a Polish space \( Y \). Let \( X = Y^\omega \) and let \( B \subset X \) be the Borel set of all elements \( x \in X \) whose range consists of pairwise \( G \)-related points. Let \( F \) be the Borel equivalence relation on \( X \) connecting points \( x_0, x_1 \) if either they are equal or else they are both in \( B \) and their ranges are subsets of the same \( G \)-class. Let \( E = \mathbb{F}_2 \cap F \). It is not difficult to check that \( E \subset F \) are Borel equivalence relations on \( X \) and \( F \) is pinned. Consider the associated \( E, F \)-transversal poset. The generic set is a countable complete section of the relation \( E \). In view of the fact that virtual \( \mathbb{F}_2 \)-classes are classified by nonempty subsets of \( Y \), Theorem 6.3.6 says that the balanced conditions are classified by arbitrary sets \( A \subset Y \) which have nonempty intersection with every \( G \)-class.

Numerous examples of uniformization posets arise in the context of countable Borel equivalence relations. If \( E \) is a countable Borel equivalence relation on a Polish space \( Y \) and \( P \) is a poset adding a choice of a structure of some type on each \( E \)-class with countable approximations, then \( P \) can be represented as a transversal poset in the sense of Definition 6.3.5. The following example spells out the details.
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Example 6.3.10. Let $G$ be a countable Borel equivalence relation on a Polish space $Y$ with all equivalence classes infinite. Let $P$ be the poset of all countable functions on the quotient space $Y/G$ assigning to each equivalence class a $\mathbb{Z}$-ordering on it; the poset $P$ serves to introduce a (discontinuous) $\mathbb{Z}$-action on $Y$ such that $G$ is its orbit equivalence relation. It is not difficult to cast $P$ as a transversal poset in the sense of Definition 6.3.5. Let $X = (Y^2)^\omega$ and let $B \subset X$ be the Borel set of all elements $x \in X$ which enumerate a linear ordering of a whole single $G$-class of ordertype $\mathbb{Z}$. Let $F$ be the equivalence relations on $X$ consisting of all pairs $x_0, x_1 \in X$ if either they are equal or else they both belong to $B$ and enumerate a linear ordering of the same $G$-class. Let $E = \mathbb{F}_2 \cap F$. It is not difficult to check that both $E, F$ are pinned Borel equivalence relations. Consider the associated $E, F$-transversal poset. The generic set selects one $\mathbb{Z}$-type ordering for each $G$-class. Theorem 6.3.6 says that the balanced conditions are classified by all functions $f$ assigning each $G$-class a linear ordering of its elements isomorphic to $\mathbb{Z}$.

Example 6.3.11. Let $X = ((2^\omega)^\omega)^2$ and consider the Borel set $B \subset X$ consisting of all pairs $\langle y_0, y_1 \rangle \in X$ such that $\text{rng}(x_0) \cap \text{rng}(x_1) = 0$. Let $F$ be the equivalence relation connecting points $\langle y_0, y_1 \rangle$ and $\langle z_0, z_1 \rangle$ if they are either equal or both in $B$ and $y_0 \not\in \mathbb{F}_2 z_0$. Let $E$ be the equivalence relation $\mathbb{F}_2 \times \mathbb{F}_2$ intersected with $F$. The resulting $E, F$-transversal poset is designed to add a function assigning to each nonempty countable set a single nonempty countable set disjoint from it. However, the equivalence relation $F$ is not pinned and so Theorem 6.3.6 does not apply to show that the uniformization poset is balanced. Indeed, in ZF one can prove that existence of a maximal $K_{E,F}$-set implies the existence of an $\omega_1$-sequence of pairwise distinct $E$-classes. To see this, let $A \subset X$ be such a maximal set, suppose $c_0 \subset 2^\omega$ is an arbitrary nonempty countable set, and by transfinite recursion on $\alpha \in \omega_1$ define $c_\alpha$ as the unique nonempty countable set such that for some enumerations $y_0$ of $\bigcup_{\beta<\alpha} c_\beta$ and $y_1$ of $c_\alpha$ it is the case that $\langle y_0, y_1 \rangle \in B$. It is clear that the sets $c_\alpha$ for $\alpha \in \omega_1$ are nonempty and pairwise disjoint and therefore distinct. Finally, note (Theorem 9.1.1) that in balanced extensions of the symmetric Solovay model there are no uncountable sequences of distinct equivalence classes of Borel equivalence relations such as $\mathbb{F}_2$.

If the larger equivalence relation $F$ in Definition 6.3.5 is not pinned, the circumstances must be quite peculiar for the $E, F$-transversal poset to be balanced. One class of examples of this type is captured by the following theorem. For Borel equivalence relations $E \subset F$ on a Polish space $X$ write $X^*_E$, $X^*_F$ to be their respective virtual quotient spaces. In addition, for a function $f: X^*_E \to X^*_F$ such that for every virtual $F$-class $c$, the virtual $E$-class $f(c)$ is forced to be a subset of $c$, let $\tau_f$ be the Coll($\omega, \bigcup_{c} \omega$)-name for the set of all countable subsets $a \subset X$ such that for each virtual $F$-class $c$, $a$ contains a realization of the virtual $E$-class $f(c)$.

Theorem 6.3.12. Let $E \subset F$ be Borel equivalence relations on a Polish space $X$ such that each $F$-class consists of countably many $E$-classes. Let $P$ be the $E, F$-transversal poset. Then...
1. for every function \( f : X^*_E^* \to X^*_E^* \) such that for every virtual \( F \)-class \( c \), the virtual \( E \)-class \( f(c) \) is forced to be a subset of \( c \), the pair \( \langle \text{Coll}(\omega, \mathcal{Z}_\omega), \tau_f \rangle \) is balanced;

2. for every balanced pair \( \langle Q, \tau \rangle \) there is a function \( f \) as in (1) such that the balanced pairs \( \langle Q, \tau \rangle, \langle \text{Coll}(\omega, \mathcal{Z}_\omega), \tau_f \rangle \) are equivalent;

3. distinct functions as in (1) yield nonequivalent balanced pairs.

Proof. The argument hinges on a simple claim of independent merit:

**Claim 6.3.13.** Let \( V[H] \) be a generic extension of \( V \), and in \( V[H] \) let \( c \) be an \( F \)-class which is a realization of a ground model virtual \( F \)-class. Then \( c \) is a union of realizations of ground model virtual \( E \)-classes.

Proof. Suppose towards contradiction that this fails. Then in \( V \) there has to be a poset \( P \) and a \( P \)-name \( \tau \) for an element of the space \( X \) such that \( \tau \) is \( E \)-pinned, but below no condition of \( P \), \( \tau \) is \( F \)-pinned. Let \( M \) be a countable elementary submodel of a large structure containing \( E, F, \tau \) and use Proposition 1.8.10 to find a perfect collection \( \{g_y : y \in 2^\omega\} \) of filters on \( P \cap M \) mutually generic over \( M \). Let \( x_y = \tau/g_y \in X \). For distinct binary sequences \( y, z \in 2^\omega \), \( M[g_y, g_z] \models x_y F x_z \) and \( \neg x_y E x_z \) by the forcing theorem applied in \( M \). By the Mostowski absoluteness for the model \( M[g_y, g_z] \), it follows that \( x_y F x_z \) and \( \neg x_y E x_z \) holds in \( V \). However, this means that the unique \( F \)-class of the points \( x_y \) for \( y \in 2^\omega \) contains perfectly many pairwise \( E \)-unrelated points, contradicting the initial assumptions on the equivalence relations \( E, F \). \( \square \)

Now, for (1) suppose that \( V[H_0], V[H_1] \) are mutually generic extensions containing respective conditions \( p_0, p_1 \) below \( \tau_f \); we must show that \( p_0, p_1 \) are compatible. This is to say, \( p_0 \cup p_1 \) is a \( K_{E,F} \)-set; in other words, if \( x_0 \in p_0 \) and \( x_1 \in p_1 \) are \( F \)-related points then they are in fact \( E \)-related. To see this, by a mutual genericity argument \( x_0, x_1 \) are realizations of some virtual \( F \)-class \( c \); by the definition of the virtual condition \( \tau_f \), they must be realizations of the virtual \( E \)-class \( f(c) \) and therefore must be \( F \)-related.

For (2), first use a balance argument and the claim to argue that for each virtual \( F \)-class \( c \) there must be a unique virtual \( E \)-class \( d \) which is forced to be a subclass of \( c \) and such that \( Q \models \tau \) contains a realization of \( d \). Let \( f : X^*_F \to X^*_E \) be the function recording the correspondence \( c \to d \) and observe that \( Q \times \text{Coll}(\omega, \mathcal{Z}_\omega) \models \tau \leq \tau_f \) holds. (2) then follows by a reference to Proposition 5.2.4.

Finally, (3) is obvious. \( \square \)

**Example 6.3.14.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) which contains no odd length cycles such that the \( G \)-path connectedness equivalence relation \( F \) on \( X \) is Borel. We wish to add a 2-coloring of the graph \( \Gamma \). To this end, let \( E \) be the Borel equivalence relation on \( X \) relating \( x \) to \( y \) if \( x E y \) holds and some (equivalently, every) \( \Gamma \)-path from \( x \) to \( y \) is of even length. It is clear that every \( F \)-class consists of exactly two \( E \)-classes. The \( E, F \)-transversal poset selects exactly one \( E \)-class from each \( F \)-class and therefore adds a bipartization
of the graph $\Gamma$. The balanced conditions are classified by functions which select exactly one virtual $E$-class from each virtual $F$-class.
Chapter 7

Ultrafilter forcings

Many applications of the axiom of choice rely on nonprincipal ultrafilters on \( \omega \) and their combinatorial properties. In this section, we will show that in several cases, natural attempts to add an ultrafilter result in balanced forcings and moreover the resulting generic ultrafilter can be characterized in a simple way. The following definition and proposition will play a central role in several arguments.

**Definition 7.0.1.** Let \( U \) be a nonprincipal ultrafilter on a set \( \text{dom}(U) \). An infinite set \( a \subset \text{dom}(U) \) in some forcing extension diagonalizes \( U \) if for every \( b \in U \), \( a \setminus b \) is finite.

**Proposition 7.0.2.** Let \( U \) be a nonprincipal ultrafilter on \( \text{dom}(U) \). Suppose that \( Q_0, Q_1 \) are posets and \( \tau_0, \tau_1 \) are respective names for infinite subsets of \( \text{dom}(U) \) which diagonalize \( U \). Then \( Q_0 \times Q_1 \Vdash \tau_0 \cap \tau_1 \) is infinite.

**Proof.** The most appealing argument uses the general Proposition 1.8.8 on product forcing extensions. Let \( H_0, H_1 \subset Q_0, Q_1 \) be mutually generic filters and let \( a_0 = \tau_0/H_0 \) and \( a_1 = \tau_1/H_1 \). Suppose towards contradiction that the intersection \( a_0 \cap a_1 \) is finite. By Proposition 1.8.8, there are sets \( b_0, b_1 \subset \text{dom}(U) \) in the ground model such that \( a_0 \subset b_0 \), \( a_1 \subset b_1 \), and \( b_0 \cap b_1 \) is finite. At most one of the sets \( b_0, b_1 \) belongs to the ultrafilter \( U \). If, say, \( b_0 \not\in U \), then \( \omega \setminus b_0 \in U \), violating the assumption that \( a_0 \) diagonalizes the ultrafilter \( U \).

7.1 A Ramsey ultrafilter

The most elementary attempt at adding a nonprincipal ultrafilter is the poset \( P \) of infinite subsets of \( \omega \) ordered by inclusion. Here, the classification of balanced conditions is particularly appealing and simple.

**Definition 7.1.1.** Let \( U \) be a nonprincipal ultrafilter on \( \omega \). The symbol \( \tau_U \) denotes the \( \text{Coll}(\omega, U) \)-name for the analytic collection of all infinite sets \( a \subset \omega \) which diagonalize the ultrafilter \( U \).
Theorem 7.1.2. Let $P$ be the partial order of infinite subsets of $\omega$, ordered by inclusion.

1. The pair $(\text{Coll}(\omega, U), \tau_U)$ is a balanced virtual condition for $P$ for every nonprincipal ultrafilter $U$ on $\omega$;

2. If $(Q, \tau)$ is a balanced pair for $P$, then there is a nonprincipal ultrafilter $U$ such that $(Q, \tau)$ is equivalent to $(\text{Coll}(\omega, U), \tau_U)$;

3. distinct nonprincipal ultrafilters yield inequivalent balanced virtual conditions.

In particular, the poset $P$ is balanced.

Proof. For (1), note that the pair in fact is a virtual condition: in $\text{Coll}(\omega, U)$, the set $U$ is countable and so $\tau_U$ is an analytic subset of $P$, and the evaluation of $\tau_U$ does not depend on the particular generic filter on the collapse poset. The balance of the pair follows immediately from Proposition 7.0.2 applied to $U$.

For (2), first use the balance of the name $\tau$ to note that for every set $a \subset \omega$, it must be the case that either $Q \Vdash \tau \subset \check{a}$ up to a finite set, or $Q \Vdash \tau \cap \check{a}$ is finite. Let $U$ be the set of all $a \subset \omega$ for which the first option occurs. It is immediate that $U$ is a nonprincipal ultrafilter on $\omega$. The equivalence of $(Q, \tau)$ with $(\text{Coll}(\omega, U), \tau_U)$ is immediately clear from Proposition 7.0.2.

For (3), not that if $U_0, U_1$ are distinct ultrafilters on $\omega$ then there is a set $a \subset \omega$ such that $a \in U_0$ and $\omega \setminus a \in U_1$. Thus the balanced names $\tau_{U_0}$ and $\tau_{U_1}$ represent incompatible virtual conditions as the former is below $a$ and the other is below $\omega \setminus a$.

To prove the last sentence of the theorem, for any infinite set $a \subset \omega$ there is a nonprincipal ultrafilter $U$ containing $a$ as an element. A reference to item (1) then completes the argument. \qed

7.2 Ramsey sequences of structures

There is a family of forcings present in several papers concerning the Rudin–Keisler order on ultrafilters with strong Ramsey-type properties [17]. The family is parametrized by sequences of structures tied by a partition property as in the following definition.

Definition 7.2.1. A Ramsey sequence of finite structures is a sequence $A = \langle A_n : n \in \omega \rangle$ of finite structures in the same language, with pairwise disjoint domains, such that for every natural number $n \in \omega$, $A_n$ contains an isomorphic copy of $A_n$. Moreover for every $k < n \in \omega$ there is $m > n$ such that the structural Ramsey property $A_m \rightarrow (A_n)^2$ holds.

For each Ramsey sequence $A = \langle A_n : n \in \omega \rangle$ of finite structures, we will use the following notation. $\text{dom}(A)$ stands for $\bigcup_n \text{dom}(A_n)$. Given a number $n \in \omega$, the symbol $D_n$ stands for the set of all finite sets $d$ such that for some $m \in \omega$,
\(d \subseteq \text{dom}(A_m)\) and \(A_m \mid d\) is isomorphic to \(A_n\). For every set \(p \subseteq \text{dom}(A)\), the symbol \(p^{A_n}\) stands for \(\mathcal{P}(p) \cap D_n\). Each Ramsey sequence of finite structures has a natural \(\sigma\)-closed poset associated to it.

**Definition 7.2.2.** Let \(A = \langle A_n : n \in \omega \rangle\) be a Ramsey sequence of finite structures. Let \(P_A\) denote the poset of all sets \(p \subseteq \text{dom}(A)\) such that \(p^{A_n}\) is nonempty for each \(n \in \omega\). The ordering is defined by \(q \leq p\) if \(q \subseteq p\) modulo finite.

The posets of the form \(P_A\) are balanced. The balanced virtual conditions are parametrized by sequences of ultrafilters as opposed to single ultrafilters in the case of \(\mathcal{P}(\omega)\) modulo finite.

**Definition 7.2.3.** Let \(A = \langle A_n : n \in \omega \rangle\) be a Ramsey sequence of finite structures. An \(A\)-sequence of filters is a sequence \(\langle F_n : n \in \omega \rangle\) such that

1. for every number \(n \in \omega\), \(F_n\) is a filter on the set \(D_n\);
2. for all numbers \(n, m \in \omega\) and every selection \(\langle a_i : i \in n \rangle\) of sets in the respective filters \(F_i\) for \(i \in n\), there is a set \(d \subseteq D_m\) such that \(d^{A_i} \subseteq a_i\) holds for all \(i \in n\).

Whenever \(F = \langle F_n : n \in \omega \rangle\) is an \(A\)-sequence of ultrafilters, define the \(\text{Coll}(\omega, 2^\omega)\)-name \(\tau_F\) for the set of all conditions \(p \in P\) such that for every \(n \in \omega\) and every set \(a \subseteq F_n\), \(p^{A_n} \subseteq a\) modulo finite.

**Theorem 7.2.4.** Let \(A = \langle A_n : n \in \omega \rangle\) be a Ramsey sequence of finite structures.

1. For every \(A\)-sequence \(F\) of ultrafilters, the pair \(\langle \text{Coll}(\omega, 2^\omega), \tau_F \rangle\) is balanced in the poset \(P_A\);
2. For every balanced pair \(\langle Q, \sigma \rangle\) there is an \(A\)-sequence \(F\) of ultrafilters such that the balanced pairs \(\langle Q, \sigma \rangle\) and \(\langle \text{Coll}(\omega, 2^\omega), \tau_F \rangle\) are equivalent;
3. distinct \(A\)-sequences of ultrafilters yield inequivalent balanced pairs.

In particular, the poset \(P_A\) is balanced.

**Proof.** Write \(P = P_A\). For the first item, it is necessary to argue that in fact \(\text{Coll}(\omega, \tau_F) \Vdash \tau_F \subseteq P\) is a nonempty set. To see this, work in the \(\text{Coll}(\omega, 2^\omega)\) extension, for each number \(n \in \omega\) let \(D_n = \text{dom}(F_n)\) and fix an enumeration \(\langle a_i : i \in \omega \rangle\) of all elements of \(F_n\). By induction on \(n \in \omega\), repeatedly using item (2) of Definition 7.2.3 build sets \(d_n \subseteq D_n\) such that for every \(i \in n\) the set \(d_n^{A_i}\) is a subset of \(\cap_{j \in n} a_i^j\). It is immediate that the condition \(p = \bigcup_n d_n \subseteq P\) is an element of \(\tau_F\).

For the balance, suppose that \(R_0, R_1\) are posets and \(\sigma_0, \sigma_1\) are their respective names for conditions in \(P\) such that \(R_0 \Vdash\) for every \(n \in \omega\) and every set \(a \in F_n\), \(\sigma_0^{A_n} \subseteq a\) modulo finite, and similarly for \(R_1, \sigma_1\). We have to show that \(R_0 \times R_1 \Vdash\) \(\sigma_0, \sigma_1\) are compatible in \(P\), which is the same as to say that
Claim 7.2.5. Let \( F \) be a nonempty family of \( Q \)-ultrafilters. For each \( k \in \omega \) find a number \( m_k \in \omega \) such that \( A_{m_k} \rightarrow (A_k)_2^{\omega} \). Find pairwise disjoint finite sets \( d_k \subseteq p \) in \( D_{m_k} \) for each \( k \in \omega \). Use the partition property to find sets \( d'_k \subseteq d_k \) such that \( (d'_k)^{\omega} \) is either a subset of \( a \) or disjoint from \( a \). One of the options prevails for infinitely many numbers \( k \). For definiteness, assume that the set \( b = \{ \langle k \in \omega : (d'_k)^{\omega} \subseteq a \} \) is infinite. Let \( q = \bigcup_{k \in b} d'_k \) and observe that \( q \leq p \) works. \( \square \)

Now let \( \langle Q, \sigma \rangle \) be a balanced pair. Replacing \( Q \) with \( Q \times \mathcal{C}(\omega, 2^{\omega}) \) and repeatedly strengthening \( p \) using the claim, we may assume that for every \( n \in \omega \) and every set \( a \subseteq D_n \), \( a^{\omega} \) is forced to be either a subset of \( a \) or disjoint from \( a \) modulo finite. By a balance argument, it must be the case that either \( Q \models a^{\omega} \subseteq \dot{a} \) modulo finite, or \( Q \models a^{\omega} \cap \dot{a} \) modulo finite. Let \( F_n \) be the collection of all sets \( a \subseteq D_n \) for which the former option prevails. It is clear that \( F_n \) is an ultrafilter and that \( F = \{ F_n : n \in \omega \} \) is an \( \dot{A} \)-sequence. Since \( Q \times \mathcal{C}(\omega, 2^{\omega}) \models \sigma \triangleleft \tau_F \), by the first item and Proposition 5.2.4 it must be the case that the balanced pairs \( \langle Q, \sigma \rangle \) and \( \langle \mathcal{C}(\omega, 2^{\omega}), \tau_F \rangle \) are equivalent balanced pairs.

The third item is immediate. The last sentence of the theorem is not an entirely formal consequence of the previous work. It needs the following application of the axiom of choice.

Claim 7.2.6. Every \( \dot{A} \)-sequence of filters can be extended to a \( \dot{A} \)-sequence of ultrafilters.

Proof. Order the \( \dot{A} \)-sequences of filters by coordinatewise inclusion: \( F \preceq G \) if for every \( n \in \omega \), \( F_n \subseteq G_n \) holds. It is immediate from the definitions that every \( \preceq \)-chain has an upper bound—namely the coordinatewise union of the \( \dot{A} \)-sequences of filters in the chain. By an application of Kuratowski–Zorn lemma, every \( \dot{A} \)-sequence of filters can be extended into a \( \preceq \)-maximal one. It only remains to prove that any \( \preceq \)-maximal \( \dot{A} \)-sequence of filters is in fact a sequence of ultrafilters.

Let \( F \) be a \( \preceq \)-maximal \( \dot{A} \)-sequence of filters, let \( n \in \omega \) be a number, and let \( b \subseteq D_n \) be a set; we claim that either \( b \) or its complement belongs to \( F_n \). Suppose towards contradiction that neither is the case. By the maximality of \( F \) it must be the case that the sequences \( G^0, G^1 \) of filters obtained from \( F \) by adding \( b \) (or the complement of \( b \), respectively) to the filter \( F_n \) are \( \dot{A} \)-sequences. This must be witnessed by some sets \( \langle a^0_i : i \in k \rangle \) and \( \langle a^1_i : i \in k \rangle \) for some number \( k \in \omega \) respectively, and some number \( m \in \omega \); these are sets in \( F_i \) such that there is no \( d \in D_m \) such that for all \( i \in k, i \neq n \) \( d^0_i \subseteq a^0_i \) and \( d^1_i \subseteq a^1_i \).
and $d^I_n \subset a_n \setminus b$. Since $F$ is a sequence of filters, for all $i \in k$ $a_i^0 \cap a_i^1 \in F_i$ holds. Let $n' \in \omega$ be a number such that $\mathcal{A}_{n'} \to (\mathcal{A}_m)^{dn'}$ holds. Since $F$ is a $\mathcal{A}$-sequence, there is $d' \in D_{n'}$ such that for all $i \in k$, $(d')^{a_i^0} \cap a_i^1$ holds. Use the partition assumption to find a set $d \subset d'$ in $D_m$ such that either $d^I_n \subset b$ or $d^I_n \cap b = 0$. The condition $q$ violates the choice of either the sets $(a_i^0 : i \in k)$ or the sets $(a_i^1 : i \in k)$.

Now, suppose that $p \in P$ is a condition. Consider the sequence $F$ of filters $F_n$ generated by sets cofinite in $p^I_n$. Clearly, $F$ is an $\mathcal{A}$-sequence. By the claim, $F$ can be extended to a $\mathcal{A}$-sequence $G$ of ultrafilters. Then $\text{Coll}(\omega, 2^\omega, \tau_G)$ is a balanced pair which is easily seen to be below $p$ as required.

For every Ramsey sequence $\mathcal{A} = \langle \mathcal{A}_n : n \in \omega \rangle$, the poset $P_\mathcal{A}$ introduces an ultrafilter on the set $D_0$ as the set of all subsets $a \subset D_0$ such that for some condition $p \in P$ in the generic ultrafilter, $p^I_0 \subset a$. Claim 7.2.5 together with an obvious density argument show that this formula indeed defines an ultrafilter $U$ on $D_0$, which will be referred to as the $P_\mathcal{A}$-generic ultrafilter. It is clear that the generic filter on $P_\mathcal{A}$ can be reconstructed from $U$, and therefore the potential for confusion is minimal. The combinatorial and Rudin–Keisler properties of the generic ultrafilter have been the main reason for the study of the partial orders of the type $P_\mathcal{A}$ [6]. It is not difficult to see that it is a P-point.

**Example 7.2.7.** Let $\mathcal{A}$ be the sequence $\langle \mathcal{A}_n : n \in \omega \rangle$ where $\mathcal{A}_n$ is the linear order on $n$ many elements. The Ramsey theorem implies that this is in fact a Ramsey sequence of structures. The $P_\mathcal{A}$-generic ultrafilter is a P-point which is weakly Ramsey and not a Q-point [6, Theorem 4.9].

**Example 7.2.8.** Let $k \in \omega$ and let $\mathcal{A}$ be a sequence enumerating all ordered finite graphs which contain no clique of size $k$. The fact that this is a Ramsey sequence of structures follows from [59]. The $P_\mathcal{A}$-generic ultrafilter is a P-point $U$ such that $U \to (U, k)^2$ but not $U \to (U, k + 1)^2$ [6, Theorem 4.11]

### 7.3 Fubini powers of the Fréchet ideal

The most natural attempt to force an ultrafilter which is not a P-point with a $\sigma$-closed subset is encapsulated in the following definition.

**Definition 7.3.1.** For the duration of this section, write $I$ to be the ideal on $\omega \times \omega$ which is the Fubini product of the Fréchet ideal on $\omega$ with itself. That is, a set $a \subset \omega \times \omega$ belongs to the ideal $I$ just in case $a$ has only finitely many infinite vertical sections. Write also $P$ for the partial order of $I$-positive sets, ordered by $q \leq p$ if $q \setminus p \in I$.

It is not difficult to see that $P$ is a $\sigma$-closed forcing; the sets in the generic filter on $P$ form an ultrafilter disjoint from $I$. This ultrafilter, by virtue of the partition of the domain set $\omega \times \omega$ into the $I$-small vertical sections such that every transversal is again in $I$, is not a P-point; its rich combinatorial properties were investigated for example in [8, 75].
It turns out that the poset $P$ is balanced, and the balanced virtual conditions allow a simple classification. Let $\omega^*$ be the set of all nonprincipal ultrafilters on $\omega$. Let $W$ be an ultrafilter on the set $\omega \times \omega^*$ which contains no set $n \times \omega^*$ for any number $n \in \omega$. Write $\tau_W$ for the name on $\text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*))$ for the set of all conditions $p \in P$ such that there is a set $\{\langle n_i, U_i \rangle: i \in \omega \} \subset (\omega \times \omega^*)^V$ diagonalizing the ultrafilter $W$, the numbers $n_i$ for $i \in \omega$ are pairwise distinct, and the set $p \subset \omega \times \omega$ has the following property: only the vertical sections $p_n$, for some $i \in \omega$ are nonempty, and each vertical section $p_n$, is infinite and diagonalizes the ultrafilter $U_i$.

**Theorem 7.3.2.**

1. For every ultrafilter $W$ on the set $\omega \times \omega^*$ containing no set $n \times \omega^*$ for $n \in \omega$, the pair $(\text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)), \tau_W)$ is balanced;

2. For every balanced pair $(Q, \sigma)$ there is an ultrafilter $W$ on the set $\omega \times \omega^*$, containing no set $n \times \omega^*$ for $n \in \omega$, such that the balanced pairs $(Q, \sigma)$ and $(\text{Coll}(\omega, \mathcal{P}(\omega \times \omega^*)), \tau_W)$ are equivalent;

3. distinct ultrafilters give rise to inequivalent balanced conditions.

In particular, the poset $P$ is balanced.

**Proof.** Towards (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions containing respective sets $p_0, p_1 \subset \omega \times \omega$ such that there are sets $a_0 = \{\langle n_i^0, U_i^0 \rangle: i \in \omega \}$ and $a_1 = \{\langle n_i^1, U_i^1 \rangle: i \in \omega \}$ in the respective models which diagonalize the ultrafilter $W$, and the vertical sections $(p_0)_n$ diagonalize the ultrafilter $U_i^0$ for all $i \in \omega$, and the vertical sections $(p_1)_n$ diagonalize the ultrafilter $U_i^1$ for all $i \in \omega$ as well. It will be enough to show that $p_0, p_1$ are compatible in the poset $P$: i.e. the set $p_0 \cap p_1$ has infinitely many infinite vertical sections.

To this end, use Proposition 7.0.2 for the sets $a_0, a_1$ and the ultrafilter $W$ to see that $a_0 \cap a_1$ is infinite. This means, that the set $b = \{n \in \omega: \exists U \langle n, U \rangle \in a_0 \cap a_1 \}$ is infinite. For each number $n \in b$, the vertical sections $(p_0)_n$ and $(p_1)_n$ diagonalize the same ultrafilter $U$ on $\omega$ for which $\langle n, U \rangle \in a_0 \cap a_1$ holds. By another application of Proposition 7.0.2, the set $(p_0)_n \cap (p_1)_n$ is infinite. It follows that $p_0 \cap p_1 \notin I$ as desired.

Towards (2), strengthen $\sigma$ if necessary so that $\sigma$ is a name for an actual single condition in the poset $P$ as oppose to an analytic set of conditions. Replacing $Q$ with $\text{Coll}(\omega, \mathcal{P}(\omega))$ and strengthening the name $\sigma$ if necessary we may assume that $Q \Vdash \text{for every } n \in \omega, \text{if the vertical section } \sigma_n \text{ is infinite, then for every set } a \subset \omega \text{ in the ground model, either } \sigma_n \subset a \text{ or } \sigma_n \cap a = 0 \text{ modulo finite.}$

Let $\mathcal{U}_n$ be the name for the set of all sets $a \subset \omega$ in the ground model for which the former alternative prevails; define $\mathcal{U}_n = 0$ if the vertical section $\sigma_n$ is finite. Passing to a condition in $Q$ and strengthening $\sigma$ again, we may assume that either $Q \Vdash \text{for all } n \in \omega \text{ such that } \sigma_n \text{ is infinite, } \mathcal{U}_n \notin V$, or $Q \Vdash \text{for all } n \in \omega \text{ such that } \sigma_n \text{ is infinite, } \mathcal{U}_n \in V$.

Now, the former alternative is impossible as it contradicts the balance of the pair $(Q, \tau)$. To see this, let $H_0, H_1 \subset Q$ be mutually generic filters and let $p_0 = \sigma/H_0$ and $p_1 = \sigma/H_1$. To reach the contradiction, we will show that all vertical sections of $p_0 \cap p_1$ are finite. Suppose not, and let $n \in \omega$ be such
that \((p_0)_n \cap (p_1)_n\) is infinite. Then \(\hat{U}_n/H_0 = \hat{U}_n/H_1\), which by the product forcing theorem implies that \(\hat{U}_n/H_0 \in V\) and violates the former alternative assumption.

Thus, the latter alternative prevails, and the set \(\eta = \{\langle n, \hat{U}_n \rangle : \sigma_n \text{ is infinite}\}\) is forced to be a subset of \((\omega \times \omega^*)^V\). Strengthening \(\sigma\) again, we may assume that \(Q \vDash\) for every set \(b \subseteq \omega \times \omega^*\) in the ground model, either \(\eta \subseteq b\) or \(\eta \cap b = 0\) modulo finite.

**Claim 7.3.3.** For every set \(b \subseteq \omega \times \omega^*\), either \(Q \vDash \eta \subseteq \hat{b} \text{ modulo finite}, or \(Q \vDash \eta \cap \hat{b} \text{ is finite}\).

**Proof.** Suppose toward contradiction that the conclusion fails for some set \(b \subseteq \omega \times \omega^*\). Then, there must be conditions \(q_0, q_1 \subseteq Q\) forcing the former and latter alternative respectively. Let \(H_0, H_1 \subseteq Q\) be mutually generic filters containing the conditions \(q_0, q_1\) respectively. Write \(p_0 = \sigma/H_0\) and \(p_1 = \sigma_1/H_1\); we will reach a contradiction by showing that all but finitely many vertical sections of the set \((p_0 \cap p_1)_n\) are finite, violating the balance assumption on \(\sigma\).

Write \(c_0 = \eta/H_0\) and \(c_1 = \eta/H_1\). By the contradictory assumption, the intersection \(c_0 \cap c_1\) is finite. Let \(n \in \omega\) be a natural number which is not in the finite projection of the set \(c_0 \cap c_1\) to \(\omega\); we will show that \((p_0 \cap p_1)_n\) is finite. Well, either \((p_0)_n\) or \((p_1)_n\) are finite sets, in which case we are done. Or, both \((p_0)_n\) and \((p_1)_n\) are infinite. In this case, there are ultrafilters \(U_0, U_1\) on \(\omega\) in the ground model such that \(\langle n, U_0 \rangle \in c_0\) and \(\langle n, U_1 \rangle \in c_1\). Since the number \(n\) is not in the projection of \(c_0 \cap c_1\) to \(\omega\), the ultrafilters \(U_0, U_1\) must be distinct. Since \((p_0)_n\) diagonalizes \(U_0\) and \((p_1)_n\) diagonalizes \(U_1\), the intersection \((p_0)_n \cap (p_1)_n\) must be finite in this case as well.

Let \(W\) be the set of all subsets of \(b \subseteq \omega \times \omega^*\) for which \(Q \vDash \eta \subseteq \hat{b} \text{ modulo finite}\); by the claim, this is an ultrafilter on \(\omega \times \omega^*\). The definitions show that \(\text{Coll}(\omega, P(\omega \times \omega^*)) \times Q \vDash \sigma \in \tau_P\). Then, Proposition 5.2.4 shows that the balanced pairs \((Q, \sigma)\) and \((\text{Coll}(\omega, P(\omega \times \omega^*))) \text{ are equivalent as desired.}

Item (3) is immediate. For the last sentence, suppose that \(p \in P\) is a condition. Let \(a = \{n \in \omega : p_n\text{ is infinite}\}\); the set \(a\) is infinite. For each \(n \in \omega\) let \(U_n\) be a nonprincipal ultrafilter on \(\omega\) containing \(p_n\) as an element. Let \(W\) be a nonprincipal ultrafilter on the set \(\omega \times \omega^*\) containing the set \(\{\langle n, U_n \rangle : n \in a\}\). It is not difficult to see that the balanced pair \(\langle \text{Coll}(\omega, P(\omega \times \omega^*)) \tau_W \rangle\) is below the condition \(p\), proving the balance of the poset \(P\).

### 7.4 Semigroup ultrafilters

In this section, we show that in a natural forcing connected with a countable semigroup, the balanced conditions exist and are classified by idempotent ultrafilters. This connects the theory of balanced forcing extensions with Ramsey theory and dynamics.

**Definition 7.4.1.** Let \(\Gamma\) be a countable semigroup. The poset \(P(\Gamma)\) is defined as follows. The conditions of \(P(\Gamma)\) are elements of \(\Gamma^\omega\). The ordering is defined
by $q \leq p$ if there are nonempty finite sets $a_n \subset \omega$ for all $n \in \omega$ such that $\max(a_n) < \min(a_{n+1})$ and $\prod_{m \in a_n} p(m) = q(n)$, where the products are always taken in the increasing order.

It is not difficult to see that given a condition in $P(\Gamma)$, shifting its entries to the left and/or changing finitely many entries does not change the separative quotient equivalence class of the condition. It follows that the separative quotient of $P(\Gamma)$ is a $\sigma$-closed poset. The purpose of the poset $P(\Gamma)$ is clear from the following definition and proposition.

**Definition 7.4.2.** Let $A$ be a collection of subsets of $\Gamma$. A sequence $p \in P(\Gamma)$ diagonalizes $A$ if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\prod_{m \in a} p(m) \in b$ holds. The sequence $p \in P(\Gamma)$ sorts out $A$ if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\prod_{m \in a} p(m) \in b$ holds, or for every nonempty finite set $a \subset \omega \setminus n$, $\prod_{m \in a} p(m) \notin b$ holds. If $b \in A$ and the former alternative occurs, we say that $p$ accepts $b$, if the latter alternative occurs then $p$ declines $b$.

**Proposition 7.4.3.** For every condition $p \in P(\Gamma)$ and every countable set $A \subset P(\Gamma)$, there is a condition $q \leq p$ which sorts out $A$.

**Proof.** Let $A = \{b_n : n \in \omega\}$. By induction on $n \in \omega$ build a descending sequence $p_n$ of conditions in $P(\Gamma)$ such that $p = p_0$ and for all $n \in \omega$, for all nonempty finite sets $a \subset \omega \setminus n$ it is the case that $\prod_{m \in a} p_{n+1}(m) \in b_n$ holds, or for all nonempty finite sets $a \subset \omega$ it is the case that $\prod_{m \in a} p_{n+1}(m) \in b_n$ holds.

To perform the induction step, use Hindman’s theorem on the partition $\pi_n$ of nonempty finite subsets of $\omega$ defined by $\pi_n(a) = 1$ if $\prod_{m \in a} p_n(m) \in b_n$ holds.

In the end, let $q \in P(\Gamma)$ be defined by $q(n) = p_n(n)$ and observe that the conclusion of the proposition is satisfied. $\square$

Suppose that $G \subset P(\Gamma)$ is a generic filter. By Proposition 7.4.3 and a density argument, the set $U \subset P(\Gamma)$ of all sets $b \subset \Gamma$ such that some condition $p \in G$ accepts $b$, is an ultrafilter. Also, $G$ can be recovered from $U$ as the set of all conditions $p \in P(\Gamma)$ such that for all $n \in \omega$, the set $\{\prod_{m \in a} p(m) : a \subset \omega$ is a nonempty set and $\min(a) > n\}$ belongs to the ultrafilter $U$.

It turns out that the balanced conditions in the poset $P(\Gamma)$ correspond to a well-known concept from topological dynamics and Ramsey theory.

**Definition 7.4.4.** [31, Section 4.1] Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup.

1. for ultrafilters $U_0, U_1$ on $\Gamma$, let $U_0 \cdot U_1 = \{A \subset \Gamma : \{\gamma \in \Gamma : \{\delta \in \Gamma : \gamma \cdot \delta \in A\} \in U_1\} \in U_0\}$.

2. an ultrafilter $U$ on $\Gamma$ is an idempotent if $U \cdot U = U$.

It turns out that $\cdot$ is a semi-continuous operation on the space $\beta \Gamma$ of all ultrafilters on $\Gamma$. The fundamental Ellis–Numakura theorem [31, Section 2.4] says that every closed subsemigroup of $\beta \Gamma$ contains an idempotent ultrafilter. The idempotent ultrafilters have an alternative diagonalization description.
Proposition 7.4.5. Let $U$ be an ultrafilter on $\Gamma$. The following are equivalent:

1. in Coll($\omega, \mathcal{P}(\Gamma)$) extension there is a condition in $\mathcal{P}(\Gamma)$ diagonalizing $U$;

2. for every countable subset of $U$, there is a condition in $\mathcal{P}(\Gamma)$ diagonalizing it;

3. $U$ is an idempotent.

Proof. For the implication (1) $\rightarrow$ (2), if $U' \subset U$ is a countable set, then in some forcing extension there is a condition $p \in \mathcal{P}$ which diagonalizes $U$ and therefore $U'$. By a Mostowski absoluteness argument between the ground model and the forcing extension, there must be a condition $p \in \mathcal{P}$ diagonalizing $U'$.

For the implication (2) $\rightarrow$ (1), consider the partial order $Q = [\mathcal{P}(U)]^{\aleph_0}$ modulo the nonstationary ideal. Let $G \subset Q$ be a filter generic over $V$ and let $j : V \rightarrow M$ be the generic ultrapower. Then $M$ is an $\omega$-model which is possibly illfounded. Moreover, $M$ contains the set $j''U$ as the equivalence class of the identity function on $[\mathcal{P}(U)]^{\aleph_0}$, and $M | G = j''U \subset U$ is a countable set by the Łoś theorem. Thus, if the ground model satisfies (2), then by elementarity $M$ (and so $V[G]$) contains a condition $p \in \mathcal{P}$ which diagonalizes $j''U$ and therefore $U$.

The existence of such a condition then transfers to any Coll($\omega, \mathcal{P}(\Gamma)$) extension of the ground model. This confirms (1).

(3) implies (2): this is the contents of Galvin–Glazer theorem or its folkloric variation for countable sets, [31, Chapter 5]. Finally, negation of (3) implies the negation of (2). To see this, if $U$ is not an idempotent, there must be a set $A \in U$ such that $A \notin U \cdot U$. This means that the set $B = \{\gamma \in \Gamma : \gamma \cdot \delta \notin A\} \in U$ belongs to $U$. Consider the countable set including $A, B$, as well as all the sets $C_\gamma = \{\delta \in \Gamma : \gamma \cdot \delta \notin A\}$ for $\gamma \in B$; we will show that it cannot be diagonalized. Suppose towards contradiction that some condition $p \in \mathcal{P}(\Gamma)$ does diagonalize it. This means that there is $m_0 \in \gamma$ such that for all $k > m \geq m_0$, $p(m) \in B$ and $p(m) \cdot p(k) \in A$. There must be also some $k_0 \in \omega$ such that for all $k \geq k_0$, $p(k) \in C_{p(m_0)}$. Then $p(m_0) \cdot p(k_0)$ should belong simultaneously to $A$ and to the complement of $A$, which is a contradiction. 

Finally, the statement of the classification theorem for the balanced virtual conditions in the poset $\mathcal{P}(\Gamma)$ is at hand.

Definition 7.4.6. Whenever $\langle \Gamma, \cdot \rangle$ is a countable semigroup and $U$ is an idempotent ultrafilter, let $\tau_U$ denote the Coll($\omega, \mathcal{P}(\Gamma)$)-name for the (nonempty analytic) set of all conditions diagonalizing $U$.

Theorem 7.4.7. Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup.

1. Whenever $U$ is an idempotent ultrafilter then $\langle \text{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$ is a balanced virtual condition;

2. every balanced pair is equivalent to $\langle \text{Coll}(\omega, \mathcal{P}(\Gamma)), \tau_U \rangle$ for some idempotent ultrafilter $U$.
3. distinct idempotent ultrafilters give rise to inequivalent balanced virtual conditions.

In particular, the poset $P(\Gamma)$ is balanced.

Proof. (1) follows immediately from the following claim:

Claim 7.4.8. Let $U$ be an ultrafilter on $\Gamma$ and $\langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle$ are posets and their respective names for elements of $P(\Gamma)$ diagonalizing $U$. Then $Q_0 \times Q_1 \VDash \tau_0, \tau_1$ are compatible in $P(\Gamma)$.

Proof. Let $q_0 \in Q_0$ and $q_1 \in Q_1$ be conditions and $n \in \omega$ be a natural number. We must find an element $\gamma \in \Gamma$ and conditions $q_0' \leq q_0$ and $q_1' \leq q_1$ and finite nonempty sets $a_0, a_1 \subset \omega$ with minimum larger than $n$ such that $q_0' \Vdash \hat{\gamma} = \Pi_{m \in a_0} \tau_0(m)$ and $q_1' \Vdash \hat{\gamma} = \Pi_{m \in a_1} \tau_1(m)$. The compatibility of $\tau_0, \tau_1$ is then granted by a genericity argument.

Let $A_0 \subset \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subset \omega$ with minimum greater than $n$ and a condition $q' \leq q_0$ in $Q_0$ forcing $\hat{\gamma} = \Pi_{m \in a} \tau_0(m)$. Since $\tau_0$ is forced to diagonalize $U$, it must be the case that $A_0 \Vdash q'$.

Similarly, let $A_1 \subset \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subset \omega$ with minimum greater than $n$ and a condition $q' \leq q_1$ in $Q_1$ forcing $\hat{\gamma} = \Pi_{m \in a} \tau_1(m)$. Since $\tau_1$ is forced to diagonalize $U$, it must be the case that $A_1 \Vdash q'$. Choose $\gamma \in A_0 \cap A_1$ and choose $q_0' \leq q_0, q_1' \leq q_1$ and sets $a_0, a_1$ witnessing the membership of $\gamma$ in $A_0, A_1$. This completes the proof.

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. Without loss of generality we may assume that $Q$ collapses the size of $P(\Gamma) \cap V$ to $\aleph_0$. Strengthening $\tau$ in the $Q$-extension repeatedly by an application of Proposition 7.4.3, we may assume that $\tau$ sorts out $P(\Gamma) \cap V$. Let $\sigma$ be the $Q$-name for the collection of all sets in $P(\Gamma) \cap V$ which $\tau$ accepts.

Claim 7.4.9. The membership of every set $A \subset \Gamma$ in $\dot{U}$ is decided by the largest condition in $Q$.

Proof. If not, then there is a set $A \subset \Gamma$ and conditions $q_0, q_1 \in Q$ such that $q_0 \Vdash \dot{A} \subset \dot{U}$ and $q_1 \Vdash \Gamma \setminus \dot{A} \subset \dot{U}$. Plainly, the condition $\langle q_0, q_1 \rangle$ forces in the product $Q \times Q$ that $\tau_{\text{left}}, \tau_{\text{right}}$ are conditions incompatible in $P(\Gamma)$, contradicting the balance assumption on the name $\tau$.

Let $U = \{A \subset \Gamma: Q \Vdash \dot{A} \in \sigma\}$. This is an ultrafilter on $\Gamma$. By Proposition 7.4.5, it is an idempotent ultrafilter. By Claim 7.4.8, the pair $\langle Q, \tau \rangle$ is equivalent to $\langle \text{Coll}(\omega, P(\Gamma)), \tau_U \rangle$. (2) follows.

(3) is immediate. To prove the last sentence, note that the Ellis–Numakura theorem yields an idempotent ultrafilter below any condition $p \in P(\Gamma)$.
Chapter 8

Other forcings

In this chapter we gather partial orders which do not fit in the previous chapters.

8.1 Chromatic numbers

Given a Polish space \( X \), a Borel hypergraph \( \Gamma \subset [X]^{<\aleph_0} \), and a number \( n \leq \omega \), one can consider the task of forcing a \( \Gamma \)-coloring \( c: X \to n \). In principle, this is a very difficult task and except for the rather simple case \( n = 2 \) we do not have a general way of resolving it. The first poset one can consider is the poset \( P_{\mathcal{K}} \) where \( \mathcal{K} \) is the Borel simplicial complex on \( X \times n \) consisting of finite partial \( \Gamma \)-colorings. The basic disadvantage of this poset is that not all maximal \( \mathcal{K} \)-sets are total \( \Gamma \)-colorings, and there are elements of \( P_{\mathcal{K}} \) which cannot be extended to total \( \Gamma \)-colorings. Thus, we have to restrict the posets \( P_{\mathcal{K}} \) in some way to get the desired effect. This is not always straightforward, and the resulting posets are usually not simplicial complex posets in the sense of Definition 6.1.1. The most natural attempt is the following.

**Definition 8.1.1.** Let \( X \) be a Polish space, \( \Gamma \subset [X]^{<\aleph_0} \) be a Borel graph on a Polish space \( X \), and \( n \leq \omega \) be a number. The \( \Gamma \), \( n \)-coloring forcing \( P_n(\Gamma) \) is the partially ordered set of all partial countable functions \( p: X \to n \) which can be extended to a \( \Gamma \)-coloring of the whole space \( X \) by \( n \) colors. The ordering is that of reverse inclusion.

It is immediate that the poset \( P_n(\Gamma) \), if nonempty, forces the union of the conditions in the generic filter to be a total \( \Gamma \)-coloring by \( n \) many colors. However, it is not at all clear whether the poset \( P_n(\Gamma) \) is Suslin, \( \sigma \)-closed, or balanced. In an interesting large class of Borel graphs \( \Gamma \), these questions have a smooth resolution.

**Theorem 8.1.2.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) with countable coloring number. Then

1. \( P_\omega(\Gamma) \) contains a dense subset which is a Suslin \( \aleph_0 \)-distributive poset;
2. for every total \(\Gamma\)-coloring \(c:\ X \to \omega\) the pair \(\langle Coll(\omega, X), \hat{c} \rangle\) is balanced;

3. for every balanced pair \(\langle Q, \tau \rangle\) there is a total coloring \(c\) such that \(\langle Q, \tau \rangle\) is equivalent to \(\langle Coll(\omega, X), \hat{c} \rangle\);

4. \(P_\omega(\Gamma)\) is balanced.

If the coloring number of \(\Gamma\) is a finite \(n \in \omega\) the above items hold with any number \(\geq 3n - 2\) replacing \(\omega\).

Proof. We will start with the case of a graph \(\Gamma\) of countable coloring number. Fix a well-ordering \(<\) on the space \(X\) such that for every element \(x \in X\), the set \(\{y \in X : y < x\} \text{ and } \{x, y\} \in \Gamma\) is finite. We start with a humble observation which will be used in several places below.

Claim 8.1.3. Let \(M\) be a countable elementary submodel of a large countable structure containing \(<\). Then every \(y \in X \setminus M\) is \(\Gamma\)-connected with at most finitely many elements of \(X \cap M\).

Proof. Suppose that \(y \in X \setminus M\) is a point. Then for every \(x \in M\) \(<\)-greater than \(y\), \(\{x, y\} \notin \Gamma\) holds: if \(\{x, y\} \in \Gamma\) then \(y\) belongs to the finite set of all points \(<\)-smaller than \(x\) and connected with \(x\) and by elementarity it would have to belong to the model \(M\), which is not the case. Thus, the only points of \(M\) connected to \(y\) in \(\Gamma\) are \(<\)-smaller than \(y\), and there are only finitely many points like that by the definitory property of the ordering \(<\).

Let \(P\) be the poset of all partial countable \(\Gamma\)-colorings \(p:\ X \to \omega\) such that for all points \(y \in X \setminus \text{dom}(p)\), the set \(\{x \in \text{dom}(p) : \{x, y\} \in \Gamma\}\) is finite. The ordering on \(P\) is that of reverse extension. The following two claims show that \(P\) has the properties required in (1).

Claim 8.1.4. \(P\) is a Suslin \(\aleph_0\)-distributive poset.

Proof. We first need to show that the set of all (enumerations of) conditions in \(P\) is Borel. Use Claim 8.1.3 to conclude that for every countable set \(a \subseteq X\), the set \(b = \{y \in X : \exists x \in a \{x, y\} \in \Gamma\}\) is countable. By the Lusin–Novikov theorem, there are Borel functions \(\{f_i : i \in \omega\}\) from \(X^\omega\) to \(X\) such that for every \(z \in X^\omega\), the list \(\{f_i(z) : i \in \omega\}\) includes all points in \(X\) which are \(\Gamma\)-connected with infinitely many points of \(\text{rng}(z)\). Now we can evaluate the complexity of the set of enumerations of conditions in the poset \(P\). A function \(r: \omega \to X \times \omega\) is an enumeration of a condition in \(P\) just in case when \(\text{rng}(r)\) is a function and a \(\Gamma\)-coloring, and, writing \(s: \omega \to X\) for the first component function of \(r\), for every \(i \in \omega\) if \(f_i(s)\) is \(\Gamma\)-connected to infinitely many points in \(\text{rng}(s)\), then \(f_i(s) \in \text{rng}(s)\). This is a Borel condition.

To show that the poset \(P\) is Suslin, note that two conditions \(p_0, p_1 \in P\) are compatible just in case they are compatible as functions: then \(p_0 \cup p_1\) will be their lower bound. To argue for the \(\aleph_0\)-distributivity, let \(p \in P\) be a condition and let \(\{D_i : i \in \omega\}\) be a countable collection of open dense subsets of \(P\). To find a condition \(q \leq p\) in the intersection \(\bigcap_i D_i\), let \(M\) be a countable elementary
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submodel of a large structure containing \( p, \Gamma \), \( < \) and \( \{ D_i : i \in \omega \} \) and use the elementarity of \( M \) to build a sequence \( p_i \) for \( i \in \omega \) by recursion so that

- \( p = p_0 \geq p_1 \geq p_1 \geq \ldots \) are all conditions in \( M \);
- \( p_{i+1} \in D_i \) and \( \text{dom}(p_{i+1}) \) contains the \( i \)-th element of \( M \cap X \) in some fixed enumeration.

In the end, let \( q = \bigcup p_i \) and use Claim 8.1.3 to argue that \( q \in P \) as desired.  

**Claim 8.1.5.** \( P \) is a dense subset of \( P_\omega(\Gamma) \).

**Proof.** It is first necessary to show that \( P \) is a subset of \( P_\omega(\Gamma) \). To this end, let \( p \in P \). Let \( \{ b_k : k \in \omega \} \) be a collection of pairwise disjoint infinite subsets of \( \omega \) and \( c : X \to \omega \) be a total \( \Gamma \)-coloring. Let \( d : X \to \omega \) be the function defined as follows: \( d(x) = p(x) \) if \( x \in \text{dom}(p) \), and \( d(x) = \) the smallest number in \( b_{c(x)} \) which is different from the finitely many numbers \( \{ p(y) : y \in \text{dom}(p) \} \) and \( \{ x, y \} \in \Gamma \). It is not difficult to verify that \( d \) is a total \( \Gamma \)-coloring extending \( p \), and so \( p \in P_\omega(\Gamma) \).

For the density, suppose that \( p \in P_\omega(\Gamma) \) is a condition. Let \( c : X \to \omega \) be a total \( \Gamma \)-coloring extending \( p \). Let \( M \) be a countable elementary submodel of a large structure containing \( p, c \), and \( \prec \). By Claim 8.1.3, the set \( \{ y \in X : \exists x \in M \{ x, y \} \in \Gamma \} \subset M \) holds. Thus, the function \( q = c \upharpoonright M \) is a condition in \( P \) extending \( p \) as desired.  

For the classification of the balanced virtual conditions, we need a claim:

**Claim 8.1.6.** Let \( R_0, R_1 \) be posets and \( \eta_0, \eta_1 \) be respective \( R_0 \)- and \( R_1 \)-names for elements of \( X \setminus V \). Then

1. \( R_0 \models \eta_0 \) is \( \Gamma \)-connected with at most finitely many elements of the ground model;
2. \( R_0 \times R_1 \models \{ \eta_0, \eta_1 \} \notin \Gamma \).

**Proof.** For the first item, suppose towards contradiction that this fails as forced by some condition \( r \in R \). Let \( M \) be a countable elementary submodel of a large structure containing all relevant information, let \( g \subset R_0 \cap M \) be a filter generic over the model \( M \), and let \( y = \eta_0 / g \in X \). By the forcing theorem, \( M[g] \models y \) is \( \Gamma \)-connected with infinitely many elements of \( X \cap M \). By the Mostowski absoluteness for the model \( M[g] \), \( y \) is indeed connected with infinitely many elements of \( M \), and this contradicts the conclusion of Claim 8.1.3.

For the second item, suppose towards contradiction that this fails as forced by some condition \( \langle r_0, r_1 \rangle \) in the product. Let \( M \in N \) be countable elementary submodels of a large structure containing all relevant information. In the model \( N \), there are filters \( \{ g_i : i \in \omega \} \) on the poset \( R_0 \cap M \) which are mutually generic over the model \( M \). By a mutual genericity argument applied in \( M \), the points \( x_i = \eta_0 / g_i \in X \) are pairwise distinct elements of \( N \). Now, let \( h \subset R_1 \cap M \) be a filter generic over the model \( N \) containing the condition \( r_1 \) and let \( y = \eta_1 / h \in \)
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Let $c: X \to \Gamma$ be a total $\Gamma$-coloring. Note that $\text{Coll}(\omega, X) \Vdash \dot{c} \in P$ by Claim 8.1.6(1). To see that the pair $\langle \text{Coll}(\omega, X), \dot{c} \rangle$ is balanced, suppose that $R_0, R_1$ are posets and $\sigma_0, \sigma_1$ are $R_i$-names for elements of $P$ extending $c$; we must show that $R_0 \times R_1 \Vdash \sigma_0, \sigma_1$ are compatible in $P$; in other words, $\sigma_0 \cup \sigma_1$ is a function and a $\Gamma$-coloring. To verify that $\sigma_0 \cup \sigma_1$ is a function, use the product forcing theorem to conclude that $\text{dom}(\sigma_0) \cap \text{dom}(\sigma_1)$ is forced to be a subset of the ground model, and both $\sigma_0 \upharpoonright V$ and $\sigma_1 \upharpoonright V$ are forced to be equal to $\dot{c}$. To see that $\sigma_0 \cup \sigma_1$ is a $\Gamma$-coloring, use Claim 8.1.3(2). This completes the proof of (2).

(4) now easily follows: if $p \in P_{\omega}(\Gamma)$ is a condition then by the definition of $P_{\omega}(\Gamma)$ it can be extended to a total $\Gamma$-coloring $c: X \to \omega$ which then yields a balanced virtual condition below $p$ by (2). To argue for (3), if $\langle Q, \tau \rangle$ is a balanced pair, then use the balance to see that for each $x \in X$ there must be a specific number $c(x) \in \omega$ such that $Q \Vdash \tau \Vdash \tau \in P(\Gamma) p(x)$ is the value of the generic coloring on $\dot{x}$. It is not difficult to check that $c: X \to \omega$ is a total $\Gamma$-coloring and $Q \times \text{Coll}(\omega, X) \Vdash \tau \leq \dot{c}$ in the separative quotient of the poset $P_{\omega}(\Gamma)$. (3) then follows from Proposition 5.2.4.

The statement regarding the finite coloring number $n$ version is obtained in the same way. We will only indicate why the blowup from $m$ to $m = 3n - 2$ is needed. Let $P$ be the poset of countable partial $\Gamma$-colorings $p: X \to m$ such that every point $y \in X \setminus \text{dom}(p)$ is $\Gamma$-connected to fewer than $n$ many elements of $X \cap \text{dom}(p)$. The ordering on $P$ is that of a reverse inclusion. We must show that the compatibility of conditions $p_0, p_1 \in P$ is equivalent to $p_0 \cup p_1$ being a function and a $\Gamma$-coloring; this is used in the computation of the complexity of compatibility in $P$ as well as in the verification that a total $\Gamma$-coloring is balanced. Clearly, if $p_0 \cup p_1$ is either not a function or not a $\Gamma$-coloring, then the conditions $p_0, p_1$ are incompatible in $P$. On the other hand, suppose that $p = p_0 \cup p_1$ is a function and a $\Gamma$-coloring; we must produce a lower bound $q \in P$ of $p_0, p_1$. Note that in this case it may not be true that $p_0 \cup p_1 \in P$ since a point $x \notin \text{dom}(p)$ may be connected to $n - 1$ many elements of $\text{dom}(p_0)$ and $n - 1$-many elements of $\text{dom}(p_1)$, for a total of $2n - 2$-many elements of $\text{dom}(p)$.

To overcome this obstacle, let $g: X \to [m]^n$ be any function such that for all $x \notin \text{dom}(p)$, $g(x)$ contains no elements of the set $\{p(y): y \in \text{dom}(p) \text{ and } \{x, y\} \in \Gamma\}$; note that the latter set has size at most $2n - 2$, leaving still at least $n$ many numbers in $m$ untouched. As the coloring number of $\Gamma$ is $n$, there must be a total $\Gamma$-coloring $c: X \to m$ such that for all $x \in X$, $c(x) \in g(x)$. Let $d: X \to m$ be the function defined by $d(x) = p(x)$ if $x \in \text{dom}(p)$ and $d(x) = c(x)$ otherwise; the choice of the function $g$ guarantees that $d$ is a $\Gamma$-coloring. In the end, find a countable set $a \subset X$ such that $\text{dom}(p) \subset a$ and for every point $y \in X \setminus a$ is connected with fewer than $n$-many elements of $X \cap a$. Let $q = d \upharpoonright a$ and observe that $q \in P$ and $q$ is a lower bound of the conditions
8.2 DISCONTINUOUS HOMOMORPHISMS

Example 8.1.7. Let \( n \leq 3 \) be a number, let \( D \subset \mathbb{R} \) be a countable set of positive reals converging to 0, and let \( \Gamma_{Dn} \) be the graph connecting two points of \( \mathbb{R}^n \) if their distance belongs to \( D \). The graph \( \Gamma_{Dn} \) has countable coloring number by [47, Theorem 7], and therefore the poset \( P_\omega(\Gamma_{Dn}) \) is balanced.

Example 8.1.8. Let \( D \subset \mathbb{R} \) be a countable set of positive reals and consider the graph \( \Gamma_D \) connecting two points if their distance belongs to \( D \). The graph \( \Gamma_D \) does not contain 2 times uncountable as a subgraph; by [19] it must have countable coloring number and so the poset \( P_\omega(\Gamma_D) \) is balanced. If \( D = \)all positive rational then the graph contains infinite cliques and so its chromatic number is infinite. If \( D \) is an algebraically independent set, then it is not known whether the chromatic number is finite or infinite, and Bukh [11] conjectured the former. In such a case, for a suitable number \( k \in \omega \), the poset \( P_k(\Gamma_D) \) is balanced.

8.2 Discontinuous homomorphisms

Another task which can be handled by balanced posets derived from simplicial complexes is the adding of discontinuous homomorphisms between topological structures. In this section, we treat the case of Polish groups. The problem of the existence of discontinuous homomorphisms from one Polish group to another has been studied for decades. In ZFC, some groups such as the unitary group [81] or the group of Homeomorphisms of the unit circle [64] cannot be homomorphically mapped to another Polish group in a discontinuous way. In a ZF+DC context, existence of discontinuous homomorphisms has interesting consequences: in general, it implies that the chromatic number of the Hamming graph \( \mathbb{H}_2 \) is 2 [63], for some specific groups it implies the existence of a nonprincipal ultrafilter [76, Theorem 4.1]. For an exposition, see [62]. In this section, we provide a balanced forcing adding a dicontinuous homomorphism of Polish groups in a certain common special case.

Definition 8.2.1. Let \( \Gamma \) and \( \Delta \) be Polish groups. The simplicial complex \( K(\Gamma, \Delta) \) on \( X \times Y \) consists of all finite sets \( a \subset X \times Y \) which are subsets of a homomorphism from \( X \) to \( Y \). We will write \( P(\Gamma, \Delta) \) for the associated \( \Gamma, \Delta \)-homomorphism poset of countable \( K(\Gamma, \Delta) \)-sets ordered by inclusion.

As it was the case in Section 8.1, it seems to be difficult to evaluate the complexity of the complex \( K(\Gamma, \Delta) \) without having additional information on the nature of the two groups. In this section, we will deal with a very special case which nevertheless resolves some interesting problems.

Theorem 8.2.2. Let \( \Gamma, \Delta \) be abelian Polish groups on respective Polish spaces \( X, Y \). Suppose that \( \Delta \) is divisible. Then

1. the simplicial complex \( K(\Gamma, \Delta) \) is Borel;
2. for every homomorphism $h: \Gamma \to \Delta$, the pair $\langle \text{Coll}(\omega, X \times Y), \hat{h} \rangle$ is balanced;

3. for every balanced pair $\langle Q, \tau \rangle$ there is a homomorphism $h: \Gamma \to \Delta$ such that the pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, X \times Y), \hat{h} \rangle$ are equivalent;

4. the poset $P(\Gamma, \Delta)$ is balanced.

Proof. Write $K = K(\Gamma, \Delta)$ and $P = P(\Gamma, \Delta)$. For (1), note that a finite partial function $a \subset X \times Y$ belongs to $K$ if and only if for every $\mathbb{Z}$-combinations $\Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ of elements of $P$, it is the case that $\Sigma_i n_i a(\gamma_i) = \Sigma_j n_j a(\gamma_j)$.

The left-to-right implication is clear, since the right hand side holds for every homomorphism from $\Gamma$ to $\Delta$. For the right-to-left implication, note that the assumption on the right hand side implies that $a$ can be extended to a homomorphism from the subgroup generated by $\text{dom}(a)$ to $\Delta$, which then can be extended to a homomorphism of all of $\Gamma$ to $\Delta$ by the divisibility assumption on $\Delta$ and Baer's criterion [3].

For (2), suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model $V$ and $a_0, a_1 \in K$ are finite functions in the respective models such that $a_0 \cup h$ and $a_1 \cup h$ are $K$-sets. We need to show that $a_0 \cup a_1 \in K$. Towards contradiction, suppose that this fails, and write $a = a_0 \cup a_1$. By the work of the previous paragraph, there must be a $\mathbb{Z}$-combinations $\Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ such that $\Sigma_i n_i a(\gamma_i) \neq \Sigma_j n_j a(\gamma_j)$. Note that elements of both $a_0, a_1$ must be used in these combinations. Rearranging the combinations, we may assume that the left combination uses only elements from $a_0$ and the right one uses only elements of $a_1$. By the product forcing theorem, $\gamma = \Sigma_i n_i \gamma_i = \Sigma_j n_j \gamma_j$ must belong to $V$. Since both $a_0$ and $a_1$ form a $K$-set with the homomorphism $h$, the outputs $\Sigma_i n_i a(\gamma_i), \Sigma_j n_j a(\gamma_j)$ must both be equal to $h(\gamma)$, and therefore cannot be distinct, a contradiction.

For (4), note that every condition $p \in P$ can be extended to a homomorphism from $\Gamma$ to $\Delta$ by the work of the first paragraph. This also means that $P$ forces that the union of the generic filter is a total homomorphism from $\Gamma$ to $\Delta$; the name for the union we will denote by $\hat{h}_{\text{gen}}$. For (3), let $\langle Q, \tau \rangle$ be a balanced pair. For each points $\gamma \in \Gamma$ and an open set $O \subset \Delta$, it must be the case that $Q \models \tau \models \hat{h}_{\text{gen}}(\gamma) \in \hat{O}$ or $Q \models \tau \not\models \hat{h}_{\text{gen}}(\gamma) \not\in \hat{O}$ by a balance argument. This means that there is a single point $\delta \in \Delta$ such that for every open neighborhood $O$ of $\delta$, $Q \models \tau \not\models \hat{h}_{\text{gen}}(\gamma) \in \hat{O}$ and therefore $Q \models \tau \models \hat{h}_{\text{gen}}(\gamma) = \delta$. The function $\gamma \mapsto \delta$ must be a homomorphism from $\Gamma$ to $\Delta$. Clearly $Q \models \tau \leq \hat{h}$ in the separative quotient of $P$ and therefore, by Proposition 5.2.4, the pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, X \times Y), \hat{h} \rangle$ are equivalent.

Example 8.2.3. Suppose that $\Gamma, \Delta$ are abelian Polish groups, with $\Gamma$ torsion free and uncountable and $\Delta$ divisible—the case $\Gamma = \mathbb{R}$ and $\Delta = \mathbb{R}/\mathbb{Z}$ is of particular interest. Then the poset $P = P(\Gamma, \Delta)$ forces the generic homomorphism to be discontinuous. To see this, let $p \in P$ be a condition. Let $\langle \gamma_i : i \leq \omega \rangle$ be a sequence of elements of $\Gamma$ which are linearly independent over $\text{dom}(p)$ and such that $\lim \gamma_i = \gamma_\omega$. Pick distinct elements $\delta, \delta_\omega$ and let
8.3 Automorphisms of $\mathcal{P}(\omega)$ modulo finite

Automorphisms of the Boolean algebra $\mathcal{P}(\omega)$ modulo finite have been investigated for decades. The simply definable ones are trivial [84] and the Proper Forcing Axiom implies that downright all automorphisms of the algebra are trivial [70]. It turns out that there is a natural balanced poset for adding a nontrivial automorphism of the Boolean algebra $B = \mathcal{P}(\omega)$ modulo finite.

**Definition 8.3.1.** The automorphism poset $P$ consists of pairs $p = (B_p, \pi_p)$ where $B_p \subset B$ is a countable subalgebra and $\pi_p$ is its automorphism. The ordering is that of coordinatewise reverse inclusion.

The automorphism poset is clearly $\sigma$-closed and Suslin. By a simple density argument, it adds a nontrivial automorphism of the algebra $B$. This follows from the following observation.

**Proposition 8.3.2.** Let $C \subset B$ be a countable algebra and $\pi: C \to C$ be an automorphism. Let $\chi: B \to B$ be an automorphism. Then there is an automorphism $\eta: B \to B$ extending $\pi$ and not equal to $\chi$.

**Proof.** This is the contents of [7, Theorem 2.3].

The balanced virtual conditions in the poset $P$ are naturally classified by automorphisms of the algebra $B$. Let $\pi: B \to B$ be such an automorphism, and consider the $\text{Coll}(\omega, B)$-name $\tau_{\pi}$ for the (set of all conditions stronger than the) condition $\langle B^V, \dot{\pi} \rangle \in P$. The following theorem provides the key classification information.

**Theorem 8.3.3.** Let $P$ be the automorphism poset.

1. For every automorphism $\pi$ of $B$, the pair $\langle \text{Coll}(\omega, B), \tau_{\pi} \rangle$ is balanced in $P$;
2. for every balanced pair $\langle Q, \tau \rangle$ there is an automorphism $\pi$ of $B$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, B), \tau_{\pi} \rangle$ are equivalent;
3. distinct automorphisms yield inequivalent balanced conditions.

In particular, in view of Proposition 8.3.2, the automorphism poset is balanced.

**Proof.** Item (1) is actually the most demanding part. Suppose that $V[H_0]$, $V[H_1]$ are mutually generic extensions of $V$, and $p_0 = (B_0, \pi_0) \in V[H_0]$ and $p_1 = (B_1, \pi_1) \in V[H_1]$ are Boolean algebras extending $B^V$ and automorphisms extending $\pi$ respectively. Let $C$ be the subalgebra of $B^V[H_0,H_1]$ generated by $B_0 \cup B_1$. We will find an automorphism of $C$ extending $\pi_0 \cup \pi_1$, providing a common lower bound of the conditions $p_0, p_1$. 

$q = p \cup \{ (\gamma_i, \delta) : i \in \omega, (\gamma, \delta), \omega \}$. It is immediate that $q$ is a $K(\Gamma, \Delta)$-set, and it forces $\gamma_\omega$ to be a point of discontinuity of the generic homomorphism.
Claim 8.3.4. 1. If \( a_0 \in B_0, \ a_1 \in B_1 \) and \( c \in B^V \) are such that \( a_0 \land a_1 \leq c \), then \( \pi_0(a_0) \land \pi_1(a_1) \leq \pi(c) \) holds;

2. If \( a_0, b_0 \in B_0 \) and \( a_1, b_1 \in B_1 \) and \( a_0 \land a_1 \leq b_0 \land b_1 \), then \( \pi_0(a_0) \land \pi_1(a_1) \leq \pi_0(b_0) \land \pi_1(b_1) \) holds;

3. If \( a_0, b^i_0 \) for \( i \in n \) belong to \( B_0 \), \( a_1, b^i_1 \) for \( i \in n \) belong to \( B_1 \), and \( a_0 \land a_1 \leq \bigvee_i (b^i_0 \land b^i_1) \), then \( \pi_0(a_0) \land \pi_1(a_1) \leq \bigvee_i (\pi_0(b^i_0) \land \pi_1(b^i_1)) \) holds.

Proof. For (1), note that \( a_0 - c \) and \( a_1 - c \) are disjoint sets in the mutually generic models \( V[H_0] \) and \( V[H_1] \), and so by Proposition 1.8.8 there are ground model sets \( d_0, d_1 \) such that \( a_0 - c \leq d_0, \ a_1 - c \leq d_1 \), and \( d_0 \land d_1 = 0 \). Then \( \pi_0(a_0) - \pi(c) \leq \pi(d_0) \) since \( \pi_0 \) is an automorphism of \( B_0 \) extending \( \pi \). For the same reason \( \pi_1(a_1) - \pi(c) \leq \pi(d_1) \) holds; moreover, \( \pi(d_0) \land \pi(d_1) = 0 \) follows from the fact that \( \pi \) is an automorphism. For (2), the inequality \( a_0 \land a_1 \leq b_0 \land b_1 \) yields \( (a_0 \land b_0) \land (a_1 - b_1) = 0 \). Since \( a_0 \land b_0 \in V[H_0] \) and \( a_1 - b_1 \in V[H_1] \) and the two extensions are mutually generic, by Proposition 1.8.8 there has to be a ground model set \( c \subseteq \omega \) such that \( a_0 \land b_0 \leq c \) and \( c \land (a_1 - b_1) = 0 \). For the same reason, there is a ground model set \( d \subseteq \omega \) such that \( a_1 \land b_1 \leq d \) and \( \pi(d_0) \land \pi(d_1) = 0 \). The choice of \( c \) shows that \( a_0 \land a_1 \leq c \) and moreover \( a_1 \land c \leq b_1 \); for the same reason, \( a_0 \land a_1 \leq d \) and \( a_0 \land d \leq b_0 \) holds. In total, writing \( e = c \land d \), we have \( a_0 \land a_1 \leq e \), \( a_0 \land e \leq b_0 \), and \( a_1 \land e \leq b_1 \). By the first item, \( \pi_0(a_0) \land \pi_1(a_1) \leq \pi(e) \) follows. Moreover, since \( \pi_0 \) and \( \pi_1 \) are automorphisms, the inequalities \( \pi_0(a_0) \land \pi(e) \leq \pi_0(b_0) \) and \( \pi_1(a_1) \land \pi(e) \leq \pi_1(b_1) \) hold. The conclusion of (2) is then at hand.

For (3), for each nonempty set \( t \subseteq n \) let \( c^t = \bigwedge_{i \in t} b^i_0 - \bigvee_{i \notin t} b^i_0 \); these are pairwise disjoint sets in \( V[H_0] \). Write also \( d^t = \bigvee_{i \in t} b^i_1 \); these are sets in \( V[H_1] \). Now, \( a_0 \land a_1 \subseteq \bigvee_i c^i \), and by (2) \( \pi_0(a_0) \land \pi_1(a_1) \leq \pi \pi_0(c^t) \) holds. For each \( t \subseteq n \), \( a_0 \land a_1 \land c^t \leq d^t \) holds, and by (2) \( \pi_0(a_0) \land \pi_1(a_1) \land \pi_0(c_t) \subseteq \pi_1(d_t) \) holds. The conclusion of (3) follows.

Now, express each element of the algebra \( C \) as a disjunction of conjunctions of elements of \( B_0 \) and \( B_1 \), and define \( \chi(\bigvee_i (a^i_0 \land a^i_1)) = \bigvee_i (\pi_0(a^i_0) \land \pi_1(a^i_1)) \) where \( a^i_0 \in B_0 \) and \( a^i_1 \in B_1 \). Claim 8.3.4(3) shows that the definition of \( \chi(c) \) does not depend on the choice of the representation of the element \( c \in C \) as a disjunction of conjunctions. The map \( \chi: C \to C \) preserves ordering by (3) of Claim 8.3.4 and therefore is an automorphism of the algebra \( C \) as desired.

For item (2) of the theorem, extend \( \tau \) if necessary so that \( Q \models \tau = \langle C, \chi \rangle \) for some algebra \( C \) containing \( B^V \) and some automorphism \( \chi \) of \( C \). By a balance argument, for each element \( b \in B \) there must be an element \( \pi(b) \in B \) such that \( Q \models \chi(b) = \pi(b) \). It is immediate that \( \pi \) is an automorphism of \( B \) and \( Q \models \langle C, \chi \rangle \leq \langle B^V, \pi \rangle \). Item (2) then follows from (1) and Proposition 5.2.4.

Finally, item (3) is obvious.
8.4 Kurepa families

The notion of a Kurepa family on a set is an old one [49]; it appears intermittently in modern set theory [80], [18, Section 7]. In this section, we show how to force a cofinal Kurepa family on a Polish space with a balanced poset.

Definition 8.4.1. Let $X$ be a set.

1. A Kurepa family is a set $A \subseteq [X]^{\aleph_0}$ such that for every countable set $b \subseteq X$, the set $\{a \cap b : a \in A\}$ is countable;
2. the family is cofinal if for every countable set $b \subseteq X$ there is a set $a \in A$ such that $b \subseteq a$.

The main task of this section is to show how a cofinal Kurepa family for a given Polish space can be added by balanced forcing.

Definition 8.4.2. Let $X$ be an uncountable Polish space. The Kurepa poset $P$ is a poset of all countable sets $p \subseteq [X]^{\aleph_0}$ closed under finite intersections. The ordering is defined by $q \leq p$ if $p \subseteq q$ and for every $a \in q$ and $b \in p$, $a \cap b \in p$.

Clearly the poset $P$ does not depend on the choice of the uncountable Polish space $X$ up to isomorphism. If $G \subseteq P$ is a generic filter, then a simple genericity argument shows that $\bigcup G$ is a cofinal Kurepa family on $X$ which is closed under finite intersections. For every set $A \subseteq \mathcal{P}(X)$ closed under finite intersections and containing the set $X$, note that $\check{A}$ is a $\text{Coll}(\omega, \mathcal{P}(\omega))$-name for an element of the Kurepa poset. It turns out that if $X \in A$ then the pair $\langle \text{Coll}(\omega, \mathcal{P}(\omega)), \check{A} \rangle$ is balanced, and all balanced pairs are up to equivalence of this form. This is the contents of the following theorem.

Theorem 8.4.3. Let $X$ be an uncountable Polish space. The Kurepa poset $P$ is a $\sigma$-closed Suslin forcing. Moreover,

1. if $A \subseteq \mathcal{P}(X)$ is a set containing $X$ and closed under finite intersections, then the pair $\langle \text{Coll}(\omega, \mathcal{P}(\omega)), \check{A} \rangle$ is balanced;
2. if $\langle Q, \tau \rangle$ is a balanced pair for $P$ then there is a set $A \subseteq \mathcal{P}(X)$ closed under finite intersections and containing $X$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, \mathcal{P}(\omega)), \check{A} \rangle$ are equivalent;
3. distinct sets as in (1) yield inequivalent balanced conditions.

Proof. The Suslinity and closure of the Kurepa forcing is immediate from the definitions. In particular, conditions $p_0, p_1 \in P$ are compatible if and only if $p_0 \cup p_1 \in P$. For (1), fix a set $A \subseteq \mathcal{P}(X)$ closed under finite intersections and containing the set $X$. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model $V$ in which the set $\mathcal{P}(X) \cap V$ is countable. Suppose that $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions stronger than $A$; we must show that $p_0, p_1 \in P$ are compatible, in other words that $p_0 \cup p_1 \in P$. To
this end, let $a_0 \in p_0$ and $a_1 \in p_1$ be sets; we must show that $a_0 \cap a_1 \in p_0 \cap p_1$. To see this, first the product forcing theorem shows that $a_0 \cap a_1 \in V \cap X$ holds. Second, both sets $a_0 \cap V \cap X$ and $a_1 \cap V \cap X$ must belong to $A$, since $V \cap X \in A$ and $A \subseteq p_0, p_1$. Since $A$ is closed under finite intersections, we see that $a_0 \cap a_1 = (a_0 \cap V) \cap (a_1 \cap V) \in A \subseteq p_0 \cap p_1$ holds. Therefore, the conditions $p_0, p_1$ are compatible as desired.

(2) is more challenging. Let $\langle Q, \tau \rangle$ be a balanced pair in the poset $P$. Without loss of generality we may assume that $Q \vDash P(X) \cap V$ is countable. Strengthening the condition $\tau$ repeatedly in the $Q$-extension if necessary, we may assume that for each set $a \in X$ in the ground model, either $a \in \tau$ or there is $b \in \tau$ such that $a \cap b \notin \tau$. A balance argument shows that for each set $a \in X$ in the ground model, the largest condition in $Q$ decides which alternative prevails. Let $A \subseteq P(X)$ be a set of all sets $a \in X$ for which the former alternative prevails. We will show that $A$ is a set closed under finite intersections containing $X$, and $Q \vDash \hat{A} \geq \tau$. The desired equivalence of $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, P(X)), \hat{A} \rangle$ then follows from Proposition 5.2.4.

The closure of the set $A$ under finite intersections follows immediately from the fact that $Q \vDash \tau$ is closed under finite intersections. To see that $X \in A$ holds, suppose towards contradiction that it fails; so there must be a $Q$-name $\sigma$ for a subset of $X$ such that $Q \vDash \sigma \in \tau$ and $\sigma \cap V \notin \tau$. Strengthen the name $\tau$ if necessary to contain a set which contains $X \cap V$ as a subset, and let $\eta$ be a $Q$-name for such a set in $\tau$. Let $H_0, H_1 \subseteq Q$ be mutually generic filters, let $p_0 = \tau/H_0$ and $p_1 = \tau/H_1$, and let $a_0 = \eta/H_0$ and $a_1 = \sigma/H_1$. By a mutual genericity argument, $a_0 \cap a_1 = (a_0 \cap V) \cap (a_1 \cap V) = a_1 \cap V$. It follows that $a_0 \cap a_1 \notin p_1$ and so $p_0, p_1$ are incompatible conditions in $P$, contradicting the balance assumption.

Finally, to show that $Q \vDash \tau \leq \hat{A}$, suppose towards contradiction that this fails. Passing to a condition in $Q$ if necessary we may find a set $b \in A$ and a $Q$-name $\sigma$ such that $Q \vDash \sigma \in \tau$ and $\sigma \cap b \notin \hat{A}$. Now proceed similarly to the previous paragraph. Strengthen the name $\tau$ if necessary to contain a set which contains $X \cap V$ as a subset, and let $\eta$ be a $Q$-name for such a set in $\tau$. Let $H_0, H_1 \subseteq Q$ be mutually generic filters, let $p_0 = \tau/H_0$ and $p_1 = \tau/H_1$, and let $a_0 = \eta/H_0$ and $a_1 = \sigma/H_1$. By a mutual genericity argument, $a_0 \cap b \cap a_1 = (a_0 \cap b) \cap (a_1 \cap b) = a_1 \cap b$. Now, if $a_1 \cap b \in V$, then $a_1 \cap b \in A$, contradicting the choice of the name $\sigma$. Thus, it must be the case that $a_1 \cap b \notin V$, and by a mutual genericity argument $a_1 \cap b \notin V[H_0]$. It follows that $a_0 \cap b \cap a_1 \notin p_0$ and so $p_0, p_1$ are incompatible conditions in $P$, contradicting the balance assumption.

Finally, (3) is obvious.

\[\Box\]

### 8.5 Saturated models on quotient spaces

One can use quotient simplicial complex forcings to add structures to the various quotient spaces as long as the structures satisfy a well-known amalgamation property from model theory. We will first review the definitions. For first order structures $M, N$ in the same language, a map $\pi: M \to N$ is an\textit{ morphism} if
it is an injection from \( \text{dom}(M) \) to \( \text{dom}(N) \) which transports all relations and functions of \( M \) to the corresponding relations and functions in \( N \).

**Definition 8.5.1.** [23] A Fraissé class is a class \( \mathcal{F} \) of finite structures in a fixed finite first order language such that

1. \( \mathcal{F} \) is closed under isomorphism and under induced substructures;
2. (joint embedding property) whenever \( N_0, N_1 \in \mathcal{F} \) are structures then there is \( M \in \mathcal{F} \) containing both \( N_0, N_1 \) as induced substructures;
3. (amalgamation) whenever \( M, N_0, N_1 \in \mathcal{F} \) are structures and \( \pi_0 : M \to N_0 \) and \( \pi_1 : M \to N_1 \) are morphisms, then there is a structure \( K \in \mathcal{F} \) and morphisms \( \chi_0 : N_0 \to K \) and \( \chi_1 : N_1 \to K \) such that \( \chi_0 \circ \pi_0 = \chi_1 \circ \pi_1 \).

We will tacitly assume that our Fraissé classes contain arbitrarily large finite structures. Fraissé classes are prominent in model theory [32, Theorems 5.3.3 and 5.3.5], Ramsey theory [58, 17], and topological dynamics [43, 89]. The classes we can handle satisfy a well-known stronger version of amalgamation:

**Definition 8.5.2.** A Fraissé class \( \mathcal{F} \) satisfies strong amalgamation if for whenever \( M, N_0, N_1 \in \mathcal{F} \) are structures and \( \pi_0 : M \to N_0 \) and \( \pi_1 : M \to N_1 \) are morphisms, then there is a structure \( K \in \mathcal{F} \) and morphisms \( \chi_0 : N_0 \to K \) and \( \chi_1 : N_1 \to K \) such that \( \chi_0 \circ \pi_0 = \chi_1 \circ \pi_1 \), and \( \text{rng}(\chi_0) \cap \text{rng}(\chi_1) = (\chi_0 \circ \pi_0)''M \).

**Definition 8.5.3.** Let \( \mathcal{F} \) be a Fraissé class in a finite relational language.

1. An \( \mathcal{F} \)-structure is a structure \( M \) in the same language as \( \mathcal{F} \) such that for every finite set \( a \subset \text{dom}(M) \), \( M \upharpoonright a \in \mathcal{F} \);
2. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). The \( E, \mathcal{F} \)-Fraissé poset \( P \) consists of conditions \( p \) where \( p \) is an \( \mathcal{F} \)-structure whose domain is a countable subset of the quotient space \( X/E \). The ordering is defined by \( q \leq p \) if \( \text{dom}(p) \subset \text{dom}(q) \) and \( p = q \upharpoonright \text{dom}(p) \).

It is obvious that the poset \( P \) introduces a \( \mathcal{F} \)-structure on the space \( X/E \), denoted by \( M_{\text{gen}} \). It is not difficult, but at the same time also not particularly natural, to describe the poset \( P(\mathcal{F}, E) \) as a quotient simplicial complex poset: the vertices of the simplicial complex \( K \) will be finite \( \mathcal{F} \)-structures on \( X/E \) and a finite set \( a \) of vertices belongs to \( K \) if all structures in \( a \) are induced substructures of a single finite \( \mathcal{F} \)-structure on \( X/E \). It is obvious that the posets \( P(\mathcal{F}, E) \) and \( P_K \) are naturally co-dense; however, we will never use the simplicial complex presentation of \( P(\mathcal{F}, E) \). To describe the balanced conditions in the poset \( P(\mathcal{F}, E) \), suppose that \( M \) is an \( \mathcal{F} \)-structure on the virtual space \( X^{**} \). Let \( \tau_M \) be a \( \text{Coll}(\omega, \exists \omega) \) name for the condition \( \pi''M \) where \( \pi \) is the map from the virtual space \( (X^{**})^V \) to the quotient space \( X/E \) of the extension which maps each virtual class to its realization. Theorem 2.5.6 provides the necessary assurance that the map \( \pi \) is well-defined and its range is a countable set.
Theorem 8.5.4. Let \( F \) be a Fraissé class in a finite relational language which satisfies strong amalgamation. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). Then in the \( E,F \)-Fraissé poset \( P \),

1. for every \( F \)-structure \( M \) on \( X^{**} \), the pair \( \langle \text{Coll}(\omega, \sigma), \tau_M \rangle \) is balanced;

2. for every balanced pair \( \langle Q, \tau \rangle \) there is a \( F \)-structure \( M \) on \( X^{**} \) such that the pairs \( \langle \text{Coll}(\omega, \sigma), \tau_M \rangle \) and \( \langle Q, \tau \rangle \) are equivalent;

3. distinct \( F \)-structures on \( X^{**} \) yield inequivalent balanced conditions.

In particular, \( P \) is balanced.

Proof. For (1), suppose that \( M \) is a \( F \)-structure on \( X^{**} \). Let \( R_0, R_1 \) be arbitrary posets and \( \sigma_0, \sigma_1 \) are respective \( R_0 \times \text{Coll}(\omega, \sigma_0) \)- and \( R_1 \times \text{Coll}(\omega, \sigma_1) \)-names for elements of \( P \) such that in the respective posets \( \sigma_0 \leq \tau_M \) and \( \sigma_1 \leq \tau \) is forced. Let \( H_0, H_1 \) be mutually generic filters on the respective posets and let \( p = \tau_M/H_0, p_0 = \sigma_0/H_0 \) and \( p_1 = \sigma_1/H_1 \); we need to show that \( p_0, p_1 \) are compatible in the poset \( P \).

For this, first note that by the mutual genericity, \( \text{dom}(p_0) \cap \text{dom}(p_1) = \text{dom}(p) \). By the assumption on the names \( \sigma_0, \sigma_1 \) it is the case that \( p_0 \upharpoonright \text{dom}(p) = p_1 \upharpoonright \text{dom}(p) = p \). Write \( c = \text{dom}(p_0) \cup \text{dom}(p_1) \). By the strong amalgamation property of the Fraissé class \( F \), for each finite set \( a \subset c \) there is a structure \( N_a \in F \) such that \( \text{dom}(N_a) = a \) and writing \( a_0 = a \cap \text{dom}(p_0) \) and \( a_1 \cap \text{dom}(p_1) \), \( N_a \upharpoonright a_0 = p_0 \upharpoonright a_0 \) and \( N_a \upharpoonright a_1 = p_1 \upharpoonright a_1 \) both hold. Let \( U \) be an ultrafilter on \( [c]^{<\aleph_0} \) such that for each finite set \( a \subset c \) the set \( \{ b : a \subset b \} \) is in the ultrafilter \( U \). Let \( N \) be the structure on \( c \) which is the \( U \)-integral of the structures \( N_a \). It is immediate that \( N \upharpoonright \text{dom}(p_0) = p_0 \) and \( N \upharpoonright p_1 = p_1 \); the closure of the Fraissé class under substructures shows that \( N \) is a \( F \)-structure. It is the desired lower bound of the conditions \( p_0, p_1 \).

For (2), note that for each relation \( R \) in the language of the Fraissé class \( F \) and every tuple \( a \) of virtual \( E \)-classes, it must be the case that either \( Q \models \tau \models a \) belongs to the relation \( R \) in the generic structure \( M_{\text{gen}} \) or \( Q \models \tau \models a \) does not to the relation \( R \) in the generic structure \( M_{\text{gen}} \) by the balance of the pair \( \langle Q, \tau \rangle \). Let \( M \) be the structure on the virtual \( E \)-quotient space consisting of those tuples for which the former alternative occurs. It is immediate that \( Q \times \text{Coll}(\omega, \sigma_0) \models \tau \leq \tau_M \) in the separative quotient of the poset \( P \). The equivalence of the two pairs follows from Proposition 5.2.4.

Now, (3) is obvious. For the last sentence, suppose that \( p \in P \) is a condition. For each finite set \( a \subset X^{**} \), choose a structure \( N_a \in F \) such that \( \text{dom}(N_a) = a \) and \( N_a \upharpoonright (\text{dom}(p) \cap a) = p \upharpoonright (\text{dom}(p) \cap a) \). Let \( U \) be an ultrafilter on \( [X^{**}]^{<\aleph_0} \) such that for each finite set \( a \subset X^{**} \) the set \( \{ b : a \subset b \} \) is in the ultrafilter \( U \). Let \( N \) be the structure on \( X^{**} \) which is the \( U \)-integral of the structures \( N_a \). It is immediate that \( N \upharpoonright \text{dom}(p) = p \); the closure of the Fraissé class under substructures shows that \( N \) is a \( F \)-structure. The pair \( \langle \text{Coll}(\omega, \sigma_0), \tau_N \rangle \) is a balanced virtual condition below \( p \). \( \square \)
**Example 8.5.5.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $P$ be the poset of all linear orderings on countable subsets of the quotient space $X/E$. The poset is designed to add a linear ordering on the quotient space. Since the Fraissé class of finite linear orderings has the strong amalgamation property, the poset $P$ is balanced, and its balanced conditions are classified by linear orders on the virtual quotient space $X^{**}$.

**Example 8.5.6.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $P$ be the poset of all tournaments on countable subsets of the quotient space $X/E$. The poset is designed to add a tournament on the quotient space. Since the Fraissé class of finite tournaments has the strong amalgamation property, the poset $P$ is balanced, and its balanced conditions are classified by tournaments on the virtual quotient space $X^{**}$.

The Fraissé posets can be viewed as a particularly well-behaved subclass of a significantly larger class obtained from standard amalgamation constructions in model theory.

**Definition 8.5.7.** Let $T$ be a complete first-order theory in countable language, with infinite models. Let $E$ be a Borel equivalence relation on a Polish space $X$. The poset $P_{TE}$ consists of structures $p$ satisfying the theory $T$ whose domain is a countable subset of the quotient space $X/E$. The ordering is that of elementary substructure: $q \leq p$ if the domain of $p$ is a subset of the domain of $q$, and $p$ is an elementary substructure of $q$.

It is immediate that the union of the structures in the generic filter is a countably saturated model of the theory $T$ on the $E$-quotient space. To classify the balanced virtual conditions, we use the following definition.

**Definition 8.5.8.** A balanced theory is a complete consistent theory $S$ in the language of $T$ plus a constant for each virtual $E$-class which contains $T$ and asserts that no elements other than the virtual $E$-class constants are algebraic over those constants.

Suppose that $S$ is a balanced theory. Let $\tau_S$ be the Coll($\omega$, $\beth_1$)-name for the set of all countable models $p \in P_{TE}$ such that $\text{dom}(p)$ contains the set $p_0$ of realizations of all ground model virtual $E$-classes and $p \models S$. It is not difficult to see that $\tau_S$ is a name for an analytic, nonempty, and open subset of the poset $P_{TE}$.

**Theorem 8.5.9.** Let $T$ be a complete first-order theory in countable language, with infinite models. Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. For every balanced theory $S$, the pair $\langle \text{Coll}(\omega, \beth_1), \tau_S \rangle$ is balanced in $P_{TE}$;

2. for every balanced pair $\langle Q, \tau \rangle$ there is a balanced theory $S$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, \beth_1), \tau_S \rangle$ are equivalent;

3. distinct balanced theories yield inequivalent balanced pairs.
In particular, the poset $P_{TE}$ is balanced.

**Proof.** For (1), suppose that $S$ is a balanced theory, $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, and $p_0, p_1 \in P_{TE}$ are conditions in the respective models in the set given by the name $\tau_S$. We must show that there is a model $q \in P_{TE}$ in which both models $p_0, p_1$ of $T$ are elementary. Note that $\text{dom}(p_0) \cap \text{dom}(p_1)$ is exactly the set of all realizations of ground model virtual $E$-classes by the mutual genericity assumption. Use the algebraicity clause of Definition 8.5.8 and a textbook amalgamation theorem [32, Theorem 5.3.5] to conclude that in $V[H_0, H_1]$ there is a countable model $q$ of $T$ and elementary embeddings $j_0: p_0 \rightarrow q$ and $j_1: p_1 \rightarrow q$ such that $j_0, j_1$ coincide on the set $r$ of realizations of virtual $E$-classes from the ground model, and the sets $j_0′(p_0 \setminus r)$ and $j_1′(p_1 \setminus r)$ are disjoint. It is easy to see that such a model $q$ can be realized in such a way that its domain is a set of $E$-classes and the embeddings $j_0, j_1$ are both identity maps. Then $q$ is the sought lower bound of the conditions $p_0, p_1$.

For (2), let $(Q, \sigma)$ be a balanced pair. Strengthening the name $\sigma$ and the poset $Q$ if necessary, we may assume that $\sigma$ is in fact a name for a single element of $P_{TE}$ as opposed to an analytic subset of $P_{TE}$ and that $\sigma$ is forced to contain the realization of every virtual $E$-class in the ground model. By a balance argument, for every formula $\phi(\bar{x})$ and every tuple $\bar{c}$ of virtual $E$-classes of the same length as $\bar{x}$, it must be the case that either $Q \vdash \sigma \models \phi(\bar{c})$ or $Q \nvdash \sigma \models \neg \phi(\bar{c})$. Let $S$ be the set of all formulas and tuples of virtual $E$-classes $\phi(\bar{c})$ for which the former option prevails. We need to show that $S$ is a balanced theory and the balanced pairs $(Q, \tau)$ and $(\text{Coll}(\omega, \exists \omega_1), \tau_S)$ are equivalent.

**Claim 8.5.10.** $Q \models \text{no element of } \sigma \text{ which is not a realization of a virtual } E\text{-class is algebraic over a tuple of virtual } E\text{-classes from the ground model.}$

**Proof.** Suppose towards contradiction that this fails, and let $q \in Q$ be a condition, let $\eta$ be a $Q$-name for an element of $\sigma$ which is not a realization of a virtual $E$-class, and let $\phi$ be a formula, $\bar{c}$ a tuple of virtual $E$-classes and $n \in \omega$ be a natural number such that $q \vdash \sigma \models \text{there are precisely } n \text{ many } x \text{ such that } \phi(x, \bar{c}) \text{ holds, and } \eta \text{ is one of them.}$ Let $H_0, H_1 \subset Q$ be mutually generic filters over the ground model, both containing $q$, and let $p_0 = \sigma/H_0$ and $p_1 = \sigma/H_1$. We will show that $p_0, p_1$ have no lower bound in the poset $P_{TE}$, reaching a contradiction with the balance assumption on the pair $(Q, \sigma)$.

Suppose then that $q$ is such a lower bound. By elementarity, $q$ must see exactly $n$ many solutions to $\phi(x, \bar{c})$, and all solutions in $p_0$ and in $p_1$ are solutions in $q$. Now, $p_0$ already sees $n$ many solutions, and $\eta/H_1$ is another solution in $p_1$ which does not belong to $p_0$ or even to $V[H_0]$. Thus, $q$ in fact sees at least $n + 1$ many solutions,a contradiction. 

It follows that $S$ is a balanced theory, and also that $\text{Coll}(\omega, \exists \omega_1) \times Q \models \sigma \in \tau_S$. The equivalence of the balanced pairs $(Q, \tau)$ and $(\text{Coll}(\omega, \exists \omega_1), \tau_S)$ then follows from Proposition 5.2.4.

(3) is obvious. For the last sentence, suppose that $p \in P_{TE}$ is a condition—a countable model of the theory $T$. By the upwards Löwenheim–Skolem theorem,
there is a model $M$ of the theory $T$ on the virtual $E$-quotient space in which $p$ is an elementary submodel. Let $S$ be the diagram of $M$ and observe that $S$ is a balanced theory and its corresponding balanced condition is below $p$. 

**Example 8.5.11.** Let $T$ be the theory of dense linear orders without endpoints. The notion of algebraicity is trivial in the theory $T$, and $T$ has elimination of quantifiers. It follows that the balanced theories are exactly the linear orders on the set of virtual $E$-classes. This should be compared to Example 8.5.5.

### 8.6 Non-DC variations

All of the partial orders exhibited so far are either $\sigma$-closed or $\aleph_0$-distributive, and therefore their corresponding extensions of the symmetric Solovay model satisfy DC, the Axiom of Dependent Choices. This is normaly viewed as highly desirable, as DC is a key tool for developing mathematical analysis and descriptive set theory as we know them today. However, balanced forcing can be used to generate extensions of the symmetric Solovay model in which DC fails. The simplicity of the partial orders and the associated balance arguments is striking.

**Definition 8.6.1.** Let $X$ be an uncountable Polish space. The finite-countable poset $P$ associated with $X$ consists of pairs $p = \langle a_p, b_p \rangle$ where $a_p \subseteq X$ is a finite set, $b_p \subseteq X$ is a countable set, and $a_p \cap b_p = 0$. The ordering is that of coordinatewise reverse inclusion.

The finite-countable poset is rather worthless in the ZFC context. The lack of control over the finite part means that it collapses $\aleph_1$. Namely, for every sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ of distinct points of $X$, the set $\{ \alpha \in \omega_1 : \langle \{ x_\alpha \}, 0 \rangle \}$ belongs to the generic filter on $P$ is forced to be cofinal in $\omega_1$ of ordertype $\omega$. However, the finite-countable poset provides a particularly easy answer to an old question of Woodin: is there a Suslin forcing which is not c.c.c. and yet has no perfect antichain? The finite-countable poset is not c.c.c. as it collapses $\aleph_1$. On the other hand, it does satisfy $\aleph_2$-c.c. in ZFC, and so by a straightforward absoluteness argument cannot contain a perfect antichain. This should be contrasted with the convoluted answer to the Woodin question given in [37].

In order to classify the balanced conditions in the finite-countable poset, we need a piece of notation. Whenever $a \subset X$ is a finite set, let $\tau_a$ be the $\text{Coll}(\omega, X)$-name for a condition in the poset $P$ which is the pair $\langle a, (X \cap V) \setminus a \rangle$.

**Theorem 8.6.2.** Let $X$ be a Polish space and $P$ the associated finite-countable poset.

1. For every finite set $a \subset X$, the pair $\langle \text{Coll}(\omega, X), \tau_a \rangle$ is balanced in $P$;

2. for every balanced pair $\langle Q, \tau \rangle$ there is a finite set $a \subset X$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, X), \tau_a \rangle$ are equivalent;

3. distinct finite sets yield inequivalent balanced pairs.
Definition 8.6.3. Let Γ be a Borel hypergraph on a Polish space $X$ and in which we do not know how to design a stronger condition. Since the poset $P$ is not designed to resolve any independence problem in particular. We will now present several examples.

Proof. For (1), suppose that $V[H_0], V[H_1]$ are mutually generic extensions of the ground model $V$ and $p_0 = \langle a_0, b_0 \rangle \in V[H_0]$ and $p_1 = \langle a_1, b_1 \rangle \in V[H_1]$ are conditions stronger than $\bar{p} = \langle a, (X \cap V) \setminus a \rangle$. We have to show that $p_0, p_1$ are compatible, which is to say that $a_0 \cap b_1 = a_1 \cap b_0 = 0$ holds. Suppose towards contradiction that for example $x \in a_0 \cap b_1$ is a point. By the product forcing theorem, $x \in V[H_0] \cap V[H_1] = V$. If $x \in a$ then $x \in b_1$ is impossible as $p_1 \leq \bar{p}$, and if $x \notin a$ then $x \in a_0$ is impossible as $p_0 \leq \bar{p}$. A contradiction!

For (2), let $\tau = \langle \dot{a}, \dot{b} \rangle$ where $\dot{a}, \dot{b}$ are Q-names for disjoint subsets of $X$. A balance argument show that for each point $x \in X$, either $Q \models \dot{x} \in \dot{a}$ or $Q \not\models \dot{x} \in \dot{b}$. Let $a \subset X$ be the set of all points $x \in X$ for which the first alternative prevails. It is immediate that $a$ is a finite set and $Q \times \text{Coll}(\omega, X) \models \tau \leq \tau_a$. Proposition 5.2.4 shows then that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, X), \tau_a \rangle$ are equivalent, proving (2). (3) is immediate. \qed

While the finite-countable poset is interesting, it is not designed to resolve any independence problem in particular. We will now present several examples of posets which do have a specific purpose (namely, to force countable chromatic number to a specific hypergraph) and in which we do not know how to design a balanced poset which preserves DC.

Definition 8.6.4. Let $\Gamma$ be a Borel hypergraph on a Polish space $X$. Let $J$ be a Borel ideal on $\omega$ which is not generated by a single infinite set over the Fréchet ideal. The poset $P(\Gamma, J)$ consists of countable partial $\Gamma$-colorings $p: X \to \omega$ such that $\text{rng}(p) \in J$. The ordering is that of inclusion.

A simple density argument shows that the union of the generic filter on the poset $P(\Gamma, J)$ is a $\Gamma$-coloring on the set $X^V$; note that below any condition, there is still an infinite set of colors which have not been used and can be used to obtain a stronger condition. Since the poset $P(\Gamma, J)$ very badly fails to be $\sigma$-closed, it may appear useless. However, in numerous cases the poset $P(\Gamma, J)$ is balanced, and then over the symmetric Solovay model it adds no new ordinal sequences and therefore no new elements of the Polish space $X$ by Theorem 9.1.1. As a result, in the $P(\Gamma, J)$-extension of the symmetric Solovay model, the chromatic number of $\Gamma$ will be countable. The generic coloring $c$ will witness the failure of DC in the sense that $\text{rng}(c) = \omega$ and for every countable set $b \subset X$, $c''b \in J$.

Definition 8.6.5. Let $\Gamma$ be a half-countable Borel hypergraph on a Polish space $X$. Let $J$ be a Borel ideal on $\omega$ which is not generated by a single infinite set over the Fréchet ideal. In the poset $P(\Gamma, J)$:

1. if $c: X \to \omega$ is a total $\Gamma$-coloring such that $\text{rng}(c) \in J$, then the pair $(\text{Coll}(\omega, X), \dot{c})$ is balanced;
2. for every balanced pair \( \langle Q, \tau \rangle \) there is a coloring \( c: X \rightarrow \omega \) such that \( \text{rng}(c) \in J \) such that \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X \times Y), \hat{c} \rangle \);

3. if the Continuum Hypothesis holds then the poset \( P(\Gamma, J) \) is balanced.

Proof. For (1), suppose that \( c \) is a coloring with \( \text{rng}(c) \in J \). First of all, \( \text{Coll}(\omega, X) \models \hat{c} \in P(\Gamma, J) \) holds. Now, choose mutually generic extensions \( V[H_0], V[H_1] \) of the ground model in \( V \) and conditions \( p_0, p_1 \in P(\Gamma, J) \) in the respective models such that \( c \subset p_0, p_1 \). We will show that \( p_0 \cup p_1 \) is their lower bound, proving the balance of \( c \). Indeed, \( p_0 \cup p_1 \) is a function, since by the product forcing theorem \( \text{dom}(p_0) \cap \text{dom}(p_1) \subset V \), and \( p_0 \upharpoonright V = p_1 \upharpoonright V = c \). The range of \( p_0 \cup p_1 \) belongs to \( J \) as \( J \) is an ideal. Finally, we must show that \( p_0 \cup p_1 \) is a \( \Gamma \)-coloring. To this end, assume that \( b \subset \text{dom}(p_0 \cup p_1) \) is a hyperedge in \( \Gamma \). Say \( |b \cap \text{dom}(p_0)| \geq |b|/2 \); the case where \( |b \cap \text{dom}(p_1)| \geq |b|/2 \) is symmetric. Write \( a = b \cap \text{dom}(p_0) \). By the half-countability assumption, the set \( e = \{ \langle d \in \Gamma: a \subset d \rangle \} \) is countable, and by a Mostowski absoluteness argument \( c \in V[H_0] \) must hold. But then, \( b \in V[H_0] \) and so \( b \subset \text{dom}(p_0) \). Since \( p_0 \) is a \( \Gamma \)-coloring, it means that \( p_0 \cup p_1 \) is not constant on \( b \) as desired.

For (2), note that the poset \( P(\Gamma, J) \) forces the union of the conditions in the generic filter to be a \( \Gamma \)-coloring from \( X^V \) to \( \omega \). If \( \langle Q, \tau \rangle \) is a balanced pair, it has to decide the value of this generic coloring at every point \( x \in X \) to be some number \( c(x) \in \omega \) by the balance. It is not difficult to see that the map \( c \) has to be a \( \Gamma \)-coloring, and an additional balance argument shows that \( \text{rng}(c) \in J \) must hold. It is immediate that \( Q \times \text{Coll}(\omega, X) \models \tau \leq \hat{c} \) in the separative quotient, and so \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X), \hat{c} \rangle \) by Proposition 5.2.4.

For (3), we will first show that under \( \text{CH} \), the chromatic number of \( \Gamma \) is countable. To this end, let \( \langle M_\alpha: \alpha \in \omega_1 \rangle \) be a continuous \( \in \)-tower of countable elementary submodels of a large structure containing \( \Gamma \). The \( \text{CH} \) assumption shows that \( X \subset \bigcup_{\alpha} M_\alpha \). Let \( c: X \rightarrow \omega \) be any map such that \( c \upharpoonright X \cap M_0 \) is an injection, and for every ordinal \( \alpha \) the map \( c \upharpoonright M_{\alpha+1} \setminus M_\alpha \) is an injection. We claim that \( c \) is a \( \Gamma \)-coloring. To show this, suppose that \( b \in \Gamma \) is a hyperedge. Let \( \alpha \in \omega_1 \) be the largest ordinal such that \( b \cap M_{\alpha+1} \setminus M_\alpha \) is nonempty; if \( b \subset M_0 \) then let \( \alpha = -1 \). If \( b \) contains more than one element of \( M_{\alpha+1} \setminus M_\alpha \) then they are colored in a different way by \( c \) and so \( b \) is not monochromatic. However, it is impossible for \( b \) to contain just one element of \( M_{\alpha+1} \setminus M_\alpha \), since then \( |b \cap M_\alpha| \geq |b|/2 \), the countable set \( \{ d \in \Gamma: b \cap M_\alpha \subset d \} \) is an element and therefore a subset of \( M_\alpha \), and so \( b \) would be an element and therefore a subset of \( M_\alpha \).

Finally, every condition \( p \in P(\Gamma, J) \) can be extending to a total coloring of \( \Gamma \) with range in \( J \). To see this, find an infinite set \( a \subset \omega \) such that \( a \subset J \) and \( a \cap \text{rng}(p) = 0 \), find a total \( \Gamma \)-coloring \( d: X \rightarrow a \) and let \( c: X \rightarrow \omega \) be the map defined by \( c(x) = p(x) \) if \( x \in \text{dom}(p) \) and \( c(x) = d(x) \) otherwise. It is immediate that \( c \) is a total \( \Gamma \)-coloring with range in \( J \), and \( p \subset c \), completing the proof of (3). \( \square \)

Example 8.6.6. Let \( X \) be a Polish space. Let \( \Gamma \) be the hypergraph on \( |X|^2 \) of arity 3 defined by \( a \in \Gamma \) if \( a \) is a triangle. The hypergraph is half-countable and
therefore the poset \( P(\Gamma, J) \) is balanced. In ZFC it is not difficult to check that the chromatic number of \( \Gamma \) is countable if and only if the Continuum Hypothesis holds.

**Example 8.6.7.** Let \( \Gamma \) be the hypergraph on \( \mathbb{R}^2 \) of arity three defined by \( a \in \Gamma \) if \( a \) is a set of vertices of an equilateral triangle. The hypergraph is half-countable and therefore the poset \( P(\Gamma, J) \) is balanced. In ZF+ there is a Hamel basis, the chromatic number of \( \Gamma \) is countable by a result of [13].

The hypergraphs of higher arity which are not half-countable present genuinely difficult challenges. We provide a single example:

**Definition 8.6.8.** Let \( X, Y \) be uncountable Polish spaces.

1. The *rectangle hypergraph* \( \Gamma \) on \( X \times Y \) is the hypergraph on \( X \times Y \) consisting of all edges of the form \( a \times b \) where \( a \subset X \), \( b \subset Y \) are sets of size 2;

2. let \( J \) be a Borel ideal on \( \omega \) which is not generated by a single infinite set over the Frechet ideal. The associated poset \( P \) consists of all partial countable \( \Gamma \)-colorings \( p : X \times Y \to \omega \) such that the domain of \( p \) is a rectangle \( \text{dom}(p)_X \times \text{dom}(p)_Y \) and \( \text{rng}(p) \in J \). The ordering is defined by \( q \leq p \) if \( p \subset q \), for every \( x \in \text{dom}(q)_X \setminus \text{dom}(p)_X \) and distinct \( y_0, y_1 \in \text{dom}(p)_Y \) holds, and for every \( y \in \text{dom}(q)_Y \setminus \text{dom}(p)_Y \) and distinct \( x_0, x_1 \in \text{dom}(p)_X \) holds.

It is well-known that in ZFC, the hypergraph \( \Gamma \) has countable chromatic number if and only if the Continuum Hypothesis holds. If CH holds, then the product \( X \times Y \) can be covered by a countable union of graphs of functions from \( X \) to \( Y \) and inverses of functions from \( Y \) to \( X \) by an old result of Sierpinski, and neither of these graphs contains a \( \Gamma \)-edge. For the converse, one can invoke the standard partition result with \( \kappa = \aleph_0 \): every partition of \( \kappa^{++} \times \kappa^{++} \) to \( \kappa \) many classes contains arbitrarily large finite monochromatic rectangles, for every infinite cardinal \( \kappa \). This implies that \( \Gamma \) has uncountable chromatic number when \( \kappa \geq \aleph_2 \).

The poset \( P(\Gamma, J) \) is a straightforward attempt to force in ZF the chromatic number of \( \Gamma \) to be countable. It is not difficult to check that in fact \( P(\Gamma, J) \) is a Suslin partial order. Investigation of the balance properties yields the following theorem.

**Theorem 8.6.9.** Let \( X, Y \) be Polish spaces and let \( \Gamma \) be the rectangle hypergraph \( X \times Y \). Let \( J \) be a Borel ideal on \( \omega \) which is not generated by a single infinite set over the Frechet ideal and let \( P \) be the poset of Definition 8.6.8.

1. If \( c : X \times Y \to \omega \) is a total \( \Gamma \)-coloring such that \( \text{rng}(c) \in J \), then the pair \( \langle \text{Coll}(\omega, X \times Y), \hat{c} \rangle \) is balanced;

2. for every balanced pair \( \langle Q, \tau \rangle \) there is a coloring \( c : X \times Y \to \omega \) such that \( \text{rng}(c) \in J \) such that \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X \times Y), \hat{c} \rangle \).
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3. If the Continuum Hypothesis holds then the poset $P$ is balanced.

Proof. For (1), suppose that $R_0, R_1$ are partial orders, let $H_0 \subset R_0$ and $H_1 \subset R_1$ be mutually generic filters, and let $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be conditions in the poset $P$ such that $p_0 \leq c$ and $p_1 \leq c$. To construct a common lower bound of the conditions $p_0, p_1$, first observe that $p_0 \cup p_1$ is a function: by the product forcing theorem, $\text{dom}(p_0) \cap \text{dom}(p_1) \subset V$, and $p_0 \upharpoonright V = p_1 \upharpoonright V = c$. Now, write $\text{dom}(p_0) = d_0 \times e_0$, $\text{dom}(p_1) = d_1 \times e_1$ and $d = d_0 \cup d_1$ and $e = e_0 \cup e_1$. Use the assumptions on the Borel ideal $J$ to find an infinite set $f \in J$ disjoint from $\text{rng}(p_0 \cup p_1)$, and find any function $q: d \times e \to \omega$ extending $p_0 \cup p_1$ such that $q \upharpoonright (p_0 \cup p_1)$ is an injection into the set $f$. We claim that $q \in P$ and moreover $q \leq p_0, p_1$.

The verification of these statements is rather elementary. To check that $q$ is a $\Gamma$-coloring, suppose that $a \subseteq d$ and $b \subseteq e$ are sets of size 2 and check that $q$ is not constant on $a \times b$. To see this, discern a number of cases:

- if there is $x \in a$ and $y \in b$ such that $x \in d_0 \setminus V$ and $y \in e_1 \setminus V$ then the node $\langle x, y \rangle$ is the unique node colored by $q$;
- a symmetric argument deals with the case where there is $x \in a$ and $y \in b$ such that $x \in d_1 \setminus V$ and $y \in e_0 \setminus V$;
- if $a \subseteq d_0$ and $b \subseteq e_0$ then $q \upharpoonright a \times b = p_0 \upharpoonright a \times b$ and so cannot be constant;
- the case $a \subseteq d_1$ and $b \subseteq e_1$ is again symmetric;
- if $a \subseteq V$ and $b \not\subseteq V$ (say there is $y \in b \cap e_0 \setminus V$) then $q \upharpoonright a \times \{y\} = p_0 \upharpoonright a \times \{y\}$ is not constant since $p_0 \leq c$;
- finally, a symmetric argument deals with the cases where $b \subseteq V$ and $a \not\subseteq V$.

To check that $q \leq p_0, p_1$ use the fact that $q \upharpoonright (p_0 \cup p_1)$ is an injection.

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. Use the balance to show that for each point $\langle x, y \rangle \in X \times Y$ there is a number $c(x, y) \in \omega$ such that $Q \Vdash \tau \Vdash \text{the generic coloring assumes the value } c(x, y) \text{ on the point } \langle x, y \rangle$. It is clear that $c: X \times Y \to \omega$ is a $\Gamma$-coloring, and another balance argument shows that $\text{rng}(c) \in J$. To conclude that the balanced pair $\langle Q, \tau \rangle$ is equivalent to $\langle \text{Coll}(\omega, X \times Y, \check{c}) \rangle$, we now need to show that $Q \times \text{Coll}(\omega, X \times Y) \Vdash \tau \leq \check{c}$ in the separative quotient.

Suppose towards contradiction that this fails. Then there must be a condition $r \in Q \times \text{Coll}(\omega, X \times Y)$ in the product, a product name $\sigma$ for a condition in the poset $P(\Gamma, J)$ stronger than $\tau$ and one of the following:

- a product name $\check{x}$ for an element of $X \setminus V$, elements $y_0, y_1 \in Y$ and a number $n \in \omega$ such that $r \Vdash \langle \check{x}, y_0 \rangle$ and $\langle \check{x}, y_1 \rangle$ are both in domain of $\sigma$ and $\sigma$ assigns the value of $n$ to both of them;
- a product name $\check{y}$ for an element of $Y \setminus V$, elements $x_0, x_1 \in X$ and a number $n \in \omega$ such that $r \Vdash \langle \check{x}_0, \check{y} \rangle$ and $\langle \check{x}_1, \check{y} \rangle$ are both in domain of $\sigma$ and $\sigma$ assigns the value of $n$ to both of them.
In either case, let \( H_0, H_1 \subset Q \times \text{Coll}(\omega, X \times Y) \) be mutually generic filters containing the condition \( \tau \) and let \( p_0 = \sigma/H_0, p_1 \in \sigma/H_1 \in P \). We will argue that the conditions \( p_0, p_1 \) are incompatible, which will contradict the balance of the pair \((Q, \tau)\).

Indeed, if the first item above occurs, then let \( x_0 = \hat{x}/H_0, x_1 = \hat{x}/H_1 \). By the product forcing theorem, these are distinct points of the space \( X \). The \( \Gamma \)-edge \( \{x_0, x_1\} \times \{y_0, y_1\} \) is monochromatic under the function \( p_0 \cup p_1 \) with color \( n \). This means that \( p_0, p_1 \) are incompatible. If the second item above occurs, the conclusion is symmetric and we leave it out.

For (3), suppose that the Continuum Hypothesis holds, and let \( p \in P \) be an arbitrary condition. Use the initial assumption on the ideal \( J \) to find infinite disjoint sets \( a, b \subset \omega \) which are also disjoint from \( \text{rng}(p) \) and belong to the ideal \( J \). Use the Continuum Hypothesis to find a \( \Gamma \)-coloring \( d \) of the rectangle \( (X \setminus \text{dom}(p)_X) \times (Y \setminus \text{dom}(p)_Y) \) with colors in \( a \). Let \( e_X \) be the coloring of the rectangle \( (X \setminus \text{dom}(p)_X) \times \text{dom}(p)_Y \) with colors in \( b \) which is one-to-one in all vertical sections. Let \( e_Y \) be the coloring of the rectangle \( \text{dom}(p)_X \times (Y \setminus \text{dom}(p)_Y) \) with colors in \( b \) which is one-to-one in all horizontal sections. Finally, let \( c = p \cup d \cup e_X \cup e_Y \). It is not difficult to check that \( c \) is a \( \Gamma \)-coloring of \( X \times Y \) and \( \text{Coll}(\omega, X \times Y) \Vdash c \leq p \) in the poset \( P \). (3) follows. \( \square \)

### 8.7 Weakly balanced variations

There is a number of restrictions on the balanced extensions of the Solovay model—Corollaries 9.1.2, 9.1.5 or Theorems 14.1.1 or 14.2.1 are good examples. Transcending these restrictions requires reaching for a weakly balanced forcing. The weakly balanced arguments are invariably more complicated than the balanced ones, and we never obtain a classification of weakly balanced classes. We include two examples. The first poset adds an injection from the \( \mathbb{F}_2 \)-quotient space to the \( \mathbb{E}_\omega \)-quotient space. This is impossible to do with balanced forcing by Theorem 9.1.1.

**Example 8.7.1.** The weakly balanced collapse forcing \( P \) is the poset of all pairs \( p = \langle a_p, b_p \rangle \) where \( a_p \) is a countable partial injective function from \( [X]^{\omega_n} \) to \( \mathbb{E}_\omega \)-classes, and \( b_p \) is a countable collection of closed nowhere dense subsets of \( (2^{\omega})^n \) for various \( n \in \omega \). The ordering is defined by \( q \leq p \) if \( a_q \subset a_p \), \( b_p \subset b_q \), and whenever \( C \in b_p \) is a closed nowhere dense subset of some \( (2^{\omega})^n \) and \( \bar{x} \in (2^{\omega})^m \) is a tuple of pairwise \( \mathbb{E}_\omega \)-inequivalent elements of \( \bigcup \text{rng}(a_q \setminus a_p) \) and \( m = n \), then \( \bar{x} \notin C \), and if \( m < n \) then the closed set \( D = \{ y \in (2^{\omega})^{n-m} : \bar{x} \cup y \in C \} \) is nowhere dense and in \( b_q \).

It is not difficult to see that \( P \) is a \( \sigma \)-closed Suslin partial order and if \( G \subset P \) is a generic filter then \( f = \bigcup_{p \in G} a_p \) is a total injection from \( [2^{\omega}]^{\omega_0} \) to the \( \mathbb{E}_\omega \)-quotient space. The proof of Example 8.7.1 requires a preliminary definition and a proposition of independent merit. For a set \( A \) in a generic extension, the symbol \( V(A) \) denotes the smallest model of ZF containing \( V \) as a subset and \( A \).
Proposition 8.7.3. Let $a$ be a set. In some generic extension $V[G]$ of $V$ let $f$ be a $C$-function over $V$.

1. The theory of the model $(V(f), f, \in)$ with ground model parameters does not depend on $f$.

2. every element of $2^{\omega} \cap V(f)$ belongs to $V[b]$ for some finite set $b \subseteq \bigcup \text{rng}(f)$.

Proof. Work in $V$. Let $P$ be the finite support product of Cohen forcing indexed by the set $a$; i.e. $P$ is the poset of all partial finite functions $p$ from $a$ to $2^{<\omega}$ ordered by coordinatewise inclusion. Let $\tau$ be the $P$-name for a function with domain $a$ which attaches to each element of $a$ the $E_0$-class of its Cohen real. We will show that for every $p \in P$, in some generic extension there is a filter $H \subseteq P$ such that $p \in H$, $\tau / H = f$, and $V[G] \cap V[H] \cap 2^{\omega} = \bigcup \{2^{\omega} \cap V[b] : b \subseteq \bigcup \text{rng}(f) \}$ is finite. This will prove both items of the proposition as (for the first item) whenever $\phi(f, z)$ is a formula with a ground model parameter $z$, then $V(f) \models \phi(f, z)$ just in case $V \models P \Vdash V(\tau) \models \phi(\tau, \dot{z})$ and (for the second item) $V(f) \subset V[G] \cap V[H]$.

For the construction of the filter $f$, let $Q$ be the poset of finite functions $q$ with domain included in $a$ such that for all $y \in a$, $q(y) \in f(y)$ holds, and if $y \in \text{dom}(p)$ then $p(y) \subset q(y)$. The ordering on $Q$ is that of reverse inclusion. Let $\dot{H}$ be the $Q$-name for the set of all conditions $p' \in P$ such that for some condition $q'$ in the $Q$-generic filter, $\text{dom}(p') \subseteq \text{dom}(q')$ and for all $y \in \text{dom}(p')$, $p'(y) \subset q'(y)$. We will show that $Q \Vdash \dot{H} \subseteq P$ is a filter with the required properties.

First of all, let $q \in Q$ be a condition and $D \subseteq P$ be a dense set in the ground model. Since the set $q$ is product Cohen generic over $V$, there must be a condition $p' \in P$ which is in $D$, it is stronger than $p$, and such that for each $y \in \text{dom}(q)$, $q(y) \in p'(y)$. Let $q' \leq q$ be any condition such that $\text{dom}(p') \subseteq \text{dom}(q')$ and for each $y \in \text{dom}(q')$, $q'(y) \in p(y)$. (This is possible since $E_0$-classes are dense in $2^{\omega}$.) It is clear that $q' \Vdash p' \in \dot{H}$. By a density argument then, $Q \Vdash \dot{H}$ meets every dense subset of $P$ in $V$, and therefore it is a filter generic over $V$.

Second, suppose that $\sigma$ is a $P$-name for an element of $2^{\omega}$ in $V$ and $z \in 2^{\omega}$ is an element in $V[G] \setminus \bigcup \{2^{\omega} \cap V[b] : b \subseteq \bigcup \text{rng}(f) \}$ is finite; we must show that $Q \Vdash \tau / H \neq \dot{z}$. Let $q \in Q$ be an arbitrary condition. Since $z \notin V[q]$, there must be $i \in \omega$ and conditions $p_0, p_1 \in P$ such that for all $y \in \text{dom}(q)$, $y \in \text{dom}(p_0) \cap \text{dom}(p_1)$ and $p_0 \Vdash \tau_0(i) = 0$ and $p_1 \Vdash \tau_0(i) = 1$. Assume for definiteness that $z(i) = 0$. Let $q' \leq q$ be any condition in $Q$ such that

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as an element. The following definition and observation is standard; we include it since the next proposition elaborates and relies on it.

Definition 8.7.2. Let $V[G]$ be a generic extension of $V$. A function $f \in V[G]$ is a $C$-function over $V$ if its domain belongs to $V$, it is an injection into the $E_0$-quotient space, and every finite subset of $\bigcup \text{rng}(f)$ of pairwise $E_0$-inequivalent elements is product Cohen-generic over $V$.
dom(p_1) \subset \text{dom}(q') and for all \( y \in \text{dom}(p_1) \), \( p_1(y) \subset q'(y) \). It follows that \( q' \models (\tau/H)(i) = 1 \neq \check{z}(i) \), completing the proof by a density argument. \( \square \)

The following definition and proposition are key generalizations of the previous concepts.

**Definition 8.7.4.** Let \( V[G] \) be a generic extension of \( V \) and let \( \alpha \) be an ordinal. A set \( T \) is a *C-tower over \( V \) of height \( \alpha \) if \( T \) is a partial injective function from \( \mathcal{P}(2^\omega) \) to \( \mathbb{E}_0 \)-classes, every finite subset of \( \bigcup \text{rng}(T) \) of pairwise \( \mathbb{E}_0 \)-inequivalent elements is product Cohen-generic over \( V \), and the following recursive construction

1. \( M_0 = V; \)
2. for every ordinal \( \beta \leq \alpha \), \( M_{\beta+1} = V(T \upharpoonright M_\beta) \);
3. for every limit ordinal \( \beta \in \alpha \), \( M_\beta = V(T \upharpoonright \bigcup_{\gamma \in \beta} M_\gamma) \);

satisfies \( \text{dom}(T) = \bigcup_{\beta \in \alpha} (\mathcal{P}(2^\omega) \cap M_\beta) \).

**Proposition 8.7.5.** Let \( \alpha \) be an ordinal. In some generic extension \( V[G] \), let \( T \) be a C-tower over \( V \) of length \( \alpha \).

1. The theory of the model \( (V(T), T, \in) \) does not depend on \( T \);
2. every element of \( 2^\omega \cap V(T) \) belongs to \( V[b] \) for some finite set \( b \subset \bigcup \text{rng}(T) \).

**Proof.** Work in the ground model \( V \). By recursion on \( \beta \leq \alpha \), define the poset \( P_\beta \) and \( P_\beta \)-name \( \tau_\beta \). If \( \beta \) is a limit ordinal then \( P_\beta \) is the direct limit of the posets \( P_\gamma \) for \( \gamma \in \beta \) and \( \tau_\beta = \bigcup_{\gamma \in \beta} \tau_\gamma \). To handle the successor step, let \( \beta \in \alpha \) be an ordinal. Let \( u_\beta \) be the \( P_\beta \)-name for the collection of all subsets of \( 2^\omega \) in the model \( V(\tau_\beta) \) but not in \( \text{dom}(\tau_\beta) \). The poset \( P_{\beta+1} \) is the iteration \( P_\beta * \check{Q}_\beta \) where \( Q_\beta \) is forced to be the finite support product of Cohen forcings indexed by elements of \( u_\beta \), adding Cohen reals \( \check{x}_y \) for \( y \in u_\beta \). The name \( \tau_{\beta+1} \) is then forced to be equal to \( \tau_\beta \) united with the set \( \{ \langle y, [\check{x}_y]_{u_\beta} \rangle : y \in u_\beta \} \).

It is rather clear that \( P_\alpha \models \tau_\alpha \) is a C-tower of length \( \beta \) over the ground model \( V \). Now, let \( p \in P_\alpha \) be a condition. We will show that in some forcing extension of \( V[G] \), there is a filter \( H \subset P_\alpha \) containing \( P \) and such that \( \tau_\alpha / H = T \) and \( V[H] \cap V[G] \cap 2^\omega = \bigcup \{ V[b] \cap 2^\omega : b \subset \bigcup \text{rng}(T) \text{ is finite} \} \). This will prove both items of the proposition as (for the first item) \( V(T) \models \phi(T, z) \) for some ground model parameter \( z \) if and only if \( V \models P_\alpha \models V(\tau_\alpha) \models \phi(\tau_\alpha, z) \), and (for the second item) \( V(T) \subset V[G] \cap V[H] \).

To find the filter \( H \), we proceed by transfinite induction on \( \beta \leq \alpha \). For every such ordinal \( \beta \) we write \( M_\beta \) for the model from Definition 8.7.4, \( T_\beta = T \upharpoonright \bigcup_{\gamma \in \beta} M_\gamma \), and for every filter \( H \subset P_\beta \) we say that \( H \) is **good** if

- it is generic over \( V \) and \( \tau_\beta / H = T_\beta; \)
- \( 2^\omega \cap V[G] \cap V[H] = \bigcup \{ 2^\omega \cap V[b] : b \subset \bigcup \text{rng}(T_\beta) \text{ finite} \}; \)
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- whenever $b \subset \bigcup(\text{rng}(T) \setminus \text{rng}(T_\beta))$ is a finite set of pairwise $E_0$-unrelated elements, then it is product Cohen-generic over the model $V[H]$.

By induction on $\beta \leq \alpha$ argue that for every ordinal $\gamma \in \beta$, whenever $H_\gamma \subset P_\gamma$ is a good filter in some generic extension and $p \in P_\beta/H_\gamma$ is a condition, then in some further extension there is a filter $H_\beta \subset P_\beta$ generic over $V$ such that $H_\gamma \subset H_\beta$ and $p \in H_\beta/H_\gamma$. This is a routine argument at successor stages using Proposition 8.7.3; we omit the details.

Proof of Example 8.7.1. Let $p \in P$ be a condition; we must find a virtual weakly balanced condition $\hat{p} \leq p$. For the brevity of notation, we assume that $p = (0,0)$. Let $\hat{p} = \text{Coll}(\omega, 2^{\omega})$, $\hat{p} = (\hat{b})$ where $b$ is the set of all ground model coded closed nowhere dense subsets of $(2^{\omega})^n$ for all $n \in \omega$. It is immediate that $\hat{p}$ is a virtual condition; It will be enough to show that $\hat{p}$ is weakly balanced.

Let $R_0 \models \sigma_0 \leq \hat{p}$ and $R_1 \models \sigma_1 \leq \hat{p}$ are posets and their respective names for conditions stronger than $\hat{p}$. In an ambient forcing extensions, we must find filters $G_0 \subset R_0$ and $G_1 \subset R_1$ separately generic over $V$ such that $\sigma_0/G_0$ and $\sigma_1/G_1$ are conditions compatible in $P$.

**Claim 8.7.6.** We can assume that there is an ordinal $\alpha$ such that $R_0 \models$ there is a C-tower $\dot{T}_0$ of height $\alpha$ over $V$ with $T_0 = a_{\sigma_0} \upharpoonright V(T_0)$ and similarly for $R_1, \dot{T}_1, \sigma_1$.

Now, in some ambient forcing extension, find a C-tower $T$ of height $\alpha$ over the ground model $V$. Comparing $V(T)$ with $V(\dot{T}_0)$ and $V(\dot{T}_1)$, we can use Proposition 8.7.5(1) to conclude that in $V(T)$, there is a poset $Q_0$ and a $Q_0$-name $\dot{G}_0$ such that $Q_0 \models \dot{G}_0 \subset R_0$ is a filter generic over $V$ and $\sigma_0/G_0 \upharpoonright V(T) = T$ and similarly for $Q_1, \dot{G}_1, \sigma_1$. (To make this conclusion, in the $R_0$-extension consider the model $V(\dot{T}_0)$ and observe that the whole extension is a forcing extension of $V(\dot{T}_0)$ by ???.) Now, let $K_0 \subset Q_0$ and $K_1 \subset Q_1$ be filters mutually generic over $V(T)$, and let $G_0 = G_0/K_0$ and $G_1 = G_1/K_1$. We claim that $p_0 = \sigma_0/G_0$ and $p_1 = \sigma_1/G_1$ are conditions compatible in $P$ as desired.

To see this, first note that $a_{\sigma_0} \cup a_{\sigma_1}$ is indeed a function. To see this, use the initial assumptions on $\dot{T}_0$ and $\dot{T}_1$ and conclude that $a_{\sigma_0} \upharpoonright V(T) = a_{\sigma_1} \upharpoonright V(T) = T$. Then, if $y_0 \in \text{dom}(a_{\sigma_0}) \setminus V(T)$ and $y_1 \in \text{dom}(a_{\sigma_1}) \setminus V(T)$, the sets $y_0, y_1 \subset 2^{\omega}$ must be distinct by mutual genericity of the filters $K_0, K_1$ over the model $V(T)$.

Now, we need to see that if ???

The following poset introduced in [51] adds a maximal disjoint family, a task impossible to perform with balanced forcing by Theorem 14.1.1.

**Example 8.7.7.** The MAD forcing $P$ consists of all pairs $p = (a_p, b_p)$ such that $a_p \subset [\omega]^{\aleph_0}$ is an infinite countable almost disjoint family, and $b_p$ is a countable set of pairs $(s, a)$ such that $s$ is a partition of $\omega$ into finite sets and $a \subset A$ is a countable set. Moreover, for every pair $(s, a) \in B$ and every finite set $d \subset a_p \setminus a$, there are infinitely many sets $e \in s$ such that $\bigcup d \cap e = 0$. The set $P$ is ordered by coordinatewise reverse inclusion. Then $P$ is a weakly balanced $\sigma$-closed Suslin forcing.
Proof. It is not difficult to see that \( P \) is a Suslin partial ordering, and \( P \) is \( \sigma \)-closed. Towards the weak balance, suppose that \( p \in P \) is a condition, and write \( b = b_p \cup \{s, a_p\} : s \) is a ground model partition of \( \omega \) into finite sets. It is clear that for any poset \( Q \) collapsing the size of the continuum to \( \aleph_0 \), \( Q \models \langle \hat{a}_p, b \rangle \) is a condition in \( P \) stronger than \( p \). It will be enough to show that the pair \( \langle Q, \langle \hat{a}_p, b \rangle \rangle \) is weakly balanced in \( P \).

To this end, suppose that \( R_0, R_1 \) are partial orders and \( \sigma_0, \sigma_1 \) are respective \( R_0 \)- and \( R_1 \)-names for conditions in \( P \) extending \( \langle a_p, b \rangle \); We must produce, in some generic extension, filters \( H_0 \subset R_0 \) and \( H_1 \subset R_1 \) separately generic over the ground model such that the conditions \( \sigma_0 / H_0 \) and \( \sigma_1 / H_1 \) are compatible in \( P \). The following claim is key.

**Claim 8.7.8.** For every \( i \in \mathbb{2} \), the following holds. Whenever \( r \in R_i \) is a condition and \( \eta_j : j \in m \) are names for elements of \( a_{\sigma_i} \setminus a_p \), then there is \( n \in \omega \) such that for every \( m \in \omega \) there is a strengthening of \( r \) forcing \( \bigcup_{j \in m} \eta_j \cap [n, m) = 0 \).

*Proof.* For definiteness let \( i = 0 \). Suppose towards contradiction that the claim fails for a condition \( r \in R_0 \) and names \( \eta_j \) for \( j \in m \). Then, there is a partition \( s \) of \( \omega \) into finitely many sets such that \( p \Vdash \bigcup_{j \in m} \eta_j \cap b \neq 0 \) for every \( b \in s \). This shows that \( r \Vdash \sigma_0 \not\leq (a_p, b) \), contradicting the initial assumptions. \( \square \)

Let \( V[K] \) be a generic extension collapsing a sufficiently large cardinal and work in \( V[K] \). An inductive application of the claim makes it possible to find filters \( H_0 \subset R_0 \) and \( H_1 \subset R_1 \) separately generic over the ground model, and a partition \( \omega = \bigcup c_j \) of \( \omega \) into finite sets such that, writing \( \sigma_0 / H_0 = \langle a_0, b_0 \rangle \) and \( \sigma_1 / H_1 = \langle a_1, b_1 \rangle \), we have:

- for every \( x \in a_0 \setminus a_p \), \( x \in \bigcup \{c_j : j \text{ even}\} \) up to finitely many exceptions and similarly for every \( x \in a_1 \setminus a_p \), \( x \in \bigcup \{c_j : j \text{ odd}\} \) up to finitely many exceptions;
- for every infinite collection \( s \in V[H_0] \) of pairwise disjoint subsets of \( \omega \), there is \( e \in s \) which is a subset of some \( c_j \) for \( j \) odd, and similarly for every infinite collection \( s \in V[H_1] \) of pairwise disjoint subsets of \( \omega \), there is \( e \in s \) which is a subset of some \( c_j \) for \( j \) even.

We claim that the filters work as required. This is clearly the same as showing that the pair \( \langle a_0 \cup a_1, b_0 \cup b_1 \rangle \) belongs to \( P \), since then it is a lower bound of both \( \langle a_0, b_0 \rangle \) and \( \langle a_1, b_1 \rangle \). First of all, the first item above immediately shows that \( a_0 \cup a_1 \) is an almost disjoint family. To confirm the “moreover” demand in the definition of the poset \( P \) for the pair \( \langle a_0 \cup a_1, b_0 \cup b_1 \rangle \), suppose that \( \langle s, a \rangle \in b_0 \cup b_1 \); for definiteness assume that \( \langle s, a \rangle \in b_0 \). Let \( d \subset (a_0 \cup a_1) \setminus a \) be a finite set; we must produce infinitely many \( e \in s \) such that \( \bigcup d \cap e = 0 \).

Write \( d_0 = d \cap a_0 \) and \( d_1 = d \setminus d_0 \). Since the pair \( \langle a_0, b_0 \rangle \) is a condition in the poset \( P \), there are infinitely many sets \( e \in s \) such that \( \bigcup d_0 \cap e = 0 \). By the second item above, there must be infinitely many \( e \in s \) such that \( \bigcup d_0 \cap e = 0 \) and \( e \subset c_j \) for some odd number \( j \in \omega \). By the first item above, there must be
infinitely many sets $e$ among them such that $\bigcup d_1 \cap e = 0$. For all such $e \in s$, it is the case that $\bigcup d \cap e = 0$. This completes the proof.

A straightforward genericity argument shows that if $G \subset P$ is a generic filter then $\bigcup_{p \in G} a_p$ is a maximal almost disjoint family. The definition of the MAD forcing may seem odd, since it skips the most obvious choice: the poset $Q$ of all infinite countable almost disjoint families ordered by reverse inclusion. We do not know if the poset $Q$ is weakly balanced.
Chapter 9

Preserving cardinalities

9.1 The well-ordered divide

The main feature of the balanced Suslin forcing is that it does not add any well-ordered sequences of elements of the symmetric Solovay extension. This has a number of cardinality corollaries for the resulting extensions.

**Theorem 9.1.1.** In a cofinally balanced extension of a symmetric Solovay model \( W \), every well-ordered sequence of elements of \( W \) belongs to \( W \).

**Proof.** Let \( P \) be a Suslin forcing and \( \kappa \) be an inaccessible cardinal such that \( P \) is balanced cofinally below \( \kappa \). Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in \( W \). For a formula \( \phi \) of the language of set theory, element \( y \in 2^\omega \), and a set \( v \in V \), write \( u(\phi, y, v) \) for the unique set \( x \) such that \( W \models \phi(x, y, v) \) if such a unique \( x \) exists; otherwise, write \( u(\phi, y, v) = 0 \). Suppose towards a contradiction that there is a condition \( p \in P \), an ordinal \( \alpha \) and a \( P \)-name \( \tau \) such that \( p \Vdash \tau \) is an \( \alpha \)-sequence of elements of \( W \) which does not belong to \( W \). The name \( \tau \) is definable from some real parameter \( z \in 2^\omega \) and some elements of the ground model. Find an intermediate model \( V[K] \) which is an extension of the ground model by a poset of size \( < \kappa \) such that \( p, z \in V[K] \) and \( P \) is balanced in \( V[K] \).

Work in \( V[K] \). Let \( \bar{p} \) be a balanced virtual condition in \( P \) below \( p \). Since in \( W \), \( \tau \) is forced not to belong to \( W \), there must be in \( V[K] \) an ordinal \( \beta \in \alpha \) and a posets \( R \) of size \( < \kappa \), \( R \)-names \( \sigma_0, \sigma_1 \) for conditions in \( P \) stronger than \( \bar{p} \), \( R \)-names \( \eta_0, \eta_1 \) for elements of \( 2^\omega \), ground models elements \( v_0, v_1 \), and formulas \( \phi_0, \phi_1 \) such that \( R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \) the following:

- \( \sigma_0 \Vdash_{p} \tau(\bar{\beta}) = u(\phi_0, \eta_0, v_0) \);
- \( \sigma_1 \Vdash_{p} \tau(\bar{\beta}) = u(\phi_1, \eta_1, v_1) \);
- \( u(\phi_0, \eta_0, v_0) \neq u(\phi_1, \eta_1, v_1) \).
Working in the model $W$, let $H_0, H_1 \subset R$ be mutually generic filters, and for bits $b, c \in 2$ let $p_{bc} = \sigma_b/H_c \in P$ and $y_{bc} = \eta_b/H_c \in 2^\omega$. By the third item above, in the model $V[K][H_0, H_1]$ there must be a bit $c \in 2$ such that $\text{Coll}(\omega, < \kappa) \forces u(\phi_0, y_{00}, v_0) \neq u(\phi_1, y_{1c}, v_1)$. The conditions $p_{00}, p_{1c} \in P$ are compatible in $W$ by the balance of the condition $\bar{p}$, with a lower bound $q \in P$. Since $W$ is the symmetric Solovay extension of the model $V[K][H_0]$, the forcing theorem applied with that model yields that $q \forces \tau(\bar{\beta}) = u(\phi_0, y_0, v_0)$. Since $W$ is the symmetric Solovay extension of $V[K][H_1]$, the forcing theorem applied with $V[K][H_1]$ yields that $q \forces \tau(\bar{\beta}) = u(\phi_1, y_{1c}, v_1)$. Finally, since $W$ is the symmetric Solovay extension of $V[K][H_0][H_1]$, $u(\phi_0, y_{00}, v_0) \neq u(\phi_1, y_{1c}, v_1)$ holds in $W$. Thus, the condition $q$ forces two distinct values to the name $\tau(\bar{\beta})$, an impossibility.

**Corollary 9.1.2.** In a cofinally balanced extension of a symmetric Solovay model, there is no transfinite uncountable sequences of pairwise distinct Borel sets of bounded rank.

**Proof.** Since cofinally balanced extensions add no countable sequences of elements of the Solovay model $W$ by Theorem 9.1.1, all Borel sets in $W[G]$ belong to $W$ and have the same Borel rank there as in $W[G]$. Thus, an uncountable sequence of distinct Borel sets of bounded Borel rank in $W[G]$ would have to belong to $W$ by Theorem 9.1.1 again. However, there are no such sequences in the Solovay model $W$ by a result of Stern [73].

**Corollary 9.1.3.** Let $E$ be a Borel equivalence relation on a Polish space $X$. In a cofinally balanced extension of a symmetric Solovay model, $\aleph_1 \not\leq |E|$ holds.

**Proof.** An $\omega_1$-sequence of distinct $E$-classes would constitute an $\omega_1$-sequence of distinct Borel sets of Borel rank bounded by the rank of $E$. Such sequences are ruled out by the previous corollary.

Corollary 9.1.3 guarantees that in balanced extensions of the symmetric Solovay model, the Friedman–Stanley jump divide is preserved. This follows from a humble ZF result of independent interest.

**Proposition 9.1.4.** (ZF) Let $X$ be a set and $E$ an equivalence relation on $X$ with all classes countable. If $|X^{<\omega}| \leq |E|$ then $|\mathbb{R}| \leq |E|$.

**Proof.** Let $g$ be an injection from the set of all countable subsets of $X$ to $E$-classes. Define a function $h$ from the collection of hereditarily countable sets to $E$-classes by $\varepsilon$-recursion: $h(a) = g(\bigcup h''a)$. By induction on the minimum of the rank $a$ and $b$ argue that $a \neq b$ implies that $h(a) \neq h(b)$. Thus the function $h$ is an injection and the statement of the proposition follows.

**Corollary 9.1.5.** Let $E$ be a Borel equivalence relation on a Polish space $X$. In a cofinally balanced extension of a symmetric Solovay model, $|E^+| \not\leq |E|$ holds.
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Proof. Suppose towards contradiction that \(|E^+| \leq |E|\) holds in the extension. The Proposition yields \(|HC| \leq |E|\). Clearly, \(\aleph_1 \leq |HC|\) holds in ZF, and the concatenation of the cardinal inequalities contradicts the conclusion of Corollary 9.1.3.

If one wishes to add well-ordered sequences of some objects in the Solovay model and maintain control, it is possible to use weakly balanced forcing. The following theorem, stated in terms of Convention 1.8.17, shows that there will be no uncountable sequences of reals in the resulting extension.

**Theorem 9.1.6.** In weakly balanced extensions of the symmetric Solovay model, every set of ordinals belongs to the Solovay model.

**Proof.** Let \(\kappa\) be an inaccessible cardinal. Let \(P\) be a Suslin poset such that \(V_\kappa \models P\) is weakly balanced in all forcing extensions. Let \(W\) be the symmetric Solovay model derived from \(\kappa\) and work in the model \(W\). Let \(p \in P\) be a partial order and let \(\tau\) be a \(P\)-name for a set of ordinals; we have to find a strengthening of the condition \(p\) which decides the membership of every ordinal in the set \(\tau\). To this end, note that both \(p, \tau\) are definable from a parameter \(z \in 2^{\omega}\) and some parameters in the ground model.

Let \(V[K]\) be an intermediate generic extension obtained by a poset of size \(< \kappa\) such that \(z \in V[K]\). Working in the model \(V[K]\), let \(\bar{p}\) be a weakly balanced virtual condition in the poset \(P\), below the condition \(p\). We claim that for every ordinal \(\alpha\), it is the case that \(Coll(\omega, < \kappa) \models \bar{p}\) decides in \(P\) the membership of the ordinal \(\check{\alpha}\) in the set \(\tau\). Suppose towards contradiction that this is not the case. Then there must be an ordinal \(\alpha\), posets \(R_0, R_1\) of size \(< \kappa\), and \(R_0\)- and \(R_1\)-names \(\sigma_0\) and \(\sigma_1\) for elements of \(p\) stronger than \(\bar{p}\) such that \(R_0 \models Coll(\omega, < \kappa) \models \sigma_0 \models \check{\alpha} \notin \tau\) and \(R_1 \models Coll(\omega, < \kappa) \models \sigma_1 \models \check{\alpha} \in \tau\).

Work in \(W\) again. Use the weak balance of \(\bar{p}\) to find filters \(H_0 \subsetneq R_0\) and \(H_1 \subsetneq R_1\) separately generic over \(V[K]\) such that the conditions \(p_0 = \sigma_0/H_0\) and \(p_1 = \sigma_1/H_1\) are compatible in \(P\). Since \(W\) is a symmetric Solovay extension of both models \(V[K][H_0]\) and \(V[K][H_1]\), \(p_0 \models \check{\alpha} \notin \tau\) and \(p_1 \check{\alpha} \in \tau\). Thus, the common lower bound of \(p_0, p_1\) must force two contradictory statements, which is impossible.

9.2 The smooth divide

A basic result in the theory of analytic equivalence relations says that there is no Borel reduction from \(E_0\) to the identity on \(2^\omega\). A basic concern of our theory then is when \(|E_0| > |2^\omega|\) holds in the balanced extensions of the symmetric Solovay model. The take home lesson of this section is that if \(P\) is a balanced Suslin poset whose balanced conditions are naturally organized into a compact Hausdorff space, then in the \(P\)-extension of the symmetric Solovay model, \(|E_0| > |2^\omega|\) indeed holds. There are other ways to preserve the inequality \(|E_0| > |2^\omega|\) discussed in Chapter 11, but the compactness arguments are the most elegant and cover a lot of ground.
Definition 9.2.1. A Suslin poset $P$ is **compactly balanced** if there is a definable compact Hausdorff topology $T$ on the set $B$ of all equivalence classes of balanced virtual conditions such that

1. for every $p \in P$ the set $\{ \bar{p} \in B : \bar{p} \leq p \} \subset B$ is nonempty and $T$-closed.

Moreover, if $V[H_0] \subset V[H_1]$ are generic extensions of $V$ then

2. for every balanced virtual condition $\bar{p}_0 \in V[H_0]$ there is a balanced virtual condition $\bar{p}_1 \in V[H_1]$ such that $\bar{p}_1 \leq \bar{p}_0$ holds;

3. the relation $\{(\bar{p}_0, \bar{p}_1) \in B^{V[H_0]} \times B^{V[H_1]} : \bar{p}_1 \leq \bar{p}_0\}$ is closed in $T^{V[H_0]} \times T^{V[H_1]}$.

The definition seems to be obscure at the first reading, but it is fully justified by the natural compact Hausdorff topologies present in the examples. One rather (intentionally) unclear feature is the definability of the topology $T$. It is central in that otherwise the definition would have no content and the key item (3) could not be stated without it. The definition is allowed to have real parameters. However, the nature of the definition of $T$ is left undetermined. In the examples, the topology $T$ is typically nonseparable, and the compact Hausdorff space $(B, T)$ can be in principle very complicated. The following theorem is the main result of this section. Its statement uses Convention 1.8.17.

**Theorem 9.2.2.** In compactly balanced extensions of the symmetric Solovay model, $|E_0| > |2^\omega|$ holds.

The proof uses a technical tool which has been studied in its own right: the Vitali forcing, also called $E_0$-forcing in [39, Section 10.9] or in [85, Section 4.7.1], where it was isolated for the first time. To define it, consider the $\sigma$-ideal $I$ on $2^\omega$ generated by Borel partial $E_0$-transversals, and let $Q$ be the poset of Borel $I$-positive sets ordered by inclusion. As every poset of this form, $Q$ adds a generic point in $2^\omega$—the unique point which belongs to all Borel sets in the generic filter [85, Proposition 2.1.2].

**Fact 9.2.3.** Let $Q$ be the Vitali forcing and let $G \subset Q$ be a generic filter. Then

1. [85, Theorem 4.7.3] every real in $V[G]$ is the image of the generic point $x_{gen}$ under a ground model Borel function ($Q$ is proper);

2. [71, Proposition 4.5] for every set $a \subset \omega$ in $V[G]$, either $a$ or its complement contain a ground model infinite set ($Q$ adds no independent reals);

3. [87, Theorem 2.11] every real definable in $V[G]$ from $[x_{gen}]_{E_0}$ and ground model parameters belongs to the ground model.

**Proof of Theorem 9.2.2.** Let $\kappa$ be an inaccessible cardinal. Suppose that $P$ is a Suslin forcing such that $V_\kappa \models$ in every forcing extension, $P$ is compactly balanced. Let $W$ be a symmetric Solovay model derived from $\kappa$ and work in $W$. 

Towards contradiction, assume that \( p \in P \) is a condition and \( \tau \) is a \( P \)-name for an injection from the \( \mathbb{E}_0 \)-classes to \( 2^\omega \). Both \( p, \tau \) are definable from some real parameter \( z \in 2^\omega \). Let \( V[K] \) be an intermediate generic extension of \( V \) obtained by a forcing of size \( < \kappa \) such that \( z \in V[K] \) holds, and work in the model \( V[K] \).

Let \( Q_0 \) be the poset of infinite subsets of \( \omega \), ordered by inclusion. Let \( Q_1 \) be the Vitali forcing of Borel subsets of \( 2^\omega \) on which \( \mathbb{E}_0 \) is not smooth, ordered by inclusion. Let \( \langle U, y \rangle \) be an object \( Q_0 \times Q_1 \)-generic over the model \( V[K] \); that is, \( U \) is an ultrafilter on \( \omega \) and \( y \in 2^\omega \) is a point. By the \( \sigma \)-closure of \( Q_0 \), the models \( V[K] \) and \( V[K][U] \) have the same reals, in particular they evaluate the definition of the Vitali forcing in the same way and so \( y \) is a Vitali-generic point over \( V[K][U] \) by the product forcing theorem. By Fact 9.2.3(2), and a genericity argument, the ultrafilter \( U \) still generates an ultrafilter on \( \omega \) in the model \( V[K][U][y] \).

Now, work in the model \( V[K][U] \). Let \( \bar{p}_0 \leq p \) be a balanced virtual condition below \( p \). Let also \( \chi \) be any \( Q_1 \)-name for a balanced virtual condition in \( P \) in the model \( V[K][U][y] \) below \( \bar{p}_0 \). This is possible by (2) of Definition 9.2.1. Note that the model \( V[K][U][y] \) has more reals than \( V[K][U] \); thus, the balance of the condition \( \bar{p}_0 \) does not necessarily transfer from \( V[K][U] \) to \( V[K][U][y] \) and \( \bar{p}_0 \) may have to be improved in a nontrivial way to get a balanced condition in \( V[K][U][y] \). For each \( n \in \omega \), let \( y_n \in 2^\omega \) be the binary sequence obtained from \( y \) by replacing its first \( n \) entries by 0. Note that for each \( n \in \omega \), \( y_n \) is still \( Q_1 \)-generic over \( V[K][U] \), and \( V[K][U][y] = V[K][U][y_n] \). In the model \( V[K][U][y] \), let \( \bar{p}_1 \) be the \( U \)-limit of the sequence \( \chi/y_n \) for \( n \in \omega \). By (3) of Definition 9.2.1, it is the case that \( \bar{p}_1 \leq \bar{p}_0 \leq p \). Note also that the definition of \( \bar{p}_1 \) does not depend on \( y \) per se, but only on the \( \mathbb{E}_0 \)-class of \( y \). The treatment divides into two cases according to the fate of the value \( \tau([y]_{\mathbb{E}_0}) \) as forced by \( \bar{p}_1 \), and a contradiction is reached in each case.

**Case 1.** In the model \( V[K][U][y] \), there is no point \( u \in 2^\omega \) such that \( \text{Coll}(\omega, \kappa) \models \bar{p}_1 \models p \tau([y]_{\mathbb{E}_0}) = \bar{u} \). In such a case, there have to be disjoint basic open subsets \( O_0, O_1 \) of \( 2^\omega \), posets \( R_0, R_1 \) of size \( < \kappa \), and respective \( R_0, R_1 \)-names \( \sigma_0, \sigma_1 \) for conditions in \( P \) stronger than \( \bar{p} \) such that \( R_0 \models \text{Coll}(\omega, \kappa) \models \sigma \models p \tau([y]_{\mathbb{E}_0}) \in O_0 \), and similarly for subscript 1. In the model \( W \), let \( H_0 \subset R_0 \) and \( H_1 \subset R_1 \) be filters mutually generic over the model \( V[K][U][y] \) and consider the conditions \( \sigma_0/H_0, \sigma_1/H_1 \in P \). By the balance of the condition \( \bar{p}_1 \), the conditions \( \sigma_0/H_0, \sigma_1/H_1 \) are compatible. The forcing theorem applied in both models \( V[K][U][y][H_0] \) and \( V[K][U][y][H_1] \) now says that in \( W \), their lower bound forces \( \tau([y]_{\mathbb{E}_0}) \) to belong simultaneously to \( O_0 \) and to \( O_1 \), an impossibility.

**Case 2.** There is a point \( u \in 2^\omega \) in \( V[K][U][y] \) such that \( \text{Coll}(\omega, \kappa) \models \bar{p}_1 \models p \tau([y]_{\mathbb{E}_0}) = \bar{u} \). This point \( u \) is clearly unique, and since \( \bar{p}_1 \) is definable from \( [y]_{\mathbb{E}_0} \), so is \( u \). By Fact 9.2.3(3), \( u \in V[K][U] \) holds. Working in \( V[K][U] \), let \( q \in Q_1 \) be a condition forcing \( \text{Coll}(\omega, \kappa) \models \bar{p}_1 \models p \tau([x_{\text{gen}}]_{\mathbb{E}_0}) = \bar{u} \), where \( x_{\text{gen}} \) is the \( Q_1 \)-name for its generic point. Let \( H_0, H_1 \subset Q \) be filters mutually generic over \( V[K][U] \) containing the condition \( q \). Let \( y^0, y^1 \in 2^\omega \) be the respective generic points; observe that \( y^0, y^1 \) are not \( \mathbb{E}_0 \)-related by mutual genericity. Write also \( p^0 = \bar{p}_1/H_0 \) and \( p^1 = \bar{p}_1/H_1 \). These are two virtual conditions in \( P \) strengthening \( \bar{p}_0 \). By the balance of \( \bar{p}_0 \), they are compatible in the poset \( P \).
Then, in the model $W$, their lower bound forces that $\tau([\check{y}^0]_{E_0}) = \tau([\check{y}^1]_{E_0}) = \check{u}$. This contradicts the assumption that $\tau$ was forced to be an injection.

**Example 9.2.4.** Let $P$ be the poset of $P(\omega)$ modulo finite. By Theorem 7.1.2, the balanced virtual conditions are classified by nonprincipal ultrafilters on $\omega$. Equip the set of balanced classes by the topology of the remainder $\beta\omega$. It is immediate that the topology witnesses the fact that $P$ is compactly balanced.

**Corollary 9.2.5.** 1. Let $P$ be the poset $P(\omega)$ modulo finite. In the $P$-extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, there is a Ramsey ultrafilter on $\omega$, and yet $|E_0| > |2^\omega|$.

**Example 9.2.6.** Let $A$ be a Ramsey sequence of finite structures and $P_A$ the associated $\sigma$-closed poset as in Definition 7.2.1. Theorem 7.2.4 provides a classification of the balanced virtual conditions: they correspond to Ramsey sequences of ultrafilters on the respective domain sets $D_n$ for $n \in \omega$. Sequences of ultrafilters of this type form a closed subset of the product $\prod_n D_n^*$ of the respective compact sequences of ultrafilters, and so form a compact Hausdorff space. Claim 7.2.6 confirms the extension property Definition 9.2.1(2). It follows that the poset $P_A$ is compactly balanced.

The conjunction of Examples 9.2.6 and 7.2.8 now yields the following:

**Corollary 9.2.7.** Let $k \in \omega$ be a number.

1. Let $P$ be the poset $P_A$ where $A$ is a sequence enumerating all finite ordered graphs with no clique of size $k$. In the $P$-extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, there is an ultrafilter $U$ on $\omega$ such that $U \rightarrow (U,k)^2$ and $U \not\rightarrow (U,k+1)^2$, and yet $|E_0| > |2^\omega|$.

**Example 9.2.8.** Let $P$ be the poset of subsets of $\omega \times \omega$ with infinitely many infinite vertical sections. The poset $P$ is compactly balanced.

**Proof.** The balanced virtual conditions of $P$ were classified in Theorem 7.3.2. They correspond to ultrafilters on $\omega \times \omega^*$ which do not contain the set $n \times \omega^*$ for any number $n \in \omega$. Thus, the space of balanced virtual conditions is naturally organized into a definable compact Hausdorff space. It is necessary to verify the conditions (2, 3) of Definition 9.2.1. Let $V[H_0] \subset V[H_1]$ be generic extensions.

We first evaluate the complexity of the order between the balanced virtual conditions in $V[H_0]$ and $V[H_1]$. Consider the partial map $f : (\omega \times \omega^*)^{V[H_1]} \rightarrow (\omega \times \omega^*)^{V[H_0]}$ defined by $f(n,U) = (n,U \cap V[H_0])$; if $U \cap V[H_0] \notin V[H_0]$, the functional value is left undefined. Unraveling the definitions, a balanced virtual condition in $V[H_0]$ corresponding to an ultrafilter $W_0$ on $\omega \times \omega^*$ is weaker than the balanced virtual condition in $V[H_1]$ corresponding to some ultrafilter $W_1$.
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on $\omega \times \omega^*$ just in case for every set $A \in W_0$, the set $f^{-1}A$ belongs to $W_1$. In view of the topology on the ultrafilter spaces, this is a closed relation.

To see that every balanced virtual condition in $V[H_0]$ can be extended to a balanced virtual condition in $V[H_1]$, suppose that $W_0$ is an ultrafilter on $\omega \times \omega^*$ in the model $V[H_0]$. Considering the function $f$ from the previous paragraph, it is obvious that the collection $\{f^{-1}A : A \in W_0\}$ of subsets of $\omega \times \omega^*$ in the model $V[H_1]$ has the finite intersection property, and therefore can be extended to an ultrafilter $W_1$. The ultrafilter $W_1$ corresponds to a balanced virtual condition in the model $V[H_1]$ stronger than the one corresponding to $W_0$. \hfill \Box

Corollary 9.2.9. 1. Let $P$ be the poset of Example 9.2.8. In the $P$-extension of the symmetric Solovay model, $|E_0| > 2^\omega$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is an ultrafilter $U$ on $\omega$ such that ???, and yet $|E_0| > 2^\omega$.

Question 9.2.10. Is there a Borel filter $F$ on $\omega$ such that ZF+DC proves that existence of a nonprincipal ultrafilter extending $F$ implies $|E_0| \leq |2^\omega|$?

Example 9.2.11. Let $F$ be a Fraissé class in a finite relational language with strong amalgamation. Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $P(F, E)$ be the poset of Definition 8.5.3 for adding a $F$-structure to the quotient space $X/E$. Then $P(F, E)$ is compactly balanced: the balanced conditions are classified by $F$-structures on the virtual quotient space $X^{**}$, which form a compact subspace of $P([X^{**}]^n)$ where $n$ corresponds to the arity of the relations in the language for $F$. It is easy to verify that the demands of Definition 9.2.1 are satisfied with the inherited topology.

Corollary 9.2.12. Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the linearization poset for $E$ of Example 8.5.5. In the $P$-extension of the symmetric Solovay model, $|E_0| > 2^\omega$;

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, the quotient space $X/E$ can be linearly ordered, and yet $|E_0| > 2^\omega$.

Another class of examples stems from posets which select a single structure on each $E$-class from a compact class of structures, where $E$ is a countable Borel equivalence relation.

Example 9.2.13. Let $X$ be a Polish space and let $\Gamma$ be a locally finite bipartite Borel graph on $X$ satisfying the Hall’s marriage condition. Consider the poset $P$ adding a perfect $\Gamma$-matching as in Example 6.2.4. The balanced virtual conditions are classified by perfect matchings. The set of perfect matchings is naturally viewed as a compact subset of $P(\Gamma)$. It is not difficult to check that the inherited compact topology satisfies the demands of Definition 9.2.1.

Corollary 9.2.14. Let $X$ be a Polish space and let $\Gamma$ be a locally finite bipartite Borel graph on $X$ satisfying Hall’s marriage condition.
1. Let $P$ be the complete matching poset for $\Gamma$. In the $P$-extension of the symmetric Solovay model, $|E_0| > |2^\omega|$.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $\Gamma$ has a perfect matching, and yet $|E_0| > |2^\omega|$.

**Example 9.2.15.** Let $X$ be a Polish space and let $\Gamma$ be a Borel, acyclic, locally finite graph on $X$ in which every vertex has degree at least 2; view $\Gamma$ as a symmetric subset of $X^2$. Consider the poset $P$ of Example 6.2.5, adding an orientation of $\Gamma$ in which every vertex has exactly one point in its outflow, or in other words which selects an end to each connected component of $\Gamma$. The balanced conditions are classified by all such orientations, which form a compact subset of $P(\Gamma)$. It is not difficult to check that the inherited compact topology satisfies the demands of Definition 9.2.1.

This example is somewhat singular in that it is the only compactly balanced poset for which we are able to confirm that it introduces some new cardinal inequalities between quotient cardinals. Namely, writing $E$ for the $\Gamma$-path-connectedness equivalence relation, in the $P$-extension of the symmetric Solovay model $|E| \leq |E_0|$ must hold. To see this, consider the equivalence relation $F$ on the set $D \subset X^\omega$ of all sequences consisting of pairwise $E$-equivalent points, making two such sequences $\vec{x}_0, \vec{x}_1$ $F$-equivalent if some tail of $\vec{x}_0$ is equal to some tail of $\vec{x}_1$. It is not difficult to see that $F$ is a hypersmooth equivalence relation [39, Theorem 8.3.1] with all classes countable and so by a standard result [39, Theorem 8.1.1], $F$ is Borel reducible to $E_0$. Consider the $P$-generic orientation $\vec{\Gamma}$ of the graph $\Gamma$: each vertex has exactly one point in its outflow. For each point $x \in X$ let $h(x) = \langle x_i : i \in \omega \rangle$ where $x_0 = x$ and $x_{i+1}$ is the unique point in the $\vec{\Gamma}$-outflow of $x_i$. Then $h$ is a (non-Borel) reduction of $E$ to $F$ and so in the $P$-extension $|E| \leq |F| \leq |E_0|$ holds.

**Corollary 9.2.16.** Let $X$ be a Polish space and let $\Gamma$ be a Borel, acyclic, locally finite bipartite Borel graph on $X$ in which every vertex has degree at least 2.

1. Let $P$ be the poset selecting an end to each connected component of $\Gamma$. In the $P$-extension of the symmetric Solovay model, $|E_0| > |2^\omega|$.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a function assigning exactly one end to each connected component of $\Gamma$, and yet $|E_0| > |2^\omega|$.

**Example 9.2.17.** Let $n \in \omega$ be a number, let $X$ be a Polish space and let $\Gamma$ be a locally countable Borel graph on $X$ such that every finite subgraph of $\Gamma$ has chromatic number $\leq n$. Consider the poset $P$ adding a $\Gamma$-coloring by $n$ colors as described in Example 6.2.6. The balanced virtual conditions are classified by total $\Gamma$-colorings by $\leq n$-colors; these form a compact subset of $n^X$. It is not difficult to see that the inherited compact product topology satisfies the demands of Definition 9.2.1.
Corollary 9.2.18. Let $X$ be a Polish space and let $\Gamma$ be a locally countable Borel graph on $X$ such that each finite subset of $\Gamma$ has chromatic number $\leq n$ for some fixed number $n \in \omega$.

1. Let $P$ be the coloring poset for $\Gamma$. In the $P$ extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $\Gamma$ has chromatic number $\leq n$, and yet $|E_0| > |2^\omega|$.

Adding finite coloring to a Borel graph which is not locally countable is a more complicated task; we offer the following.

Example 9.2.19. Let $\Gamma$ be a Borel graph on a Polish space $X$ such that the coloring number of $\Gamma$ is $\leq n \in \omega$. Consider the poset $P = P_{3n-1}(\Gamma)$ adding a $\Gamma$-coloring of the space $X$ by $3n - 1$ many colors as in Definition 8.1.1. The balanced conditions of $P$ are classified by the $\Gamma$-colorings of $X$ by $k$ colors by Theorem 8.1.2. They form a compact subset of the space $k^X$ in the product topology. It is not difficult to check that the demands of Definition 9.2.2 are met. The extension item (2) is handled by an argument similar to those in the proof of Theorem 8.1.2: whenever $V[H_0] \subset V[H_1]$ are transitive models of ZFC, then in $V[H_1]$ any $\Gamma$-coloring of $X \cap V[H_0]$ by $3n - 1$ colors can be extended to a coloring of $X \cap V[H_1]$ by $3n - 1$ colors.

Corollary 9.2.20. Let $X$ be a Polish space and let $\Gamma$ be a Borel graph on $X$ of coloring number $\leq n$.

1. Let $P$ be the coloring poset for $\Gamma$ by $3n-1$-many colors. In the $P$ extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $\Gamma$ has chromatic number $\leq 3n - 1$, and yet $|E_0| > |2^\omega|$.

Question 9.2.21. Is there a Borel graph $G$ on a Polish space $X$ such that each finite subgraph of $G$ has chromatic number $\leq n$, and ZF+DC proves that if $G$ has finite chromatic number then $|E_0| \leq |2^\omega|$?

Example 9.2.22. Let $\Gamma, \Delta$ be abelian Polish groups, with $\Delta$ divisible and compact. Let $P(\Gamma, \Delta)$ be the poset adding a homomorphism from $\Gamma$ to $\Delta$ as isolated in Definition 8.2.1. As proved in Theorem 8.2.2, the balanced conditions are classified by homomorphisms from $\Gamma$ to $\Delta$. The space of homomorphisms is a closed subset of $\Delta^\Gamma$ equipped with the product topology. It is not difficult to see that the demands of Definition 9.2.1 are met. Let us elaborate on the extension property (2). If $h: \Gamma \rightarrow \Delta$ is a homomorphism in some generic extension $V[H_0]$ then it can be extended to a homomorphism in any larger forcing extension $V[H_1]$ by the divisibility of $\Delta$ and Baer’s criterion [3]. Note that the group $\Delta$ remains abelian and divisible in all generic extensions by Mostowski absoluteness.
Corollary 9.2.23. Let $\Gamma, \Delta$ be Polish abelian groups, with $\Gamma$ uncountable and torsion free, and $\Delta$ divisible and compact.

1. In the $P(\Gamma, \Delta)$ extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a discontinuous homomorphism from $\Gamma$ to $\Delta$, and yet $|E_0| > |2^\omega|$.

Proof. Recal from Example 8.2.3 that the generic homomorphism added by the poset $P(\Gamma, \Delta)$ is discontinuous. Other than that, the corollary follows from Theorem 9.2.2. □

The assumption that $\Delta$ be compact cannot be dropped entirely from the statement of the corollary by the following ZF observation:

Proposition 9.2.24. (ZF+DC) Let $Y$ be a separable Banach space. If there is a discontinuous homomorphism $h: Y \rightarrow Y$ then there is an $E_0$-transversal, in particular $|E_0| \leq |2^\omega|$.

Proof. The discontinuity of the homomorphism and the DC assumption yield a sequence $\langle y_n: n \in \omega \rangle$ of elements of $Y$ such that $|y_n| > \Sigma_{m>n}|y_m|$ and $|h(y_{n+1})| > 2|h(y_n)|$. Now, for every $x \in 2^\omega$ let $g(x) = \Sigma\{y_n: x(n) = 1\} \in Y$. Let $d \subset 2^\omega$ be any $E_0$-class. The homomorphism assumptions on $h$ show that the function $g \upharpoonright d$ is injective and also, the norms of the points in $h \circ g(d)$ diverge to infinity. Thus, $d$ contains a finite subset of points whose $h \circ g$-images have the smallest possible norm, and one can let $x_d \in d$ be the lexicographically smallest point in $d$ the norm of whose $h \circ g$-image is as small as possible. The set $\{x_d: d \text{ is an } E_0\text{-class}\}$ is an $E_0$-transversal. □

Example 9.2.25. The Kurepa poset $P$ on a Polish space $X$ of Definition 8.4.2 is compactly balanced. To see this, equip $2^X$, identified with $P(X)$, with the usual product topology, refer to Theorem 8.4.3 to argue that the balanced conditions are classified by subsets of $P(X)$ closed under intersections and containing $X$, and observe, that sets of this type form a closed and therefore compact subset of $P(X)$. The properties (1-3) of Definition 9.2.1 are easily verified for this topology.

Corollary 9.2.26. Let $X$ be a Polish space.

1. In the Kurepa poset extension of the symmetric Solovay model, $|E_0| > |2^\omega|$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on $X$, and yet $|E_0| > |2^\omega|$.

As a final remark in this section, the consistency results obtained here can be combined using the countable support product. The balanced virtual conditions in a countable product of posets are simply sequences of balanced virtual conditions in each coordinate (Theorem 5.2.10). A product of Hausdorff compact spaces is Hasudorff compact again; thus, a product of countably many compactly
balanced forcings with full support is compactly balanced again. However, the machinery of Chapter 11 (which can also be used to show that the smooth divide is preserved in certain extensions) seems to be incompatible with the compactly balanced approach. This brings up a question:

**Question 9.2.27.** Are there $\Sigma^2_1$ sentences $\phi_0, \phi_1$ such that each of them is consistent with $\text{ZF}+(\text{DC}+)\mid \mathbb{E}_0 > |2^\omega|$ but their conjunction implies in $\text{ZF}(+\text{DC})$ that $|\mathbb{E}_0| \leq |2^\omega|$?

### 9.3 The turbulent divide

We wish to transfer the ergodicity theorem 3.3.5 to cardinal inequalities in generic extensions of the Solovay model. The following variations of balance will be central in this effort.

**Definition 9.3.1.** Let $\mathbb{P}$ be a Suslin forcing.

1. A pair $\langle Q, \tau \rangle$ is **placid** if $Q \Vdash \tau \subset \mathbb{P}$ is an analytic set and whenever $R_0, R_1$ are posets and in some ambient forcing extension, $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ are filters separately mutually generic over $V$ such that $V[H_0] \cap V[H_1] = V$, then any two conditions $p_0 \leq \Sigma \tau/H_0$ and $p_1 \leq \Sigma \tau/H_1$ in the respective models $V[H_0], V[H_1]$ are compatible in $\mathbb{P}$.

2. The poset $\mathbb{P}$ is **placid** if below every condition $p \in \mathbb{P}$ there is a virtual condition $\bar{p} \leq p$ which is placid.

We will find use for a slight weakening as well:

**Definition 9.3.2.** Let $\mathbb{P}$ be a Suslin poset.

1. A pair $\langle Q, \tau \rangle$ is **weakly placid** if $Q \Vdash \tau \subset \mathbb{P}$ is an analytic set and whenever $R$ is a poset and in some ambient forcing extension in which $\mathcal{P}(R \times Q)^V$ is countable, $y \mapsto H_y$ is a continuous function from $2^\omega$ to $\mathcal{P}(R \times Q)$ such that for any pair of distinct points $y_0, y_1 \in 2^\omega$ the sets $H_{y_0}, H_{y_1} \subset R \times Q$ are filters separately generic over $V$ such that $V[H_{y_0}] \cap V[H_{y_1}] = V$, then the set $\{(y_0, y_1) \in 2^\omega \times 2^\omega : \text{whenever } p_0 \leq \Sigma \tau/H_0 \text{ and } p_1 \leq \Sigma \tau/H_1 \text{ are conditions in } \mathbb{P} \text{ in the respective models } V[H_{y_0}], V[H_{y_1}] \text{, then } p_0, p_1 \text{ are compatible in } \mathbb{P} \}$ is co-meager in $2^\omega \times 2^\omega$.

2. The poset $\mathbb{P}$ is **weakly placid** if below every condition $p \in \mathbb{P}$ there is a virtual condition $\bar{p} \leq p$ which is weakly placid.

These properties may appear technical until they are fleshed out in examples. In the meantime, we connect them with the principal notion of balance.

**Proposition 9.3.3.** Let $\mathbb{P}$ be a Suslin forcing.

1. Every placid pair is weakly placid;
Figure 9.1: A weakly placid virtual condition.
2. every weakly placid pair is balanced;

3. the properties of placidity and weak placidity are invariant under the equivalence of balanced pairs.

In particular, if a Suslin poset $P$ is placid then it is weakly placid.

Proof. (1) follows immediately by a careful reading of the definitions. For (2), suppose that a pair $(Q, \tau)$ is not balanced and work to show that it is not weakly placid. There must be posets $R_0, R_1$ and respective $R_0 \times Q$ and $R_1 \times Q$-names $\sigma_0, \sigma_1$ for elements of $P$ such that $R_0 \times Q \models \sigma_0 \leq \Sigma \tau$, $R_1 \times Q \models \sigma_1 \leq \Sigma \tau$, and $R_0 \times R_1 \models \sigma_0, \sigma_1$ are incompatible in $P$. Move to an ambient extension in which $\mathcal{P}(R_0 \times R_1 \times Q)$ is countable and use Proposition 1.8.10 to produce a continuous function $y \mapsto H_y$ from $2^\omega$ to $\mathcal{P}(R_0 \times R_1 \times Q)$ such that the sets $H_y$ are mutually generic over the ground model. Now, whenever $y_0, y_1 \in 2^\omega$ are distinct points, the conditions $\sigma_0/H_{y_0}, \sigma_1/H_{y_1} \in P$ must be incompatible, while they are respectively stronger than $\Sigma \tau/H_0, \Sigma \tau/H_1$. Thus, the map $y \mapsto H_y$ violates the weak placid balance of the pair $(Q, \tau)$.

For (3), we will verify it only for placidity for brevity. Let $(Q_0, \tau_0), (Q_1, \tau_1)$ be equivalent balanced pairs, and suppose that $(Q_0, \tau_0)$ is placid. To show that $(Q_1, \tau_1)$ is placid, suppose that $V[H_0], V[H_1]$ are separately generic extensions respectively containing some filters $G_0 \subset Q_1$ and $G_1 \subset Q_1$ generic over the ground model and conditions $p_0 \leq \tau_1/G_0$ and $p_1 \leq \tau_1/G_1$, and $V[H_0] \cap V[H_1] = V$. We must show that $p_0, p_1$ are compatible in $P$. Let $K_0, K_1 \subset Q_0$ be filters mutually generic over the model $V[H_0, H_1]$. It follows that $V[H_0][K_0] \cap V[H_1][K_1] = V$, the filters $G_0, K_0$ are mutually generic, and the filters $G_1, K_1$ are mutually generic. By the equivalence of the balanced pairs $(Q_0, \tau_0)$ and $(Q_1, \tau_1)$, the conditions $\tau_0/K_0$ and $p_0$ must be compatible and they have a lower bound $q_0 \in V[H_0][K_0]$. On the same account, the conditions $\tau_0/K_1$ and $p_1$ must be compatible and they have a lower bound $q_1 \in V[H_1][K_1]$. By the placidity of the pair $(Q_0, \tau_0)$, the conditions $q_0, q_1$ must have a lower bound, which is also the desired lower bound of $p_0, p_1$. 

To state the main theorem of this section in a brief form, we establish standard parlance.

**Definition 9.3.4. (ZF)** The phrase “the turbulent divide is preserved” denotes the following statement: Let $E$ be an analytic equivalence relation on a Polish space induced as an orbit equivalence relation of a turbulent Polish group action. Let $F$ be a virtually placid analytic equivalence relation on a Polish space. Then $|E| \not\leq |F|$.

In addition, the theorem is stated using the standard Convention 1.8.17.

**Theorem 9.3.5.** In weakly placid extensions of the symmetric Solovay model, the turbulent divide is preserved.
Case 1 fails. There must be a poset \( P \) is forced to be an injection.

For each \( \gamma \), there is a nonempty open set \( P \) of size \(< \kappa \), contains \( z \) and in which the poset \( P \) is weakly placid.

Work in the model \( V[K] \). Let \( \bar{p} \leq p \) be a virtual condition in \( P \) which is weakly placid. Let \( P_X \) be the poset for adding a Cohen point of the space \( X \).

That is, \( P_X \) is the poset of all nonempty open subsets of \( X \) ordered by inclusion, adding a point \( \dot{x} \in X \). There are two cases.

Case 1. There is a nonempty open set \( O \subset X \) and a poset \( R \) of size \(< \kappa \) and a \( P_X \times R \)-name \( \sigma \) for an element of \( P \) stronger than \( \bar{p} \) and a virtual \( F \)-class \( y \in V[K] \) such that \( (O,1) \models Q \times R \text{Coll}(\omega,\kappa) \models \sigma \models_p \tau([\dot{x}]_E) = [\bar{y}]_F \). In this case, let \( x_0, x_1 \in X \) and \( H_0, H_1 \subset R \) be points in \( O \) and filters on \( R \) mutually generic over the model \( V[K] \).

Case 2. Case 1 fails. There must be a poset \( R \) of size \(< \kappa \) and \( P_X \times R \)-names \( \sigma \) for an element of the poset \( P \) stronger than \( \bar{p} \) and \( \eta \) for an element of \( Y \) such that \( P_X \times R \models \text{Coll}(\omega,\kappa) \models \sigma \models_p \tau([\dot{x}]_E) = [\bar{y}]_F \). The case assumption shows that \( \eta \) is forced not to be a realization of any ground model \( F \)-pinned class.

Let \( x_0 \in X \) be a \( P_X \)-generic point. In some large extension of \( V[x_0] \), use Proposition 1.8.10 to find a continuous map \( y \mapsto \gamma_y, H_y \) from \( 2^\omega \) to \( \Gamma \times \mathcal{P}(R) \) such that the for each \( y \in 2^\omega \), \( \gamma_y \) and \( H_y \subset R \) are mutually generic for \( P_\Gamma \times R \) over \( V[K][x_0] \) and for distinct points \( y_0, y_1 \in 2^\omega \), the objects \( \gamma_{y_0}, H_{y_0}, \gamma_{y_1}, H_{y_1} \) are mutually generic for the poset \( P_\Gamma \times R \times P_\Gamma \times R \). For each point \( y \in 2^\omega \), let \( x_y = \gamma_y \cdot x_0 \).

Claim 9.3.6. For each \( y \in 2^\omega \), the pair \( x_y, H_y \) is product generic for \( P_X \times R \) over \( V[K] \). For distinct points \( y_0, y_1 \in 2^\omega \), \( V[K][x_{y_0}][H_{y_0}] \cap V[K][x_{y_1}][H_{y_1}] = V[K] \).

Proof. For the first sentence, fix a point \( y \in 2^\omega \). Since the action of the group is a continuous open map from \( \Gamma \times X \) to \( X \), the point \( x_y \) is \( P_X \)-generic over...
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the model $V[K]$ by Proposition 3.1.1 applied in $V[K]$. By the product forcing theorem, the filter $H_y \subset R$ is generic over the model $V[K][x_0][\gamma_y]$ and so also over the smaller model $V[K][x_y]$. The first sentence of the claim then follows from another application of the product forcing theorem.

For the second sentence, fix distinct points $y_0, y_1 \in 2^\omega$. By Theorem 3.2.2 (this is the only place where we use the turbulence assumption) we have $V[K][x_0] \cap V[K][x_{y_0}] = V[K]$. Now, by the mutual genericity of the points $\gamma_{y_0}, \gamma_{y_1} \in \Gamma$ over the model $V[K][x_0]$, the point $\gamma_{y_1} \in \Gamma$ is $\Pi_1$-generic over the model $V[K][x_0][x_{y_0}]$ and so by a density argument $V[K][x_0][\gamma_{y_1}] \cap V[K][x_{y_0}] = V[K][x_0] \cap V[K][x_{y_0}] = V[K]$. It follows that $V[K][x_{y_0}] \cap V[K][x_{y_1}] = V[K]$. By the product forcing theorem, the filters $H_{y_0}, H_{y_1} \subset R$ are mutually generic over the model $V[K][x_0][\gamma_{y_0}, \gamma_{y_1}]$ which contains both points $x_{y_0}, x_{y_1} \in X$. Another density argument with the $R \times R$-product shows that $V[K][x_{y_0}][H_{y_0}] \cap V[K][x_{y_1}][H_{y_1}] = V[K]$ as desired.

Finally, it is time to use the weak placidity of the condition $\bar{p}$. By the claim, there must be distinct points $y_0, y_1 \in 2^\omega$ such that the conditions $p_0 = \sigma/H_{y_0}$ and $p_1 = \sigma/H_{y_1} \in P$ are compatible. Let $q$ be their lower bound. Consider the points $\eta/H_{y_0}, \eta/H_{y_1} \in Y$. These points do not realize any virtual $F$-class of the model $V[K]$, and so by the claim and the virtual placid assumption on the equivalence relation $F$, they are not $F$-related. However, the forcing theorem applied in the models $V[K][x_{y_0}][H_{y_0}]$ and $V[K][x_{y_1}][H_{y_1}]$ shows that in the model $W$, the condition $q$ forces in $P$ that $\tau([x_{y_0}, E]) = [\eta/H_{y_0}]_F$ and $\tau([x_{y_1}, E]) = [\eta/H_{y_1}]_F$. Since the two $E$-classes are identical and the two $F$ classes are distinct, this violates the assumption that $\tau$ is forced to be a function.

Many commonly occurring balanced Suslin posets are in fact placid. This is the contents of the following list of examples.

**Example 9.3.7.** Suppose that $K$ is a Borel simplicial complex on a Polish space $X$ of Borel coloring number $\aleph_1$. Then the poset $P_K$ is placid, and every balanced virtual condition is placid. To see this, revisit Theorem 6.2.9 and observe that its proof uses the product forcing theorem only to ascertain that if $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, then $V[H_0] \cap V[H_1] = V$.

**Example 9.3.8.** Suppose that $K$ is a modular Borel complex on a Polish space $X$. Every balanced virtual condition in the poset $P_K$ is placid. To see this, revisit the proof of Theorem 6.2.26 and note that $V[H_0] \cap V[H_1] = V$ was the only feature used of the mutually generic extensions $V[H_0], V[H_1]$.

**Corollary 9.3.9.** Let $X$ be a Polish vector space over a countable field.

1. In the extension of the symmetric Solovay model by the basis poset for $X$ of Example 6.2.16, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that ZF+DC holds, $X$ has a basis, and yet the turbulent divide is preserved.
Corollary 9.3.10. Let $\Gamma$ be a Borel graph on a Polish space $X$.

1. In the extension of the symmetric Solovay model by the spanning poset for $\Gamma$ of Example 6.2.17, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that $ZF+DC$ holds, $\Gamma$ contains a maximal acyclic subgraph, and yet the turbulent divide is preserved.

Example 9.3.11. Suppose that $E, F$ are Borel virtually placid equivalence relations on the respective Polish spaces $X, Y$. Suppose that $\lambda(E) \leq \lambda(F)$. Then the $E, F$-collapse poset of Definition 6.3.2 is placid, and every balanced condition is placid.

Example 9.3.12. Suppose that $E \subset F$ are Borel equivalence relation on a Polish spaces $X$, with $F$ placid. The $E, F$-transversal poset of Definition 6.3.5 is placid, and every balanced condition is placid.

Proof. The proof of Theorem 6.3.6 uses only one consequence of mutual genericity: the two mutually generic extensions $V[H_0], V[H_1]$ do not share any $F$-class which is not represented in the ground model. In the case of a placid equivalence relation $E$, this is implied by the assumption that $V[H_0] \cap V[H_1] = V$ by the definition of placidity.

Corollary 9.3.13. Suppose that $E$ is a Borel placid equivalence relation on a Polish space $X$.

1. In the extension of the symmetric Solovay model by the transversal poset for $E$ of Example 6.3.8, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that $ZF+DC$ holds, $E$ has a transversal, and yet the turbulent divide is preserved.

The posets which select a structure on each $E$-class for a countable Borel equivalence relation $E$ are placid by Example 9.3.12, resulting of many corollaries of the following kind.

Corollary 9.3.14. Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$ with infinite classes.

1. In the extension of the symmetric Solovay model by the $Z$-action poset of Example 6.3.10, the turbulent divide is preserved;

2. it is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, $E$ is an orbit equivalence of a (discontinuous) action of $Z$, and yet the turbulent divide is preserved.

Example 9.3.15. Suppose that $\mathcal{F}$ is a Fraissé class of structures in finite relational language, with strong amalgamation. Let $E$ be a virtually placid equivalence relation on a Polish space $X$. Let $P$ be the poset for adding an $\mathcal{F}$-structure on the quotient space $X$ with countable approximations of Definition 8.5.3. Then $P$ is placid, and every balanced condition is placid.
9.3. THE TURBULENT DIVIDE

Corollary 9.3.16. Let $E$ be a Borel virtually placid equivalence relation on a Polish space $X$.

1. In the extension of the symmetric Solovay model by the linearization poset for $E$ of Example 8.5.5, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that ZF+DC holds, the quotient space $X/E$ is linearly ordered and yet the turbulent divide is preserved.

Example 9.3.17. Suppose that $X$ is a Polish space and $P$ is the Kurepa poset on it as isolated in Definition 8.4.2. Then $P$ is placid, and every balanced condition is placid. To see this observe that the proof of Theorem 8.4.3 used only the consequence $V[H_0] \cap V[H_1] = V$ for mutually generic filters $H_0, H_1$.

Corollary 9.3.18. Let $X$ be a Polish space.

1. In the extension of the symmetric Solovay model by the Kurepa poset, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on $X$, and yet the turbulent divide is preserved.

Finally, we include a class of examples which are weakly placid but in general not placid.

Example 9.3.19. Let $\Gamma$ be a Borel graph on a Polish space $X$ with countable coloring number. Let $P = P_\omega(\Gamma)$ be the poset of Definition 8.1.1. Then $P$ is weakly placid and every balanced virtual condition is weakly placid.

Proof. We will identify the poset $P$ with its dense Suslin subposet as in Theorem 8.1.2. That same theorem shows that the balanced virtual conditions are up to equivalence the pairs $\langle \text{Coll}(\omega, X), \check{c} \rangle$ where $\check{c}: X \to \omega$ is a $\Gamma$ coloring. Fix the coloring $c$ and work to prove that the resulting balanced condition is in fact weakly placid.

Let $R$ be a poset. Let $V[G]$ be a generic extension such that $P(R)^V$ is countable in $V[G]$, and work in $V[G]$. Suppose that $y \mapsto H_y$ is a continuous map from $2^\omega$ to $P(R)$ such that for each $y \in 2^\omega$, the set $H_y \subset R$ is a filter generic over $V$ and for distinct points $y_0, y_1 \in 2^\omega$, $V[H_{y_0}] \cap V[H_{y_1}] = V$ holds.

Claim 9.3.20. The set $\{\langle y_0, y_1 \rangle \in 2^\omega \times 2^\omega \text{ whenever } x_0, x_1 \in X \text{ are points in } V[H_{y_0}] \setminus V \text{ and } V[H_{y_1}], \text{ respectively, then } \langle x_0, x_1 \rangle \notin \Gamma \}$ is co-meager.

Proof. Suppose towards contradiction that this is not the case. By the Baire category theorem, there must be $R$-names $\eta_0, \eta_1$ for non-ground-model elements of $X$ such that the Borel set $B = \{\langle y_0, y_1 \rangle \in 2^\omega \times 2^\omega : \langle \eta_0/H_{y_0}, \eta_1/H_{y_1} \rangle \in \Gamma \}$ is non-meager. By the Kuratowski-Ulam theorem, there must be a basic open neighborhood $O \subset 2^\omega$ such that the set $C = \{y \in 2^\omega : B_y \text{ is comeager in } O \}$ is non-meager. Let $a \subset C$ be any countable set. The points $\eta_0/y$ for $y \in a$
are pairwise distinct since the models $V[H_y]$ for $y \in a$ pairwise intersect in the ground model $V$. The set $D = \bigcap_{y \in a} B_y$ is nonmeager, and for each $y_0 \in a$ and $y_1 \in D$, the points $\eta_0/H_{y_0}$ and $\eta_1/H_{y_1}$ are $\Gamma$-connected by the definition of the set $B$. Moreover, the points $\eta_1/H_y$ for $y \in D$ are pairwise distinct, since the models $V[H_y]$ for $y \in D$ pairwise intersect in $V$. Thus, the set $\{\eta_1/H_y : y \in D\}$ is uncountable and analytic, and all of its points are connected to all points in the infinite set $\{\eta_0/H_y : y \in a\}$. However, no graph with countable coloring number can contain an infinite times uncountable complete bipartite graph. A contradiction.

Fix distinct points $y_0, y_1 \in 2^\omega$ as in the claim and suppose that $p_0, p_1 \in P$ are conditions in the respective models $V[H_0], V[H_1]$ which extend the coloring $c$; we must show that $p_0, p_1$ are compatible. The claim immediately shows that $p_0 \cup p_1$ is a partial $\Gamma$-coloring, which verifies the compatibility.

**Corollary 9.3.21.** Let $\Gamma$ be the graph on $\mathbb{R}^2$ connecting two points if they are at a rational distance from each other.

1. In the $P(\Gamma)$ extension of the Solovay model, the turbulent divide is preserved;

2. it is consistent relatively to an inaccessible cardinal that ZF+DC holds, the chromatic number of $\Gamma$ is countable and yet the turbulent divide is preserved.

It turns out that for the rational distance graph $\Gamma$ on $\mathbb{R}^2$, the poset $P(\Gamma)$ is not placid. In fact, in placid extensions of the symmetric Solovay model the chromatic number of $\Gamma$ remains uncountable by Example 12.2.10.

**Example 9.3.22.** The ultrafilter poset of Subsection 7.1 is balanced but not placid or weakly placid. This observation should be compared with Example 12.2.5 below.

**Question 9.3.23.** Let $P$ be the poset of Subsection 7.1 for adding an ultrafilter. Let $W$ be a symmetric Solovay extension and $G \subset P$ be a filter generic over $W$. Does $P$ preserve the turbulent divide?

We conclude this section with a question. Both placidity and weak placidity of Suslin posets are closed under countable products. This brings up the following question:

**Question 9.3.24.** Are there $\Sigma^2_1$ sentences $\phi_0, \phi_1$ such that each of them is consistent with ZF+DC+the turbulent divide is preserved while ZF+DC+\(\phi_0+\phi_1\) proves that the turbulent divide is not preserved?
9.4 The orbit divide

It is well-known that $E_1$ is not Borel reducible to any orbit equivalence relation. It is then tempting to think that in many models which we study, $|E_1|$ cannot be smaller than $|E|$ where $E$ is an orbit equivalence relation. This question is with some success addressed in the present section. The following definition connects coherent sequences of generic extensions with Suslin forcings.

Definition 9.4.1. Let $P$ be a Suslin forcing. A nest below a condition $p \in P$ is a choice-coherent sequence $(V_n : n \in \omega)$ of generic extensions of $V$ and a sequence $(\bar{p}_n : n \leq \omega)$ so that

1. $2^\omega \cap V_n \neq 2^\omega \cap V_{n+1}$ for all $n \in \omega$;
2. $\bar{p}_0 \leq \bar{p}_1 \leq \cdots \leq \bar{p}_\omega \leq p$ where $\bar{p}_n$ for each $n \in \omega$ is a balanced virtual condition in $V_n$ and $\bar{p}_\omega$ is a balanced virtual condition in the intersection model $V_\omega = \bigcap_n V_n$.

Definition 9.4.2. Let $P$ be a Suslin forcing. The poset $P$ is nested balanced below $\kappa$ if it has a nest below every condition.

To state the results of this section succinctly, we establish a standard parlance.

Definition 9.4.3. (ZF) The phrase “orbit divide is preserved” denotes the following statement. Let $E$ be an orbit equivalence relation on a Polish space induced as an orbit equivalence relation of a Polish group action. Then $|E_1| \nleq |E|$.

The central theorem can now be stated easily using Convention 1.8.17.

Theorem 9.4.4. In nested balanced extensions of the symmetric Solovay model, the orbit divide is preserved.

Proof. Suppose that $Y$ is a Polish space, $\Gamma$ is a Polish group and $\Gamma$ continuously acts on $Y$, inducing an orbit equivalence relation $E$. Suppose also that $W$ is a symmetric Solovay model derived from $\kappa$ and in $W$, $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \Vdash \tau$ is a function from the $E_1$-quotient space to the $E$-quotient space. We must find a stronger condition and two distinct $E_1$ classes which are forced by the stronger condition to be mapped to the same $E$-class.

The objects $\Gamma$, $Y$, $p$ and $\tau$ are all definable in the model $W$ from parameters in the ground model plus a parameter $z \in 2^\omega$. Use the assumptions to find an intermediate forcing extension $V[K]$ by a poset of size $< \kappa$ such that $z \in V[K]$ and $V[K]$ has a nest below $p$. Still working in $W$, find the nest. This is a sequence $(V_n : n \in \omega)$ of generic extensions of $V[K]$ together with a sequence $(\bar{p}_n : n \leq \omega)$ of balanced virtual conditions below $p$ satisfying the demands of Definition 9.4.1.

Use the coherence of the sequence $(V_n : n \in \omega)$ to find a well-ordering $\prec$ of $2^\omega$ in $V_0$ such that for each $n \in \omega$, $\prec| V_n \in V_n$ holds. Let $z_n \in 2^\omega$ be the $\prec$-least element of $V_n \setminus V_{n+1}$ for every $n \in \omega$ and let $x_n \in X = (2^\omega)^\omega$ be the...
Figure 9.2: A nest of models and balanced virtual conditions.
point defined by \( x_n(m) = \) the zero binary sequence if \( m < n \) and \( x_n(m) = z_m \) if \( m \geq n \); thus \( x_n \in V_n \). The points \( x_n \in X \) for \( n \in \omega \) are pairwise \( \mathbb{E}_1 \)-equivalent; at the same time, they have no \( \mathbb{E}_1 \)-equivalent in the model \( V_\omega = \bigcap_n V_n \), since none of the points \( z_n \) belong to \( V_\omega \).

By the forcing theorem, for each number \( n \in \omega \), in the model \( V_n \) there must be a poset \( R_n \) of size \( < \kappa \), an \( R_n \)-name \( \sigma_n \) for a condition in \( P \) stronger than \( \bar{p}_n \) and an \( R_n \)-name \( \eta_n \) for an element of \( Y \) such that \( R_n \models \text{Coll}(\omega, < \kappa) \models \sigma_n \upharpoonright P \models \tau(\bar{x}_n) = [\eta_n]_E \).

**Claim 9.4.5.** For \( n \in \omega \), the pair \( \langle R_n, \eta_n \rangle \) is an \( E \)-pin.

**Proof.** If this failed for some number \( n \in \omega \), then in the model \( W \) there would be filters \( H_0, H_1 \subset R_n \) mutually generic over the model \( V_n \) such that the points \( y_0 = \sigma_n/H_0, y_1 = \sigma_n/H_1 \in Y \) are \( E \)-unrelated. Let \( p_0 = \sigma_n/H_0 \) and \( p_1 = \sigma_n/H_1 \) be conditions in the poset \( P \); they are compatible by the balance of \( \bar{p}_n \). By the forcing theorem applied in the models \( V_n[H_0] \) and \( V_n[H_1] \), their lower bound forces in \( P \) that \( \tau(\bar{x}_n) = [\eta_0]_E \) and \( [\eta_1]_E \). This is impossible.

**Claim 9.4.6.** For \( n \in m \in \omega \), the \( E \)-pins \( \langle R_n, \eta_n \rangle \) and \( \langle R_m, \eta_m \rangle \) are equivalent.

**Proof.** If this failed for some numbers \( n \in m \in \omega \), then in the model \( W \) there would be filters \( H_0, H_1 \subset R_n \) mutually generic over the model \( V_n \) such that the points \( y_0 = \sigma_n/H_0, y_1 = \sigma_m/H_1 \in Y \) are \( E \)-unrelated. Let \( p_0 = \sigma_n/H_0 \) and \( p_1 = \sigma_m/H_1 \) be conditions in the poset \( P \); they are compatible by the balance of \( \bar{p}_n \) since \( p_0 \leq \bar{p}_n \leq \bar{p}_m \) by demand (2) of Definition 9.4.1. By the forcing theorem applied in the models \( V_n[H_0] \) and \( V_n[H_1] \), their lower bound forces in \( P \) that \( \tau(\bar{x}_n) = [\eta_0]_E \) and \( [\eta_1]_E \). This is impossible since \( x_n \mathbb{E}_1 x_m \).

By Theorem 4.3.7, there is an \( E \)-pin \( \langle R, \eta \rangle \in V_\omega \) which is equivalent to all the pins \( \langle R_n, \eta_n \rangle \) for \( n \in \omega \). Still in \( V_\omega \), find a poset \( Q \) generating the model \( V_0 \), a \( Q \)-name \( \dot{x} \) for the sequence \( x_0 \), and a \( Q \)-name \( \dot{R}_0 \) for the poset \( R_0 \). By the forcing theorem, there must be a condition \( q \in Q \) which forces the following: \( \dot{x} \) has no \( \mathbb{E}_1 \)-equivalent in \( V_\omega \), \( \dot{R}_0 \models \sigma_0 \leq p_\omega \) and \( \dot{R} \times \dot{R}_0 \models \eta E \eta_0 \). Note that the last statement means that the name \( \eta_0 \) is \( E \)-pinned on the iteration \( Q \restriction q \ast \dot{R}_0 \). In the model \( W \), let \( H_0, H_1 \subset Q \restriction q \ast \dot{R}_0 \) be filters mutually generic over the model \( V_\omega \), and write \( p_0 = \sigma_0/H_0, p_1 = \sigma_0/H_1, \bar{x}_0 = \dot{x}/H_0, \bar{x}_1 = \dot{x}/H_1, y_0 = \eta_0/H_0 \) and \( y_1 = \eta_0/H_1 \). The balance of the condition \( p_\omega \) implies that \( p_0, p_1 \in P \) are compatible conditions. Since \( \mathbb{E}_1 \) is a pinned equivalence relation, the points \( \bar{x}_0, \bar{x}_1 \in X \) are \( \mathbb{E}_1 \)-unrelated. Since the name \( \eta_0 \) was pinned on the iteration \( Q \restriction q \ast \dot{R}_0 \), the points \( y_0, y_1 \) are \( E \)-related. In total, in the model \( W \), the lower bound of the conditions \( p_0, p_1 \) forces that \( \tau([\bar{x}_0]_{\mathbb{E}_1}) = \tau([\bar{x}_1]_{\mathbb{E}_1}) = [\eta_0]_E \), completing the proof.

It turns out that one large class of nested balanced posets has already been isolated in Definition 9.2.1:
Example 9.4.7. Every compactly balanced Suslin poset is nested balanced.

Proof. Let $P$ be a compactly balanced Suslin poset and $p \in P$ be a condition. Let $U \subset P(\omega)$ modulo finite and $(x_n : n \in \omega)$ be mutually generic ultrafilter and Sacks reals added with full support product. Let $(V_n : n \in \omega)$ be the sequence of models determined by $V_n = V[U, y_m : m \geq n]$; this is a choice-coherent sequence by Theorem 4.3.6 or Example 4.3.5 applied in the model $V[U]$. We will produce a descending sequence of balanced virtual conditions as required by Definition 9.4.1.

It is well known [71] that the Sacks real product adds no independent reals, and therefore, by a genericity argument, $U$ generates an ultrafilter on $\omega$ in the model $V_0$. The generating set of $U$ is present already in the model $V_0 = \bigcap_n V_n$. For each number $n \in \omega$, use the extension property (2) of Definition 9.2.1 repeatedly to find a sequence $\langle \bar{p}(\omega, n), \bar{p}(m, n) : m \leq n \rangle$ such that $\bar{p}(\omega, n)$ is a balanced virtual condition in $V_\omega$, each $\bar{p}(m, n)$ is a balanced virtual condition in the model $V_m$, and $p \geq \bar{p}(\omega, n) \geq \bar{p}(n, n) \geq \bar{p}(n-1, n) \geq \bar{p}(n-2, n) \geq \ldots$; also, demand that each virtual condition $\bar{p}(m, n)$ is selected $\prec$-least where $\prec$ in $V_0$ is some coherent well-ordering of the set of virtual conditions on $P$. It is not difficult to see that the resulting system is coherent in the sense that for each $n \leq \omega$, the system $\langle p(m, k) : m = \omega \text{ and } k \in \omega \text{ or } k \geq m \geq n \rangle$ belongs to the model $V_n$.

Finally, use the compactness of the space of balanced virtual conditions to define $\bar{p}_n$ for $n \leq \omega$ to be the $U$-limit of the sequence $\langle \bar{p}(n, k) : k \geq n \rangle$. Each $\bar{p}_n$ is a balanced virtual condition in the model $V_n$, $m \leq n \leq \omega$ implies that $\bar{p}_m \leq \bar{p}_n \leq p$ by the closedness demands (1) and (3) of Definition 9.2.1. It follows that the sequence $\langle \bar{p}_n : n \leq \omega \rangle$ is as required in Definition 9.4.1.

Corollary 9.4.8.

1. Let $P$ be the poset $P(\omega)$ modulo finite. In the $P$-extension of the symmetric Solovay model, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Ramsey ultrafilter on $\omega$, and the orbit divide is preserved.

Proof. This is a conjunction of Example 9.2.4 and Theorem 9.4.4.

Question 9.4.9. Is there a Borel (or even $F_\sigma$) ideal $I$ on $\omega$ such that the existence of an ultrafilter disjoint from $I$ implies in ZF+DC that $|E_1| \leq |E|$ for some orbit equivalence relation $E$?

Corollary 9.4.10. Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. In the extension of the Solovay model by the $E$-linearization poset of Example 8.5.5 for the quotient $X/E$, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the quotient $X/E$ is linearly ordered, and the orbit divide is preserved.

Proof. This is a conjunction of Example 8.5.5, Example 9.2.11 and Theorem 9.4.4.
Corollary 9.4.11. Let $\Gamma, \Delta$ be abelian Polish groups, with $\Gamma$ uncountable and torsion-free and $\Delta$ compact, nontrivial, and divisible.

1. Let $P(\Gamma, \Delta)$ be the poset of Definition 8.2.1. In the $P(\Gamma, \Delta)$-extension of the symmetric Solovay model, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that $\text{ZF+DC}$ holds, there is a discontinuous homomorphism from $\Gamma$ to $\Delta$, the orbit divide is preserved.

Proof. This is a conjunction of Example 9.2.22 and Theorem 9.4.4.

Question 9.4.12. Does the existence of a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ imply in $\text{ZF+DC}$ that $|E_1| \leq |E|$ holds for some orbit equivalence relation $E$, or for $E = \mathbb{F}_2$ in particular? That $E_1$ has a countable complete section?

Corollary 9.4.13. Let $X$ be a Polish space.

1. In the extension of the Solovay model by the Kurepa poset of Definition 8.4.2, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that $\text{ZF+DC}$ holds, there is a Kurepa family on $X$, and the orbit divide is preserved.

Proof. This is a conjunction of Example 9.2.25 and Theorem 9.4.4.

Another group of nested balanced posets is obtained from orbit equivalence relations:

Example 9.4.14. Let $E \subset F$ be Borel equivalence relations on a Polish space $X$ such that $F$ is pinned and Borel reducible to an orbit equivalence relation. Let $P$ be the $E, F$-transversal poset of Definition 6.3.5. Then $P$ is nested balanced.

Proof. By Theorem 6.3.6, the balanced virtual conditions are classified by functions selecting, for each $F$-class, a single virtual $E$-class which is forced to be a subset of the $F$-class. Let $p \in P$ be a condition. To find a nest below $p$, consider any choice-coherent sequence $\langle V_n : n \in \omega \rangle$ consisting of generic extensions of the ground model such that for every $n \in \omega$, $2^n \cap V_n \setminus V_{n+1} \neq 0$—the choice-coherent sequence of models obtained from the infinite product of Sacks reals will do. To find a coherent sequence of balanced virtual conditions, in the model $V_0$ consider the function $f$ on $X/F$ defined as follows: if $c$ is an $F$-class mentioned in $p$, let $f(c)$ be the $E$-class $d \subset c$ which $p$ selects. If $c$ is a class represented in $V_0$ but not mentioned in $p$, then let $f(c)$ be the $E$-class of the $\prec$-first representative of $c$ in the model $V_0$. If $n \in \omega$ is the largest number such that $c$ is represented in $V_n$, then let $f(c)$ be the $E$-class of the $\prec$-first representative of $c$ in the model $V_n$. Now, by Theorem 4.3.7, this defines the function $f$ on all $F$-equivalence classes represented in $V_0$. It is also clear that for each $n \leq \omega$, $f \upharpoonright V_n \in V_n$. For each $n \leq \omega$, let $p_n$ be the virtual balanced condition in the model $V_n$ associated with the function $f \upharpoonright V_n$ by Theorem 6.3.6. The system $\langle V_n : n \in \omega, p_n : n \leq \omega \rangle$ is the sought orbit charm below $p$. 

\qed
Corollary 9.4.15. Let $E$ be a Borel pinned orbit equivalence relation on a Polish space $X$.

1. Let $P$ be the transversal poset of Example 6.3.8. In the $P$-extension of the symmetric Solovay model, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $E$ has a transversal, and the orbit divide is preserved.

Proof. This is a conjunction of Example 9.4.14 and Theorem 9.4.4. □

The posets selecting a structure on each $E$-class for a countable Borel equivalence relation $E$ are always nested balanced by virtue of Example 9.4.14. The following is a sample corollary.

Corollary 9.4.16. Let $E$ be a countable Borel equivalence relation on a Polish space $X$, with all orbits infinite.

1. Let $P$ be the $Z$-action poset of Example 6.3.10. In the $P$-extension of the symmetric Solovay model, the orbit divide is preserved.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $E$ is induced as an orbit equivalence relation of a discontinuous $Z$-action, and the orbit divide is preserved.

The determination of the nested balanced status of the posets adding a maximal $K$-set for a Borel simplicial complex $K$ is much more challenging. The ultimate limitation is the surprising Corollary 9.4.31 below, showing in ZF+DC that if $\mathbb{R}$ has a Hamel basis, then $\mathbb{E}_1$ has a complete countable section, and therefore $|\mathbb{E}_1| \leq |\mathbb{F}_2|$ holds. Thus, we have to resort to partial results.

Example 9.4.17. Let $\Gamma$ be a Borel graph on a Polish space $X$ with Borel coloring number $\aleph_1$. Let $P = P_\Gamma$ be the poset of countable $\Gamma$-anticliques ordered by reverse inclusion. Then $P$ is nested balanced.

Proof. Let $p \in P$ be a condition; we must produce a nest below $p \in P$. Let $\langle V_n : n \in \omega \rangle$ be a choice-coherent sequence of generic extensions such that $V_0$ is an extension of the ground model by proper forcing, and for all $n \in \omega$ the set $2^\omega \cap V_n \setminus V_{n+1}$ is nonempty. We will show that there is a sequence $\langle p_n : n \in \omega \rangle$ of balanced conditions satisfying the demands of Definition 9.4.1.

Theorem 6.2.9 classified the balanced conditions in the poset $P$ in terms of weakly maximal $\Gamma$-anticliques. Thus, it will be enough to produce a set $A \subset X$ in the model $V_0$ which contains $p$ as a subset and such that for each $n \leq \omega$, the set $A \cap V_n$ belongs to $V_n$ and is a weakly maximal $\Gamma$-anticlique there. To cut the notational clutter, we ignore the condition $p \in P$. Let $\bar{\Gamma}$ be a Borel orientation of $\Gamma$ such that the $\bar{\Gamma}$-outflow $o(y)$ of every vertex $y \in X$ is countable. Consider the set $B \subset X$ of all points $x \in X$ for which there is a countable set $a \subset X$ such that
• either for every countable set \( b \subseteq X \) there is \( y \notin b \) such that \( x \in o(y) \subseteq a \);
• or for every countable set \( b \subseteq X \) there is \( y \in X \) such that \( \langle y, x \rangle \in \Gamma \) and \( o(y) \setminus a \) is a nonempty set disjoint from \( b \).

A careful complexity computation shows that the set \( B \) is \( \Sigma^1_2 \). The following claim uses the properness assumption:

**Claim 9.4.18.** Whenever \( n \in \omega \) is a number and \( y \in X \cap V_n \setminus V_{n+1} \) is a point and \( \langle y, x \rangle \in \Gamma \), then \( x \in B \) or \( x \in V_n \setminus V_{n+1} \).

**Proof.** Suppose that \( x \in V_{n+1} \); we must show that \( x \in B \). The set \( o(y) \cap V_{n+1} \) is countable in \( V_n \); by the properness assumption, it is covered by a set \( a \in V_{n+1} \) which is countable in \( V_{n+1} \). We claim that \( a \) is a witness to \( x \in B \). Indeed, if \( o(y) \subseteq V_{n+1} \) then \( x \in B \) by the first item in the definition of \( B \)–the point \( y \) is not in \( V_{n+1} \) and so it does not belong to any countable set in \( V_{n+1} \). On the other hand, if \( o(y) \subseteq V_{n+1} \) then \( x \in B \) by the second item in the definition of \( B \)–the nonempty set \( o(y) \setminus a \) is disjoint from any countable set in \( V_{n+1} \). \( \square \)

Now, work in \( V_0 \). Let \( \prec \) be a coherent well-ordering of \( P(X) \), and for every number \( n \in \omega \) let \( C_n \in V_n \) be the \( \prec \)-least maximal \( \Gamma \)-anticlique in the set \( (X \setminus B) \cap V_n \setminus V_{n+1} \). Let also \( C_\omega \in V_\omega \) be the \( \prec \)-maximal \( \Gamma \)-anticlique in \( (X \cap V_\omega) \setminus B \) extending \( p \). Let \( A = \bigcup_{n \in \omega} C_n \); we claim that for each \( n \leq \omega \), the set \( A \cap V_n \) belongs to \( V_n \) and is weakly maximal there. Note that \( A \cap V_n \in V_n \) follows immediately from the coherence of the choices above. The claim implies that \( A \cap V_n \) is a \( \Gamma \)-anticlique and so a maximal \( \Gamma \)-anticlique in the set \( (X \setminus B) \cap V_n \). It follows directly from the definition of a weakly maximal set (Definition 6.2.8) that \( A \cap V_n \) is a weakly maximal set in \( V_n \) as desired. \( \square \)

**Corollary 9.4.19.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) of Borel coloring number \( \aleph_1 \).

1. In the \( P_\Gamma \) extension of the symmetric Solovay model and the orbit divide is preserved;
2. it is consistent relative to an inaccessible cardinal that \( ZF + DC \) holds, there is a maximal \( \Gamma \)-anticlique and the orbit divide is preserved.

**Example 9.4.20.** Let \( X \) be a Polish space and let \( G \) be a Borel graph on \( X \). Let \( K \) be the Borel simplicial complex on \( G \) of finite acyclic subsets of \( G \), and let \( P = P_K \). Then \( P \) is nested balanced.

**Proof.** Let \( Q_m \) for each \( m \in \omega \) be the countable support product of \( \aleph_1 \) many Sacks posets, and for each \( n \in \omega \) let \( P_n \) be the countable support product \( \Pi_{m \geq n} Q_m \), with the natural projection maps. Let \( H_0 \subseteq P_0 \) be a generic filter, and for each \( n \in \omega \) let \( H_n = H \cap P_n \). By Example 4.3.5 and Theorem 4.3.6, the sequence \( \langle V[H_n] : n \in \omega \rangle \) is a choice-coherent sequence of models of \( ZFC \); write \( V_n = V[H_n] \) and \( V_\omega = \bigcap_n V_n \).

Write \( X_\omega = X \cap V_\omega \), and for each \( m < n \) write \( X_{m:n} = X_\omega \cup (X \cap V_m \setminus V_n) \). The following elusive claim is the main reason for the choice of the posets \( Q_m \).
Claim 9.4.21. Let \( l < m < n \) be natural numbers and \( x_0, x_1 \in X_{mn} \) be vertices. If \( x_0, x_1 \) are connected by a path in the graph \( G \upharpoonright X_m \), then they are connected by a path in the graph \( G \upharpoonright X_{mn} \).

Proof. It will be enough to show that if \( x_0, x_1 \) are connected by a path in the graph \( G \upharpoonright X_m \) whose vertices except for \( x_0, x_1 \) all belong to \( X_m \), then they are connected by a path in the graph \( G \upharpoonright X_{mn} \). To this end, work in \( V_m \) and let \( A = \{ a \in [X]^{<\aleph_0} : \text{there is a G-path connecting } x_0, x_1 \text{ using only the vertices in } a \} \).

The set \( A \subseteq [X]^{<\aleph_0} \) is Borel. It cannot be punctured by a countable set by a Mostowski absoluteness argument: in the model \( V_l \) there is a \( G \)-path between \( x_0, x_1 \) using no vertices in \( V_m \) and therefore no vertices in any given countable set in \( V_m \). By the work on puncture sets [12, Theorem 21] applied in the model \( V_m \), there is a perfect set \( B \subseteq A \) in the model \( V_m \) consisting of pairwise disjoint sets. By a standard fusion argument with the product of Sacks forcing, there is a countable set \( c \) of Sacks reals added by the filter \( H_m \) such that \( B \) has a code in the model \( V[l] \subseteq V_m \). Since the poset \( Q_m \) is a product of uncountably many copies of Sacks forcing, in \( V_m \) there has to be an element \( a \in B \) which does not belong to the model \( V_n \subseteq V_m \). Since the sets in \( B \) are pairwise disjoint, every element of \( a \) would reconstruct \( a \) over the model \( V_n \). Therefore, no element of \( a \) belongs to the model \( V_n \); in particular, \( a \cap V_n = \emptyset \) and \( a \) yields the desired \( G \)-path between \( x_0 \) and \( x_1 \) using only vertices in \( X_{mn} \). \( \square \)

By Theorem 6.2.28, it will be enough to find sets \( A_m \subseteq G \) for \( m \leq \omega \) forming an inclusion-descending sequence such that \( V_m \models A_m \subseteq G \) is a maximal acyclic subset of \( G \). To this end, let \( < \) be a coherent well-ordering of all subsets of \( G \) in \( V_0 \). Let \( A_\omega \subseteq G \) be a maximal acyclic subset of \( G \) in the model \( V_\omega \). For each \( m \in \omega \), let \( A_{mn+1} \subseteq V_m \) be the \(<\)-first set in the model \( V_m \) which is a maximal acyclic subset of \( G \upharpoonright X_{mn+1} \) and extends \( A_\omega \).

Now, by induction on \( n - m \), for \( m < n \) define \( A_{mn} \subseteq V_m \) to be the \(<\)-first set in the model \( V_m \) which is a maximal acyclic subset of \( G \upharpoonright X_{mn} \) and extends both \( A_{m+1,n} \) and \( A_{m,n-1} \). To see that this is possible, simultaneously argue by induction on \( n - m \) that if \( m \leq m' < n \) then \( A_{m',n'} \subseteq A_{mn} \) and moreover, the set \( A_{m+1,n} \cup A_{m,n-1} \) does not contain a cycle. The former statement is clear. The latter statement requires the claim. A putative cycle \( c \subseteq A_{m+1,n} \cup A_{m,n-1} \) has to contain some edges from both sets by the acyclicity induction hypothesis. Choose an inclusion-maximal contiguous part \( d \subseteq c \) of the cycle consisting of edges in \( A_{m,n-1} \setminus A_{m+1,n} \). The end-nodes of \( d \), denote them by \( x_0, x_1 \), must be distinct because the cycle must use some edges from the set \( A_{m+1,n} \) as well. It must also be the case that \( x_0, x_1 \in X_{m,n-1} \cap X_{m+1,n} = X_{m+1,n-1} \). The path \( d \) connects the nodes \( x_0, x_1 \) in the graph \( G \upharpoonright X_{m,n-1} \); by the claim then, they have to be connected by a path in the graph \( G \upharpoonright X_{m+1,n-1} \) as well, and therefore by a path \( e \) in the set \( A_{m+1,n-1} \). Then \( e \cup d \) forms a cycle in the set \( A_{m,n-1} \), violating the induction hypothesis.

In the end, let \( A_m = \bigcup_{n \geq m} A_{mn} \). It is clear from the coherence of the well-ordering \( < \) that \( A_m \subseteq V_m \). The construction also guarantees that \( V_m \models A_m \subseteq G \).
is a maximal acyclic subset, and \( m < n \) implies \( A_n \subset A_m \). The proof is complete.

**Corollary 9.4.22.** Let \( X \) be a Polish space and let \( G \) be a Borel graph on \( X \).

1. Let \( P \) be the poset of Example 6.2.17 for adding a maximal acyclic subset of \( G \). In the \( P \)-extension of the symmetric Solovay model, the orbit divide is preserved;

2. it is consistent relative to an inaccessible cardinal that \( \text{ZF+DC} \) holds, \( G \) has a maximal acyclic subset, and the orbit divide is preserved.

**Example 9.4.23.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) of countable coloring number. The coloring poset \( P = P_\omega(\Gamma) \) of Definition 8.1.1 is nested balanced.

**Proof.** We identify \( P \) with its dense Suslin subset identified in Theorem 8.1.2. Let \( p \in P \) be a condition, and let \( \langle V_n \rangle_{n \in \omega} \) be a nontrivial choice coherent sequence of generic extensions of \( V \), with \( V_\omega = \bigcap_n V_n \). We will produce a decreasing sequence \( \langle \bar{p}_n \rangle_{n \leq \omega} \) of balanced virtual conditions below \( p \) for the respective models \( V_n \). Let \( \langle a_n \rangle_{n \leq \omega} \) be a recursive sequence of pairwise disjoint infinite subsets of \( \omega \).

**Claim 9.4.24.** Fix \( n \leq \omega \). In the model \( V_n \), there is a total \( \Gamma \) coloring \( d: X \to \omega \) which extends \( p \) and at all points \( x \in X \setminus \text{dom}(p) \), \( d(x) \in a_n \).

**Proof.** Work in \( V_n \). Fix a coloring \( e: X \to \omega \). Let \( a_n = \bigcup_i b_i \) be a partition of \( a_n \) into infinitely many sets of size \( m \). Let \( d: X \to \omega \) be a function such that \( p \subset d \) and for all \( x \notin X \setminus \text{dom}(p) \), \( d(x) \) is a number in \( e(x) \) which is not one of the (fewer than \( m \) many) colors \( \{ p(y): \langle x, y \rangle \in \Gamma \} \). This is the desired coloring.

Let \( \prec \) be a coherent wellordering of \( P(X \times \omega) \), and for each \( n \leq \omega \) let \( d_n \) be the \( \prec \)-least coloring as in the claim in the model \( V_n \). Finally, in the model \( V_0 \), define a map \( c: X \to \omega \) as follows: \( c(x) = d_n(x) \) if \( x \notin \text{dom}(p) \) and \( n \leq \omega \) is the largest number such that \( x \in V_n \), and \( c(x) = p(x) \) if \( x \in \text{dom}(p) \). It is not difficult to check that \( c \) is a \( \Gamma \)-coloring extending \( p \). Moreover, the coherence of the well-ordering \( \prec \) guarantees that for each \( n \leq \omega \), \( c \upharpoonright V_n \) holds. In each model \( V_n \) for \( n \leq \omega \), the coloring \( c \upharpoonright V_n \) is associated with a balanced virtual condition \( \bar{p}_n \) for the poset \( P(\Gamma) \) by Theorem 8.1.2. The sequence \( \langle \bar{p}_n \rangle_{n \leq \omega} \) is as desired.

**Corollary 9.4.25.** Let \( \Gamma \) be the graph on \( \mathbb{R}^2 \) connecting points \( x_0, x_1 \) if they have a nonzero rational distance.

1. In the \( P(\Gamma) \)-extension of the symmetric Solovay model, the orbit divide is preserved;

2. it is consistent relative to an inaccessible cardinal that \( \text{ZF+DC} \) holds, the chromatic number of \( \Gamma \) is countable, and the orbit divide is preserved.
Example 9.4.26. Let $\Gamma$ be a Borel hypergraph on a Polish space $X$ which is half-countable in the sense of Definition 8.6.4. Let $J$ be a Borel ideal on $\omega$ which is not generated by a single infinite set over the Fréchet ideal. The poset $P(\Gamma, J)$ of Definition 8.6.3 is nested balanced.

Proof. Let $p \in P(\Gamma, J)$ be an arbitrary condition. Consider the poset $P = \text{Coll}(\omega_1, 2^\omega) \times \prod n Q_n$ where each $Q_n$ is a single copy of Sacks forcing and the product is taken with full support. Let $G, y_n : n \in \omega$ be the generic objects for the poset $P$ and let $V_n = V[G, y_m : m \geq n]$ for each number $\omega$ and $V_\omega = \bigcap_n V_n$.

Then $\langle V_n : n \in \omega \rangle$ is a choice-coherent, generic and nontrivial sequence of models by Example 4.3.5 and Theorem 4.3.6; note that each of the models satisfies the Continuum Hypothesis. We will produce a decreasing sequence $\langle \bar{p} : n \leq \omega \rangle$ of balanced virtual conditions below $p$ for the respective models $V_n$. Let $\langle a_n : n \leq \omega \rangle \in V$ be a sequence of pairwise disjoint infinite subsets of $\omega$ which are disjoint from $\text{dom}(p)$ and such that $\bigcup_n a_n \in J$.

Claim 9.4.27. For each $n \leq \omega$, in the model $V_n$ there is a $\Gamma$-coloring $d : X \rightarrow \omega$ such that $p < d$ and for each $x \in X \setminus \text{dom}(p)$, $d(x) \in a_n$.

Proof. Use the fact that the chromatic number of $\Gamma$ is countable (as in the proof of Theorem 8.6.5(3)) to find a $\Gamma$-coloring $e : X \rightarrow a_n$, and let $d$ be defined by $d(x) = p(x)$ if $x \in \text{dom}(p)$ and $d(x) = e(x)$ if $x \notin \text{dom}(p)$. \hfill \Box

Let $\prec$ be a coherent wellordering of $\mathcal{P}(X \times \omega)$, and for each $n \leq \omega$ let $d_n$ be the $\prec$-least coloring as in the claim in the model $V_n$. Finally, in the model $V_0$, define a map $c : X \rightarrow \omega$ as follows: $c(x) = d_n(x)$ if $x \notin \text{dom}(p)$ and $n \leq \omega$ is the largest number such that $x \in V_n$, and $c(x) = p(x)$ if $x \in \text{dom}(p)$. We need to check that $c$ is a $\Gamma$-coloring extending $p$. Let $a \in \Gamma$ be a hyperedge. If $a \subset V_\omega$, then $a$ is not $c$-monochromatic since it is not $d_\omega$-monochromatic. If $a$ is a subset of some $V_n \setminus V_{n+1}$, then it is not $c$-monochromatic since it is not $d_n$-monochromatic. Otherwise, $a$ is not monochromatic since the sets $a_n$ for $n \in \omega$ are pairwise disjoint. Moreover, the coherence of the well-ordering $\prec$ guarantees that for each $n \leq \omega$, $c \upharpoonright V_n \in V_n$ holds. In each model $V_n$ for $n \leq \omega$, the coloring $c \upharpoonright V_n$ is associated with a balanced virtual condition $\bar{p}_n$ for the poset $P(\Gamma)$ by Theorem 8.6.5. The sequence $\langle \bar{p}_n : n \leq \omega \rangle$ is as desired. \hfill \Box

Corollary 9.4.28. Consider the hypergraph $\Gamma$ on $\mathbb{R}^2$ of arity 3 defined by $a \in \Gamma$ just in case $a$ consists of the vertices of an equilateral triangle.

1. In the $P(\Gamma, J)$-extension of the symmetric Solovay model, the orbit divide is preserved;

2. it is consistent relative to an inaccessible cardinal that $ZF$ holds, the chromatic number of $\Gamma$ is countable, and the orbit divide is preserved.

To conclude this section, we show that several natural posets introduced earlier in the book are not nested balanced because of an underlying $ZF$ reason: the object they add implies the inequality $|E_1| \leq |F|$ for some orbit equivalence
relation \( F \). In fact, it implies the existence of a \textit{countable complete section} for \( E_1 \): a set \( C \subset (2^\omega)^\omega \) which meets each \( E_1 \)-class in a nonempty countable set. It is immediate that the existence of a countable complete section \( C \) implies the inequality \( |E_1| \leq |F_2| \) since one can send a given \( E_1 \)-class to the \( F_2 \)-class of all enumerations of the intersection of \( C \) with the given \( E_1 \)-class. Despite appearances, in ZF+DC the existence of a countable complete section does not imply that \( |E_1| \leq |F| \) for a countable Borel equivalence relation \( F \).

In the following theorem and corollaries, we neglect the real parameter necessary to define the Polish spaces in question. For an element \( x \in (2^\omega)^\omega \), write \( x \setminus n \) for the element of \( (2^\omega)^\omega \) defined by \((x \setminus n)(m) = x(m)\) if \( m \geq n \), and \((x \setminus n)(m) = \) the infinite sequence of zeros if \( m < n \).

**Theorem 9.4.29.** (ZF) Suppose that there is a Polish space \( Y \) and a set \( b \subset Y \) such that for each \( x \in X = (2^\omega)^\omega \), the sequence \( \langle X \cap L[b, x \setminus n] : n \in \omega \rangle \) is eventually constant. Then there is a countable complete section for \( E_1 \).

**Proof.** Let \( x \in X \), and for each number \( m \in \omega \) write \( V_m(x) \) for the model \( L[b, x \setminus m] \). As an initial observation, note that by a standard condensation argument the model \( V_m(x) \) is a model of ZFC+CH, and its constructibility ordering orders its elements of \( X \) in ordertype \( \omega_1^{V_m(x)} \).

For natural numbers \( n \leq m \in \omega \) define \( \rho_{nm}(x) \) to be the index of \( x \setminus n \) in the constructibility well-ordering of \( V_m(x) \) if \( x \setminus n \in V_m(x) \), and let \( \rho_{nm}(x) = 0 \) otherwise. Let \( \alpha(x) = \limsup_n \sup_{m \geq n} \rho_{nm}(x) \). The definition of the ordinal \( \alpha(x) \) does not depend on the choice of \( x \) within its equivalence class by the stabilization assumption. Also, \( \alpha(x) \) is a countable ordinal. To see this, work in the model \( V_0(x) \) and evaluate the ordinal \( \alpha(x) \) there. Since \( V_0(x) \) is a model of choice, its \( \omega_1 \) is regular and so \( \alpha(x) \in \omega_1^{V_0(x)} \). Since \( \omega_1^{V_0(x)} \leq \omega_1 \), the countability of \( \alpha(x) \) follows.

Finally, let \( C = \{ x \in X : \exists n \ x = x \setminus n \text{ and for all } m \geq n, x \setminus n \in V_m(x) \text{ and } \rho_{nm}(x) \leq \alpha(x) \} \). To see that the set \( C \) meets every \( E_1 \) class in a nonempty countable set, let \( z \in X \) be arbitrary. The definition of the ordinal \( \alpha(z) \) shows that there is a number \( n \in \omega \) such that for all \( m \geq n, z \setminus n \in V_m(z) \) and is enumerated before \( \alpha(z) \) there; clearly, letting \( x = z \setminus n \) we get \( x \in C \cap [z]_{E_1} \). Also, note that \( C \cap [z]_{E_1} \subset \{ x \in X : \exists n \ x \in V_n(z) \} \) and the rank of \( x \) in the constructibility well-ordering of \( V_n(z) \) is \( \leq \alpha(z) \) and observe that the latter set is countable as the ordinal \( \alpha(z) \) is countable.

**Corollary 9.4.30.** (ZF) Let \( \Gamma \) be the rectangle hypergraph on the product \( Y \times Z \) of uncountable Polish spaces as in Definition 8.6.8. If the chromatic number of \( \Gamma \) is countable, then \( E_1 \) has a countable complete section.

In particular, in ZF the chromatic number of \( \Gamma \) equal to \( \aleph_0 \) implies \( |E_1| \leq |F_2| \). In ZFC, the poset \( P(\Gamma, J) \) of Definition 8.6.8 designed to add a countable \( \Gamma \)-coloring over the symmetric Solovay model is not nested balanced.

**Proof.** Let \( c : Y \times Z \to \omega \) be the coloring. Fix a countable basis for the space \( Y \) and let \( d = \{ (z_0, z_1, O, m) : z_0, z_1 \in Z \text{ are distinct points, } O \subset Y \text{ is a basic} \)
open set, \( m \in \omega \), and there is \( y \in O \) such that \( c(y, z_0) = c(y, z_1) = m \). We shall show that for each point \( x \in (2^\omega)^\omega \), the sequence \( \langle X \cap L[c, d, x \setminus n] : n \in \omega \rangle \) is eventually constant; a reference to Theorem 9.4.29 will then complete the argument.

Write \( V_n \) for the model \( L[c, d, x \setminus n] \). Suppose towards contradiction that the sequence \( \langle X \cap V_n : n \in \omega \rangle \) never stabilizes. The sequence \( \langle V_1^n : n \in \omega \rangle \) must stabilize though by a wellfoundedness argument. Let \( n \in \omega \) be such that \( V_1^n = V_1^{n+1} \) and \( X \cap V_n \neq X \cap V_{n+1}(x) \). Let \( y_0 \in Y \cap V_n \setminus V_{n+1} \) be an arbitrary point. Since the set \( Z \cap V_{n+1} \) is uncountable in the model \( V_n \), by a counting argument in the model \( V_n \) there must be a number \( m \in \omega \) and distinct points \( z_0, z_1 \in Z \cap V_{n+1} \) such that \( c(y_0, z_0) = c(y_0, z_1) = m \). If \( y_0 \in Y \) was the unique point satisfying these equalities then it can be constructed from \( d \) and the points \( z_0, z_1 \); as such, it would belong to the model \( V_{n+1} \), which is not the case. Thus, there is a point \( y_1 \in Y \) distinct from \( y_0 \) such that \( c(y_1, z_0) = c(y_0, z_1) = m \). However, then \( \{y_0, y_1\} \times \{z_0, z_1\} \) is a monochromatic \( \Gamma \)-edge, a contradiction.

**Corollary 9.4.31.** (ZF) Let \( Y \) be a separable Banach space. If there is a basis for \( Y \) over the rationals then there is a countable complete section for \( E_1 \).

In particular, in ZF the existence of a Hamel basis for \( \mathbb{R} \) implies \( |E_1| \leq |\mathcal{P}[2]| \).

In ZFC, the poset of Example 6.2.16 designed to add a Hamel basis over the symmetric Solovay model is not nested balanced.

**Proof.** Let \( b \subset Y \) be a basis for \( Y \) over the rationals. Let \( c = \{\langle y, O_i : i \in k \rangle : y \in Y, O_i \subset Y \) are basic open sets and there are points \( y_i \in O_i \cap b \) such that \( y \) is a linear combination of \( y_i : i \in k \) with nonzero rational coefficients\}. Let \( x \in X = (2^\omega)^\omega \) be an arbitrary point. We will show that the sequence \( \langle X \cap L[b, c, x \setminus n] : n \in \omega \rangle \) is eventually constant. A reference to Theorem 9.4.29 then concludes the proof.

Write \( V_n \) for the model \( L[b, c, x \setminus n] \). As an initial observation, note that the set \( b \cap V_n \) is a basis in \( V_n \); for every \( y \in Y \setminus V_n \), the unique linear combination of elements of \( b \) with nonzero rational coefficients yielding \( y \) can be constructed from \( y \) and \( c \) and so belongs to \( V_n \). Now, suppose towards contradiction that the sequence \( \langle X \cap V_n : n \in \omega \rangle \) is not eventually constant. Thus, the set \( a = \{n \in \omega : X \cap V_n \neq X \cap V_{n+1}\} \subset \omega \) is infinite. For every \( n \in a \) the set \( b \cap V_n \cap V_{n+1} \) is nonempty. To see this, note that \( b \cap V_n \) spans all of \( Y \cap V_n \), which is a strictly larger set than \( Y \cap V_{n+1} \). Let \( y_n \in b \cap V_n \cap V_{n+1} \) be the first such element in the constructibility well-ordering of \( V_n \). Let \( q_n \) be the first nonzero rational in some fixed enumeration such that the norm of \( q_n \cdot y_n \) is smaller than \( 2^{−n} \). Let \( z_n = \sum(q_m \cdot y_m : m \geq n, m \in a) \). It is clear that \( z_n \in V_n \) holds. Note that if \( n \in a \), then also \( z_n \notin V_{n+1} \) holds, since \( y_n = \frac{1}{q_n}(z_n - z_m) \), \( z_m \in V_{n+1} \), and \( y_n \notin V_{n+1} \) where \( m \in a \) is the smallest number greater than \( n \).

Let \( n_0 \in a \) be any number. The point \( z_{n_0} \in Y \) can be expressed as a linear combination \( \phi \) of elements of \( b \) belonging to \( V_{n_0} \). Let \( k \in \omega \) be a number so large that all elements of \( b \) used in \( \phi \) belonging to \( V_k \) belong to the intersection \( \cap_n V_n \); therefore \( \phi \) decomposes into two linear combinations \( \psi_0 + \psi_1 \), \( \psi_0 \) using
only elements of \( A \setminus V_k \) and \( \psi_1 \) using only elements of \( b \cap \bigcap n V_n \). Let \( \langle n_j : j \leq i \rangle \) be an increasing enumeration of all numbers in the set \( a \) between \( n_0 \) and some \( n_i > k \). Then, \( \psi_0 + \psi_1 = z_n = \sum_{j < i} q_{n_j} \cdot y_{n_j} + z_{n_i} \). In other words, the point \( z_{n_i} \in V_{n_i} \) can be expressed as a linear combination of elements of \( b \setminus V_{n_i} \) and elements of \( b \cap V_{n_i+1} \). The first part of the combination must be empty since \( b \cap V_{n_i} \) is a basis in \( V_{n_i} \). This indicates that \( z_{n_i} \in V_{n_i+1} \), contradicting the last sentence of the previous paragraph.

We conclude this section with a question. The nested balance of Suslin posets does not seem to be closed under countable products, even though no counterexamples can be found among the posets discussed in this section. This leads to the following:

**Question 9.4.32.** Are there \( \Sigma^2_1 \) sentences \( \phi_0, \phi_1 \) such that each of them is consistent with \( ZF + DC + \) the orbit divide is preserved while \( ZF + DC + \phi_0 + \phi_1 \) proves that the orbit divide is not preserved?

### 9.5 The \( \mathbb{E}_{K_\sigma} \) divide

Recall that \( \mathbb{E}_{K_\sigma} \) is the equivalence relation on the space of all functions in \( \omega^\omega \) pointwise dominated by the identity function, connecting functions \( x_0, x_1 \) if the function \( x_0 - x_1 \) is bounded. The central position of this equivalence relation is documented by the fact that it is \( K_\sigma \), and that every \( K_\sigma \)-equivalence relation is Borel reducible to it by a result of Rosendal [61]. It is well-known that the equivalence relation \( \mathbb{E}^0_0 \) is not Borel reducible to \( \mathbb{E}_{K_\sigma} \) [39, Lemma 6.1.1], and among the equivalence relations classifiable by countable structures, it even has a central place in this regard. In this section, we show that in certain extensions of the symmetric Solovay model, this non-reducibility result is translated to a cardinality preservation result. The reader should note that in \( ZF \), the inequality \( |\mathbb{E}_0| \leq |2^\omega| \) implies \( |\mathbb{E}_0| \leq |(2^\omega)^\omega| = |2^\omega| \), and therefore preserving the \( \mathbb{E}_{K_\sigma} \)-divide is strictly more difficult than preserving the smooth divide.

The preservation theorem uses a geometric tool dual to the nested sequences of models used in the preservation of the orbit divide.

**Definition 9.5.1.** A pod is a sequence \( \langle V_n : n \in \omega \rangle \) of generic extensions of the ground model such that \( V \subset V_0 \subset V_1 \subset V_2 \subset \ldots \). Models \( V[H_0] \) and \( V[H_1] \) are mutually generic over the pod if \( \bigcup_n V_n \subset V[H_0] \cap V[H_1] \) and for disjoint sets \( a_0 \in V[H_0] \) and \( a_1 \in V[H_1] \) of ordinals there are disjoint sets \( b_0, b_1 \subset \bigcup_n V_n \) of ordinals such that \( a_0 \subset b_0 \) and \( a_1 \subset b_1 \).

To appreciate the import of this definition, consider the fact that unless the sequence \( \langle V_n : n \in \omega \rangle \) is eventually constant, the union \( \bigcup_n V_n \) is not a model of \( ZF \). Thus, the mutual genericity would be impossible to formulate were it not for the parallel with Proposition 1.8.8. The main feature of mutually generic extensions used in this section is the following:
Proposition 9.5.2. Let $V[H_0], V[H_1]$ be extensions mutually generic over some pod $\langle V_n : n \in \omega \rangle$. Every $E_{\kappa^*}$-class represented in both $V[H_0], V[H_1]$ is represented in $\bigcup_n V_n$.

Proof. Let $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ be points in $\omega^\omega$ below the identity function which are $E_{\kappa^*}$-related. Find a natural number $m \in \omega$ such that for all $k \in \omega$, $|x_0(k) - x_1(k)| \leq m$. Let $a_0 \subset \omega \times \omega$ be the set $x_0$, and let $a_1 \subset \omega \times \omega$ be the set $\{(k, l) : k \in \omega, |l - x_1| > m\}$. Thus, $a_0 \in V[H_0]$ and $a_1 \in V[H_1]$ are disjoint sets, and therefore can be separated by some disjoint sets $b_0, b_1 \subset \omega \times \omega$ in $\bigcup_n V_n$. The vertical sections of the set $b_0$ are all nonempty and so there must be a function $y \in \bigcup_n V_n$ uniformizing it. Since $y \cap a_1 = 0$, it must be the case that $y$ is $E_{\kappa^*}$-related to both $x_0, x_1$ as desired. \qed

The conclusion of the proposition fails for such simple pinned equivalence relations as $E^*_0$, and this is exactly the point exploited in this section. We will need a notion of balance for a pod.

Definition 9.5.3. A Suslin poset $P$ is pod balanced if for every pod $\langle V_n : n \in \omega \rangle$, every pair $V[H_0], V[H_1]$ of extensions mutually generic over the pod, every sequence $\langle p_n : n \in \omega \rangle$ of balanced virtual conditions in $P$ in the respective models $V_n$, and all conditions $p_0 \in V[H_0], p_1 \in V[H_1]$ such that for all $n \in \omega$, $p_0, p_1 \leq p_n$ holds, it must be the case that $p_0, p_1$ are compatible conditions.

Theorem 9.5.4. In compactly balanced, pod balanced extensions of the symmetric Solovay model, $|\mathbb{E}_0^\omega| \leq |E_{\kappa^*}|$ holds.

Proof. To simplify the expressions, write $E = \mathbb{E}_0^\omega$ and $F = E_{\kappa^*}$. The main technical tool used in the proof is the countable support product $Q$ of countably many copies of the Vitali forcing. We first study the product $Q$ in its own right.

Fact 9.5.5. The poset $Q$ is proper, bounding, and adds no independent reals.

Proof. \(??\) \qed

The poset $Q$ is designed to add an interesting pod. In the $Q$-extension, let $\langle x_n : n \in \omega \rangle$ be the sequence of points Vitali-generic over the ground model added by the product $Q$. Let $A \subset \omega \times 2^\omega$ be the relation consisting of all pairs $(n, x)$ such that $x$ is $E_{\kappa^*}$-related to $x_n$. For each number $n \in \omega$, let $V_n$ be the model of all sets hereditarily definable from $x_n$ for $m \in n$, $A$, and some parameters in the ground model. Also, let $M$ be the model of all sets hereditarily definable from the set $A$ and its elements, and the parameters in the ground model. We have the following:

Claim 9.5.6. 1. $V_0 \subset V_1 \subset V_2 \subset \cdots \subset M$;

2. the sequence $\langle V_n : n \in \omega \rangle$ belongs to $M$;

3. every set of ordinals in $M$ belongs to $\bigcup_n V_n$;

4. the model $M$ does not contain any uniformization of the set $A$. 


Figure 9.3: A balanced pod.
Note also that the models $V_n$ satisfy the axiom of choice, while the model $M$ fails even DC, as item (3) shows.

**Proof.** The first item follows directly from the definitions of the models $V_n$ and $M$: if $m \in n \in \omega$ are numbers then there are fewer definitions of elements of the model $V_m$ than of elements of the model $V_n$, and certainly fewer than the definitions of elements the model $M$. For (2), the sequence $\langle V_n : n \in \omega \rangle$ is definable from the set $A$, since in the definition of the model $V_n$ it certainly does not matter which $E_0$-equivalents of the points $x_m$ for $m \in n$ one uses as the parameters. For (3), any definition of a set of ordinals in the model $M$ uses only finitely many parameters, and then must belong to the model $V_n$ for a number $n$ large enough so that the parameters of the definition are definable from $x_m$ for $m \in n$, the set $A$, and some elements of the ground model.

Lastly, for (4), return to the ground model and assume towards contradiction that some condition in $Q$ forces (4) to fail. Then, in the poset $Q$, there must be a condition $q$, numbers $n, k \in \omega$ and a formula $\phi$ such that $q \models$ the formula $\phi$ uses only parameters $\dot{x}_m$ for $m \in n$, $A$, and some parameters in the ground model, it defines a uniformization $y$ of the set $A$, and $y(n)$ agrees with $\dot{x}_n$ from $k$ on. Let $\langle x_m : m \in \omega \rangle$ be a $Q$-generic sequence of points meeting the condition $q$. By a basic analysis of the Vitali forcing, there is also a point $x'_n \in 2^\omega$ $E_0$-related to $x_n$ which differs from $x_n$ at some point past $k$ and such that $\langle x_m : m \neq n, x'_n \rangle$ is a $Q$-generic sequence meeting the condition $q$. Applying the forcing theorem to the two sequences $\langle x_m : m \in \omega \rangle$ and $\langle x_m : m \neq n, x'_n \rangle$, we see that the formula $\phi$ should define a uniformization $y$ of the set $A$ such that $y(n)$ is equal to both $x_n$ and $x'_n$ beyond $k$. This is impossible though as the two sequences $x_n, x'_n$ do differ at some point past $k$. 

Now, towards the proof of the theorem. Let $P$ be a Suslin forcing. Let $\kappa$ be an inaccessible cardinal such that $V_\kappa \models P$ is compactly balanced and pod balanced in every forcing extension. Let $W$ be a symmetric Solovay model derived from $\kappa$ and work in the model $W$. Let $p \in P$ be a condition and $\tau$ be a $P$-name such that $p \models \tau$ is a function from the $E$-quotient space to the $F$-quotient space. We must find a condition $q \leq p$ and distinct $E$-classes which are forced by $q$ to have the same $\tau$-image.

The condition $p$ and the name $\tau$ must be definable from a parameter $z \in 2^\omega$ and additional parameters from the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of size $< \kappa$ such that $z \in V[K]$. Working in the model $V[K]$, let $Q_0$ be the poset $\mathcal{P}(\omega)$ modulo finite, and let $Q_1$ be the product of countably many copies of the Vitali forcing as discussed in the preamble of the present proof. Let $\langle U, x_n : n \in \omega \rangle$ be objects generic over $V[K]$ for the poset $Q_0 \times Q_1$; in particular, $U$ is an ultrafilter on $\omega$, and $\langle x_n : n \in \omega \rangle$ is a sequence of points in $2^\omega$. Since the poset $Q_0$ is $\sigma$-closed, the models $V[K]$ and $V[K][U]$ compute the poset $Q_1$ in the same way, in particular $V[K][U][x_n : n \in \omega]$ is a $Q_1$-extension of $V[K][U]$ and $2^\omega \cap V[K][U][x_n : n \in \omega] = 2^\omega \cap V[K][x_n : n \in \omega]$. Since (Fact 9.5.5) the poset $Q_1$ adds no independent reals, a density argument shows that $U$ generates an ultrafilter in the model $V[K][U][x_n : n \in \omega]$. 


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Let $A \subseteq \omega \times 2^\omega$ be the relation consisting of all pairs $\langle n, x \rangle$ such that $x$ is $E_0$-related to $x_n$. In the model $V[K][U][x_n : n \in \omega]$ form the models $V_n$ of all sets hereditarily definable from parameters $x_m$ for $m \in n$, $A$, and parameters in the model $V[K][U]$, and let $M$ be the model of all sets hereditarily definable from the set $A$ and its elements and some other parameters in $V[K][U]$. Note that by Vopěnka’s theorem [36, page 249], the sequence $\langle x_n : n \in \omega \rangle$ is generic over the model $M$ and the forcing adding it restores the axiom of choice in its generic extension of $M$.

The following claim makes a critical use of the compact balance assumption.

Claim 9.5.7. In the model $M$:

1. there is a sequence $\langle \bar{p}_n : n \in \omega \rangle$ such that $\bar{p}_n$ is a balanced virtual condition for $V_n$, and $p \geq \bar{p}_0 \geq \bar{p}_1 \geq \bar{p}_2 \geq \ldots$;

2. any sequence as in (1) must have a lower bound.

Proof. Start with item (1). Working in $V[K][U]$, fix $R_n$-names $\chi_n$ so that $R_n$ is the product of the first $n$-many copies of the Vitali forcing in the product $Q_1$ and $\chi_n$ is a name for a virtual balanced condition stronger than $p$ and than all $\chi_m$ for $m \in n$. This is possible by Definition 9.2.1(2). Now, in the model $V[K][U][x_n : n \in \omega]$ for each $n \in \omega$ let $\bar{p}_n$ be the $U$-limit of the sequence $\langle \chi_n / \langle x_m^i : m \in n \rangle : i \in \omega \rangle$ in the compact space of virtual balanced conditions in the model $V_n$, where $x_m^i \in 2^\omega$ is the binary sequence obtained from $x_m$ by replacing its first $i$ many entries with 0. The limit exists since $U$ generates an ultrafilter in all models $V_n$. The balanced virtual conditions $\bar{p}_n$ for $n \in \omega$ form a decreasing sequence by Definition 9.2.1(3). Also, the definition of the limit does not really depend on the sequence $\langle x_n : n \in \omega \rangle$ but only on the set $A$, and therefore $\langle \bar{p}_n : n \in \omega \rangle \subseteq M$ as desired.

Item (2) is somewhat trickier. Let $N$ be some generic extension of $M$ which restores the axiom of choice. In the model $N$, let $B$ be the compact Hausdorff space of balanced conditions for $P$. For each number $n \in \omega$, consider the set $C_n = \{ \bar{p} \in B : \bar{p} \leq \bar{p}_n \}$. This is a nonempty and closed subset of $B$ by Definition 9.2.1(2) and (3). Since the closed sets $C_n$ for $n \in \omega$ form a decreasing sequence, a compactness argument shows that there must be some $\bar{p} \in \bigcap_n C_n$. This is the required lower bound of the sequence $\langle \bar{p}_n : n \in \omega \rangle$.

Work in the model $M$. By the forcing theorem, there must be a poset $R$ of rank $< \kappa$ restoring the axiom of choice and $R$-names $\sigma$ for a condition in the poset $P$ stronger than all the balanced virtual conditions $\bar{p}_n$ for $n \in \omega$ and a name $\eta$ for an element of $\omega^\omega$ such that $R \models \text{Coll}(\omega, < \kappa) \models \sigma \equiv \bar{p} \tau([y]E) = [\eta]F$ for all functions $y$ uniformizing the relation $A$. The following claim uses the pod balance assumption.

Claim 9.5.8. There is a condition $r \in R$ and a point $u \in \bigcup_n V_n$ such that $r \models \eta$ is $F$-related to $\bar{u}$. 
Proof. Suppose towards contradiction that this fails. Let $H_0, H_1 \subset R$ be filters mutually generic over the model $M$. Observe that the models $M[H_0], M[H_1]$ are mutually generic over the pod $\langle V_n : n \in \omega \rangle$: if $a_0 \in M[H_0]$ and $a_1 \in M[H_1]$ are disjoint sets of ordinals, they must be separated by disjoint sets $b_0, b_1$ of ordinals in the model $M$ by the mutual genericity over $M$ and Proposition 1.8.8. By Claim 9.5.6 though, there must be a number $n \in \omega$ such that $b_0, b_1 \in V_n$. Now, let $p_0 = \sigma/H_0$, $u_0 = \eta/H_0$, $p_1 = \sigma/H_1$ and $u_1 = \eta/H_1$. By the pod balance of the poset $P$, the conditions $p_0, p_1$ have a lower bound in the poset $P$. By the forcing theorem in $M$ and the initial contradictory assumption, $u_0, u_1$ are not $F$-related to any element of $M$. By Proposition 9.5.2, $u_0$ cannot be $F$-related to $u_1$. Thus, the lower bound of $p_0, p_1$ would have to force the unique $E$-class of all uniformizations of the set $A$ to be mapped by $\tau$ simultaneously to $[u_0]_F$ and to $[u_1]_F$, which is impossible. 

Now, the claim together with the forcing theorem means that there is a natural number $n \in \omega$, a point $u \in V_n$ in $\omega^\omega$ and in the model $V_n$ a poset $S$ of size $< \kappa$ and an $S$-name $\chi$ for a condition in the poset $P$ stronger than $\bar{p}$ and an $S$-name $\xi$ for an element of $(2^\omega)^\omega$ which is not $E$-related to any point in $V_n$ such that $S \forces \operatorname{Coll}(\omega, \kappa) \forces \chi \forces \tau(\bar{\xi}) = \bar{u}$. Namely, the poset $S$ is a remainder of the iteration of $Q$ and $R$ after the first $n$-many $Q$-generic reals are added. Let $H_0, H_1 \subset S$ be filters mutually generic over the model $V_n = V[K][U][\vec{m} : m \in n]$ and write $p_0 = \chi/H_0$, $y_0 = \xi/H_0$, $p_1 = \chi/H_1$, and $y_1 = \xi/H_1$. By the balance of the virtual condition $\bar{p}_n$, the conditions $p_0, p_1$ have a lower bound. Since the equivalence relation $E$ is pinned, the points $y_0, y_1$ are not $E$-related. By the forcing theorem, the lower bound of the conditions $p_0, p_1$ forces $\tau([y_0]_E) = \tau([y_1]_E) = [u]_F$. This shows that $\tau$ cannot be an injection and completes the proof of the theorem. 

Example 9.5.9. The poset $P = \mathcal{P}(\omega)$ modulo finite is compactly balanced and pod balanced. Compact balance was proved in Example 9.2.4. Towards the pod balance, recall that balanced conditions are exactly nonprincipal ultrafilters on $\omega$. Now, if $V[H_0], V[H_1]$ are models mutually generic over a pod $\langle V_n : n \in \omega \rangle$ and $p_0, p_1 \in P$ are incompatible conditions in the respective models, then by the mutual genericity there must be a number $n \in \omega$ and incompatible conditions $q_0, q_1 \in V_n$ such that $q_0 \geq p_0$ and $q_1 \geq p_1$. This means that $p_0, p_1$ cannot strengthen the same balanced virtual condition in $V_n$, since that condition would have to be stronger than both $q_0, q_1$.

Corollary 9.5.10. 1. Let $P = \mathcal{P}(\omega)$ modulo finite. Then in the $P$-extension of the symmetric Solovay model, $|E^0_\omega| \not\leq |E_{K_\omega}|$ holds.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a nonprincipal ultrafilter on $\omega$ and yet $|E^0_\omega| \not\leq |E_{K_\omega}|$.

Example 9.5.11. Let $\Gamma$ be a locally finite Borel bipartite graph on a Polish space $X$ satisfying the Hall’s marriage condition. Let $P$ be the poset adding a perfect matching to $\Gamma$ as in Example 6.2.4. Then the poset $P$ is compactly balanced and pod balanced. The compact balance was proved in Example 9.2.13.
For the pod balance, recall that balanced conditions are exactly the perfect matchings of the graph $\Gamma$. Let $E$ be the $\Gamma$-path connectivity equivalence relation. Let $E$ be the $\Gamma$-path connectivity equivalence relation. Let $V[H_0], V[H_1]$ be models mutually generic over a pod $\langle V_n : n \in \omega \rangle$. Since $E$ is a countable Borel equivalence relation, it is Borel reducible to $E_{K_s}$, and by Proposition 9.5.2 all $E$-classes represented in both $V[H_0]$ and $V[H_1]$ are represented in $\bigcup_n V_n$, and in fact are subsets of $\bigcup_n V_n$. Now, if $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ extend the same balanced virtual conditions for the models $V_n$, they must identify the same perfect matching on $\Gamma \cap \bigcup_n V_n$. Then, they can have no point of disagreement anywhere else and so $p_0, p_1$ are compatible in the poset $P$.

**Corollary 9.5.12.** Let $\Gamma$ be a Borel locally finite graph on a Polish space $X$ satisfying the Hall’s marriage condition.

1. Let $P$ be the poset adding a perfect matching to $\Gamma$ with countable approximations. Then in the $P$-extension of the symmetric Solovay model, $|E^*_{\omega_0}| \not\leq |E_{K_s}|$ holds.

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, $\Gamma$ has a perfect matching, and yet $|E^*_{\omega_0}| \not\leq |E_{K_s}|$.

**Example 9.5.13.** Let $P$ be the poset adding a linear ordering on the $E_{K_s}$-quotient space as in Example 8.5.5. Then the poset $P$ is compactly balanced and pod balanced. The compact balance was proved in Example 9.2.11. For the pod balance, note that the balanced conditions correspond exactly to linear orderings of the $E_{K_s}$-quotient space. Now, suppose that $\langle V_n : n \in \omega \rangle$ is a pod and $V[H_0], V[H_1]$ are extensions mutually generic over it. By Proposition 9.5.2, the only $E_{K_s}$-classes represented in both $V[H_0]$ and $V[H_1]$ are represented in $\bigcup_n V_n$. Thus, if $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions extending the same balanced virtual conditions in the models $V_n$ for all $n \in \omega$, then $p_0, p_1$ both linearly order all the $E_{K_s}$-classes appearing in both $V[H_0]$ and $V[H_1]$, and order them in the same way. It follows that $p_0, p_1$ are compatible as desired.

**Corollary 9.5.14.**

1. Let $P$ be the $E_{K_s}$-linearization poset. Then in the $P$-extension of the symmetric Solovay model, $|E^*_{\omega_0}| \not\leq |E_{K_s}|$ holds.

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, the $E_{K_s}$-quotient space is linearly ordered and yet $|E^*_{\omega_0}| \not\leq |E_{K_s}|$.

This section leaves perhaps more questions open than it resolves. We content ourselves with quoting one glaring case:

**Question 9.5.15.** Does the conclusion of Theorem 9.5.4 stay in force if the assumption of pod balance on $P$ is dropped?

### 9.6 The pinned divide

No unpinned equivalence relation can be reduced by a Borel function to a pinned one, see Fact 2.3.2. This feature persists to the cardinality computations in balanced extensions of the choiceless Solovay model, with a small proviso.
Theorem 9.6.1. Let $E, F$ be Borel equivalence relations on respective Polish spaces $X, Y$, $E$ pinned and $F$ unpinned. In a balanced extension of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, $|F| \not\leq |E|$ holds.

We do not know if the increase in the large cardinal strength of the large cardinal hypothesis is necessary. We do know though that one has to consider Suslin posets which are balanced everywhere below $\kappa$ as opposed to just cofinally balanced. This is clear from Theorem 9.6.2 below.

Proof. Let $\kappa$ be an inaccessible cardinal which is a limit of inaccessibles. Let $P$ be a Suslin poset which is balanced below $\kappa$. This is clear from Theorem 9.6.2 below. Towards a contradiction, suppose that $p \in P$ is a condition and $\tau$ a $P$-name for an injection from $F$-classes to $E$-classes. Both $p, \tau$ are definable from some parameter $z \in 2^\omega$ and parameters in the ground model. The assumptions imply that there is an inaccessible cardinal $\lambda < \kappa$ of $V$ and a filter $K \subseteq \text{Coll}(\omega, < \lambda)$ generic over $V$ such that $z \in V[K]$.

Work in the model $V[K]$. The balanced assumption on the poset $P$ implies that there is a balanced virtual condition $\bar{p} \leq p$ in $P$. Theorem 9.6.1 shows that $E$ is pinned in $V[K]$ and therefore has at most $\mathfrak{c} = \aleph_1$ many virtual classes, while Theorem 2.5.11 shows that $F$ has at least $2^{\aleph_1}$ many virtual classes. The argument splits into two cases:

Case 1. For every poset $R$ of size $< \kappa$, every $R$-name for a condition $\sigma \leq \bar{p}$ in the poset $P$, every $F$-pinned $R$-name $\eta$ for an element of $X$ and every $R$-name $\chi$ for an element of $X$ such that $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash \tau((\eta)_E) = [\chi]_E$, the name $\chi$ is $E$-pinned.

In this case, by a counting argument with the virtual $E$- and $F$-classes, it must be the case that there are posets $R_0, R_1$ and respective names $\sigma_0, \eta_0, \chi_0$ and $\sigma_1, \eta_1, \chi_1$ on them such that $\eta_0, \eta_1$ are $F$-pinned, $\chi_0, \chi_1$ are $E$-pinned, $R_0 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash \tau(\eta_0) = [\chi_0]_E$ and $R_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash \tau(\eta_1) = [\chi_1]_E$ and $(R_0, \chi_0) \Vdash (R_1, \chi_1)$ holds while $(R_0, \eta_0) \Vdash (R_1, \eta_1)$ fails. Let $H_0 \subseteq R_0, H_1 \subseteq R_1$ be filters generic over $V[K]$, and write $r_0 = \sigma_0/H_0 \in P, r_1 = \sigma_1/H_1 \in P, x_0 = \chi_0/H_0 \in X, x_1 = \chi_1/H_1 \in X, y_0 = \eta_0/H_0 \in Y$ and $y_1 = \eta_1/H_1 \in Y$. The balance of the virtual condition $\bar{p}$ in $P$ implies that $r_0, r_1$ are compatible in $P$ with some lower bound $r \in P$, and the pinned assumptions imply that $x_0 E x_1$ holds and $y_0 F y_1$ fails. Since $W$ is the symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem in these two models implies that in $W, r \Vdash \tau([\eta_0]_F) = [\bar{x}_0]_E$ and $\tau([\eta_1]_F) = [\bar{x}_1]_E$; this contradicts the assumption that $\tau$ is forced to be an injection.

Case 2. Case 1 fails. Then, there has to be poset $R$ of size $< \kappa$, every $R$-name for a condition $\sigma \leq \bar{p}$ in the poset $P$, every $F$-pinned $R$-name $\eta$ for an element of $X$ and every $R$-name $\chi$ for an element of $X$ such that $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash \tau(\eta) = [\chi]_E$, such that the name $\chi$ is not $E$-pinned. In such a case, there must be filters $H_0, H_1 \subseteq R$ mutually generic over $V[K]$ such that the points $x_0 = \chi/H_0, x_1 = \chi/H_1 \in X$ are $E$-unrelated. Let $r_0 = \sigma/H_0, r_1 = \sigma/H_1$, and
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Let \( y_0 = \eta_0/H_0, y_1 = \eta_1/H_1 \in Y \). The balance of the virtual condition \( \bar{p} \) implies that the conditions \( r_0, r_1 \in P \) are compatible with a lower bound \( r \), and the pinned assumptions imply that \( y_0 F y_1 \) holds. Since \( W \) is the symmetric Solovay extension of both models \( V[K][H_0] \) and \( V[K][H_1] \), the forcing theorem in these two models implies that in \( W, r \Vdash^P \tau([\bar{y}_0]_E) = [\bar{x}_0]_E \) and \( \tau([\bar{y}_1]_E) = [\bar{x}_1]_E \); this contradicts the assumption that \( \tau \) is forced to be a function.

The following two characterization theorems illustrate the possibilities of violating the pinned divide if the Suslin poset in question is only cofinally balanced as opposed to balanced in every forcing extension.

**Theorem 9.6.2.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \), Borel reducible to \( \mathbb{F}_2 \). The following are equivalent:

1. \( \mathbb{F}_2 \) is Borel reducible to \( E \);
2. in any cofinally balanced extension of a symmetric Solovay model, \( |E| \nleq |2^\omega| \) holds.

We do not know if it is possible to eliminate the assumption that \( E \) is Borel reducible to \( \mathbb{F}_2 \).

**Proof.** If (1) occurs then (2) holds by Corollary 9.1.5. Thus, assume that (1) fails. Let \( P \) be the collapse poset of Definition 6.3.2, collapsing \( |E| \) to \( |2^\omega| \). To confirm the failure of (2), we only have to show that \( P \) is cofinally balanced below \( \kappa \), where \( \kappa \) is any inaccessible cardinal. To do this, given an ordinal \( \lambda < \kappa \), consider the generic extension \( V[K] \) obtained by Coll(\( \omega, \lambda \)) followed by adding \( \sum_\omega \) many Cohen reals. In the model \( V[K] \), \( E \) has only \( \epsilon \) many virtual classes by Theorem 2.7.14. The poset \( P \) is then balanced in \( V[K] \) by Corollary 6.3.4.

**Theorem 9.6.3.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \).

The following are equivalent:

1. \( E \) is pinned;
2. in any cofinally balanced extension of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, \( |\mathbb{F}_2| \nleq |E| \) holds.

**Proof.** Suppose first that (1) fails. To confirm the failure of (2), consider the poset \( P \) from Definition 6.3.2 collapsing the cardinality of \( \mathbb{F}_2 \) to \( |E| \). Let \( \kappa \) be any inaccessible limit of inaccessible cardinals. To show that \( P \) witnesses the failure of (2), let \( \lambda < \kappa \) be an inaccessible cardinal and consider the generic extension \( V[K] \) by a generic filter \( K \subset \text{Coll}(\omega, < \kappa) \). Since \( E \) is unpinned, Theorem 2.5.11 shows that it has at least \( 2^{\aleph_1} \) many virtual classes in \( V[K] \). The equivalence relation \( \mathbb{F}_2 \) has likewise \( 2^\epsilon = 2^{\aleph_1} \) many virtual classes in \( V[K] \). Corollary 6.3.4 shows that the poset \( P \) is balanced in the model \( V[K] \) as desired.

Suppose now (1) holds. To confirm (2), let \( \kappa \) be any inaccessible cardinal and suppose that \( P \) is a Suslin forcing balanced cofinally below \( \kappa \); we must show that in the symmetric Solovay model \( W \), \( P \) forces \( |\mathbb{F}_2| \nleq |E| \). To this aim, suppose...
towards contradiction that some condition in $P$ forces the opposite. Consider the poset $Q$ collapsing the cardinality of $|E|$ to $|2^\omega|$ as isolated in Definition 6.3.2. In every forcing extension, the virtual quotient space of $E$ is identical to its quotient space, so it has size $\mathfrak{c}$ and by Corollary 6.3.4 the poset $Q$ is balanced. Thus $P \times Q$ is a Suslin forcing balanced cofinally below $\kappa$ by Theorem 5.2.10 and over the symmetric Solovay model it forces $|\mathcal{P}_2| \leq |E| \leq |2^\omega|$. This contradicts Corollary 9.1.5. \hfill $\square$
Chapter 10

Uniformization

The question whether various forms of uniformization hold in the models within purview of this book is one of the more slippery issues we set out to resolve.

10.1 Adequate Suslin forcing

In order to prove all forms of uniformization in a clean sweep, several definitions must be stated.

**Definition 10.1.1.** Let \( P \) be a Suslin forcing and \( \lambda \) be a cardinal.

1. Let \( Q \) be a poset and \( \tau \) be a \( Q \)-name for an element of \( P \). Say that the pair \( \langle Q, \tau \rangle \) is \( \lambda \)-adequate if for every virtual condition \( \langle R, \sigma \rangle \) in \( P \) such that \( |R| < \lambda \), either \( Q \times R \models \tau \leq \Sigma \sigma \) or \( Q \times R \models \tau \) is incompatible with every condition in the set \( \sigma \) in \( P \);

2. \( P \) is \( \lambda \)-adequate if every \( \lambda \)-adequate pair is balanced. \( P \) is adequate if it is \( \lambda \)-adequate for some \( \lambda \).

Note that every balanced pair is \( \lambda \)-adequate for every cardinal \( \lambda \). Thus, a Suslin poset is adequate if every reasonable attempt at constructing a balanced condition actually works. In the way of an initial example, consider the poset \( P \) of all infinite subsets of \( \omega \) ordered by inclusion and \( \lambda = \aleph_0 \). If \( \langle Q, \tau \rangle \) is a \( \lambda \)-adequate condition, then for every infinite set \( a \subset \omega \) it is the case that either \( Q \models \tau \setminus \hat{a} \) is finite or \( Q \models \tau \cap \hat{a} \) is finite. Let \( U = \{ a \subset \omega : Q \models \tau \setminus \hat{a} \text{ is finite} \} \) and observe that \( U \) is a nonprincipal ultrafilter on \( \omega \). Then, Theorem 7.1.2 shows that \( \langle Q, \tau \rangle \) is balanced as required.

It is important to understand that Suslin forcing can be adequate for more complex reasons than the ultrafilter example in the previous paragraph. The \( \lambda \)-adequacy statement becomes weaker as \( \lambda \) grows, and some Suslin forcings become adequate only at higher values of \( \lambda \). The following upper bound question remains open.
Question 10.1.2. Let $P$ be an adequate Suslin forcing. Is $P$ $\lambda$-adequate for some $\lambda < \beth_1$?

One immediate corollary of adequacy is that it places an upper bound on the number of balanced classes, which we do not know how to obtain in general.

Proposition 10.1.3. Let $P$ be a Suslin forcing and $\lambda$ be an infinite cardinal. If $P$ is $\lambda$-adequate then there are at most $2^{2^\lambda}$ many balanced classes.

Proof. Let $A$ be the set of all virtual conditions of $P$ represented on posets of size $< \lambda$. It is a matter of elementary cardinal arithmetic to conclude that $|A| \leq 2^\lambda$. Whenever $\langle Q, \tau \rangle$ is a balanced pair, write $B_\tau = \{ \sigma \in A : Q \models \tau \leq \sigma \}$.

To prove the proposition, it will be enough to show that the set $B_\tau$ is a complete invariant of the equivalence of balanced pairs.

It is clear that if balanced pairs $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ are equivalent, then the sets $B_{\tau_0}, B_{\tau_1}$ are equal. For the opposite implication, if the sets $B_{\tau_0}, B_{\tau_1}$ are equal, then consider the poset $Q_2$ which is the disjoint union of $Q_0, Q_1$, and the $Q_2$-name $\tau_2$ which is the disjoint union of $\tau_0, \tau_1$. Clearly, the pair $\langle Q_2, \tau_2 \rangle$ is $\lambda$-adequate and therefore balanced by the initial assumption on the poset $P$. It follows that $Q_0 \times Q_1 \models \tau_0, \tau_1$ are compatible in $P$ and therefore the balanced pairs $\langle Q_0, \tau_0 \rangle$ and $\langle Q_1, \tau_1 \rangle$ are equivalent as desired. \qed

10.2 Pinned uniformization

In this section, we show that sets whose vertical sections are $E$-classes for a suitably regular equivalence relation $E$ can be uniformized.

Definition 10.2.1. Let $E$ be an equivalence relation on a Polish space $X$. The $E$-uniformization is the statement: if $B \subset 2^\omega \times X$ is a set whose vertical sections are $E$-classes, then there is a function $f \subset B$ such that for every $y \in 2^\omega$, if $B_y \neq 0$ then $f(y)$ is defined (and is an element of $B_y$).

The central theorem of this section can now be stated using Convention 1.8.17.

Theorem 10.2.2. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$. In adequate balanced extensions of the symmetric Solovay model, the $E$-uniformization holds.

Proof. Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin poset such that $V_\kappa \models P$ is balanced and adequate in all forcing extensions. Let $W$ be a symmetric Solovay model derived from $\kappa$. Let $G \subset P$ be a filter generic over $W$ and work in $W[G]$. The following claim contains the key use of the adequacy assumption.

Claim 10.2.3. Let $y \in W[G]$ be any set and let $M$ be the class of all sets hereditarily definable from $G, y$, and parameters in the ground model. Then $G$ contains a realization of a balanced virtual condition in $M$.

A tacit part of the statement of the claim is that $M$ is a model of ZFC, and therefore it belongs to $W$ by Theorem 9.1.1; in fact, it is a generic extension of the ground model by a poset of size $< \kappa$. 

Proof. Use the adequacy assumption to find a cardinal $\lambda \in \kappa$ such that $M \models P$ is $\lambda$-adequate. Working in $M$, let $D$ be the set of all virtual conditions in $P$ represented on posets of size $< \lambda$. In $W[G]$, let $C \subset D$ be the set of all virtual conditions in the set $D$ whose realization belongs to the generic filter $G$. The set $C$ has just been defined from the ordinal $\lambda$ and the ultrafilter $G$, so $C \in M$.

By the genericity of the filter $G$, there must be a condition $p \in G$ which is below all conditions in $C$ and incompatible with all the conditions in $D \setminus C$.

By the forcing theorem, in $M$ there has to be a poset $Q$ of size $< \kappa$ and a $Q$-name $\tau$ such that $Q \models \tau$ is below all conditions in $C$ and incompatible with all conditions in $D \setminus C$, and such that there is a filter $H \subset Q$ generic over the model $M$ with $p = \tau/H$. Thus, in the model $M$, the pair $(Q, \tau)$ is $\lambda$-adequate; by the adequacy assumption, it is balanced. Still working in the model $M$, let $R$ be the poset $\text{Coll}(\omega, \mathcal{P}(Q))$ and let $\sigma$ be the $R$-name for the analytic set $\{p' \in P : \exists H' \subset Q K \text{ meets all the open dense sets on the } R\text{-generic enumeration and } p' = \tau/H'\}$. Then $(R, \sigma)$ is a balanced virtual condition in $M$ and the filter $G$ contains a realization of it.

Return to the model $W$ for a moment. Let $p \in G$ be a condition and let $\tau$ be a $P$-name for a subset of $2^\omega \times X$ such that each vertical section of $\tau$ is a single $E$-class. Let $z \in 2^\omega$ be a point such that both $p, \tau$ are definable from $z$ and a ground model parameter. Back in the model $W[G]$, write $B = \tau/G$. Let $y \in 2^\omega$ be an arbitrary point and let $M_y$ be the model of all sets hereditarily definable from $G, y$, and parameters in the ground model. By Claim 10.2.3, there is a balanced virtual condition $\bar{p}$ in $M_y$ whose realization belongs to the filter $G$. Necessarily, $\bar{p} \leq p$.

In the model $M$, we will show that $\text{Coll}(\omega, < \kappa) \models \bar{p} \models \exists y \in M_y. \tau_y \cap M_y \neq 0$. Once this is done, we can define a uniformization of the set $B$ to be the set of all pairs $(y, x)$ such that $x \in M_y$ is the first point in the canonical well-ordering of the model $M_y$ which belongs to $B_y$.

To prove the forcing statement in the previous paragraph, suppose towards contradiction that it fails. Move to the model $M_y$. There has to be a poset $R$ of size $< \kappa$ and $R$-names $\sigma$ for a condition in $P$ and $\eta$ for an element of $X$ such that $R \models \sigma \leq \bar{p}$, $R \models \eta$ has no ground model $E$-equivalent and $R \models \text{Coll}(\omega, < \kappa) \models \exists \sigma \models P \langle \bar{y}, \bar{\eta} \rangle \in \tau$. In $W$, let $H_0, H_1 \subset R$ be filters mutually generic over the model $M_y$. Write $p_0 = \sigma/H_0, p_1 = \sigma/H_1, x_0 = \eta/H_0$ and $x_1 = \eta/H_1$. The balance of the condition $\bar{p}$ implies that $p_0, p_1 \in P$ are compatible conditions with a lower bound $q$. The pinned assumption on the equivalence relation implies that $x_0, x_1 \in X$ are not $E$-related.

Since $W$ is also a symmetric Solovay extension of both $M_y[H_0]$ and $M_y[H_1]$, the forcing theorem applied in these two models implies that $q \models \bar{x}_0, \bar{x}_1 \in \tau_y$. This is impossible as $\tau$ is forced to be an $E$-class and $x_0, x_1$ are not $E$-related. $

Theorem 10.2.2 is the strongest possible result of its kind, as the following observation shows.

Theorem 10.2.4. Let $E$ be an unpinned Borel equivalence relation on a Polish space $X$. Then $E$-uniformization fails in balanced extensions of the Solovay.
Proof. Let $W[G]$ be a balanced extension of the Solovay model. For each parameter $z \in 2^{\omega}$ the model $M_z = \text{HOD}_{V,z,G}$ is well-ordered and therefore by Theorem 9.1.1, it is a well-ordered subclass of $W$ and so an extension of $V$ by a poset $< \kappa$. The Borel equivalence relation $E$ is unpinned in $M_z$ by Theorem 2.6.1. By Theorem 2.8.3, $M_z \models$ there is a nontrivial $E$-pinned name on the poset $\text{Coll}(\omega,\omega_1)$. Note that $(\mathcal{P}(\omega_1))^{M_z}$ is a countable set in $W$. Thus, one can successfully define the set $B \subset 2^{\omega} \times X$ by setting $\langle z,x \rangle \in B$ if there is a filter $g \subset \text{Coll}(\omega,\omega_1^{M_z})$ generic over $M_z$ such that $\tau_z/g \ E x$, where $\tau_z$ is the first $E$-unpinned name in the canonical well-ordering of the model $M_z$. Every vertical section of the set $B$ then consists of precisely one $E$-equivalence class.

To see that the set $B$ cannot be uniformized, note that every $P$-name in the model $W$ is definable from a real parameter and some additional parameters in the ground model, and therefore every element of $W[G]$ is definable from a real parameter, the generic filter $G$, and some additional parameters in the ground model. Thus, if $f : 2^{\omega} \to X$ is a putative uniformization of the set $B$, one can find a real parameter $z \in 2^{\omega}$ such that $f$ is definable from $z, G$ and some elements of the ground model. Then $f(z)$ should be an element of the model $M_z$ by the definition of the model $M_z$. At the same time, the vertical sections $B_z$ contains no elements of $M_z$ by the definition of the set $B$. \hfill $\square$

10.3 Well-orderable uniformization

In this section, we will prove that a strong version of the countable-to-one uniformization statement holds in the extensions of the Solovay model by adequate forcings.

Definition 10.3.1. Let $E$ be an equivalence relation on a Polish space $X$. The $E$-well-orderable uniformization is the statement: if $E$ is a Borel equivalence relation on a Polish space $X$ and $B \subset 2^{\omega} \times X$ is a set, then the following are equivalent:

1. for every $y \in 2^{\omega}$, $B_y$ is a union of a well-orderable collection of $E$-classes;

2. $B = \bigcup_\alpha B_\alpha$ where each $B_\alpha$ is a set whose vertical sections are either empty or $E$-classes and the variable $\alpha$ ranges over ordinals.

Clearly, only the implication (1)$\implies$(2) has nontrivial content. As a very special case, one can consider a set $B \subset 2^{\omega} \times X$ with all vertical sections countable and $E=$ the identity on $X$. Then (2) yields in particular a function uniformizing the set $B$: for each $y$ with $B_y$ nonempty find the least $\alpha = \alpha_y$ for which the vertical section $(B_\alpha)_y$ is nonempty and then let $f(y) = x$ for the unique $x \in X$ such that $(y, x) \in B_\alpha_y$. This is the familiar countable-to-one uniformization.

The central theorem of this section can now be stated using Convention 1.8.17.
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**Theorem 10.3.2.** Let $E$ be a Borel equivalence relation on a Polish space $X$. In adequate, balanced extensions of the symmetric Solovay model, $E$-well-orderable uniformization holds.

**Proof.** Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin poset such that $V_\kappa \models P$ is balanced and adequate in all forcing extensions. Let $W$ the symmetric Solovay model derived from $\kappa$. In the model $W$, let $p \in P$ be a condition and $\tau$ be a $P$-name such that $p \Vdash \tau \subseteq 2^\omega \times X$ is a set whose vertical sections are well-orderable unions of $E$-classes. Let $z \in 2^\omega$ be a parameter such that $p, \tau$ are both definable from $z$ and some additional ground model parameters. Let $G \subseteq P$ be a generic filter meeting the condition $p$ and let $B = \tau/G$. Work in the model $W[G]$. Fix a point $y \in 2^\omega$ and consider the model $M_y$ of all sets hereditarily definable from parameters $y, z, G$ and parameters in the ground model. Note that $M_y$ is a model of ZFC and therefore finds itself already in the model $W$ by Theorem 9.1.1; in fact, it is a generic extension of the ground model by a poset of size $< \kappa$. We claim that the vertical section $B_y$ consists of realizations of virtual $E$-classes of the model $M_y$. Once this is proved, one can decompose the set $B$ into the union $B = \bigcup \alpha B_\alpha$ by setting $(y, x) \in B_\alpha$ if $(y, x) \in B$ and $x$ belongs to the realization of $\alpha$-th virtual $E$-class in $M_y$ in the canonical well-ordering of the model $M_y$.

To this end, use Claim 10.2.3 to argue that there is a balanced virtual condition $\bar{p}$ in the model $M_y$ such that its realization belongs to the generic filter $G$. Necessarily $\bar{p} \leq p$ must hold. Now, move to the model $M_y$ and argue that in this model, $\text{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash p \Vdash \tau_y$ consists of realizations of virtual $E$-classes of the model $M_y$. The proof of the theorem is then concluded by an appeal to the forcing theorem applied in $M_y$.

Suppose towards contradiction that the forcing statement in the previous paragraph fails and work in $M_y$. In view of Corollary 9.1.3, there is a poset $R_0$ of size $< \kappa$, an $R_0$-name $\eta$ for an element of $X^\omega$ and an $R_0$-name $\sigma_0$ for an element of $P$ stronger than $\bar{p}$ such that $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash \eta$ enumerates representatives of all $E$-classes in $\tau_y$. Also there must be a poset $R_1$ of size $< \kappa$, an $R_1$-name $\chi$ for an element of $X$ which is forced not to realize any virtual $E$-class in $V$, and an $R_1$-name $\sigma_1$ for a condition in $P$ stronger than $\bar{p}$ such that $R_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash \chi \in \tau_y$.

Working in $W$, let $H_0 \subseteq R_0$ and $H_1 \subseteq R_1$ be filters mutually generic over $V$, let $u = \eta/H_0 \in X^\omega$, $x = \chi/H_1 \in X$, $p_0 = \sigma_0/H_0 \in P$ and $p_1 = \sigma_1/H_1 \in P$. The balance of the condition $\bar{p}$ implies that $p_0, p_1$ are compatible conditions in the poset $P$ with a lower bound $q$. The choice of the name $\chi$ implies that $x$ is not $E$-related to any element on the sequence $u$. Since $W$ is a symmetric Solovay extension of the model $M_y[H_0]$, the forcing theorem applied in that model says that $q \Vdash \tau = [\text{rng}(u)]_E$. Since $W$ is a symmetric Solovay extension of the model $M_y[H_1]$, the forcing theorem applied in that model says that $q \Vdash \bar{x} \in \tau$. Since $x \notin [\text{rng}(u)]_E$, this is a contradiction. \qed
10.4 Saint Raymond uniformization

In this section, we will prove that a strong version of Saint Raymond uniformization holds in the generic extensions of the Solovay model by adequate balanced forcing.

**Definition 10.4.1.** Let \( I \) be a downward closed collection of closed subsets of a Polish space \( X \). The \( I \)-Saint Raymond uniformization is the following statement: for every set \( B \subset 2^{\omega} \times X \), the following are equivalent:

1. for every \( y \in 2^{\omega} \), \( B_y \) is a union of a well-ordered collection of sets in \( I \);
2. \( B = \bigcup \alpha B_\alpha \) where for each \( \alpha \) and each \( y \in 2^{\omega} \), the vertical section \( (B_\alpha)_y \subset X \) is closed and belongs to \( I \).

As in the previous section, only the \((1) \rightarrow (2)\) implication has content. In the case of \( I = \) the collection of singletons (plus the empty set), we can again derive the familiar countable-to-one uniformization as a very special case. The following theorem is stated using Convention 1.8.17.

**Theorem 10.4.2.** Let \( I \) be an analytic collection of closed subsets of a Polish space \( X \). In the balanced, adequate extensions of the symmetric Solovay model, the \( I \)-Saint-Raymond uniformization holds.

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that \( V_\kappa \models P \) is balanced and adequate in every forcing extension. Let \( W \) be the symmetric Solovay model derived from \( \kappa \) and work in \( W \). Let \( p \in P \) be a condition and \( \tau \) a \( P \)-name for a subset of \( 2^{\omega} \times X \) whose vertical sections are well-ordered unions of sets in \( I \). Let \( z \in 2^{\omega} \) be a parameter such that \( p, \tau \) are both definable from \( z \) and some additional ground model parameters. Let \( G \subset P \) be a generic filter meeting the condition \( p \) and let \( B = \tau/G \).

Work in \( W[G] \). Fix a point \( y \in 2^{\omega} \) and consider the model \( M_y \) of sets hereditarily definable from parameters \( G, z, y \) and parameters in the ground model. Note that \( M_y \) is a model of choice and therefore belongs to \( W \) already; in fact, it is an extension of the ground model by a poset of size \( < \kappa \). We claim that every nonempty vertical section \( B_y \) is a union of sets in \( I \) which are coded in the model \( M_y \). Once this is proved, one can decompose the set \( B \) into the union \( B = \bigcup \alpha B_\alpha \) by setting \( (y, x) \in B_\alpha \) if \( (y, x) \in B \) and \( x \) belongs to the \( \alpha \)-th set in \( I \) in the model \( M_y \), in the canonical well-ordering of the model \( M_y \), and this \( \alpha \)-th set is a subset of \( B \).

To prove this, first use Claim 10.2.3 to find a balanced virtual condition \( \bar{p} \) in \( M_y \) whose realization belongs to the generic filter \( G \). Work in the model \( M_y \) and argue that \( \text{Coll}(\omega, < \kappa) \models \bar{p} \Vdash \tau \) is a union of ground model coded elements of \( I \). By the forcing theorem applied in the model \( M_y \), this will complete the proof.

To prove the forcing statement in the previous paragraph, work in \( M_y \). Suppose towards contradiction that it fails. By the assumptions on the name \( \tau \), there must be a poset \( R_0 \) of size \( < \kappa \) and \( R_0 \)-name \( \eta \) for a countable sequence
of elements of \( I \) and an \( R_0 \)-name \( \sigma_0 \) for a condition in \( P \) stronger than \( \bar{p} \) such that \( R_0 \models \text{Coll}(\omega, < \kappa) \models \sigma_0 \upharpoonright_p \tau_y = \bigcup \text{rng}(\eta) \). By the contradictory assumption, there also must be a poset \( R_1 \) of size \( < \kappa \) and \( R_1 \)-name \( \chi \) for an element of \( X \) and an \( R_1 \)-name \( \sigma_1 \) for a condition in \( P \) stronger than \( \bar{p} \) such that \( R_1 \models \text{Coll}(\omega, < \kappa) \models \sigma_1 \upharpoonright (\bar{y}, \chi) \in \tau \) and \( \chi \) does not belong to any ground model element of \( I \) which is a subset of \( \tau_y \).

Move to the model \( W \). Let \( H_0, H_1 \subseteq R_0, R_1 \) be filters mutually generic over \( M_y \), and let \( p_0 = \sigma_0/H_0 \) and \( p_1 = \sigma_1/H_1 \). By the balance of the condition \( \bar{p}, p_0, p_1 \in P \) are compatible conditions with a lower bound \( q \in P \). Let \( x = \chi/H_1 \in X \). The contradiction is now reached by a split into cases.

Case 1. There is a condition \( r \in H_1 \) and a closed set \( C \subseteq \text{rng}(\eta/H_0) \) such that the ground model closed set \( D = X \setminus \bigcup \{O: O \subseteq X \text{ is a basic open set such that } r \upharpoonright \chi \notin O\} \) is a subset of \( C \). By the closure of the collection \( I \) under subsets, \( D \in I \) holds; by the definition of the set \( D, x \in D \) holds. Now, \( W \) is the symmetric Solovay extension of both models \( M_y[H_0] \) and \( M_y[H_1] \). The forcing theorem applied in \( M_y[H_0] \) shows that \( W \models q \upharpoonright D \subseteq C \subseteq \tau \). The forcing theorem applied in \( M_y[H_1] \) shows that \( W \models q \upharpoonright \bar{x} \in D \) and \( \bar{x} \) belongs to no ground model closed set in \( I \) which is a subset of \( \tau_y \). Thus, the same condition \( q \) forces two contradictory statements.

Case 2. Case 1 fails. Then, by the mutual genericity of the filters \( H_0, H_1 \) it must be the case that \( x \notin \bigcup \text{rng}(\eta/H_0) \). Now, \( W \) is the symmetric Solovay extension of both models \( M_y[H_0] \) and \( M_y[H_1] \). The forcing theorem applied in \( M_y[H_0] \) shows that \( W \models q \upharpoonright \tau_y = \bigcup \text{rng}(\eta/H_0) \). The forcing theorem applied in \( M_y[H_1] \) shows that \( W \models q \upharpoonright \bar{x} \in \tau \setminus \bigcup \text{rng}(\eta/H_0) \). So again, the same condition \( q \) forces two contradictory statements. The proof is complete.

10.5 Examples

In order to justify the extensive development in the earlier sections of this chapter, the class of adequate Suslin forcings better contain as many posets as possible. In this section, we show that this is indeed the case. The careful classification of balanced conditions in Chapter 5 will come handy over and over again. In all examples below, the conclusion is that in view of Theorems 10.2.2, 10.3.2, and 10.4.2, the extensions of the symmetric Solovay model by the posets in question satisfy the pinned uniformization, the well-orderable uniformization, and the Saint Raymond uniformization.

We start with simplicial complex posets which naturally live on Polish spaces as opposed to quotient spaces. These posets are \( \aleph_0 \)-adequate in all cases that we can check.

**Example 10.5.1.** Let \( \mathcal{K} \) be a Borel simplicial complex on a Polish space \( X \) of Borel coloring number \( \aleph_1 \). Then the poset \( P = P_{\mathcal{K}} \) is \( \aleph_0 \)-adequate.

**Proof.** Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we must show that it is balanced. For each \( x \in X \) let \( B_x = \{p \in P: x \in p\} \). By the adequacy assumption, for each \( x \in X \), either \( Q \models \tau \leq \Sigma B_x \) or \( Q \models \tau \) is incompatible with every element of \( B_x \).
Let $A \subset X$ be the set of all $x \in X$ for which the former option prevails. We will show that the $A \subset X$ is a weakly maximal $\mathcal{K}$-set as in Definition 6.2.8. Then, $\langle \text{Coll}(\omega, X), \mathcal{A} \rangle$ is a balanced pair by Theorem 6.2.9, clearly $Q \times \text{Coll}(\omega, X) \models \tau \leq \mathcal{A}$, and so the pair $(Q, \tau)$ is balanced by Proposition 5.2.4.

It is clear that the set $A \subset X$ is a $\mathcal{K}$-set since $Q \models \tau \models \mathcal{P}$. Let $\mathcal{K}$ be the intersection of the generic $\mathcal{K}$-set with the ground model. To see its weak maximality, let $x \in X \setminus A$ be a point. Then $Q \models \tau \models \mathcal{P}$ there is $c \in \mathcal{K}$ such that in the generic $\mathcal{K}$-set such that $c \cup \{x\} \notin \mathcal{K}$. By a Mostowski absoluteness argument between the ground model and its $Q$-extension, given a countable set $b \subset X$, there must be a set $c \in \mathcal{K}$ disjoint from $b$ such that $(A \cap b) \cup c$ is a $\mathcal{K}$-set while $(A \cap b) \cup c \cup \{x\}$ is not. This confirms the weak maximality of the set $A$ as required. 

Example 10.5.2. Let $\mathcal{K}$ be a modular Borel complex on a Polish space $X$. Then the poset $P = P_{\mathcal{K}}$ is $\aleph_0$-adequate.

Proof. Let $f : \mathcal{K} \to [Y]^\aleph_0$ be a Borel function witnessing the modularity of the complex $\mathcal{K}$. Let $(Q, \tau)$ be an $\aleph_0$-adequate pair; we will show that it is a balanced pair.

For every point $x \in X$ let $A_x = \{p \in P : x \in p\}$ and for all $y \in Y$ let $B_y = \{p \in P : y \in f(p)\}$. By the adequacy assumption, for each $x \in X$ it must be the case that either $Q \models \tau \leq \Sigma A_x$ or $Q \models \tau$ is incompatible with $\Sigma A_x$ in the completion of the poset $P$. Let $A \subset X$ be the collection of all $x \in X$ for which the former case occurs. Similarly, for all $y \in Y$ it must be the case that either $Q \models \tau \leq \Sigma B_y$ or $Q \models \tau$ is incompatible with $\Sigma B_y$ in the completion of the poset $P$. Let $B \subset Y$ be the set of all $y \in Y$ for which the former case occurs. In view of Theorem 6.2.26, it will be enough to show that $B = f(A)$.

It is immediate that $f(A) \subseteq B$ holds; the other implication is the crux of the proof. Let $y \in B$ be an arbitrary point. The set $D = \{a \in \mathcal{K} : y \in f(a) \text{ and } a \text{ is inclusion minimal such}\}$ is analytic; let $g : 2^{<\omega} \to [X]^{<\aleph_0}$ be a continuous function whose image is $D$. For each finite string $t \in \omega^{<\omega}$, let $C_t = \{p \in P : \exists z \in \omega^{\omega} \text{ such that } t \subseteq z \text{ and } g(z) \subseteq a\}$. This is an analytic subset of the poset $P$, and $C_t = \bigcup_n C_{t \cdot n}$.

By the adequacy assumption, for every finite string $t \in T$, either $Q \models \tau \leq \Sigma C_t$ or $Q \models \tau$ is incompatible with $\Sigma C_t$ in the completion of the poset $P$. Let $T \subset \omega^{<\omega}$ be the set of all $t \in \omega^{<\omega}$ in which the former case occurs. Since $C_0 = B_y$, $0 \in T$. It is also immediate that $T$ is a tree with no terminal nodes. Thus, there must be a branch $z \in \omega^{<\omega}$ through the tree $T$. Let $a = g(z)$. It will be enough to show that $a \subset A$ since $y \in f(a)$ is a part of the definition of the set $D = \text{rng}(g)$.

Suppose towards contradiction that this fails. Then, in the $Q$-extension, $\tau$ can be extended to a condition $p$ such that $y \not\in f(p)$, there is an inclusion minimal set $a' \subset p$ such that $y \in f(a')$, and $a' \neq a$. There must be a point $z' \in \omega^\omega$ such that $f(z') = a'$. By the definition of the tree $T$, $z'$ is a branch of $T$. By the continuity of the function $g$, there must be initial segments $t \subseteq z$ and $t' \subseteq z'$ such that $g''[t] \cap g''[t'] = 0$. Since $t, t'$ are both nodes in $T$, $p$ can be extended further to contain another inclusion minimal set $a'' \subset p$ such that $y \in f(a'')$ and such that there is $z'' \in \omega^\omega$ extending $t$ with $a'' = g(z'')$. 


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Now, for the final push, note that \( g''[t] \cap g''[t'] \) implies that \( a' \neq a'' \). The set \( a' \cup a'' \) is a subset of \( p \) and therefore an element of the complex \( \mathcal{K} \). By the modularity of the function \( f, f(a') \cap f(a'') = f(a' \cap a'') \). Thus, \( y \in f(a' \cap a'') \). This contradicts the assumptions that \( a', a'' \) were distinct inclusion minimal elements containing \( y \) in their \( f \)-image.

\[ \blacksquare \]

**Example 10.5.3.** Let \( \mathcal{K} \) be a co-\( F_\omega \)-matroid as in Definition 6.2.31. Then the poset \( P = P_\mathcal{K} \) is \( \aleph_0 \)-adequate.

**Proof.** Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we will show that it is a balanced pair. Fix a point \( x \in X \). Note that by the exchange property of the pre-geometry associated with \( \mathcal{K} \), if \( a \subset X \) is a free set and \( x \) belongs to its algebraic closure, there is an inclusion-smallest set \( b \subset a \) such that \( x \) is in the algebraic closure of \( b \).

For every number \( n \in \omega \) let \( A_n = \{ p \in P : x \) is in the algebraic closure of \( p \) and the inclusion-smallest set \( b \subset p \) such that \( x \) is in the algebraic closure of \( b \) has size \( n \} \). An immediate density argument shows that \( \{ \Sigma A_n : n \in \omega \} \) is a partition of the unit in the poset \( P \) and so by the adequacy assumption there is \( n \in \omega \) such that \( Q \models \tau \leq \Sigma A_n \). Now, fix a complete compatible metric \( d \) for \( X \) and a rational \( \varepsilon > 0 \). Let \( B_\varepsilon \) be the set of all \( n \)-tuples \( \vec{O} \) of basic open subsets with disjoint closures of diameter \( < \varepsilon \). For each \( \vec{O} \in B_\varepsilon \), let \( A_{\vec{O}} = \{ p \in P : \) the inclusion smallest \( b \subset p \) such that \( x \) is in the algebraic closure of \( b \) has one point in the closure of each set in \( \vec{O} \} \). Since the set \( \{ \Sigma A_{\vec{O}} : \vec{O} \in B_\varepsilon \} \) sums up to \( \Sigma A_n \), an adequacy argument shows that there must be \( \vec{O}_\varepsilon \in B_\varepsilon \) such that \( Q \models \tau \leq \Sigma A_{\vec{O}_\varepsilon} \). By the completeness of the metric \( d \), there is exactly one set \( b_x \subset X \) of size \( n \) which has exactly one point in the closure of each set on \( \vec{O}_\varepsilon \), and this for every positive rational number \( \varepsilon > 0 \). It is clear that \( Q \models \tau \leq b_x \) in the separative quotient of the poset \( P \), and \( x \) belongs to the algebraic closure of \( b_x \).

Let \( A = \bigcup b_x : x \in X \). Since for every \( x \in X \), \( Q \models \tau \leq b_x \), it must be the case that \( A \) is a \( \mathcal{K} \)-set, and moreover \( \text{Coll}(\omega, A) \times Q \models \tau \leq \check{A} \) in the separative quotient of \( P \). In addition, the algebraic closure of \( A \) is all of \( X \) since any \( x \in X \) is in the algebraic closure of \( b_x \), and so \( A \) is in fact a maximal \( \mathcal{K} \)-set. The argument is concluded by a reference to Theorem 6.2.32: the pair \( (\text{Coll}(\omega, A), \check{A}) \) is balanced in the poset \( P \).

\[ \blacksquare \]

**Example 10.5.4.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) of countable coloring number. The \( \Gamma \)-\( \omega \)-coloring poset \( P \) of Definition 8.1.1 is adequate.

**Proof.** We identify \( P \) with its Suslin dense subset isolated in Theorem 8.1.2. Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we will show that it is balanced in \( P \). Indeed, for each point \( x \in X \) and each number \( m \in \omega \), write \( A_{x_m} = \{ p \in P(\Gamma) : x \in \text{dom}(p), p(x) = m \} \). By the adequacy assumption, either \( Q \models \tau \) is incompatible with \( \Sigma A_{x_m} \) or \( Q \models \tau \leq \Sigma A_{x_m} \). Since the sums \( \{ \Sigma A_{x_m} : m \in \omega \} \) form a partition of the unit, there must be exactly one \( m = c(x) \in \omega \) such that the latter option occurs. The map \( c : X \to \omega \) is a \( \Gamma \)-coloring and \( Q \models \tau \leq \check{c} \) holds.
By Theorem 8.1.2, total $\Gamma$-colorings naturally identify with balanced virtual conditions. In conclusion, the pair $\langle Q, \tau \rangle$ is balanced as desired. □

The quotient complex posets are more complicated; in some cases they do fail to satisfy the adequacy demands and the resulting forcing extensions of the symmetric Solovay model may fail countable-to-one uniformization. Here is what we know in this direction:

**Example 10.5.5.** Let $E \subset F$ be Borel equivalence relations on a Polish space $X$ such that every $F$ class consists of countably many $E$-classes. The $E,F$-transversal poset $P$ of Definition 6.3.5 is adequate.

**Proof.** Let $\lambda = \beth_\omega$. We will show that the poset $P$ is $\lambda$-adequate. Let $\langle Q, \tau \rangle$ be a $\lambda$-adequate pair; we must argue that it is balanced.

For each virtual $E$-class $c$, in the $\text{Coll}(\omega, \lambda)$-extension consider the analytic set $A_c = \{ p \in P : p \text{ contains a realization of } c \}$. Let $d$ be a virtual $F$-class. In view of Claim 6.3.13, the virtual conditions $\{ \Sigma A_c : c \text{ is a virtual } E \text{-class forced to be a subset of } d \}$ are forced to form a partition of the unit in $P$. By the adequacy assumption, there must be exactly one virtual $E$-class $c = f(d)$ which is forced to be a subset of $d$ and such that $Q \Vdash \tau \leq \Sigma A_c$. Consider the virtual condition $\langle \text{Coll}(\omega, \lambda), \sigma_d \rangle$ where $\sigma_d$ is the name for the analytic set of all conditions $p \in P$ which contain a realization of the virtual class $d$. In view of Theorem 6.3.12, the function $f$ from the virtual $F$-quotient space to the virtual $E$-quotient space generates a balanced virtual condition. By the definitions, $Q$ forces $\tau$ to be stronger than this balanced condition and therefore is balanced itself. □

The following example covers all posets which associate a structure to each $G$-class where $G$ is a countable Borel equivalence relation on a Polish space, as in Example 6.3.10.

**Example 10.5.6.** Let $X$ be a Polish space and let $E \subset F$ be Borel equivalence relations on $X$ such that $F$ is pinned and for each $E$-equivalence class $C \subset X$, $E \upharpoonright C$ is smooth. Then the $E,F$-transversal poset $P$ of Definition 6.3.5 is adequate.

**Proof.** Let $\langle Q, \tau \rangle$ be an $\aleph_0$-adequate pair for $P$; we will show that it is a balanced pair.

For each $F$-class $C$, let $h : C \to 2^\omega$ be a Borel map which is a reduction of $E \upharpoonright C$ to the identity on $2^\omega$. For each binary string $t \in 2^{<\omega}$, let $A_{ht} \subset P$ be the analytic set of all conditions $p \in P$ such that there exists $x \in C \cap \text{dom}(p)$ with $t \subset h(x)$. The adequacy of the pair shows that either $Q \Vdash \tau \leq \Sigma A_{ht}$ or $Q \Vdash \tau$ is incompatible with all elements of $A_{ht}$. Let $k(C) \subset 2^{<\omega}$ be the set of those $t \in 2^{<\omega}$ for which the former alternative prevails.

Now, observe that if $s, t$ are inclusion-incompatible binary strings then $\Sigma A_{hs}$ and $\Sigma A_{ht}$ are incompatible conditions in the completion of $P$, and if $n \in \omega$ is any number, then $\{ \Sigma A_{ht} : t \in 2^n \}$ is a partition of unity in the completion of $P$. It follows that the set $k(C) \subset 2^{<\omega}$ is an infinite path. There must be an $E$-class...
D ⊂ C such that for all x ∈ D, h(x) = ⋃k(C), and then Q ⊨ τ ≤ Σ{p ∈ P: p ∩ D ≠ ∅}. The map g: C ⏐→ D generates a balanced virtual condition ⟨Coll(ω, X), τg⟩ by Theorem 6.3.6. It is immediate that Q × Coll(ω, X) ⊨ τ ≤ τg and so the pair ⟨Q, τ⟩ is balanced as required.

Example 10.5.7. Let E be a non-smooth Borel pinned equivalence relation on a Polish space X. The collapse poset P of |E| to 2ω of Definition 6.3.2 is not adequate. The countable-to-one uniformization fails in the resulting extension of the choiceless Solovay model.

Proof. We prove the last sentence. The poset P is balanced by the pinned assumption on E and Corollary 6.3.4. In the P-extension of the symmetric Solovay model, |E| ≤ |2ω| holds by the definition of the poset P. In addition, |E0| ≤ |E| holds by the Glimm–Effros dichotomy as E is assumed to be non-smooth. In sum, |E0| ≤ |2ω| holds, as witnessed by some function g: 2ω → 2ω.
In addition, Corollary 11.4.3 shows that in the P-extension of the symmetric Solovay model, E0 has no transversal, and therefore the function g has no left inverse. This feature of g stands witness to the failure of the countable-to-one uniformization.

Example 10.5.8. Let F be a Fraïssé class in a finite relational language with strong amalgamation. Let E be a Borel equivalence relation on a Polish space X. Then the E, F-Fraïssé poset P of Definition 8.5.3 is adequate. In addition, if E is pinned, then P is ℵ0-adequate.

Proof. Start with the general case. Let λ = ℵ0. We will show that the poset P is λ-adequate. To this end, let ⟨Q, τ⟩ be a λ-adequate pair; we must argue that it is balanced.

For brevity, assume that F has just one relational symbol of its language of some finite arity n ∈ ω; so P forces ⟨X/E, Agen⟩ to be a F-structure. Let Agen be the P-name for the generic relation on the quotient space X/E. Use the adequacy and Theorem 2.5.6 to show that for each x ∈ (X**)n, Q ⊨ τ decides in P whether a realization of x belongs to Agen or not. Let A = {x ∈ (X**)n: Q ⊨ τ ⊨ p a realization of x belongs to Agen}. It is clear that (X**, A) is an F-structure, since the realization map embeds it in the generic F-structure as an induced substructure. Then, by Theorem 8.5.4, the pair ⟨Coll(ω, λ), A⟩ is balanced; clearly Coll(ω, λ) × Q ⊨ A ≤ τ in the separative quotient of the poset P and so ⟨Q, τ⟩ must be a balanced pair equivalent to ⟨Coll(ω, λ), A⟩.

The case of a pinned equivalence relation E proceeds in the same way, noting that in this case X** is equal to the E-quotient space.

The ultrafilter posets typically do satisfy the adequacy demands.

Example 10.5.9. The poset P of all infinite subsets of ω ordered by inclusion is ℵ0-adequate.

Proof. Let ⟨Q, τ⟩ be an ℵ0-adequate pair; we will show that it is a balanced pair. For every set x ⊂ ω let Ax = {p ∈ P: p ⊂ x}. By the adequacy
assumption, for each $x \subset \omega$ it must be the case that either $Q \models \tau \leq \Sigma A_x$ or $Q \models \tau$ is incompatible with $\Sigma A_x$ in the completion of the poset $P$. Let $U$ be the collection of those $x \subset \omega$ for which the former alternative occurs. It is easy to see that $U$ is a nonprincipal ultrafilter on $\omega$. It is also clear that $Q \models \tau$ is modulo finite included in every element of $U$. The balance of $\tau$ is now proved by a reference to Theorem 7.1.2.

Example 10.5.10. Let $\mathcal{A}$ be a Ramsey sequence of finite structures. The poset $P = P_\mathcal{A}$ of Definition 7.2.2 is $\aleph_0$-adequate.

Proof. Let $(Q, \tau)$ be an $\aleph_0$-adequate pair; we must show that it is balanced. Use the terminology of Section 7.2. Let $n \in \omega$ be a natural number. For every set $b \in D_n$ let $A^n_b = \{ p \in P : p^n \cap b \text{ modulo finite} \}$ and $A^1_b = \{ p \in P : p^1 \cap b = 0 \text{ modulo finite} \}$. The sets $A^n_b, A^1_b \subset P$ are analytic and no condition in $A^1_b$ can be compatible with any condition in $A^1_b$. A straightforward Ramsey arguments shows that any condition can be strengthened to one in the set $A^n_b \cup A^1_b$; i.e. the sums $\Sigma A^n_b, \Sigma A^1_b$ form a partition of the unit in the completion of the poset $P$.

By the adequacy assumption, it must be the case that either $Q \models \sigma \leq \Sigma A^n_b$ or $Q \models \Sigma A^1_b$. Let $F_n$ be the set of those subsets $b \in D_n$ for which the former alternative prevails; thus, $F_n$ is a nonprincipal ultrafilter on the set $D_n$. It is immediate that $F = \langle F_n : n \in \omega \rangle$ is an $\mathcal{A}$-sequence of ultrafilters in the sense of Definition 7.2.3. The definition of the balanced name $\tau_F$ (see Theorem 7.2.4) immediately shows that $Q \times \Coll(\omega, 2^\omega) \models \sigma \in \tau_F$ and therefore the pair $(Q, \sigma)$ is balanced and equivalent to the pair $(\Coll(\omega, 2^\omega), \tau_F)$.

Example 10.5.11. Let $(\Gamma, \cdot)$ be a countable semigroup. The poset $P = P(\Gamma)$ of Section 7.4 is adequate.

Proof. Let $(Q, \tau)$ be an $\aleph_0$-adequate pair; we will show that it is a balanced pair.

Let $b \subset \Gamma$ be a set. Let $A_b = \{ p \in P : p \text{ accepts } b \}$. This is an analytic subset of $P$. By the adequacy assumption, it must be the case that either $Q \models \tau \leq \Sigma A_b$ or $Q \models \tau$ is incompatible with $\Sigma A_b$ in the completion of the poset $P$. Let $U$ be the collection of those $b \subset \Gamma$ for which the former alternative occurs. It is easy to see that $U$ is a nonprincipal ultrafilter on $\Gamma$, and Proposition 7.4.5 shows that $U$ is in fact an idempotent ultrafilter. Theorem 7.4.7 then implies that $Q$ forces $\tau$ to be smaller than the balanced condition derived from $U$, showing that $(Q, \tau)$ is balanced.

Question 10.5.12. Let $I$ be the ideal of all subsets of $\omega \times \omega$ generated by vertical sections and sets with all vertical sections finite. Is the poset $P$ of $I$-positive sets ordered by inclusion adequate?

We conclude this chapter by several questions. Is it possible to separate the various versions of uniformization present in this chapter?

Question 10.5.13. Let $E, F$ be analytic equivalence relations. Is it consistent with ZF+DC that $E$-countable-to-one uniformization holds while $F$-countable-to-one uniformization fails?
Question 10.5.14. Let $I, J$ be analytic downward closed collections of closed sets. Is it consistent with ZF+DC that $I$-uniformization holds and at the same time $J$-uniformization fails?
Chapter 11

Locally countable structures

This chapter introduces a methodology for proving that certain models of ZF+DC we produce do not contain locally countable structures of certain type. In particular, we learn how to produce models of ZF+DC which contain locally countable structures of one type but not of another type. The whole enterprise should be viewed as a parallel to the extensive field of descriptive set theory of locally countable structures. There are many striking similarities present, and many others are sure to be found in the future.

11.1 Central objects and notions

In this section, we introduce certain basic locally countable graphs and hypergraphs and the associated concerns of the descriptive set theory of locally countable structures. Some of the critical issues deal with the comparison of chromatic and Borel chromatic numbers of various analytic hypergraphs on Polish spaces. An analytic hypergraph on a Polish space $X$ is a set $G \subseteq [X]^{<\aleph_0}$ which is an analytic subset of the hyperspace $K(X)$ with the Vietoris topology. The elements of a hypergraph are referred to as hyperedges. Thus, our hypergraphs contain finite hyperedges only—they are finitary; it may occur though that there is no fixed $n \in \omega$ (arity) such that $G \subseteq [X]^n$. As a matter of convention, our hypergraphs contain no singleton sets. A graph is a hypergraph of arity 2. If $G, H$ are hypergraphs on respective spaces $X, Y$, a function $h: X \to Y$ is a homomorphism if the $h$-image of any $G$-hyperedge is an $H$-hyperedge. An ant clique is a set $a \subseteq X$ such that $[a]^{<\aleph_0} \cap G = 0$. The chromatic number of $G$ is the smallest cardinality of a set of anticliques covering the whole space $X$; in the choiceless context, where cardinalities are not well-ordered, we distinguish only between countable and uncountable chromatic number and different values of countable chromatic numbers.

Most of the central analytic hypergraphs in this section are in fact products of hypergraphs on finite or countable domain. The following definition explains the typical construction.
Definition 11.1.1. Let \( \langle a_n, H_n, t_n : n \in \omega \rangle \) be a sequence such that \( a_n \) is a nonempty countable (often finite) set, \( H_n \) is a hypergraph on \( a_n \) and \( t_n \in \prod_{m \in n} a_m \) are finite strings such that \( \{ t_n : n \in \omega \} \) is a dense subset of \( \bigcup_n \prod_{m \in n} a_m \). Let \( X = \prod_n a_n \) and define:

1. (the skew product) \( \prod_n H_n, t_n \) is the hypergraph on \( X \) containing all sets \( b \subset X \) such that for some \( n \in \omega \), \( \forall m \neq n \forall x_0, x_1 \in b \ x_0(m) = x_1(m), \forall x \in b \ t_n \subset b, \) and \( \{ x(n) : x \in b \} \in H_n \);

2. (the product) \( \prod_n H_n \) is the hypergraph on \( X \) containing all sets \( b \subset X \) such that for some \( n \in \omega \), \( \forall m \neq n \forall x_0, x_1 \in b \ x_0(m) = x_1(m) \), and \( \{ x(n) : x \in b \} \in H_n \).

In all cases, the sets \( a_n \) are implicit in the notation. The definition of the product (as opposed to the skew product) does not depend on the strings \( t_n \).

Among the product graphs, the Hamming graphs are prominent.

Definition 11.1.2. Let \( n \geq 2 \) be a natural number. \( \mathbb{H}_n \), the Hamming graph on \( n^\omega \) is the product of infinitely many cliques of size \( n \). \( \mathbb{H}_<\omega \), the diagonal Hamming graph on the space \( \prod_n (n + 1) \), is the product of cliques of all possible nonzero finite cardinalities. Finally, \( \mathbb{H}_\omega \), the Hamming graph on \( \omega^\omega \), is the product of infinitely many cliques on \( \omega \).

The skew products are useful as minimal examples of various phenomena. One standard example is the uncountable Borel chromatic number of graphs:

Definition 11.1.3. Let \( G \) be an analytic hypergraph on a Polish space \( X \). \( G \) has countable Borel chromatic number if there are Borel \( G \)-anticliques \( B_n \subset X \) for \( n \in \omega \) such that \( X = \bigcup_n B_n \).

Definition 11.1.4. For each \( n \in \omega \), let \( a_n = 2 \), let \( H_n \) be the graph on \( a_n \) containing only the whole set \( a_n \) as an edge and let \( t_n \in 2^n \) be a string such that \( \{ t_n : n \in \omega \} \subset 2^{<\omega} \) is dense. The graph \( G_0 \) is defined as \( \prod_n H_n, t_n \).

Fact 11.1.5. [44](ZF+DC) For every analytic graph \( G \) on a Polish space \( X \), exactly one of the following occurs:

1. the Borel chromatic number of \( G \) is countable;
2. there is a continuous map from \( 2^\omega \) to \( X \) which is a homomorphism of \( G_0 \) to \( G \).

Uncountable chromatic numbers of locally countable graphs can be used in ZF+DC to rule out seemingly unrelated phenomena. The following humble proposition will be useful at several points of this book:

Proposition 11.1.6. (ZF+DC) If the chromatic number of \( G_0 \) is greater than two then

1. there is no linear ordering of the \( E_0 \)-quotient space;
2. $|E_0| \leq |2^\omega|$;

3. there is no discontinuous homomorphism between Polish groups.

Proof. For the first item, consider the equivalence relation $E$ on $2^\omega$ connecting points $x, y \in 2^\omega$ if they differ in finite, even number of entries. Then $E \subset E_0$; as a subset of a hyperfinite equivalence relation, it is itself hyperfinite and therefore Borel reducible to $E_0$ [39, Theorem 8.1.1]. Now, suppose that the $E_0$-quotient space is linearly orderable; then so is the $E$-quotient space, and one can define the function $c \colon 2^\omega \to 2$ by setting $c(x) = 0$ if among the two $E$-classes which constitute the $E_0$-class of $x$, $x$ belongs to the smaller one in the fixed linear ordering of the $E$-quotient space. It is not difficult to show that $c$ is a $G_0$-coloring with two colors.

The second item follows from the first, since the inequality $|E_0| \leq |2^\omega|$ yields a linear ordering on the $E_0$-space by simply pulling back the usual lexicographical order on $2^\omega$ to the $E_0$-quotient space via the assumed injection from the $E_0$-quotient space to $2^\omega$. The third item is much harder. [63] constructs, in ZF+DC, a coloring of the Hamming graph (a superset of the $G_0$-graph) with two colors from the assumption that there is a discontinuous homomorphism between Polish groups.

It is not difficult to extend this result to arbitrary finitary analytic hypergraphs. In this generality, one has to allow for uncountably many basis hypergraphs, which are nevertheless easy to describe:

Definition 11.1.7. Suppose that $\langle a_n, H_n, t_n : n \in \omega \rangle$ is a sequence such that each $a_n$ is a finite set of cardinality at least two, $H_n$ is the hypergraph on it containing $a_n$ as its only hyperedge, and $t_n \in \prod_{m \in n} a_m$ is a string such that the set $\{t_n : n \in \omega \}$ is dense in $\bigcup_n \prod_{m \in n} a_m$. Then the hypergraph $\prod_n H_n, t_n$ on $\prod_n a_n$ is called a principal skew product.

The following can be proved in the same way as the $G_0$-dichotomy. We suppress the standard proof.

Fact 11.1.8. (ZF+DC) Let $G$ be a finitary analytic hypergraph on a Polish space $X$. Exactly one of the following occurs:

1. the Borel chromatic number of $G$ is countable;

2. there is a principal skew product and a continuous homomorphism of it to the hypergraph $G$.

In this chapter, we will also use more involved variations of the uncountable Borel chromatic number.

Definition 11.1.9. Let $G$ be an analytic finitary hypergraph on a Polish space $X$. The hypergraph $G$ has Borel $\sigma$-finite chromatic number if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that $X = \bigcup_n B_n$ and for every $n \in \omega$, every finite subset of $B_n$ has $G$-chromatic number less than $n + 2$. 
It is immediate that countable Borel chromatic number implies Borel \( \sigma \)-finite chromatic number. The opposite implication does not hold as the following examples show.

**Example 11.1.10.** The Hamming graph \( H_2 \) has uncountable Borel chromatic number but it does have Borel \( \sigma \)-finite chromatic number. To see this, observe that it contains no odd length cycles and therefore \( H_2 \) on each finite subset of \( 2^\omega \) has chromatic number 2. The sequence \( B_n \) defined by \( B_1 = X \) and \( B_n = 0 \) for all \( n \neq 1 \) exemplifies the Borel \( \sigma \)-finite chromatic number of \( H_2 \).

**Example 11.1.11.** The diagonal Hamming graph \( H_{<\omega} \) does not have Borel \( \sigma \)-finite chromatic number. To see this, by the Baire category theorem it is enough to show that any non-meager Borel subset of \( X = \prod_{n \in \omega} (n + 1) \) contains arbitrarily large finite \( H_{<\omega} \)-cliques. Thus, let \( B \subset X \) be non-meager Borel and let \( m \in \omega \) be a number. Let \( t \) be a finite string of natural numbers such that \( B \) is comeager in \( t \) and \( |t| = n > m \). A simple construction yields a point \( x \in X \) such that \( t \subset x \) and for every \( i \in n \) the points \( x_i \), obtained from \( x \) by rewriting its \( n \)-th entry with \( i \), belong to the set \( B \). Since the set \( \{x_i : i \in n\} \subset B \) is an \( H_{<\omega} \)-clique, the argument is complete.

The class of analytic finitary hypergraphs which do not have Borel \( \sigma \)-finite chromatic number has a simple basis.

**Definition 11.1.12.** Let \( \langle a_n, H_n, t_n : n \in \omega \rangle \) be a sequence such that \( a_n \) is a nonempty finite set, \( H_n \) is a hypergraph on \( a_n \) of chromatic number > \( n \), \( t_n \in \prod_{m \in \omega} a_m \), and the set \( \{t_n : n \in \omega\} \) is dense in \( \bigcup_n \prod_{m \in \omega} a_m \). The skew product \( \prod_{n} H_n, t_n \) is then called a large skew product.

A straightforward Baire category argument shows that a large skew product does not have Borel \( \sigma \)-finite chromatic number. The following fact is proved in the same way as the \( G_0 \)-dichotomy, and we omit the standard argument.

**Fact 11.1.13.** Let \( G \) be a finitary analytic hypergraph on a Polish space \( X \). Exactly one of the following occurs:

1. \( G \) has Borel \( \sigma \)-finite chromatic number;
2. there is a large skew product \( H \) on a Polish space \( Y \) and a continuous homomorphism \( h : Y \to X \) of \( H \) to \( G \).

Many interesting natural examples of hypergraphs which do not have Borel \( \sigma \)-finite chromatic number actually have a stronger property encapsulated in the following definitions.

**Definition 11.1.14.** Let \( a \) be a finite set and \( G \) be a hypergraph on \( a \). The fractional chromatic number of \( G \) is the maximum of all numbers \( \sum_{v \in a} f(v) \) where \( f : a \to [0, 1] \) ranges over all functions such that \( \sum_{v \in b} f(v) \leq 1 \) for all \( G \)-anticliques \( b \subset a \).
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A word about the terminology is in order. In fractional graph theory [65], the fractional chromatic number is typically defined for graphs only. In the graph context, the definition above corresponds to the fractional clique number of $G$, which is equal to the standard fractional chromatic number of $G$ by a linear programming duality argument [65, Section 3.1]. In order to maintain coherent terminology, we neglect this important but for our ends irrelevant point. A trivial restatement of the fractional chromatic number will be used below: it is the smallest real number $r$ such that there is a probability measure on $a$ in which every $G$-anticlique has mass $\leq 1/r$.

Note that the fractional chromatic number is no greater than the chromatic number. Just as in the case of the chromatic number, if $G_0, G_1$ are hypergraphs on $a_0, a_1$ respectively and $h: a_0 \to a_1$ is a homomorphism of $G_0$ to $G_1$ then the fractional chromatic number of $G_1$ is not smaller than the fractional chromatic number of $G_0$. The following examples further elucidate the relationship between the chromatic number and its fractional counterpart.

**Example 11.1.15.** For every $n \in \omega$ let $G_n$ be the graph on $[n]^2$ connecting pairs $a, b$ if the smaller element of $a$ is equal to the larger element of $b$ or vice versa. Then the chromatic numbers of $G_n$ tend to infinity with $n$, but the fractional chromatic number remains bounded by 4.

For the former statement, note that if $n$ is such that $n \to (3)_{n}^{2}$ then the chromatic number of $G_n$ is greater than $m$: if $[n]^2 = \bigcup_{m \in \mathbb{N}} B_m$, then by the Ramsey theorem there is $i \in m$ and a triple $c \subset n$ such that $[c]^2 \subset B_i$; however, $[c]^2$ clearly contains a $G$-edge.

For the latter statement, suppose that $\mu$ is a probability measure on $[n]^2$, and let $\lambda$ be the normalized counting measure on $\mathcal{P}(n)$. Let $B = \{\langle a, x \rangle \in [n]^2 \times \mathcal{P}(n) : x \text{ contains the smaller number in } a \text{ but not the larger number in } a \}$. The vertical sections of $B$ have $\lambda$-mass $1/4$, so by the Fubini theorem there must be a horizontal section $B^x$ whose $\mu$-mass is $\geq 1/4$. It is not difficult to check that $B^x$ is a $G$-anticlique. Thus, if $f : [n]^2 \to [0, 1]$ is any function, then

**Example 11.1.16.** Let $\varepsilon > 0$ be a fixed real number. For each $n \in \omega$, let $G_n$ be the graph on $2^n$ which connects $x, y$ if the set $\{m \in n : x(m) = y(m)\}$ has size at most $\varepsilon n$. The fractional chromatic numbers of $G_n$ tend to infinity with $n$. To see this, consider the normalized counting measure $\mu_n$ on $2^n$, the normalized Hamming metric $d_n$ on $2^n$, and the automorphism $\pi_n: 2^n \to 2^n$ defined by $\pi_n(x)(m) = 1 - x(m)$. Let $b \subset 2^n$ be a set of $\mu$-mass $\delta > 0$. The set $\pi^n b$ has $\mu$-mass $b$ again. By the concentration of measure results for the Hamming cubes [60, Theorem 4.3.19], if the number $n$ is large enough, the $\varepsilon/2$-neighborhoods of the sets $b$ and $\pi^n b$ in the sense of the metric $d_n$ have both $\mu_n$ mass greater than $1/2$ and therefore intersect. It follows that there must be points $x \in b$ and $y \in \pi^n b$ with $d_n(x, y) \leq \varepsilon_n$, and then $x, \pi^{-1} y$ are $G_n$-connected elements of the set $b$.

Note that if the number $\varepsilon > 0$ is small, the previous example provides graphs with large fractional chromatic number and no cycles of small even length.
Example 11.1.17. For every $n \in \omega$, let $H_n$ be the hypergraph on $n$ of arity 3 consisting of all arithmetic progressions of length 3. Then the fractional chromatic numbers of $H_n$ tend to infinity with $n$—just consider the normalized counting measure on $n$ and the density van der Waerden’s theorem [74].

Definition 11.1.18. Let $G$ be an analytic finitary hypergraph on a Polish space $X$. We say that $G$ has Borel $\sigma$-finite fractional chromatic number if there are Borel sets $B_n \subset X$ such that $X = \bigcup_n B_n$, and for each $n \in \omega$, the hypergraph $G$ has fractional chromatic number less than $n + 2$ on every finite set $a \subset B_n$.

Clearly, Borel $\sigma$-finite chromatic number implies Borel $\sigma$-finite fractional chromatic number but the opposite implication fails. The most natural class of Borel graphs exemplifying the distinction is the following:

Example 11.1.19. Let $E$ be a Borel non-smooth equivalence relation on a Polish space $X$. Let $Q$ be the poset of all pairs $q = \langle a_q, b_q \rangle$ of finite subsets of $X$ such that $(a_q \times b_q) \cap E = \emptyset$; the ordering is that of coordinatewise reverse inclusion. Let $G$ be the graph connecting two conditions in $Q$ if they are incompatible. Then $G$ does not have Borel $\sigma$-finite chromatic number, yet it does have Borel $\sigma$-finite fractional chromatic number.

Proof. To show that $G$ does not have Borel $\sigma$-finite chromatic number, first use the Glimm–Effros dichotomy to find a Borel reduction $h: 2^\omega \rightarrow X$ of $\mathbb{E}_0$ to $E$. For $\mathbb{E}_0$-unrelated points $y_0, y_1 \in 2^\omega$ let $q(y_0, y_1) = \langle \{h(y_0), h(y_1)\} \rangle$. By the Baire category theorem, it will be enough, for any Borel nonmeager set $B \subset 2^\omega \times 2^\omega$ and every $n \in \omega$, to produce a finite subset $b \subset B$ such that the chromatic number of $G$ on $q''b$ is greater than $n$. To this end, let $t_0, t_1 \in 2^{<\omega}$ be two binary strings of the same length such that $B \subset [t_0] \times [t_1]$ is comeager. Let $k \in \omega$ be a number such that $k \rightarrow (3)_n^2$. A simple construction yields points $y_i \in 2^\omega$ for $i \in k$ such that for any $i \in j \in k$, the pair $p_{ij} = \langle t_0^i y_i, t_1^i y_j \rangle$ belongs to the set $B$. It will be enough to show that the graph $G$ on the set $c = \{q(p_{ij}) : i \in j \in k\}$ has chromatic number greater than $n$. To see this, suppose that $c = \bigcup_{m \in n} d_m$. By the choice of the number $k$, there is $m \in n$ and numbers $i_0 \in i_1 \in i_2$ such that any pair of them gives rise to a condition in the set $d_m$. Now note that the conditions $q(p_{i_0 i_1})$ and $q(p_{i_1 i_2})$ are incompatible in $Q$ and so form a $G$-edge.

To show that $G$ has Borel $\sigma$-finite fractional chromatic number, let $B_n = \{q \in Q : |a_q \cup b_q| \leq n\}$, observe that the set $B_n$ is Borel and $Q = \bigcup_n B_n$, and argue that for every finite set $c \subset B_n$, the fractional chromatic number on $G$ on $c$ is $\leq 2^{-n}$. Indeed, assume that $\mu$ is any probability measure on the set $c$; we will produce a set of $\mu$-mass $\geq 2^{-n}$ consisting of pairwise compatible conditions, i.e. a $G$-anticlique. To do this, let $e$ be the finite set of all $E$-classes with nonempty intersection with $\bigcup_{q \in c} (a_q \cup b_q)$ and let $\lambda$ be the normalized counting measure on $P(e)$. Let $A = \{\langle q, u \rangle \in c \times P(e) : a_q \subset \bigcup(u) \wedge b_q \cap \bigcup(u) = \emptyset\}$. Since $c \subset B_n$, the vertical sections of the set $A$ have $\lambda$-mass $\geq 2^{-n}$. By the Fubini theorem, there is a horizontal section of $A$ of $\mu$-mass $\geq 2^{-n}$. This horizontal section is the desired $G$-anticlique of $\mu$-mass $\geq 2^{-n}$.  \[\square\]
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Example 11.1.11 actually shows that the diagonal Hamming graph does not have Borel \(\sigma\)-finite fractional chromatic number, since the density of a clique of size \(n\) is \(n\). Many other examples arise from various density versions of Ramsey-type theorems.

Example 11.1.20. Suppose that \(\mathbb{Z}\) acts in a Borel, free and measure-preserving way on a Polish probability measure space \(\langle X, \mu \rangle\). Let \(G\) be the Borel hypergraph on \(X\) of arity 3 containing a triple \(\{x, y, z\}\) if there is \(n \in \omega\) such that \(n \cdot x = y\) and \(n \cdot y = z\). The hypergraph \(G\) does not have Borel \(\sigma\)-finite fractional chromatic number.

Proof. Suppose towards contradiction that \(X = \bigcup_n B_n\) is a decomposition of \(X\) into Borel sets witnessing the Borel \(\sigma\)-finite fractional chromatic number of \(G\).

Pick \(n \in \omega\) and \(\varepsilon > 0\) such that \(\mu(B_n) > \varepsilon\). Use the density version of van der Waerden theorem \([74]\) to find a number \(m_0 \in \omega\) such that in every subset of \(m_0\) of size \(> m_0/n\), one of the classes contains an arithmetic progression of length three. Use the density van der Waerden theorem again to find a number \(m_1\) such that every subset of \(m_1\) of size \(> \varepsilon m_1\) contains an arithmetic progression of length \(m_0\). Let \(C \subset m_1 \times \mathbb{Z}\) be the set \(C = \{(i, x) : x \in i \cdot B_n\}\). As every vertical section of the set \(C\) has \(\mu\)-mass \(> \varepsilon\), by the Fubini theorem there must be a point \(x \in X\) such that the horizontal section \(C_x \subset m_1\) contains more than \(\varepsilon m_1\) many numbers and so an arithmetic progression \(\{i + jk : k \in m_0\}\) for some choice of \(i, k\). Let \(x_j = (-i - jk) \cdot x\) for \(j \in m_1\) and note that \(\{x_j : j \in m_0\} \subset B_n\) holds. By the choice of \(m_0\), the fractional chromatic number of \(G\) on the set \(\{x_j : j \in m_0\}\) is larger than \(n\), contradicting the choice of the set \(B_n\).

The class of analytic finitary hypergraphs which do not have Borel \(\sigma\)-finite fractional chromatic number has a simple basis.

Definition 11.1.21. Suppose that \(\langle a_n, H_n, t_n : n \in \omega \rangle\) is a sequence such that \(a_n\) is a nonempty finite set, \(H_n\) is a hypergraph on \(a_n\) such that every \(H_n\)-anticlique has fewer than \(|a_n|/(n + 2)\) many elements, and \(t_n \in \prod_{m \in n} a_m\) and the set \(\{t_n : n \in \omega\}\) is dense in \(\bigcup_n \prod_{m \in n} a_m\). The skew product \(\prod_n H_n, t_n\) is then called a large measured skew product.

A straightforward Baire category argument shows that a large measured skew product does not have Borel \(\sigma\)-finite fractional chromatic number. The following fact is proved in the same way as the \(\mathbb{G}_0\)-dichotomy, and we omit the standard argument.

Fact 11.1.22. Let \(G\) be a finitary analytic hypergraph on a Polish space \(X\). Exactly one of the following occurs:

1. \(G\) has Borel \(\sigma\)-finite fractional chromatic number;
2. there is a large measured skew product \(H\) on a Polish space \(Y\) and a continuous homomorphism \(h : Y \to X\) of \(H\) to \(G\).

Finally, we include a definition which deals with infinite sets.
Definition 11.1.23. Let $G$ be an analytic graph on a Polish space $X$. We say that $G$ has \emph{Borel $\sigma$-finite clique number} if there are Borel sets $B_n \subseteq X$ for $n \in \omega$ such that no $B_n$ contains an infinite $G$-clique.

Fact 11.1.24. Let $G$ be an analytic graph on a Polish space $X$. Exactly one of the following occurs:

1. $G$ has Borel $\sigma$-finite clique number;

2. there is a continuous homomorphism from a skew product of infinitely many infinite cliques to $G$.

11.2 Very Suslin forcings

The preservation arguments in this chapter are in spirit quite different than those in other chapters. All of them use definable c.c.c. \emph{control forcings}. These are posets that serve to build interesting balanced conditions in the central $\sigma$-closed poset. We will need to learn how to iterate the control forcings with finite support while preserving their definability properties. We will also need to isolate some definable regularity properties of the control forcings and learn how to preserve those. This last task is quite reminiscent of Stevo Todorcevic’s emphasis on regularity properties of c.c.c. posets as exhibited in numerous papers of his–[79, 78]. However, we must stress that in our case the regularity properties must be witnessed in a definable way, otherwise they are worthless. Similarly, our c.c.c. control forcings are nearly all isomorphic to the product of continuum many Cohen reals in ZFC; their worth for our purpose stems from their definability properties as opposed to their forcing properties.

Definition 11.2.1. [35] A pre-ordering $(P, \leq)$ is a \emph{Suslin forcing} if there is an ambient Polish space $X$ such that $P \subseteq X$ is an analytic set, $\leq$ is an analytic subset of $X^2$, and the incompatibility relation $\perp$ on $P$ is an analytic subset of $X^2$. To ease the notational clutter, in this section we only require that $\leq$ is a transitive relation on $P$ containing the diagonal. In addition, the poset $P$ is required to have a largest element.

A given Suslin poset $(P, \leq)$ can be reinterpreted in every forcing extension. Note that the demands on the analytic definitions of $\leq$ and $\perp$ are $\Pi^1_2$ and so the reinterpretation is again a Suslin poset. Importantly, the c.c.c. property of Suslin forcings persists to forcing extensions.

Fact 11.2.2. [35] Let $(P, \leq)$ be a Suslin c.c.c. poset. The reinterpretation of $P$ in any forcing extension is c.c.c.

One has to enter the much more restrictive class of \emph{very Suslin forcings} to guarantee that the finite support iterations, separative quotients, and completions have desirable descriptive properties.
Definition 11.2.3. Suppose that $P$ is a Suslin c.c.c. poset on an ambient Polish space $X$. The poset is very Suslin if the set \( \{ a \in P^\omega : \text{rng}(a) \text{ is predense in } P \} \) is an analytic subset of $X^\omega$.

Note that being a very Suslin partial order is a \( \Pi^1_2 \) statement in the code for the analytic set of maximal antichains and therefore absolute among all forcing extensions. Note that if the poset $P$ and the ordering $\leq$ are Borel, which is the case for all applications in this book and elsewhere as well, then both the incompatibility relation and the set of maximal antichains are coanalytic and therefore Borel by the Suslin theorem.

Example 11.2.4. The Cohen forcing, random forcing, and the eventually different real forcing are very Suslin.

Proof. The Suslinity of the definition of the posets is elementary and left for the reader to check; we only verify the very Suslin property. The Cohen forcing is the poset $P$ of finite binary strings ordered by reverse inclusion. A set $a \subset P$ is predense if every condition in $P$ is compatible with some element of $a$, which in view of the fact that $P$ is countable is a Borel condition. If the random forcing $P$ is realized as the collection of compact \( \mu \)-positive mass subsets of $X$ ordered by inclusion, where $\mu$ is some Borel probability measure on a Polish space $X$, then a countable set $a \subset P$ is pre-dense just in case $\mu(\bigcup a) = 1$, which is a Borel condition. The case of the eventually different real forcing is somewhat more involved, and addressed in [85, Proposition 3.8.12].

It is highly relevant to mention that the very Suslin posets used in this chapter below are never generated by a single real as the examples above. The reader

Example 11.2.5. The Hechler forcing is Suslin c.c.c. but not very Suslin.

The Suslinness and c.c.c. of Hechler forcing are well-known and easily checked. The failure of the the very Suslin property follows from the basic distinction between Suslin and very Suslin c.c.c. posets: the latter cannot add dominating reals.

Proposition 11.2.6. Let $P$ be a very Suslin c.c.c. forcing. Then $P$ does not add dominating reals.

Proof. Let $p \in P$ and $\tau$ be a $P$-name for an element of $\omega^\omega$. Consider the set $A = \{ T \subset \omega^{<\omega} : \exists q \leq p \ q \vdash \hat{T} \text{ has no infinite branch modulo finite dominated by } \tau \}$. By the evaluation of the complexity of the forcing relation in Proposition 11.2.9 below, $A$ is an analytic set of trees. At the same time, if $p \models \tau$ is a dominating real, then $A$ is the set of all well-founded trees, which is coanalytic and not analytic. Since the conclusion is impossible, the assumption must be false as well. The proposition follows.

Many independence results achieved in this chapter can be combined. The combinations are enabled by the fact that the relevant classes of c.c.c. posets are closed under product. Interestingly enough, in a general setting this appears to be open:
Question 11.2.7. The class of Suslin c.c.c. forcings is closed under countable, finite support product. Is this true for the class of very Suslin c.c.c. forcings as well?

An important general feature of very Suslin forcings used over and over below is that their forcing relation is $\Sigma_1^1$ on $\Sigma_1^1$ in a precise sense we now proceed to explain.

Definition 11.2.8. Suppose that $P$ is a very Suslin c.c.c. poset and $X$ is a Polish space.

1. A *layered $P$-name* for an element of $X$ is a tuple $\tau = \langle A_n, f_n : n \in \omega \rangle$ such that $A_n \subseteq P$ is a maximal antichain, $f_n$ is a map from $A_n$ to basic open subsets of $X$ of radius $< 2^{-n}$, and if $m < n$ then $A_m$ refines $A_n$ and if $p_n \in A_n$ and $p_m \in A_m$ are such that $p_m \leq p_n$, then the closure of $f_m(p_m)$ is a subset of $f_n(p_n)$.

2. $X^P$ is the set of all layered $P$-names for elements of $X$.

Note that the set $X^P$ is analytic in a suitable ambient space. The definition of $X^P$ appears to depend on the choice of the metric for $X$, a dependence that we happily suppress. It is clear that for every name for an element of the space $X$ there is a layered name which is forced to be equal to the original name.

Proposition 11.2.9. Suppose that $P$ is a very Suslin c.c.c. poset, $X$ is a Polish space, and $A \subseteq X$ is an analytic set. The set $\{ (p, \tau) \in P \times X^P : p \Vdash \tau \in \dot{A} \}$ is an analytic subset of $P \times X^P$.

Proof. Let $\tau = \langle A_n, f_n : n \in \omega \rangle$. Let $h : \omega^\omega \to A$ be a continuous surjection. The statement $p \Vdash \tau \in \dot{A}$ is equivalent to the existence of a system $(B_n, g_n : n \in \omega)$ where for each $n \in \omega$, $B_n$ is a maximal antichain of $P$ of conditions which are either below $p$ or incompatible with $p$, $g_n : B_n \to \omega^\omega$, $B_{n+1}$ refines $B_n$ and $A_n$, whenever $r \leq q$ are conditions in $B_{n+1}$ and $B_n$ respectively then $g_{n+1}(r)$ properly extends $g_n(q)$, and if $r \leq q$ is an element of $B_{n+1}$ below $p$ and an element of $A_n$ respectively then $[g_{n+1}(r)] \subseteq h^{-1}f_n(q)$. This is an analytic statement.

In the following subsections we isolate several iterable classes of very Suslin c.c.c. forcing notions. This may be relevant for the independence results concerning cardinal invariants of the continuum; our aims are different, but still the iteration theorems must be used in the course of our proofs.

11.2a Finite support iterations

The purpose of this section is to set up the finite support iteration machinery for very Suslin c.c.c. forcings. We emphasise that no such treatment is possible for general Suslin forcings. The complexity of the forcing relation on e.g. the Hechler forcing causes the complexity of the iterated posets to explode in ways that are unmanageable without large cardinal assumptions.
**Definition 11.2.10.** Let $P, Q$ be posets. A *projection* of $Q$ to $P$ is a pair of order-preserving functions $\pi : Q \to P$ and $\xi : P \to Q$ such that

1. $\pi \circ \xi$ is the identity on $P$;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

The following definition will be used at the successor stage of the iterations.

**Definition 11.2.11.** Let $P$ be a very Suslin poset on a Polish space $X$ and $Q$ be a very Suslin poset on a Polish space $Y$. Define $P \ast \dot{Q}$ to be the set of all ordered pairs $\langle p, \dot{q} \rangle$ where $p \in P$ and $q$ is a nice $P$-name for an element of $Y$ such that $p \Vdash \dot{q} \in \dot{Q}$. The ordering is defined by $\langle p_1, \dot{q}_1 \rangle \leq \langle p_0, \dot{q}_0 \rangle$ if $p_1 \leq p_0$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_0$. The iteration $P \ast \dot{Q}$ also includes the iteration maps $\pi : P \ast \dot{Q} \to P$ defined by $\pi(p, \dot{q}) = p$ together with the function $\xi : P \to P \ast \dot{Q}$ defined by $\xi(p) = (p, 1_Q)$.

The limit stages will be defined in the following way.

**Definition 11.2.12.** A very Suslin system is a tuple $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle$ where

1. each $P_n$ is a very Suslin c.c.c. forcing;
2. for each $m \leq n \in \omega$, the functions $\pi_{nm} : P_n \to P_m$ and $\xi_{mn} : P_m \to P_n$ are analytic and form a projection of $P_n$ to $P_m$, with $\pi_{nn}$ and $\xi_{nn}$ equal to the identity on $P_n$;
3. the functions $\pi_{nm}$ commute, as do the functions $\xi_{mn}$.

The limit of the system is the poset $P_\omega$ of all pairs $\langle p, n \rangle$ where $p \in P_n$, ordered by $\langle q, n \rangle \leq \langle p, m \rangle$ if $m \leq n$ and $\pi_{nm}(q) \leq p$ in $P_n$. The limit also includes the limit maps $\pi_{\omega m} : P_\omega \to P_m$ and $\xi_{m\omega} : P_m \to P_\omega$ defined by $\xi_{m\omega}(p) = (p, m)$ and $\pi_{\omega m}(\langle p, n \rangle) = \pi_{nm}(p)$ if $n > m$ and $\pi_{\omega m}(\langle p, n \rangle) = \xi_{nm}(p)$ if $n \leq m$.

**Theorem 11.2.13.** Let $\alpha \in \omega_1$ be a countable ordinal and $P$ a very Suslin c.c.c. forcing. The finite support iteration of $P$ of length $\alpha$ is naturally isomorphic to a very Suslin c.c.c. forcing.

**Proof.** The argument proceeds by a standard transfinite induction argument using the following claims:

**Claim 11.2.14.** Let $P, Q$ be very Suslin c.c.c. posets. Then

1. $P \ast \dot{Q}$ is a very Suslin c.c.c. poset;
2. the iteration maps are analytic and form a projection from $P \ast \dot{Q}$ to $P$. 
Proof. This is a routine complexity calculation. The underlying set $P \ast \dot{Q}$ is analytic by Proposition 11.2.9, since $p \Vdash \dot{q} \in \dot{Q}$ is an analytic statement. The ordering on $P \ast \dot{Q}$ is analytic for the same reason. To check the analyticity of the incompatibility relation, observe that conditions $\langle p_0, \dot{q}_0 \rangle$ and $\langle p_1, \dot{q}_1 \rangle$ are incompatible just in case there is a (countable) maximal antichain $A \subset P$ such that for every $p \in A$, either $p$ is incompatible with $p_0$, or it is incompatible with $p_1$, or it is below both $p_0, p_1$, and in the latter case, $p \Vdash \dot{q}_0, \dot{q}_1$ are incompatible in the poset $\dot{Q}$. This is an analytic statement by Proposition 11.2.9 and the assumption that the incompatibility relation on $\dot{Q}$ is analytic.

The poset $P \ast \dot{Q}$ is c.c.c. because $\dot{Q}$ remains c.c.c. in the $P$-extension by Fact 11.2.2 and so $P \ast \dot{Q}$ is an iteration of two c.c.c. forcings. Finally, we have to check that for a countable set $B = \{ \langle p_n, \dot{q}_n \rangle : n \in \omega \} \subset P \ast \dot{Q}$, the statement that $B$ is predense is analytic. To see this, let $Y_0$ be the Polish space resulting from adding an isolated point 0 to $Y$, and consider the following formula $\phi$: there exists a name $\tau$ for an element of $(Y_0)^\omega$ and maximal antichains $A_n \subset P$ for $n \in \omega$ such that for each $n$, $A_n$ consists of elements which are either incompatible with $p_n$ or stronger than $p_n$, if they are stronger than $p_n$ then they force $\tau(n) = \dot{q}_n$, if they are incompatible with $p_n$ then they force $\tau(n) = 0$, and $1_P$ forces $\operatorname{rng}(\tau) \cap Y$ is predense in $\dot{Q}$. Parsing the formula $\phi$, we see that it is analytic by Proposition 11.2.9, and that it says that $1_P \Vdash \{ \dot{q}_n : p_n \text{ belongs to the generic filter} \}$ is predense in $\dot{Q}$, which is exactly equivalent to the set $B$ being predense in $P \ast \dot{Q}$.

Checking the properties of the projection functions is routine and left to the reader. \qed

Claim 11.2.15. Suppose that $\{ P_n : n \in \omega, \pi_{nm}, \xi_{m} : m \leq n \in \omega \}$ is a very Suslin system. The limit $P_\omega$ is a very Suslin c.c.c. forcing. The limit maps $\pi_{nm}$ and $\xi_{m\omega}$ form analytic projections and commute with the projection maps of the very Suslin system.

Proof. This is a straightforward complexity computation. Let $X_n$ be the ambient Polish space of each poset $P_n$; the ambient space of the poset $P_\omega$ is then the disjoint union $\bigcup_n X_n$ in product with $\omega$. The ordering of $P_\omega$ is analytic by the definition. The incompatibility relation is analytic as well: suppose that $\langle q, n \rangle$ and $\langle p, m \rangle$ are conditions in $P_\omega$, say with $n \geq m$. These two conditions are incompatible just in case $\pi_{nm}(q)$ is incompatible with $p$ in $P_{nm}$ by the definitory properties of projections.

The poset $P_\omega$ is c.c.c. since it is a direct limit of c.c.c. forcings. To check the very Suslin property of the poset $P_\omega$, note that a countable set $A \subset P_\omega$ is predense if and only if for every number $n \in \omega$, there is a maximal antichain $B_n \subset P_\omega$ of conditions such that for each $p \in B_n$, either there is a condition $\langle q, m \rangle \in A$ such that $m \leq n$ and $\pi_{nm}(p) \leq q$ in $P_{nm}$, or there is a condition $\langle q, m \rangle \in A$ such that $m \geq n$ and $p \leq \pi_{mn}(q)$ in $P_{nm}$. This is an analytic statement as all the posets $P_n$ are very Suslin.

Checking the projection properties of the functions $\pi_{nm}$ and $\xi_{m\omega}$ is routine and left to the reader. \qed
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Theorem 11.2.13 follows.

11.2b Suslin \(\sigma\)-linked posets

The posets we use later in this chapter are fairly innocent from ZFC point of view. However, we need their regularity properties to be witnessed in a Suslin way, which immediately brings up many questions. The following is the first very Suslin regularity property considered in this chapter.

Definition 11.2.16. Let \(P\) be a Suslin forcing.

1. A set \(A \subset P\) is \textit{linked} if any two conditions in \(A\) have a lower bound;
2. \(P\) is Suslin \(\sigma\)-linked if \(P\) can be written as a countable union \(\bigcup_n A_n\) of linked analytic sets.

Note that the definitory properties of the cover \(P = \bigcup_n A_n\) are \(\Pi^1_2\) and therefore persist to all forcing extensions. The main general result of this section is the iteration preservation theorem for Suslin \(\sigma\)-linkedness:

Theorem 11.2.17. Let \(P\) be a very Suslin c.c.c. forcing which is Suslin \(\sigma\)-linked. Let \(\alpha \in \omega_1\) be a countable ordinal. Then the finite support iteration of \(P\) of length \(\alpha\) is a Suslin \(\sigma\)-linked forcing.

Proof. The argument proceeds by a straightforward transfinite induction argument given the following two claims.

Claim 11.2.18. Let \(P,Q\) be very Suslin c.c.c. forcings, both of which are Suslin \(\sigma\)-linked. Then \(P \ast Q\) is Suslin \(\sigma\)-linked.

Proof. Let \(P = \bigcup_n A_n\) and \(Q = \bigcup_m B_m\) be the covers of \(P,Q\) by analytic centered (or linked) sets. Let \(C_{nm} \subset P \ast Q\) be the set of all conditions \((p,\dot{q})\) such that there exists a condition \((\dot{p}',\dot{q}') \leq (p,\dot{q})\) such that \(p' \in A_n\) and \(p' \Vdash \dot{q}' \in B_m\). It is not difficult to check that \(P \ast Q = \bigcup_{nm} C_{nm}\), the sets \(C_{nm}\) are linked. Moreover, the sets \(C_{nm}\) are analytic by Proposition 11.2.9.

Claim 11.2.19. Let \((P_n: n \in \omega, \pi_{nm}, \xi_{mn}: m \leq n \in \omega)\) be a very Suslin system consisting of Suslin \(\sigma\)-linked forcings. Then the limit is Suslin \(\sigma\)-linked.

Proof. Let \(P_\omega\) be the limit of the system as described in Definition 11.2.12. Let \(P_n = \bigcup_m A_{nm}\) be a cover by analytic linked sets for each \(n \in \omega\). For each \(n,m \in \omega\) let \(B_{nm} \subset P_\omega\) be the set \(\{(p,n) \in P_\omega: p \in A_{nm}\}\). It is immediate that these are linked analytic sets covering \(P_\omega\).

The theorem follows.

Theorem 11.2.20. The finite support product of countably many very Suslin, Suslin \(\sigma\)-linked posets, if very Suslin, is Suslin \(\sigma\)-linked.
Proof. Let $P_n$ for $n \in \omega$ be the very Suslin posets, with their Suslin $\sigma$-linked property witnessed by sets $A_{nm}$ for $m \in \omega$. Consider the finite support product $Q = \prod_n P_n$ and assume it is very Suslin. For each number $n_0 \in \omega$ and each function $h : n_0 \to \omega$ let $B_h = \{ q \in Q : \text{dom}(q) = n_0 \land \forall m \in n_0 \ q(m) \in A_{m,h(m)} \}$. It is not difficult to verify that the sets $B_h$ witness the Suslin $\sigma$-linked property of the poset $Q$.

11.2c Suslin $\sigma$-centered posets

Definition 11.2.21. Let $P$ be a Suslin forcing.

1. A set $A \subset P$ is \textit{centered} if any finite subset of $A$ has a lower bound in $P$;
2. The poset $P$ is \textit{Suslin $\sigma$-centered} if $P$ can be written as a countable union $\bigcup_n A_n$ and the sets $A_n$ are analytic and centered.

The key feature of the Suslin $\sigma$-centered property is that it is preserved under the finite support iterations of very Suslin forcings. The proof of the following theorems are literally copied from Theorems 11.2.17 and 11.2.20, replacing the word “linked” with “centered”.

Theorem 11.2.22. Let $P$ be a very Suslin c.c.c. forcing which is Suslin $\sigma$-centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of $P$ of length $\alpha$ is a Suslin $\sigma$-centered forcing.

Theorem 11.2.23. The finite support product of countably many very Suslin, Suslin $\sigma$-centered posets, if very Suslin, is Suslin $\sigma$-centered.

11.2d Suslin measured posets

The existence of finitely additive measures on complete Boolean algebras is a useful tool in forcing theory, see [38]. The Suslin version of it needs to reflect the fact that finitely additive measures on uncountable sets are typically not Suslin in any sense. We use the following poor man’s version of a finitely additive measure:

Definition 11.2.24. Let $P$ be a very Suslin c.c.c. poset. We say that $P$ is \textit{Suslin measured} if there are analytic sets $A_n \subset P$ and positive rationals $\varepsilon_n > 0$ such that $P = \bigcup_n A_n$ and for every $n \in \omega$ and every finite sequence $\langle p_i : i \in j \rangle$ of elements of $A_n$ with possible repetitions there is a set $b \subset j$ of size at least $\varepsilon_n j$ such that the set $\{ p_i : i \in b \}$ has a lower bound in $P$.

Note that the definitory properties of the cover $P = \bigcup_n A_n$ are $\Pi^1_2$ and therefore persist to all forcing extensions.

Theorem 11.2.25. Let $P$ be a very Suslin c.c.c. poset. If $P$ is Suslin measured, then so are all of its finite support iterations of countable length.

Proof. As in the previous subsections, we proceed with claims dealing with the successor and limit stage of the iteration.
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Claim 11.2.26. Suppose that \( P, Q \) are very Suslin, c.c.c. posets which are Suslin measured. Then \( P \ast Q \) is Suslin measured.

Proof. Let \( A_n, \alpha_n \) for \( n \in \omega \) and \( B_n, \beta_n \) for \( n \in \omega \) witness the Suslin measured property of \( P, Q \) respectively. Let \( C_{nm} \subseteq P \ast Q \) be the upward closure of the set of all pairs \( \langle p, q \rangle \) such that \( p \in A_n \) and \( p \Vdash \dot{q} \in B_m \). Let \( \gamma_{nm} = \alpha_n \beta_m \). We claim that the sets \( C_{nm} \) and the numbers \( \gamma_{nm} \) witness the Suslin measured property of the iteration.

It is clear from Proposition 11.2.9 that the sets \( C_{nm} \) are analytic; they also clearly exhaust all of \( P \ast \dot{Q} \). Now, fix numbers \( n, m \in \omega \). Suppose that \( \langle \langle p_i, \dot{q}_i \rangle : i \in j \rangle \) is a sequence of conditions in the set \( C_{nm} \); strengthening the conditions we may assume that \( p_i \in A_n \) and \( i \models \dot{q}_i \in B_m \) holds for every \( i \in j \).

By the properties of the set \( A_n \), there is a set \( b \subset j \) such that \( |b| > \alpha_n j \) and the conditions \( p_i \) for \( i \in b \) have a lower bound \( p \). By the properties of the set \( B_m \) applied in the generic extension, it is possible to strengthen \( p \) and identify a set \( c \subset b \) such that \( |c| > \beta_m |b| \) in such a way that \( p \Vdash \{ \dot{q}_i : i \in c \} \) has a lower bound in \( \dot{Q} \). Then the conditions \( \langle p_i, \dot{q}_i \rangle \) for \( i \in c \) have a common lower bound in \( P \ast \dot{Q} \). As the set \( c \subset j \) has size \( > \alpha_n \beta_m |j| \), this verifies the desired properties of the set \( C_{nm} \).

Claim 11.2.27. Let \( \langle P_n : n \in \omega, \pi_{nm}, \xi_{nm} : m \leq n \in \omega \rangle \) be a very Suslin system consisting of Suslin measured forcings. Then the limit is Suslin measured.

Proof. Let \( P_\omega \) be the limit of the system as described in Definition 11.2.12. For each \( n \in \omega \), let \( A_{nm}, \varepsilon_{nm} \) for \( m \in \omega \) be analytic subsets of \( P_n \) and positive rational numbers witnessing the measured property of \( P_n \). For each \( n, m \in \omega \) let \( B_{nm} \subseteq P_\omega \) be the set \( \{ \langle p, n \rangle \in P_\omega : p \in A_{nm} \} \). It is immediate that the sets \( B_{nm} \) with the numbers \( \varepsilon_{nm} \) witness the Suslin measured property of the limit \( P_\omega \).

A routine transfinite induction argument now completes the proof.

Theorem 11.2.28. The finite support product of countably many very Suslin, Suslin measured posets, if very Suslin, is Suslin measured.

Proof. Let \( P_n \) for \( n \in \omega \) be the very Suslin posets, with their Suslin measured property witnessed by sets \( A_{nm} \) and numbers \( \varepsilon_{nm} \) for \( m \in \omega \). Consider the finite support product \( Q = \prod_n P_n \) and assume it is very Suslin. For each number \( n_0 \in \omega \) and each function \( h : n_0 \to \omega \) let \( B_h = \{ q \in Q : \text{dom}(q) = n \land \forall m \in n_0 \ q(m) \in A_{m,h(m)} \} \), and let \( \varepsilon_h = \prod_{m \in \omega} \varepsilon_{m,h(m)} \). It is not difficult to verify that the sets \( B_h \) and the numbers \( \varepsilon_h \) witness the Suslin measured property of the poset \( Q \).

11.2e Suslin \( \sigma \)-liminf-centered posets

In this section, we elaborate on a notion which has been used in several contexts to guarantee that various posets do not add dominating reals \([57,1]\); we will use it for a different purpose.
Definition 11.2.29. Let $P$ be a partial order. We say that a set $A \subset P$ is liminf-centered if for every collection $\{p_n : n \in \omega\} \subset A$ there is a condition $q \in P$ such that every condition stronger than $q$ is compatible with $p_n$ for infinitely many numbers $n \in \omega$.

It is not difficult to see that the property of the condition $q$ demanded by the definition can be stated in the forcing language as $q \Vdash$ there are infinitely many $n \in \omega$ such that $p_n$ belongs to the generic filter.

Definition 11.2.30. Let $P$ be a Suslin poset. We say that $P$ is Suslin $\sigma$-liminf centered if $P = \bigcup_n A_n$ where each set $A_n \subset P$ is analytic and liminf-centered.

It is important and instructive to note that if $P$ is a very Suslin c.c.c. poset then its Suslin $\sigma$-liminf-centeredness will persist to all generic extensions. To see this, it is enough to observe that for an analytic set $A \subset P$, the statement “$A$ is liminf-centered” is $\Pi^1_2$; then, an application of Shoenfield absoluteness gives the required conclusion. Now, the statement “$A$ is liminf-centered” is equivalent to the following: for all $\langle p_n : n \in \omega \rangle$, either for some $n \in \omega$, $p_n \notin A$, or there is $q \in P$ such that $q \Vdash$ there are infinitely many $n \in \omega$ such that $p_n$ belongs to the generic filter. The forcing statement is analytic by Proposition 11.2.9 and the very Suslin assumption on $P$. Thus, the total statement is $\Pi^1_3$ as required.

Theorem 11.2.31. Let $P$ be a very Suslin c.c.c. forcing which is Suslin $\sigma$-liminf centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support of $P$ of length $\alpha$ is a very Suslin c.c.c. Suslin $\sigma$-liminf centered forcing.

Proof. The argument is a routine variation of the proofs of previous preservation theorems.

Claim 11.2.32. Let $P,Q$ be very Suslin c.c.c. forcings, both of which are Suslin $\sigma$-liminf-centered. The $P \ast Q$ is Suslin $\sigma$-liminf-centered.

Proof. Find covers $Q = \bigcup_n B_m$ and $P = \bigcup_n A_n$ by analytic liminf-centered sets. Let $C_{nm} \subset P \ast Q$ be the set of all conditions $\langle p,q \rangle$ such that there exists a condition $\langle p',q' \rangle \leq \langle p,q \rangle$ such that $p' \in A_n$ and $p' \Vdash q' \in B_m$. It is not difficult to check that $P \ast Q = \bigcup_{nm} C_{nm}$ and the sets $C_{nm}$ are analytic by Proposition 11.2.9. To check the liminf-centered property, suppose that $\{\langle p_i,q_i \rangle : i \in \omega\}$ are conditions in $C_{nm}$. Strengthening if necessary, we may assume that $p_i \in A_n$ and $p_i \Vdash q_i \in B_m$. Use the liminf-centeredness of $A_n$ to find a condition $p$ forcing that for infinitely many $i \in \omega$, $p_i$ is in the generic filter. Using the liminf-centeredness of the set $B_m$, find a name $\dot{q}$ for a condition in $Q$ such that $p \Vdash \dot{q} \Vdash$ for infinitely many $i \in \omega$ such that $p_i$ belongs to the generic filter on $P$, $\dot{q}_i$ belongs to the generic filter on $Q$. The condition $\langle p,q \rangle$ is as required.

Claim 11.2.33. Let $\langle p_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle$ be a very Suslin system consisting of Suslin $\sigma$-liminf-centered forcings. Then the limit is Suslin $\sigma$-liminf-centered.
Proof. Let $P_\omega$ be the limit of the system as described in Definition 11.2.12. Let $P_n = \bigcup_m A_{nm}$ for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subseteq P_\omega$ be the set $\{ (p, n) \in P_\omega : p \in A_{nm} \}$. It is immediate that these are analytic liminf-centered analytic sets exhausting $P_\omega$.

The theorem follows by a routine transfinite induction argument.

The following question probably has a negative answer; still, products of the very Suslin $\sigma$-liminf-centered posets used later in the chapter are again Suslin $\sigma$-liminf-centered.

**Question 11.2.34.** Is the class of Suslin $\sigma$-liminf-centered posets closed under product?

### 11.3 Locally countable simplicial complexes

In this section, we concentrate on tasks of a very specific form common to many concerns of modern descriptive set theory. These tasks start with a countable Borel equivalence relation $E$ on a Polish space $X$ and attempt to assign a structure of some type to each $E$-class; at the same time, distinct $E$-classes do not communicate with each other in any way. Reviewing the tasks of this type, it becomes obvious that often there is a locally countable simplicial complex $K$ on $X$ such that the associated equivalence relation $E_K$ (as identified in Definition 6.1.3) is $E$, and we seek a maximal $K$-set. Sometimes, the domain of the simplicial complex $K$ is instead the collection of finite subsets of $X$ consisting of pairwise $E$-related points, and sometimes we actually seek a maximal $K$-set with some additional largeness properties. The variations considered in this section all fall under the following definition:

**Definition 11.3.1.** A *locally countable pair* is a pair $(K, L)$ such that

1. $K$ is a Borel locally countable simplicial complex on a Polish space $X$;
2. $L \subseteq K$ is a cofinal Borel subset, i.e. $\forall a \in K \exists b \in L \ a \subset b$;
3. for every finite set $a \subset X$, $a \in L$ just in case $a \cap c \in L$ holds for every $E_K$-class $c \subseteq X$.

A $K$-set $a \subseteq X$ is *$L$-regular* if every finite subset of $a$ is a subset of a finite subset of $a$ which is in $L$.

The role of the set $L$ deserves a comment. Note that item (3) holds with $K$ replacing $L$ by the definition of the equivalence relation $K$; thus, distinct $E_K$-classes do not communicate regarding the membership of finite sets in $K$ and $L$. In many cases, the equality $L = K$ will occur; some tasks seem to be impossible to achieve in this way though. For example, let $G$ be a graph on $\omega$ and let $K$ be the simplicial complex of all finite $G$-colorings with colors coming from $\omega$. Not
every maximal $K$-set needs to be a total $G$-coloring. To alleviate this unwanted effect, let $L \subset K$ be the set of those finite colorings whose domain is a natural number; then every $L$-regular maximal $K$-set is in fact a total $G$-coloring.

**Definition 11.3.2.** Let $(K, L)$ be a locally countable pair on a Polish space $X$. The poset $P_{KL}$ is the poset of all sets $p \subset K$ for which there is a countable $E_K$-invariant set $\text{dom}(p) \subset X$ such that $p$ is a $L$-regular, maximal $K$-subset of $\text{dom}(p)$. The ordering is that of reverse inclusion. If $K = L$, we refer to the poset as $P_K$ instead.

It is immediate that the poset $P_{KL}$ is Suslin and $\sigma$-closed; in fact, it is a special case of the uniformization posets of Definition 6.3.5 with the space $X \times X^\omega$, the equivalence relations $E$ on $X$ and $F_2$ on $X^\omega$, and the invariant Borel set $B \subset X \times X^\omega$ consisting of all pairs $\langle x, y \rangle$ where $y$ enumerates an $L$-regular maximal $K$-subset of $[x]_E$. The classification of balanced conditions then follows from Theorem 6.3.6:

**Theorem 11.3.3.** Let $(K, L)$ be a locally countable pair on a Polish space $X$, with the associated poset $P_{KL}$.

1. For every $L$-regular maximal $K$-set $A \subset X$, the pair $\langle \text{Coll}(\omega, X), \bar{A} \rangle$ is balanced;

2. for every balanced pair $\langle Q, \sigma \rangle$ there is a set $A$ as in (1) such that the balanced pairs $\langle Q, \sigma \rangle$ and $\langle \text{Coll}(\omega, X), \bar{A} \rangle$ are equivalent;

3. distinct sets as in (1) yield inequivalent balanced pairs.

In order to identify properties of the poset $P_{KL}$ in the Solovay model, we have to analyze its very Suslin c.c.c. control poset which in this case is simply the partial order $K$ ordered by reverse inclusion. Note that the set $L$ does not enter the definition of the control poset at all. The following proposition verifies the basic property of the control poset.

**Proposition 11.3.4.** Let $K$ be a Borel locally countable simplicial complex on a Polish space $X$. Then $K$ is a very Suslin c.c.c. forcing.

**Proof.** Write $E = E_K$; this is a Borel countable equivalence relation on the space $X$. Observe that for an $E$-invariant set $C \subset X$, $K \cap \mathcal{P}(C)$ is a regular subposet of $K$. It is certainly closed under unions in $K$. Now suppose that $a \in K$ is any condition and let $b = a \cap C$. Whenever $c \supset b$ is some condition in $K \cap \mathcal{P}(C)$, $a \cup c \in K$ holds. To see this, recall that $a \cup c$ belongs to $K$ just in case its intersection with every $E$-class is. Now, if $d$ is an $E$-class, if $d \subset C$ then $(a \cup c) \cap d = c \in K$, and if $d \cap C = 0$ then $(a \cup c) \cap d = a \in K$ holds again. The regularity of $K \cap \mathcal{P}(C)$ in $K$ has just been proved.

To verify the c.c.c. part of the proposition, let $A \subset K$ is a maximal antichain and let $M$ be a countable elementary submodel of a large structure containing $A, E$, and $K$. By the elementarity of $M$, the set $A \cap M$ is a maximal antichain in the poset $K \cap M = K \cap \mathcal{P}(X \cap M)$. Since $E$ is an equivalence relation with
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all classes countable, the elementarity of $M$ shows that $X \cap M$ is an $E$-invariant set. By the previous paragraph then, $A \cap M$ is a maximal antichain in $K$. As a result, $A \cap M = A$ and so $A$ is countable.

To verify the very Suslin part of the proposition, let $A \subseteq K$ be a countable set. To check that $A$ is a maximal antichain in $K$, it is only necessary to check that it is a maximal antichain in the poset $K \cap P(C)$, where $C$ is the $E$-saturation of $\bigcup A$ by the first paragraph of the proof. Such a verification is a Borel procedure as all $E$-classes are countable. \qed

The key point in all subsections below will be the identification of some Suslin forcing preservation properties of the control poset and their connection to the forcing properties of the poset $P_{KL}$ in the Solovay model.

11.3a Suslin $\sigma$-centered complexes

The results of this section show that in certain extensions of the Solovay model, uncountable chromatic numbers of analytic hypergraphs of finite arity do not change at all. This is the contents of the following theorem.

Theorem 11.3.5. Let $(K, L)$ be a locally countable pair. Suppose that the simplicial complex $K$ is Suslin $\sigma$-centered. Suppose that $G$ is an analytic finitary hypergraph of uncountable Borel chromatic number. Then in the $P_{KL}$-extension of the Solovay model, the chromatic number of $G$ is uncountable.

Proof. As a preliminary consideration, it is clear from Fact 11.1.8 that it is enough to consider the case of $G$ which is a principal skew product as in Definition 11.1.7. Let $(a_n, H_n, t_n; n \in \omega)$ be a sequence such that $H_n = \{a_n\}$ and $G = \prod_n H_n, t_n$.

Write $X$ for the Polish space which is the domain of the Borel simplicial complex $K$, write $E = E_K$ and $P = P_{KL}$. Let $\kappa$ be an inaccessible cardinal, let $W$ be the Solovay model derived from $\kappa$. Let $n \geq 2$ be a natural number and work in the model $W$. Suppose that $p \in P$ is a condition and $\tau$ is a $P$-name for a function from $n^\omega$ to $\omega$. We must find a stronger condition $\bar{p} \leq p$, a number $m \in \omega$ and a $G$-hyperedge $e$ such that $\bar{p} \Vdash \tau(y) = \bar{m}$ for all vertices $y \in \bar{e}$.

The condition $p \in P$ as well as the name $\tau$ are definable from some parameters in the ground model and a parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate model obtained by a poset of size $< \kappa$ containing the parameter $z$, and work in the model $V[K]$ until further notice. Consider the iteration $Q * \hat{R}$ where $Q$ is the Cohen poset on $\prod_n a_n$ adding a generic point $y_{gen} \in \prod_n a_n$, and in the $Q$-extension, $\hat{R}$ is the finite support iteration of length $\omega_1$ of the poset $K$. As usual, the poset $K$ is reinterpreted at each stage of the iteration; we denote the model obtained after $\alpha$-th stage by $V_\alpha$, so $V_0$ is the $Q$-extension of $V[K]$. Thus, the $\hat{R}$ poset adds a sequence of $K$-sets, $\langle A_\alpha; \alpha < \omega_1 \rangle$ derived from the iteration components of $R$. By a density argument, for each $E$-class $c$ represented in the model $V_\alpha$, the set $A_\alpha \cap c$ is a maximal $K$-subset of $c$ which is $L$-regular. Consider the $Q * R$-name $\bar{p}$ for the $K$-set defined in the following way: if $c$ is an $E$-class represented in $\text{dom}(p)$, then $c \cap \bar{p} = c \cap p$, if $c$ is an $E$-class represented
in $V_0$ but not in $\text{dom}(p)$ then $c \cap \bar{p} = \bar{A}_0 \cap c$, and if $\alpha \in \omega_1$ is an arbitrary nonzero ordinal and $c$ is an $E$-class represented in $V_\alpha$ but not in $\bigcup_{\beta < \alpha} V_\beta$ then $c \cap \bar{p} = c \cap \bar{A}_\alpha$. It is not difficult to see that $\bar{p}$ is a $Q \ast \bar{R}$-name for a maximal $\mathcal{K}$-set which is $\mathcal{L}$-large, and therefore a balanced condition for the poset $P$ by Theorem 11.3.3. Note that the condition does not depend on the point $\dot{y}_{\text{gen}}$ but only on the model $V[K][\bar{y}_{\text{gen}}]$.

Now, by a standard balance argument, $Q \ast \bar{R} \models \exists m \text{ Coll}(\omega, < \kappa) \models \bar{p} \Vdash_{P} \tau(\bar{y}_{\text{gen}}) = \bar{m}$. Let $\langle q, \bar{r} \rangle$ be a condition in the iteration $Q \ast \bar{R}$ deciding the value of $m$; we abuse the notation and call the specific value $m$ again. Strengthening the condition $q \in Q$ if necessary, we may assume that

- there is an ordinal $\alpha \in \omega_1$ such that $q \Vdash_{Q} \bar{r}$ is a name for a condition in the initial segment $R_\alpha$ of the iteration $\bar{R}$;
- by the iteration theorem 11.2.22, $R_\alpha$ is a very Suslin forcing which is Suslin $\sigma$-centered. Let $\{B_k : k \in \omega\}$ be analytic centered sets covering $R_\alpha$. We may find a specific number $k \in \omega$ such that $q \Vdash \bar{r} \in B_k$;
- $q = t_j$ for some number $j \in \omega$.

Let $y \in \prod_a a_n$ be a Cohen-generic element over the model $V[K]$ extending $t_j$ and move into the model $V[K][y]$. For each $i \in a_n$ let $y_i \in \prod_{a} a_n$ be the element obtained from $y$ by rewriting the $j$-th entry with $i$. Thus, each $y_i$ is a Cohen generic element over $V[K]$ meeting the condition $q$, $V[K][y_i] = V[K][y]$, and $\{y_i : i \in n\}$ is a $G$-hyperedge. Let $r_i = \bar{r}/y_i$. Thus, each $r_i$ is a condition in the poset $R_\alpha$, and even in the analytic centered set $B_k$. Thus, the conditions $r_i$ for $i \in a_n$ have a common lower bound. Now, let $H \subset R$ be a filter generic over the model $V[K][y]$ and containing all conditions $r_i$ for $i \in a_n$. Consider the balanced condition $\bar{p} \leq p$ in the poset $P$ in the model $V[K][y][H]$ described in the third paragraph of the present proof. The forcing theorem applied for each $i \in a_n$ shows that for each $i \in a_n$, $V[K][y][H] \models \text{Coll}(\omega, < \alpha) \models \bar{p} \Vdash_{P} \tau(\bar{y}_i) = \bar{m}$. Thus, $W \models \bar{p} \Vdash \tau(\bar{y}_i) = \bar{m}$ for each $i \in a_n$, completing the proof.\hfill $\square$

The verification of Suslin $\sigma$-centeredness for a given locally countable Borel simplicial complex $\mathcal{K}$ may look like a formidable challenge. However, there is a standard trick, recorded in the following theorem, which makes short work of it in all cases we are aware of.

**Theorem 11.3.6.** Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space $X$ and write $E = E_\mathcal{K}$. Suppose that there is a Borel support function $\text{supp} : \mathcal{K} \to [X]^{< \omega_0}$ such that

1. $\text{supp}(a) \subset [a]_E$;
2. whenever $\{a_i : i \in j\}$ are sets in $\mathcal{K}$, subsets of the same $E$-class with pairwise disjoint supports, then $\bigcup_i a_i \in \mathcal{K}$.

Then $\mathcal{K}$ is Borel $\sigma$-centered.
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Proof. Fix a countable basis $B$ for the space $X$. Let $\Gamma$ be a countable group acting on $X$ in a Borel way such that $E$ is the resulting orbit equivalence relation. Let $a \in K$, and write $\text{supp}^*(a) = \bigcup \{\text{supp}(a \cap c) : c \text{ is an } E\text{-class}\}$. We say that a tuple $(b, f)$ is a descriptor of $a$ if

- $b \subset B$ is a finite set consisting of pairwise disjoint sets such that every element of $\text{supp}^*(a)$ belongs to $\bigcup b$ and no element of $b$ contains more than one element of $\text{supp}^*(a)$;
- $f : b \to [\Gamma]^{<\aleph_0}$ is a function such that whenever $O \in b$ is an open set, $c$ is an $E$-class, and $x \in O \cap \text{supp}(a \cap c)$ is a point then $a \cap c = \{\gamma \cdot x : \gamma \in f(O)\}$.

Note that $a$ indeed has a descriptor since the set $\text{supp}^*(a)$ is finite (making the construction of $b$ as in the first item possible), and for any $E$-class $c$ the set $\text{supp}(a \cap c)$ is a subset of $c$ by (1) (making the construction of $f$ as in the second item possible). We will show that if $\{a_i : i \in j\}$ are elements of $K$ with the same descriptor $(b, f)$ then $\bigcup_i a_i \in K$. Since there are only countably many possible descriptors, and the set of elements of $K$ with a given descriptor is Borel, this will conclude the proof of the theorem.

Let $c$ be an $E$-class; we must show that $\bigcup_i (a_i \cap c) \in K$. In view of property (2) of the supp function, it will be enough to show that whenever $i_0 \neq i_1$ are elements of $j$ then either $a_{i_0} \cap c = a_{i_1} \cap c$ or $\text{supp}(a_{i_0} \cap c)$ is disjoint from $\text{supp}(a_{i_1} \cap c)$. Towards this end, suppose that $x \in \text{supp}(a_{i_0} \cap c) \cap \text{supp}(a_{i_1} \cap c)$ is a point; we have to show that $a_{i_0} \cap c = a_{i_1} \cap c$ holds. Let $O \in b$ be an open set such that $x \in O$, and use the second property of a descriptor to see that $a_{i_0} \cap c = a_{i_1} \cap c = \{\gamma \cdot x : \gamma \in f(O)\}$. The theorem follows.

Our first example is motivated by a ZF+DC result of [15]: for every Borel locally finite graph on a Polish space $X$, if the chromatic number of $G$ is $\leq n$, then there is a meager set $B \subset X$ and a decomposition of the set $X \setminus B$ into $2n-1$ many Borel $\text{G}$-anticliques. We have:

**Example 11.3.7.** Let $G$ be a Borel locally finite graph on a Polish space $X$ of chromatic number $n$. Let $K$ be the simplicial complex of all finite partial $G$ colorings with $\leq 2n - 1$ colors which can be extended to a total coloring with $\leq 2n - 1$ many colors in which at most $n$ many colors are attained infinitely many times. Then $K$ is a Borel locally countable simplicial complex which is Suslin $\sigma$-centered.

**Proof.** Note that $K$ is in fact a simplicial complex on $X \times (2n-1)$. The various assertions of the example follow from two simple general graph theoretic claims. Let $H$ be a locally finite graph on a set $V$ of vertices and let $c : V \to m$ be a $H$-coloring with finitely many colors. Define $F(c) = \{i \in m : \text{ the set } \{v \in V : c(v) = i\} \text{ is finite}\}$.

**Claim 11.3.8.** For every finite set $a \subset V$ and every set $e \subset m$ of size $|F(c)|$ there is a $H$-coloring $d : V \to m$ such that $c \upharpoonright a = d \upharpoonright a$ and $F(d) = e$. 
Claim 11.3.9. For every finite set $a \subseteq V$ there is a finite partial $H$-coloring $d : V \to m$ such that $a \subseteq \text{dom}(d)$, $c \upharpoonright a = d \upharpoonright a$ and for every edge $\{u, v\} \in H$ with $u \in \text{dom}(d)$ and $v \notin \text{dom}(d)$, $d(u) \notin |F(c)| + 1$ holds.

Proof. By the previous claim, we can adjust $c$ so that $F(c)$ is the set of the first $F(c)$-many natural numbers. Let $a'$ be the union of the finite sets $a$, $\{v \in V : c(v) \in F(c)\}$, and $\{v \in V : \exists u \{u, v\} \in H$ and $c(u) \in F(c)\}$. Let $b$ be the union of the finite sets $a'$ and $\{v \in V : \exists u \in a' \{u, v\} \in H$ and $c(u) = |F(c)|\}$. Let $d = c \upharpoonright b$ and observe that $d$ works. \qed

Now, we are ready to complete the proof of the example. Write $E$ for the Borel equivalence relation of $G$-path connectedness on the space $X$. First of all, the simplicial complex $\mathcal{K}$ is Borel and locally countable. Applying the last claim to each connected component of the graph $G$, it is clear that a finite partial $G$-coloring $a$ belongs to $\mathcal{K}$ just in case for every $E$-class $c \subseteq X$ there is a finite partial $G$-coloring $d$ such that $\text{dom}(a) \cap c \subseteq \text{dom}(d) \cap c$ and for every $G$-edge $\{x, y\}$ with $x \in \text{dom}(d)$ and $y \in c \setminus \text{dom}(d)$ it is the case that $d(u) \notin n$ holds. (Such a coloring $d$ can be completed by coloring the rest of the class $c$ with the first $n$ many colors, which is possible by the chromatic number assumption on $G$.) This is a Borel condition.

To prove the Suslin $\sigma$-centeredness of the simplicial complex $\mathcal{K}$, use the Lusin–Novikov theorem to find a Borel function $g$ which, for each finite coloring $a \in \mathcal{K}$ whose domain consists of elements of a single $E$-class $c$, assigns a $G$-coloring $g(a) \supseteq a$ such that $\text{dom}(a) \subset \text{dom}(g(a)) \subset c$ and for every $G$-edge $\{x, y\}$ with $x \in \text{dom}(g(a))$ and $y \in c \setminus \text{dom}(g(a))$ it is the case that $g(a)(u) \notin n$ holds. Let $\text{supp}(a)$ be the set of all pairs $(e, i)$ such that $e \in \text{dom}(g(a))$ and $i \in |\text{dom}(a)|$. In view of Theorem 11.3.6, it will be enough to show that if $\{a_i : i \in j\}$ are colorings of $\mathcal{K}$ whose domain is a subset of one and the same $E$-class $c$, with pairwise disjoint supports, then $\bigcup_i a_i \in \mathcal{K}$. This is nearly trivial, however. The colorings $g(a_i)$ for $i \in j$ have pairwise disjoint domains. Let $f : c \setminus \bigcup_i \text{dom}(g(a_i)) \to n$ be any $G$-coloring—its existence is guaranteed by the chromatic number assumption on $G$. Then, $f \cup \bigcup_i g(a_i)$ is a $G$-coloring of the class $c$ which extends $\bigcup_i a_i$ and uses no colors $\geq n$ infinitely many times, witnessing that $\bigcup_i a_i \in \mathcal{K}$. \qed

Corollary 11.3.10. Let $G$ be a locally finite Borel graph on a Polish space $X$ of finite chromatic number $n$. Let $\mathcal{K}$ be the simplicial complex as in Example 11.3.7. Let $H$ be a finitary analytic hypergraph of uncountable Borel chromatic number. Then

1. in the $P_\mathcal{K}$-extension of the Solovay model, the chromatic number of $H$ is uncountable;
2. it is consistent relative to an inaccessible cardinal that ZF+DC holds and the chromatic number of $G$ is $\leq 2n - 1$ and the chromatic number of $H$ is uncountable.

The following example is motivated by a question of Marks and Unger (private communication): given a countable Borel equivalence relation $E$, is it consistent with ZF+DC that $E$ is a union of an increasing countable sequence of equivalence relations with all equivalence classes finite and every set of reals has the Baire property? While that question remains open, we do have partial information:

**Example 11.3.11.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$ with all classes infinite. Let $K$ be the simplicial complex of all finite directed subgraphs of $E$ which are acyclic and every vertex has in-degree $\leq 1$ and out-degree $\leq 1$. Note that $K$ is in fact a simplicial complex on $E$ rather than on $X$. The simplicial complex $K$ is Suslin $\sigma$-centered by Theorem 11.3.6. To see this, let $\Gamma$ be a countable group acting on $X$ in a Borel way such that $E$ is the resulting orbit equivalence relation. Let $\Gamma = \{\gamma_k : k \in \omega\}$ be an arbitrary enumeration. Whenever $a \in K$ is a graph whose domain is a subset of a single $E$-class, let $n(a)$ be the smallest number such that for any two points $x, y \in \text{dom}(a)$ there is $k \in n(a)$ such that $x = \gamma_k \cdot y$, and let $\text{supp}(a) = \{\{\gamma_l : x, \gamma_k : x\} : x \in \text{dom}(a), k, l \in n(a)\}$. If two graphs $a_0, a_1 \in K$ have disjoint support, they must have disjoint domains. It follows that if $a_i \in K$ are sets for $i \in j$ which are subsets of the same $E$-class and have pairwise disjoint supports, then $\bigcup_i a_i \in K$ as required in Theorem 11.3.6.

We plan to use the simplicial complex $K$ to introduce an ordering of ordertype $\mathbb{Z}$ to each $E$-class. However, not every maximal $K$-set does that. Apparently, a cofinal subset $L \subset K$ is needed so that every $L$-large maximal $K$-set comes close the the purpose in mind. For one simple solution, let $L \subset K$ be the collection of all sets $a \in K$ which in each $E$-class have only one vertex of in-degree 0 and only one vertex of out-degree 0. Note that $L \subset K$ is cofinal, and if $c \subset X$ is an $E$-class and $d \subset c^2$ is an $L$-regular maximal $K$-set, then $d$ is a ray of ordertype $\mathbb{Z}$ or $\mathbb{N}$ or inverted $\mathbb{N}$.

**Corollary 11.3.12.** Let $E$ be a countable Borel equivalence relation with all classes infinite as in Example 11.3.11. Let $H$ be a finitary analytic hypergraph of uncountable Borel chromatic number.

1. In the $P_{K\mathcal{L}}$-extension of the Solovay model, the chromatic number of $H$ is uncountable;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds and $E$ is the orbit equivalence relation of a (discontinuous) $\mathbb{Z}$-action and the chromatic number of $H$ is uncountable.

**Proof.** The first item is the consequence of the fact that $K$ is Suslin $\sigma$-centered as proved in Example 11.3.11. For the second item, let $W$ be the Solovay model derived from some inaccessible cardinal and let $G \subset P_{K\mathcal{L}}$ be a generic filter over
the model $W$. A genericity argument shows that $\bigcup G$ is an $\mathcal{L}$-regular maximal $\mathcal{K}$-set. For every $E$-class $c$, $\bigcup G$ on the class $c$ indicates either an ordering of type $\mathbb{Z}$ or of type $\mathbb{N}$ or of type inverted $\mathbb{N}$. It is a matter of minor surgery in ZF+$\text{DC}$ to turn this into a relation which is an ordertype $\mathbb{Z}$ on every class. This immediately provides an action of $\mathbb{Z}$ on $X$ which yields $E$ as an orbit equivalence relation. The chromatic number part of the second item follows from the first item.

As a final observation, we show that the forcings of the type discussed in Theorem 11.3.5 can be placed in countable support product, and the conclusion of Theorem 11.3.5 remains in place. Suppose that $\langle X_i : i \in \omega \rangle$ are pairwise disjoint Polish spaces, and $\langle \mathcal{K}_i, \mathcal{L}_i \rangle$ are locally countable pairs on each. Suppose that each simplicial complex $\mathcal{K}_i$ is Suslin $\sigma$-centered. Let $X$ be the Polish space which is the disjoint union of $X_i$ for $i \in n$, let $\mathcal{K}$ be the simplicial complex of those finite sets $a \subset X$ such that $a \cap X_i \in \mathcal{K}_i$ holds for every $i \in \omega$, and let $\mathcal{L} = \{a \in \mathcal{K} : \forall i : a \cap X_i \neq \emptyset \rightarrow a \cap X_i \in \mathcal{L}_i\}$. Then $P_{\mathcal{K},\mathcal{L}}$ is a countable support product of the posets $P_{\mathcal{K}_i,\mathcal{L}_i}$, and the simplicial complex $\mathcal{K}$ is Suslin $\sigma$-centered by Theorem 11.2.23, since it is the finite support product of the posets $\mathcal{K}_i$ for $i \in \omega$. Therefore, by Theorem 11.3.5, in the $\prod_i P_{\mathcal{K}_i,\mathcal{L}_i}$-extension of the Solovay model, the chromatic numbers of the hypergraphs $G_n$ are uncountable.

### 11.3b Suslin $\sigma$-linked complexes

In fairly common situations, one wishes to preserve uncountable chromatic numbers of analytic graphs while a task of higher arity is preserved. This can be handled by Suslin $\sigma$-linked complexes as explained in the following theorem.

**Theorem 11.3.13.** Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a locally countable pair. Suppose that the simplicial complex $\mathcal{K}$ is Suslin $\sigma$-linked. Let $G$ be an analytic graph with uncountable Borel chromatic number. In the $P_{\mathcal{K},\mathcal{L}}$-extension of the Solovay model, the chromatic number of $G$ is uncountable.

The verification of the Suslin $\sigma$-linked property of Borel simplicial complexes is invariably achieved via the following result.

**Theorem 11.3.14.** Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space $X$ and write $E = E_\mathcal{K}$. Suppose that there is a Borel support function $\text{supp} : \mathcal{K} \rightarrow [X]^{<\aleph_0}$ such that

1. $\text{supp}(a) \subset [a]_E$;

2. whenever $a_0, a_1$ are sets in $\mathcal{K}$, subsets of the same $E$-class with disjoint supports, then $a_0 \cup a_1 \in \mathcal{K}$.

Then $\mathcal{K}$ is Borel $\sigma$-linked.

The proofs of Theorems 11.3.13 and 11.3.14 are nearly identical to the proofs of Theorem 11.3.5 and 11.3.6, and we omit them.
Our first example is motivated by a result of Marks and Unger [55] which found perfect matchings with the Baire property in many Borel bipartite graphs. In order to do this, they isolated the following strengthening of the well-known Hall’s condition of [30]:

**Definition 11.3.15.** Let $\Gamma$ be a locally finite bipartite Borel graph on a Polish space $X$. We say that $\Gamma$ satisfies the Marks–Unger condition if there is a real number $\varepsilon > 0$ such that for every finite set $a \subset X$ on one side of the bipartition, the number of $\Gamma$-neighbors of $a$ is at least $(1 + \varepsilon)|a|$. 

**Example 11.3.16.** Let $G$ be a locally finite bipartite Borel graph on a Polish space $X$ satisfying the Marks–Unger condition. Let $\mathcal{K}$ be the simplicial complex on the edges of $G$ consisting of all sets $b \subset G$ which can be completed to a perfect matching of $G$. Then $\mathcal{K}$ is Suslin $\sigma$-linked.

**Proof.** Let $\varepsilon > 0$ be a real number witnessing the Marks–Unger property of the graph $G$. Let $E = E_\mathcal{K}$ be the $G$-path equivalence relation on $X$. Let $a \in \mathcal{K}$ be a set which is a subset of a single $E$-class; we must calculate the support $\text{supp}(a)$ such that the assumptions of Theorem 11.3.14 are satisfied.

Let $\text{supp}(a)$ be the set of all $G$-edges which are within $G$-distance at most $8|a|/\varepsilon$ from some edge in $a$. We must verify the amalgamation condition, i.e. if $a, b \in \mathcal{K}$ are disjoint finite sets in the same $E$-class such that $\text{supp}(a) \cap \text{supp}(b) = 0$, then $a \cup b \in \mathcal{K}$. By Hall’s marriage theorem [30], it is enough to verify that the graph $G'$ obtained from $G$ by erasing all vertices mentioned in $a$ or $b$ satisfies the Hall’s marriage condition: for each finite set $c$ of vertices in the same equivalence class as $a, b$ on one side of the bipartition, the set of $G'$-neighbors of $c$ has size at least $|c|$. There are two cases:

**Case 1.** There is a $G^2$-path from some vertex in $a$ to some vertex in $b$ using only nodes in $c$ as the intermediate steps. In this case, without loss of generality suppose that $|a| \geq |b|$, and use the definition of $\text{supp}(a)$ to argue that the $G^2$-path from $a$ to $b$ must contain at least $4|a|/\varepsilon$-many steps. This means that $|c| > 4|a|/\varepsilon$. By the initial condition on the graph $G$, the set $c$ has at least $(1 + \varepsilon)|c|$ many $G$-neighbors and so at least $(1 + \varepsilon)|c| - 2|a| - 2|b|$ many $G'$-neighbors. However, $2|a| + 2|b| < \varepsilon|c|$ and so the set $c$ has at least $|c|$-many $G'$-neighbors as required.

**Case 2.** Case 1 fails. In this case, the set $c$ can be written as a union of disjoint sets $c = c_a \cup c_b$ such that the set $c_a$ has no common $G$-neighbors with $c_b$, the sets $c_a$ and $\bigcup b$ are $G$-disconnected, and the sets $c_b$ and $\bigcup a$ are $G$-disconnected. Then, since $a \in \mathcal{K}$, the set $c_a$ has at least $|c_a|$-many $G'$-neighbors; since $b \in \mathcal{K}$, the set $c_b$ has at least $|c_b|$-many $G'$-neighbors. The $G'$-neighborhoods of $c_a, c_b$ do not overlap by the choice of $c_a$ and $c_b$, and so the set $c$ has at least $|c|$-many $G'$-neighbors as desired. \qed

Note that as the graph $G$ is locally finite, maximal $\mathcal{K}$-sets are in fact perfect $G$-matchings.

**Corollary 11.3.17.** Let $G$ be a locally finite Borel bipartite graph on a Polish space $X$ satisfying the Marks–Unger condition. Let $\mathcal{K}$ be the simplicial complex
of partial $G$-matchings that can be extended to a perfect matching. Let $H$ be an analytic graph of uncountable Borel chromatic number.

1. In the $P_{\mathcal{K}}$-extension of the Solovay model, the chromatic number of $H$ is uncountable;

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a perfect $G$-matching, and the chromatic number of $H$ is uncountable.

**Question 11.3.18.** Is there a locally finite, bipartite Borel graph $G$ satisfying the Marks–Unger condition such that the associated control poset is not Suslin $\sigma$-centered? Such that in ZF, the existence of perfect matching is equivalent to some Borel hypergraph having countable chromatic number?

To many hypergraphs of arity greater than two it is possible to add a coloring while preserving uncountable chromatic numbers of analytic graphs purely as a matter of arity. Consider the following:

**Example 11.3.19.** Let $\mathbb{Z}$ act on a Polish space $X$ in a Borel free way. Let $G$ be the hypergraph of all triples $\{x_0, x_1, x_2\}$ such that there is $n \in \omega$ such that $n \cdot x_0 = x_1$ and $n \cdot x_1 = x_2$. Let $\mathcal{K}$ be the simplicial complex of partial finite colorings of the hypergraph $G$ with colors coming from $\omega$. The complex $\mathcal{K}$ is Suslin $\sigma$-linked.

**Proof.** Note that $\mathcal{K}$ is in fact a simplicial complex on $X \times \omega$. It is easy to observe that $E = E_\mathcal{K}$ is the equivalence relation on $X \times \omega$ connecting $\langle x, n \rangle$ and $\langle y, m \rangle$ just in case $x, y \in X$ are orbit-equivalent. Let $a \in \mathcal{K}$ be a nonempty finite partial $G$-coloring in a single $E$-class. We will compute a support $\text{supp}(a)$ such that if $a, b \in \mathcal{K}$ are subsets of the same $E$-class and $\text{supp}(a) \cap \text{supp}(b) = \emptyset$ then $a \cup b \in \mathcal{K}$. This will be enough by Theorem 11.3.14.

Let $n(a) \in \omega$ be the smallest number larger than all elements of $\text{rng}(a)$ and all numbers $n$ such that there are points $x_0, x_1 \in \text{dom}(a)$ with $n \cdot x_0 = x_1$. Let $\text{supp}(a) = \{\langle m \cdot x, k \rangle : x \in \text{dom}(a), k \in n(a) \}$ and $m \in \mathbb{Z}$ is of absolute value smaller than $n(a)$. We claim that the function $\text{supp}$ works as required in Theorem 11.3.14. Suppose that $a, b \in \mathcal{K}$ are colorings in the same $E$-class with disjoint supports; we must show that $a \cup b \in \mathcal{K}$. It is not hard to check that $a \cup b$ is in fact a function. To see that it is a $G$-coloring, let $\{x_0, x_1, x_2\} \in G$ be a hyperedge which is a subset of $\text{dom}(a) \cup \text{dom}(b)$; we must show that the coloring $a \cup b$ is not constant on it. If it is a subset of $\text{dom}(a)$ or $\text{dom}(b)$, then we are done as both $a, b$ are $G$-colorings. Otherwise, two of the points (say $x_0, x_1$) would have to be in the domain of one of the colorings (say $a$) and the remaining point ($z$) has to be in the domain of the other coloring ($b$). If $b(z) \neq a(x_0)$ then we are done. However, $b(z) = a(x_0)$ is impossible: then $b(z) \in n(a)$ and so $\langle z, b(z) \rangle \in \text{supp}(a) \cap \text{supp}(b)$, violating the initial assumptions on $a, b$. \qed

In order to force a total $G$-coloring, we need to choose a cofinal subset $\mathcal{L} \subset \mathcal{K}$ such that $\mathcal{L}$-regular maximal $K$-sets are in fact total colorings. One possible choice is $\mathcal{L} = \{a \in \mathcal{K} : \text{for every infinite } E\text{-class } c \subset X, \text{dom}(a) \cap c \text{ is a} \}$.
subinterval of the shift order on \( c \), and the values of \( a \) at the endpoints of this interval are larger than the length of the interval}. The verification of the requisite property of \( L \) is easy and left to the reader.

**Corollary 11.3.20.** Let \( G \) be the hypergraph and \( \langle K, \mathcal{L} \rangle \) be the locally countable pair of Example 11.3.19. Let \( H \) be an analytic graph of uncountable chromatic number.

1. In the \( P_{K,L} \)-extension of the Solovay model, the chromatic number of \( H \) is uncountable;
2. it is consistent relative to an inaccessible cardinal that ZF+DC holds and the chromatic number of \( G \) is countable while the chromatic number of \( H \) is uncountable.

### 11.3c Suslin \( \sigma \)-liminf-centered complexes

The purpose of this subsection is to show how the chromatic numbers of hypergraphs with some degree of compactness can be separated from chromatic numbers of hypergraphs which have non-compactness of some sort built directly into their definitions.

**Theorem 11.3.21.** Let \( \langle K, \mathcal{L} \rangle \) be a locally countable pair, and suppose that \( K \) is Suslin \( \sigma \)-liminf centered. Let \( H \) be an analytic graph on a Polish space \( Y \) which does not have Borel \( \sigma \)-finite clique number. In the \( P_{K,L} \)-extension of the Solovay model, the chromatic number of \( H \) is uncountable.

**Proof.** As a preliminary observation, note that in view of Fact 11.1.24 it is enough to consider the case of \( G \) being equal to the skew product of infinitely many infinite cliques. Thus, let \( \{t_n : n \in \omega\} \) be a set of finite strings of integers which is dense in \( \omega^{<\omega} \) and such that \( t_n \in \omega^n \), and let \( G \) be the graph on \( \omega^\omega \) connecting \( y_0, y_1 \) if there is \( n \in \omega \) such that \( \forall m \neq n \, y_0(m) = y_1(m) \), \( t_n \subset y_0 \), and \( y_0(n) \neq y_1(n) \).

Write \( X \) for the Polish space which is the domain of the Borel simplicial complex \( K \), write \( E = E_K \) and \( P = P_{K,L} \). Let \( \kappa \) be an inaccessible cardinal, let \( W \) be the Solovay model derived from \( \kappa \). Let \( n \geq 2 \) be a natural number and work in the model \( W \). Suppose that \( p \in P \) is a condition and \( \tau \) is a \( P \)-name for a function from the Baire space to \( \omega \). We must find a stronger condition \( p \vdash \tau(\bar{y}_0) = \tau(\bar{y}_1) = \bar{m} \).

The condition \( p \in P \) as well as the name \( \tau \) are definable from some parameters in the ground model and a parameter \( z \in 2^\omega \). Let \( V[K] \) be an intermediate model obtained by a poset of size \( < \kappa \) containing the parameter \( z \), and work in the model \( V[K] \) until further notice. Consider the iteration \( Q * \dot{R} \) where \( Q \) is the Cohen poset on the Baire space adding a generic point \( \bar{y}_{gen} \in n^\omega \), and in the \( Q \)-extension, \( \dot{R} \) is the finite support iteration of length \( \omega_1 \) of the poset \( K \).

As usual, the poset \( K \) is reinterpreted at each stage of the iteration; we denote the model obtained after \( \alpha \)-th stage by \( V_\alpha \), so \( V_0 \) is the \( Q \)-extension of \( V[K] \).
Thus, the $\dot{R}$ poset adds a sequence of $\mathcal{K}$-sets, $\langle \dot{A}_\alpha : \alpha \in \omega_1 \rangle$ derived from the iteration components of $\dot{R}$. By a density argument, for each $E$-class $c$ represented in the model $V_\alpha$, the set $A_\alpha \cap c$ is a maximal $\mathcal{K}$-subset of $c$ which is $\mathcal{L}$-large. Consider the $Q \ast \dot{R}$-name $\sigma$ for the $\mathcal{K}$-set defined in the following way: if $c$ is an $E$-class represented in $\text{dom}(p)$, then $c \cap \sigma = c \cap p$, if $c$ is an $E$-class represented in $V_\alpha$ but not in $\text{dom}(p)$ then $c \cap \sigma = A_\alpha \cap c$, and if $\alpha \in \omega_1$ is an arbitrary nonzero ordinal and $c$ is an $E$-class represented in $V_\alpha$ but not in $\bigcup_{\beta \in \alpha} V_\beta$ then $c \cap \sigma = c \cap A_\alpha$. It is not difficult to see that $\sigma$ is a $Q \ast \dot{R}$-name for a maximal $\mathcal{K}$-set which is $\mathcal{L}$-regular, and therefore a balanced condition for the poset $P$ by Theorem 11.3.3. Note that the condition does not depend on the point $\dot{y}_{\text{gen}}$ but only on the model $V[K][\dot{y}_{\text{gen}}]$.

Now, by a standard balance argument, $Q \ast \dot{R} \Vdash \exists m \text{ Coll}(\omega, < \kappa) \Vdash \sigma \Vdash p \tau(\dot{y}_{\text{gen}}) = \dot{m}$. Let $\langle q, \dot{r} \rangle$ be a condition in the iteration $Q \ast \dot{R}$ deciding the value of $m$; we abuse the notation and call the specific value $m$ again. By a standard c.c.c. argument, there is an ordinal $\alpha \in \omega_1$ such that $q \Vdash \dot{r}$ is a name for a condition before the $\alpha$-th level of the iteration $\dot{R}$. Let $R_\alpha$ be finite support iteration of $\mathcal{K}$ of length $\alpha$. By the iteration theorem 11.2.31, this is a very Suslin forcing which is Suslin $\sigma$-centered. Let $\{B_k : k \in \omega \}$ be analytic liminf-centered sets covering $R_\alpha$. Strengthening the condition $q$ if necessary, we may find a specific number $k \in \omega$ such that $q \Vdash \dot{r} \in B_k$.

Now, $q$ is a nonempty open subset of the Baire space; thinning it out, we may find $n \in \omega$ such that $q = [t_n]$. Let $y \in \omega^\omega$ be a Cohen-generic element over the model $V[K]$ extending $t_n$ and move into the model $V[K][y]$. For each $i \in \omega$, let $y_i \in \omega^\omega$ be the element obtained from $y$ by rewriting the $n$-th entry with $i$. Thus, each $y_i$ is a Cohen generic element over $V[K]$ meeting the condition $q$, $V[K][y_i] = V[K][y]$, and $\{y_i : i \in \omega \}$ is a $G$-clique. Let $r_i = \dot{r}/y_i$. Thus, each $r_i$ is a condition in the poset $R_\alpha$, and even in the analytic liminf-centered set $B_k$. In conclusion, there is a condition $r \in R_\alpha$ which forces the set of $i \in \omega$ such that $r_i$ belongs to the generic filter to be infinite. Now, let $H \subset R$ be a filter generic over the model $V[K][y]$ and meeting the condition $r$. Consider the balanced condition $\bar{p} = \sigma/H \subseteq p$ in the poset $P$ in the model $V[K][y][H]$ described in the second paragraph of the present proof. The forcing theorem applied for each $i \in \omega$ shows that for infinitely many $i \in \omega$, $V[K][y][H] \Vdash \text{Coll}(\omega, < \alpha) \Vdash \bar{p} \Vdash p \tau(\dot{y}_i) = \dot{m}$. Thus, $W \Vdash \bar{p} \Vdash \tau(\dot{y}_i) = \dot{m}$ for infinitely many $i \in \omega$, completing the proof. \hfill $\Box$

It may seem difficult to check whether a given simplicial complex is Suslin $\sigma$-liminf centered or not. The following theorem gives a rather straightforward criterion for a positive answer to this question.

**Theorem 11.3.22.** Suppose that $\mathcal{K}$ is a locally countable Borel simplicial complex on a Polish space $X$. Suppose that there is a Borel function $d : X^2 \to \mathbb{R}_0^+$ such that

1. $d$ is a metric on each $E_\mathcal{K}$-class;
2. for each natural number \( n \in \omega \) and each nonempty \( a \in \mathcal{K} \), the set \( \{ b \in \mathcal{K} : |b| \leq n \text{ and } a \cup b \text{ is an inclusion minimal set not in } \mathcal{K} \} \) is either finite or it converges to \( a \) in the Hausdorff metric induced by \( d \).

**Proof.** Let \( A_n = \{ a \in \mathcal{K} : |a| \leq n \} \) for every number \( n \in \omega \). It will be enough to show that if \( \{ a_i : i \in \omega \} \) is an infinite subset of \( A_n \), then there is an infinite set \( u \subset \omega \) and a condition \( b \in \mathcal{K} \) such that for all \( c \in \mathcal{K} \) with \( b \subset c \), for all but finitely many \( i \in u \), \( a_i \cup c \in \mathcal{K} \). Write \( E = E_\mathcal{K} \).

First, we assume that all sets \( a_i \) are a subset of a single \( E \)-class \( C \subset X \). To get the conclusion, let \( b_0 \subset C \) be an inclusion-maximal set such that for some infinite \( u_0 \subset \omega \) and sets \( a'_i \subset a_i \), the sequence \( \langle a'_i : i \in u_0 \rangle \) converges to \( b_0 \).

This is possible since all subsets of \( C \) of this type have to have cardinality \( \leq n \). Now, find an infinite set \( u \subset u_0 \) such that for every \( b' \subset b \), either for all \( i \in u \) \( a_i \cup b' \in \mathcal{K} \) or for all \( i \in u \) \( a_i \cup b' \notin \mathcal{K} \). Let \( b_1 \subset b_0 \) be an inclusion-maximal set for which the former option holds, and let \( b = b_1 \cup a_i \) where \( i = \min(u) \). We claim that \( b \) works as desired.

To see this, towards contradiction assume that there is \( c \in \mathcal{K} \) with \( b \subset c \subset C \) such that for infinitely many \( i \in u \), \( a_i \cup c \notin \mathcal{K} \). This means that we can find an infinite set \( v \subset u \), a set \( c' \subset c \), and sets \( a'_i \subset a_i \) such that \( a'_i \cup c' \in \mathcal{K} \) is an inclusion minimal set not in \( \mathcal{K} \). Further thinning out the set \( v \), we may assume that the set \( a'_i \) for \( i \in v \) are either all the same or all distinct. The treatment then divides into two cases.

**Case 1.** All sets \( a'_i \) for \( i \in v \) are the same and equal to some \( e \subset C \). The maximal choices of \( b_0 \) and \( b \) then imply that \( e \subset b_0 \) and even \( e \subset b \). Then \( e' \cup e \subset c \in \mathcal{K} \) and so \( e' \cup a'_i \in \mathcal{K} \) for all \( i \in v \), contradicting the choices of \( a'_i \) and \( e' \).

**Case 2.** The sets \( a'_i \) for \( i \in v \) are all distinct. In such a case, by the demand (2) on the metric \( d \) it must be the case that they converge to \( c' \) in the Hausdorff metric on finite subsets of \( X \) induced by \( d \). By the maximal choice of \( b_0 \), it must be the case that \( c' \subset b_0 \). Since for all \( i \in v \) \( e' \cup a_i \notin \mathcal{K} \), the choice of the set \( u \) implies that \( e' \cup a_i \notin \mathcal{K} \) where \( i = \min(u) \). However, for this choice of \( i \) \( e' \cup a_i \subset c \in \mathcal{K} \) holds, which is again a contradiction. This completes the proof in the case that all sets \( a_i \) are subsets of a single \( E \)-class.

Now, we deal with the general case. Since the sets \( a_i \) are of cardinality \( \leq n \), we may find an infinite set \( u_0 \subset \omega \) and a finite set \( D \) of \( E \)-classes such that for each \( i \in u \) the set \( a_i \) has nonempty intersection with every class in \( D \), and no \( E \)-class which is not in \( D \) is visited by more than one set \( a_i \) for \( i \in u_0 \). Using the special case repeatedly for each class \( C \in D \) and the sets \( a_i \cap C \), we can find an infinite set \( u \subset u_0 \) and a set \( b \subset \mathcal{K} \) such that for each \( b' \subset b \) in \( \mathcal{K} \) and each \( E \)-class \( C \), \( (b' \cap C) \cup (a_i \cap C) \subset \mathcal{K} \) holds for all but finitely many \( i \in u \). It follows that \( b \vDash \) for infinitely many \( i \in u \), \( a_i \) belongs to the generic filter as desired.

**Example 11.3.23.** Consider the simplicial complex \( K \) of finite \( H_{<\omega} \)-colorings with colors in \( \omega \). Then \( K \) is Suslin \( \sigma \)-(liminf)-centered. To see this, let \( X = \prod_{n \in \omega} n + 1 \) be the domain of the graph \( H_{<\omega} \) and let \( d \) be the metric on \( X \times \omega \).
defined by \(d(\langle x, i \rangle, \langle y, j \rangle) = 1\) if \(i \neq j\) and \(= 2^{-n}\) if \(i = j\) and \(n\) is the least number such that \(x(n) \neq y(n)\). It is not difficult to see that the metric \(d\) satisfies the demands of Theorem 11.3.22. Note that the only inclusion-minimal sets not in \(K\) are sets \(\{\langle x, i \rangle, \langle y, i \rangle\}\) where \(x, y \in X\) differ in exactly one entry, and fixing the point \(\langle x, i \rangle\), these sets converge to \(\{\langle x, i \rangle\}\) in the Hausdorff metric.

To use \(K\) to force a total \(H^{<\omega}\)-coloring, we apparently need an auxiliary cofinal Borel subset \(L \subset K\) to make sure that all \(L\)-large maximal \(K\)-sets yield a total coloring. One possible option is to set \(L = \{a \subset K : \text{if } x, y \in \text{dom}(a) \text{ and the points } x, y \text{ differ in finitely many entries with maximum } n \in \omega \text{ then each } z \in X \text{ which differs from } x \text{ only at entries } \leq n \text{ belongs to } \text{dom}(a) \}\).

**Corollary 11.3.24.** Let \(\langle K, L \rangle\) be a locally countable pair of Example 11.3.23.

1. In the \(P_{K,L}\)-extension of the Solovay model, the graph \(H^\omega\) has uncountable chromatic number;
2. it is consistent relative to an inaccessible cardinal that \(ZF+DC\) holds and the chromatic number of \(H^{<\omega}\) is countable and the chromatic number of \(H^\omega\) is uncountable.

### 11.4 Collapses

In this section, we consider the problems of introducing an injection from \(|F|\) to \(2^\omega\) or to \(E_0\), for a given pinned Borel equivalence relation \(F\). It turns out that the natural forcings preserve uncountable chromatic numbers of many graphs and hypergraphs. In particular, they do not introduce a total \(E_0\)-selector. We start with the collapse to \(2^\omega\).

Let \(E\) be a pinned equivalence relation on a Polish space \(X\). Recall the definition of the poset collapsing \(|E|\) to \(2^\omega\) from Definition 6.3.2: \(P_E\) be the partial order of pairs \(p = (a_p, b_p)\) where \(a_p : X \to 2^\omega\) is a countable partial function such that for \(x_0, x_1 \in \text{dom}(a_p)\), \(x_0 E x_1\) iff \(a_p(x_0) = a_p(x_1)\), and \(b_p \subset 2^\omega\) is a countable set disjoint from \(\text{rng}(a_p)\). The ordering is that of coordinatewise reverse inclusion. It is immediate in \(ZF+DC\) that the ordering \(P_E\) is Suslin and \(\sigma\)-closed, and the union of the \(P_E\)-generic filter is an injection from the set of \(E\)-classes to \(2^\omega\). The balanced conditions are classified by all such injections as proved in Theorem 6.3.3.

**Theorem 11.4.1.** Let \(E\) be a pinned Borel equivalence relation on a Polish space \(X\). Suppose that \(G\) is an analytic finitary hypergraph on a Polish space \(Y\) which does not have Borel \(\sigma\)-finite fractional chromatic number. In the \(P_E\)-extension of the Solovay model, the chromatic number of \(G\) is uncountable.

**Proof.** In view of Fact 11.1.22 it is enough to consider the case of a large measured skew product \(G\). Find a sequence \(\langle a_n, H_n, t_n : n \in \omega \rangle\) which generates \(G\) as in Definition 11.1.21. Thus, \(Y = \prod_n a_n\) and \(G = \prod_n H_n, t_n\).

Now, consider the control poset \(Q\): it conditions are finite functions \(a : X \to 2^{<\omega}\) such that for some \(n \in \omega\), \(\text{rng}(a) \subset 2^n\), and for points \(x_0, x_1 \in \text{dom}(a),\)
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$x_0 E x_1$ iff $a(x_0) = a(x_1)$. The ordering is defined by $b \leq a$ if $\text{dom}(a) \subset \text{dom}(b)$ and for all $x \in \text{dom}(a)$, $a(x) \subset b(x)$. The following is the key observation.

**Claim 11.4.2.** $Q$ is a very Suslin c.c.c. poset. In addition, it is Suslin measured.

**Proof.** It is clear that $Q$ is very Suslin. To check the very Suslin property of $Q$, suppose that $a \subset Q$ is a countable set. It is not difficult to see that if there is a condition $r \in Q$ incompatible with all elements of $a$, then there is one whose support is a subset of $\bigcup_{q \in a} \text{supp}(q)$. The search for an incompatible condition among the countably many conditions of this type is a Borel procedure.

To verify the Suslin measured property of $Q$, consider the function $m: Q \rightarrow (0,1]$ defined by $m(q) = 2^{-n}2^{-k}$ where $n \in \omega$ is such that $\text{rng}(q) \subset 2^n$, and $k \in \omega$ is such that $\text{dom}(q)$ has nonempty intersection with exactly $k$ many $E$-classes. For each positive rational number $\varepsilon > 0$ let $A_\varepsilon = \{q \in Q: m(q) > \varepsilon\}$. Since the sets $A_\varepsilon \subset Q$ are all analytic and $Q = \bigcup_{\varepsilon > 0} A_\varepsilon$, it is enough to show, given $\varepsilon > 0$ and a finite sequence $(q_i: i \in j)$ of elements of the set $A_\varepsilon$, there is a set $b \subset j$ of size greater than $\varepsilon j$ such that the conditions $p_i$ for $i \in b$ have a common lower bound.

To this end, let $a$ be the set of all $E$-classes with nonempty intersection with $\bigcup_i \text{dom}(q_i)$. Let $\mu$ be the usual probability measure on $2^\omega$ and let $\lambda$ be the product measure on $Y = (2^\omega)^{\omega}$. Define the set $B \subset j \times Y$ as the set of all pairs $\langle i, y \rangle$ such that $\forall x \in \text{dom}(q_i)$ $q_i(x) \subset y([x]_E)$. By the definition of the function $m$, the vertical sections $B_i$ have $\lambda$-mass $> \varepsilon$ each. By the Fubini theorem applied to the counting measure on $j$ and $\lambda$, there must be a point $y \in Y$ such that the horizontal section $B^y$ has size greater than $\varepsilon j$. It is easy to check that the set $\{q_i: i \in B^y\}$ has a common lower bound in $Q$. 

Note that a generic filter $G \subset Q$ induces a function defined by $f(x) = \bigcup \{p(x): p \in G\}$ whenever $x \in X$ is a ground model point. It is clear that $\text{dom}(f)$ consists of the ground model elements of $X$, for all such points $x_0, x_1 \in X$ $x_0 E x_1$ iff $f(x_0) = f(x_1)$ holds, and moreover the range of $f$ consist of elements of $2^\omega$ which are not in the ground model.

Now, suppose that $p \in P$ is a condition. Consider the poset $R$ which is the finite support iteration of the posets $Q$ of length $\omega_1$. Let $f_\alpha: X \rightarrow 2^\omega$ be the $\alpha$-th iterand for the partial function defined by the generic filter on the $\alpha$-th iterand. Let $V_\alpha$ be the $R$-name for the model obtained from the first $\alpha$ many iterands. Consider the $R$-name $\sigma$ for a function from $E$-classes to $2^\omega$ such that if $c$ is an $E$-class in $\text{dom}(p)$ then $\sigma(c) = p(c)$, and if $c$ is an $E$-class not in $\sigma$ and $\alpha \in \omega_1$ is the first ordinal such that $c$ is represented in $V_\alpha$, then $f(c) = y$ for the unique $y \in 2^\omega$ such that for all points $x \in c \cap V_\alpha$, $f_\alpha(x) = y$. It is not difficult to see that $R \models \sigma$ is an injection of the set of $E$-classes to $2^\omega$. By Theorem 6.3.3, $\sigma$ represents a balanced condition in the collapse poset $P$, and it is not hard to see that this balanced condition is below $p$.

Finally, we are ready for the main body of the proof. Let $\kappa$ be an inaccessible cardinal, let $W$ be a Solovay model derived from $\kappa$, and work in $W$. Suppose that $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \models \tau: Y \rightarrow \omega$ is a function. We must find a condition $\bar{p} \leq p$, a number $n \in \omega$, and a hyperedge
Corollary 11.4.3. Let $P$ be the collapse poset of $E_0$ to $2^\omega$.

1. In the $P$-extension of the Solovay model, the chromatic number of the diagonal Hamming graph is uncountable.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds and $|E_0| \leq |2^\omega|$ and yet the chromatic number of the diagonal Hamming graph is uncountable; in particular, there is no Vitali set.
Note that the collapse does change some chromatic numbers to their lowest possible value. Consider the Hamming graph $\mathbb{H}_2$ on $2^\omega$. Let $E$ be the equivalence relation on $2^\omega$ connecting two points if they differ at finite, even cardinality set of entries. The equivalence relation $E$ is easily seen to be reducible to $\mathbb{E}_0$. Thus, if $|\mathbb{E}_0| \leq |2^\omega|$, then the $E$-quotient space can be linearly ordered by some ordering $\leq$. Consider the map $f: 2^\omega \to 2$ by putting $f(x) = 0$ just in case $x$ belongs to the $\leq$-first of the two $E$-classes whose union is the $\mathbb{E}_0$-class of $x$. This is a $\mathbb{H}_2$-coloring of $2^\omega$ with two colors.

The following example shows a somewhat more dramatic case of this behavior, as the hypergraph in question does not have Borel $\sigma$-finite chromatic number. The example also shows that the use of the fractional chromatic number in place of the usual chromatic number is to some extent necessary in the assumption of the theorem.

**Example 11.4.4.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $Q$ be the poset of all pairs $q = (a_q, b_q)$ of finite subsets of $X$ such that $(a_q \times b_q) \cap E = 0$; the ordering is that of coordinatewise reverse inclusion. Let $G$ be the graph connecting two conditions in $Q$ if they are incompatible. Then in ZF, $|E| \leq |2^\omega|$ if and only if the chromatic number of $G$ is countable.

Note that the graph $G$ does not have Borel $\sigma$-finite chromatic number as soon as the equivalence relation $E$ is non-smooth as per Example 11.1.19. In particular, there is a Borel graph which does not have Borel $\sigma$-finite chromatic number, yet the collapse of $|\mathbb{E}_0|$ to $|2^\omega|$ forces countable chromatic number to it.

**Proof.** For the left-to-right direction, let $h: X \to 2^\omega$ be a reduction of $E$ to the identity on $2^\omega$. For every $n \in \omega$ and every set $c \subset 2^n$ let $B_c = \{ q \in Q : \forall x \in a_q \ h(x) \upharpoonright n = c \wedge \forall x \in b_q \ h(x) \upharpoonright n \notin c \}$. Since the function $h$ is $E$-invariant, the sets $B_c \subset Q$ are $G$-anti-cliques. Now, if $q \in Q$ is a condition, there is $n \in \omega$ such that for all $x_0 \in a_q$ and $x_1 \in b_q$, $h(x_0) \upharpoonright n \neq h(x_1) \upharpoonright n$. Thus, writing $c = \{ h(x) \upharpoonright n : x \in a_q \}$, we conclude that $q \in B_c$. It follows that $Q = \bigcup_c B_c$ is a cover of $Q$ by countably many $G$-anti-cliques.

For the right-to-left direction, let $Q = \bigcup_n B_n$ be a cover of $Q$ by $G$-anti-cliques. For each $n \in \omega$, let $A_n = \bigcup_{q \in B_n} a_q$ and let $h: X \to 2^\omega$ be the function defined by $h(x)(n) = 0$ if $x \in A_n$. We claim that this is a reduction of $E$ to the identity on $2^\omega$, thus inducing the desired inequality $|E| < |\text{cantor}|$. To see this, let $x_0, x_1 \in X$ be arbitrary points. If $x_0 E x_1$ holds then $h(x_0) = h(x_1)$ since the sets $A_n$ are $E$-invariant. On the other hand, if $x_0 E x_1$ fails, then we have $q = \{ \{x_0\}, \{x_1\}\}$ is a condition in $Q$ and there must be $n$ such that $q \in B_n$. It is not difficult to see that then $h(x_0)(n) = 0 \neq h(x_1)(n)$ and so $h(x_0) \neq h(x_1)$. \hfill \Box

Now we move to the case of a collapse of $|E|$ to $|\mathbb{E}_0|$ for a Borel pinned equivalence relation $F$. Again, let $P_E$ be the collapse as defined in Definition 6.3.2. Recall that its balanced conditions are classified by injections from the $E$-quotient space to the $\mathbb{E}_0$-quotient space by Theorem 6.3.3.
Theorem 11.4.5. Let E be a pinned Borel equivalence relation on a Polish space X. Let G be a finitary analytic hypergraph on a Polish space Y of uncountable Borel chromatic number. In the $P_E$-extension of the Solovay model, G has uncountable chromatic number.

Proof. Consider the control poset Q: it consists of conditions $q$ such that for some number $n_q \in \omega$, $q$ is a finite partial function from $X$ to $2^{n_q}$. The ordering is defined by $r \leq q$ if $n_q \leq n_r$, $\text{dom}(q) \subseteq \text{dom}(r)$, for every $x \in \text{dom}(q)$, $q(x) \subseteq r(x)$, and for every pair $x_0, x_1 \in \text{dom}(q)$ of $E$-related points and for every $m \in n_r \setminus n_q$, $r(x_0)(m) = r(x_1)(m)$.

Claim 11.4.6. Q is a very Suslin, Suslin $\sigma$-centered poset.

Proof. It is immediate that Q is a Suslin poset. For the very Suslin part, suppose that $a \subseteq Q$ is a countable set. It is not difficult to check that if there is a condition $q \in Q$ which is incompatible with all conditions in $a$, then there must be such a condition $q$ with $\text{dom}(q) \subseteq \bigcup_{r \in a} \text{dom}(r)$. There are only countably many conditions satisfying the latter formula, and the search in this countable set for one which is incompatible with all elements of $a$ is a Borel procedure.

As far as the Suslin $\sigma$-centered property of the poset Q, fix a countable basis $\mathcal{B}$ for the space X. A descriptor of a condition $q \in Q$ is a function $h$ whose domain is a finite collection of pairwise disjoint sets in $\mathcal{B}$ such that for each $O \in \text{dom}(h)$ there is exactly one point $x_O$ of $\text{dom}(q)$ in $O$, and for each point of $\text{dom}(q)$ belongs to one set in $\text{dom}(h)$; moreover, $h(O) = q(x_O)$. It is immediate that the set of all conditions with a given descriptor is Borel and $\sigma$-centered, every condition has many descriptors, and there are only countably many possible descriptors in all. Therefore, Q is Suslin $\sigma$-centered as required.

Note that the poset $Q$ adds an injection from the ground model $E$-classes to the collection of $E_0$-classes which are not represented in the ground model. We denote the name for the injection by $\eta$; thus $\eta([x]_E)$ is forced to be the $E_0$-class of the point $y = \bigcup\{r(x): r \text{ is in the generic filter}\}$. Now, let $R$ be the finite support iteration of the poset $Q$ of length $\omega_1$, adding functions $\eta_\alpha$ at each stage of the iteration; we denote by $V_\alpha$ the generic extension obtained after the $\alpha$-th stage of the iteration. Let $p \in P_E$ be any condition. Let $\sigma$ be the $R$-name for the injection from the $E$-quotient space to the $E_0$-quotient space defined by $\sigma(c) = a_p(c)$ whenever $c$ is an $E$-class in the domain of $a_p$, and $\sigma(c) = \eta_\alpha(c)$ whenever $c$ is an $E$-class which is not in the domain of $a_p$ and $\alpha \in \omega_1$ is the first ordinal such that $c$ has a representative in the model $V_\alpha$. It is clear that $\sigma$ is a name for an injection from the $E$-quotient space to the $E_0$-quotient space compatible with the condition $p$.

The rest of the proof follows closely the argument for Theorem 11.3.5 and we omit it.

Corollary 11.4.7. Let E be a pinned Borel equivalence relation. Let G be an analytic finitary hypergraph on a Polish space, with uncountable Borel chromatic
number. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $|E| \leq |E_0|$ and the chromatic number of $G$ is uncountable.

11.5 Compactly balanced posets

In this section we prove an additional preservation property of compactly balanced posets of Definition 9.2.1 which concerns the locally countable structures.

**Definition 11.5.1.** Let $G$ be an analytic finitary hypergraph on a Polish space $X$. The hypergraph is actionable if there is a countable group $\Gamma$ acting on a Polish space $X$ such that all hyperedges of $G$ consist of pairwise orbit-related elements and for every $\gamma \in \Gamma$, $\gamma \cdot G = G$.

**Theorem 11.5.2.** Let $G$ be an analytic, finitary, actionable hypergraph on a Polish space $X$ which does not have Borel $\sigma$-finite chromatic number. Then in the compactly balanced extensions of the Solovay model, $G$ has uncountable chromatic number.

The actionable assumption is necessary, see the example below. The main difference between the general and actionable hypergraphs we exploit is the following routine strengthening of Fact 11.1.13; instead of skew products we can deal with the usual product.

**Fact 11.5.3.** Let $G$ be a finitary analytic actionable hypergraph on a Polish space $X$. Exactly one of the following occurs:

1. $G$ has Borel $\sigma$-finite chromatic number;
2. there is a large product $H$ on a Polish space $Y$ and a continuous homomorphism $h: Y \to X$ of $H$ to $G$.

**Proof.** In view of Fact 11.5.3, it is enough to prove the theorem for large product hypergraphs. Thus, let $(a_n, H_n: n \in \omega)$ be a sequence such that for every $n \in \omega$, $H_n$ is a hypergraph on the finite set $a_n$, and $|a_n| \geq 2$ and the chromatic number of $H_n$ is at least $n$, and assume that $G$ is in fact the product hypergraph $\prod_n H_n$ on the space $Y = \prod_n a_n$. With a large product of this form, we associate the poset $R$ of all functions $r$ with domain $\omega$ such that for every $n \in \omega$, $r(n) \subseteq a_n$ is a nonempty set and the chromatic numbers of $H_n$ on $r(n)$ are unbounded as $n$ tends to infinity. The ordering on $R$ is that of coordinatewise inclusion. Clearly, the poset $R$ adds a point $\dot{y}_{\text{gen}} \in Y$ defined by $\dot{y}_{\text{gen}}(n)$ is the only element of the set $\bigcap \{r(n): r \text{ belongs to the generic filter}\}$. The following claim is key.

**Claim 11.5.4.** The poset $R$ is proper, bounding, and adds no independent reals.

**Proof.** The first two assertions of the following claim are standard and proved by the usual fusion arguments. The last assertion is the heart of the present proof. Suppose that $r \in R$ is a condition and $\tau$ is an $R$-name for a subset of $\omega$; we have to find a condition $s \leq r$ and an infinite set $c \subseteq \omega$ such that...
s \models \check{c} \subset \tau \text{ or } \check{c} \cap \tau = 0. \text{ Strengthening the condition } r \text{ if necessary, a standard fusion argument will yield a continuous function } f : \prod_n r(n) \to \mathcal{P}(\omega) \text{ such that } r \models \tau = f(\check{y}_{\text{gen}}).

Now, let } B \subseteq \prod_n r(n) \times \omega \text{ be the Borel set given by } (y, n) \in B \text{ if } n \in f(y) \text{ holds. A partition result [71, Theorem 1.4] applied with an infinite subsequence of the sets } r(n) \text{ and the chromatic numbers of the hypergraphs } H_n \text{ as submeasures on the sets } r(n) \text{ shows that there is a condition } s \leq r \text{ and an infinite set } c \subset \omega \text{ such that either } \prod_n s(n) \times c \subseteq B \text{ or } (\prod_n s(n) \times c) \cap B = 0. \text{ In the former case, } s \models \check{c} \subset \tau; \text{ in the latter case, } s \models \check{c} \cap \tau = 0. \text{ The claim has just been proved.} \hfill \square

Let } \kappa \text{ be an inaccessible cardinal. Let } P \text{ be a Suslin poset such that } V_\kappa \models \text{in every forcing extension, the poset } P \text{ is compactly balanced. Let } W \text{ be the Solovay model derived from } \kappa \text{ and work in the model } W. \text{ Let } p \in P \text{ be a condition and } \tau \text{ be a } P\text{-name such that } p \models \tau : X \to \omega \text{ is a function. We will find a number } m \in \omega, \text{ finite sets } a_n \subset X \text{ for } n \in \omega, \text{ and a condition } \check{p} \leq p \text{ such that for all } n, \text{ } a_n \text{ cannot be covered by } n\text{-many } G\text{-anticliques, and } \check{p} \Vdash \forall x \in \bigcup_n a_n \tau(x) = m.

The condition } p \in P \text{ as well as the name } \tau \text{ are definable in the model } W \text{ from parameters in the ground model and a parameter } z \in 2^\omega. \text{ Let } V[K] \text{ be an intermediate forcing extension obtained by a poset of size } < \kappa, \text{ containing the point } z. \text{ Work in } V[K] \text{ and consider the poset } Q \times R \text{ where } Q \text{ is the poset of infinite subsets of } \omega \text{ ordered by inclusion. Let } U \subseteq Q \text{ and } y \in Y \text{ be mutually generic filter objects for the product; so } U \text{ is in fact a nonprincipal ultrafilter on } \omega \text{ in the model } V[K][U]. \text{ The poset } Q \text{ is } \sigma\text{-closed, and therefore } R \text{ computed in the model } V[K][U] \text{ is the same as } R \text{ computed in the model } V[K] \text{ and } y \text{ is in fact } R\text{-generic over the model } V[K][U]. \text{ By the continuous reading of names for the poset } R, \text{ the sets } \mathcal{P}(\omega) \cap V[K][U][y] \text{ and } \mathcal{P}(\omega) \cap V[K][y] \text{ are the same. By a mutual genericity argument and the fact that } R \text{ adds no independent reals, } U \text{ generates an ultrafilter on } \omega \text{ in the model } V[K][U][y]. \text{ Work in the model } V[K][U].

Claim 11.5.5. There is an } R\text{-name } \sigma \text{ for a balanced condition in } P \text{ such that } C \models \forall y \in Y \ y = \check{y}_{\text{gen}} \text{ up to finitely many entries} \rightarrow \sigma / \check{y}_{\text{gen}} = \sigma / \gamma \cdot \check{y}_{\text{gen}}.

To parse the claim correctly, note that any point of the space } Y \text{ which is up to finitely many entries equal to } \check{y}_{\text{gen}} \text{ is in fact } R\text{-generic again and yields the same forcing extension. The claim therefore says that the evaluation of the name } \sigma \text{ does not depend on the specific generic point, but only on its modulo finite equivalence class.}

Proof. Choose an arbitrary } R\text{-name } \chi \text{ for a balanced condition below the condition } p. \text{ We now use the ultrafilter } U \text{ and the compact balance of the poset } P \text{ to integrate the name } \chi. \text{ Choose any point } y_\omega. \text{ For each } n \in \omega, \text{ let } y_n \text{ be the } R\text{-name below } C \text{ for the element of } Y \text{ such that } y_n \upharpoonright n = y_\omega \upharpoonright n \text{ and } y_n \upharpoonright [n, \omega) = \check{y}_{\text{gen}} \upharpoonright [n, \omega); \text{ note that } y_n \text{ is forced to be a point } R\text{-generic over the model } V[K][U], \text{ generating the same model as } \check{y}_{\text{gen}}. \text{ Let } \chi_n \text{ be the name for}
the balanced condition $\chi/y_n$. Let $\sigma$ be the name for the $U$-limit of the sequence $\langle \chi_n : n \in \omega \rangle$ in the definable compact Hausdorff topology on the space of all balance classes for $P$. It is immediate that the name $\sigma$ works as desired.

Still working in the model $V[K][U]$, find a name $\sigma$ as in the claim. By a standard balance argument, $R \models \text{Coll}(\omega, < \kappa) \models \sigma$ decides in $P$ the value of $\tau(\dot{y_{gen}})$. Passing to a condition $r$ if necessary, we may assume that there is a specific number $m \in \omega$ such that $r \models \text{Coll}(\omega, < \kappa) \models \sigma \models P \tau(\dot{y_{gen}}) = \dot{m}$. Now, let $y \in Y$ be a point $R$-generic over the model $V[K][U]$, meeting the condition $r$ and let $\dot{p} = \sigma/y$. Find a number $n \in \omega$ such that the set $r(n)$ contains an $H_n$-hyperedge $b \in H_n$ and for each $i \in b$ let $y_i \in Y$ be the point obtained from $y$ by replacing the $n$-th entry of $y$ with $i$. Thus, the set $\{y_i : i \in b\}$ is a $G$-hyperedge (this is exactly the point which does not work if a skew product variation of $G$ is considered) and for each $i \in n$ we have $\dot{p} = \sigma/y_i$ by the choice of the name $\sigma$. By the forcing theorem, in the model $W$ $\dot{p} \models P \forall y \in a_n \tau(y) = \dot{m}$, completing the proof.

**Corollary 11.5.6.** Let $G, H$ be analytic locally countable hypergraphs on respective Polish spaces $X,Y$ such that $G$ has Borel $\sigma$-finite chromatic number and $H$ is actionable and does not have Borel $\sigma$-finite chromatic number. Then it is consistent relative to an inaccessible cardinal that ZF+DC holds, $G$ has countable chromatic number and yet $H$ has uncountable chromatic number.

**Proof.** Let $X = \bigcup_n B_n$ be a partition into Borel sets witnessing the Borel $\sigma$-finite chromatic number of $G$. Let $E$ be a countable Borel equivalence relation on $X$ such that all hyperedges of $G$ consist of pairwise $E$-related elements. For each $n \in \omega$ let $P_n$ be the poset of functions $p$ whose domain is a countable relatively $E$-invariant subset of $B_n$, range is a subset of $n$, and $p$ is a partial $G$-coloring. The ordering on $P_n$ is that of reverse inclusion. Let $P = \prod_n P_n$. Then we see that in $P_n$, the balanced conditions are classified by total $G$-colorings of the set $B_n$ whose range is a subset of $n$. Such colorings naturally form a compact subset of the compact Hausdorff space $n^{B_n}$, therefore the poset $P_n$ is compactly balanced. As a result, even the full support product $P = \prod_n P_n$ is compactly balanced, and therefore in the $P$-extension of the symmetric Solovay model, the chromatic number of $H$ is uncountable by Theorem 11.5.2. At the same time, the poset $P_n$ adds a total $G$-coloring on the set $B_n$ with $n$ many colors and therefore the product $P$ forces $G$ to have countable chromatic number.

**Corollary 11.5.7.** Let $Z$ act freely and in a measure preserving Borel way on a Polish probability space $\langle X, \mu \rangle$. Let $G$ be the hypergraph of arity three on $X$ containing a triple $\{x_0, x_1, x_2\}$ if there is a number $n \in \omega$ such that $n \cdot x_0 = x_1$ and $n \cdot x_1 = x_2$. Then

1. it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $G_0$ is uncountable and the chromatic number of $G$ is countable;
2. it is also consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $G_0$ is two and the chromatic number of $G$ is uncountable.

**Proof.** For the first item, use Corollary 11.3.20. For the second item, use Example 11.1.20 to show that the hypergraph $G$ does not have Borel $\sigma$-finite chromatic number; it is clearly actionable. Then, use Theorem 11.5.2 to the poset $P$ adding a $G_0$-coloring with two colors by countable approximations whose domain is $\mathbb{E}_0$-invariant. It is not difficult to see and follows from Theorem 6.2.2 that the balanced conditions for $P$ are classified by total $G_0$-colorings with two colors, which naturally form a closed subset of the compact Hausdorff space $2^{2^\omega}$. Thus the poset $P$ is compactly balanced.

**Corollary 11.5.8.** 1. Let $P$ be the poset of infinite subsets of $\omega$ ordered by inclusion. In the $P$-extension of the Solovay model, the chromatic number of $H_{<\omega}$ is uncountable.

2. It is consistent relative to an inaccessible cardinal that there is a nonprincipal ultrafilter on $\omega$ and the chromatic number of $H_{<\omega}$ is uncountable.

This is immediate from the fact that the poset $P$ is compactly balanced—Example 9.2.4. The corollary can be viewed as a commentary on a result of Rosendal [63]: in ZF+DC, if there is a discontinuous homomorphism between Polish groups then the chromatic numbers of the Hamming graphs $H_n$ are finite. Since a nonprincipal ultrafilter on $\omega$ yields a discontinuous homomorphism from the Cantor group $2^\omega$ to 2, this result cannot be extended to the diagonal Hamming graph.

The following two corollaries use compactly balanced posets from Examples 9.2.11 and 9.2.13.

**Corollary 11.5.9.** Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the linearization poset for the $E$-quotient space as in Example 8.5.5. In the $P$-extension of the Solovay model, the chromatic number of $H_{<\omega}$ is uncountable.

2. It is consistent relative to an inaccessible cardinal that there is a linear ordering on the $E$-quotient space and the chromatic number of $H_{<\omega}$ is uncountable.

**Corollary 11.5.10.** Let $G$ be a locally finite Borel graph on a Polish space $X$ satisfying the Hall condition.

1. Let $P$ be the poset adding a perfect matching for $G$ as in Example 6.2.4. In the $P$-extension of the Solovay model, the chromatic number of $H_{<\omega}$ is uncountable.

2. It is consistent relative to an inaccessible cardinal that $G$ has a perfect matching and the chromatic number of $H_{<\omega}$ is uncountable.
The following example presents a (necessarily) non-invariant hypergraph which does not have Borel $\sigma$-finite chromatic number, yet it has countable chromatic number in some compactly balanced extension of the Solovay model.

**Example 11.5.11.** The actionable assumption cannot be removed from the assumptions of Theorem 11.5.2.

**Proof.** Let $\Gamma$ be the free group on two generators $\gamma, \delta$. It acts on the space $2^\Gamma$ by shift: that is, $(\beta \cdot x)(\alpha) = x(\beta^{-1}\alpha)$ holds for all $\beta, \alpha \in \Gamma$ and $x \in 2^\Gamma$. Let $X \subset 2^\Gamma$ be the dense $G_\delta$ set on which the action is free. Let $E_0$ be the $\Gamma$-orbit equivalence relation on $X$. Let $E_1$ be the orbit equivalence relation induced by the subgroup of $\Gamma$ generated by $\delta$. It is not difficult to see that $E_1 \subset E_0$ are Borel equivalence relations. Let $G$ be the hypergraph on $X$ of arity three containing triples $\{x_0, x_1, x_2\}$ consisting of pairwise $E_0$-related points and containing two $E_1$-related and two $E_1$-unrelated points.

To see how a compactly balanced poset can make the chromatic number of $G$ countable, consider the Cayley graph $H$ on $X$: it connects points $x_0, x_1$ if $x_1 = \gamma \cdot x_0$ (the $\gamma$-edges of $H$) or $x_1 = \delta \cdot x_0$ (the $\delta$-edges of $H$) or vice versa. This is an acyclic 4-regular graph whose connectedness components are exactly the $E_0$-classes. In ZFC, every acyclic graph without vertices of degree 0 or 1 has an orientation in which every vertex has out-degree one, constructed component by component. The following claim shows the impact of such an orientation on the hypergraph $G$:

**Claim 11.5.12.** (ZF) Suppose that the graph $H$ has an orientation in which every vertex has out-degree one. Then the chromatic number of $G$ is countable.

**Proof.** Let $\vec{H}$ be the orientation. For each $x \in X$ the color $h(x)$ is the pair $\langle b = b(x), k = k(x) \rangle$ such that $k$ is the largest possible number such that $\vec{H}$ contains an oriented path from $x$ to $\delta^c \cdot x$ for some $c \in \{-1, 1\}$. If $k = 0$ then $b = 0$, otherwise $b = c$. If such $k$ does not exist, then the color $h(x)$ is $\infty$. We claim that $h$ is a $G$-coloring.

To see this, consider any two $E_0$-related points $x_0, x_1 \in X$. In the $H$-path between $x_0, x_1$ no vertex gets an out-degree two, and that leaves only three cases: either the whole path is oriented towards $x_0$, or towards $x_1$, or towards some point in the middle. Now, if $h(x_0) = h(x_1) = \infty$ then all three cases show that all the edges on the path must be $\delta$-edges and so $h(x_0) E_1 h(x_1)$. If $x_0, x_1$ are $E_1$-related, say $x_0 = \delta^m \cdot x_1$ for some $m > 0$, and $h(x_0) \neq \infty$, then in the first two cases $k(x_0) \neq k(x_1)$ and in the third case $b(x_0) = 1 \neq -1 = b(x_1)$. It follows that no $G$-hyperedge can be monochromatic: the homogeneous color cannot be $\infty$ on the account of the two $E_1$-unrelated points in the hyperedge, and the homogeneous color cannot be different from $\infty$ on the account of the two $E_1$-related points in the hyperedge.

**Claim 11.5.13.** The hypergraph $G$ does not have Borel $\sigma$-finite fractional chromatic number.
Proof. For a point \( x \in X \) and a number \( m \in \omega \), write \( a_{xm} = \{ \delta^{im}\gamma^{jm} : i, j \in m \} \). Note that the set \( a_{xm} \subset X \) is a subset of a single \( E_0 \)-class and visits \( n \) many distinct \( E_1 \)-classes, each in \( n \) many elements. A simple counting argument then shows that every subset of \( a_{xm} \) of size \( m + 1 \) contains a \( G \)-hyperedge, and so the fractional chromatic number of \( G \) on the set \( a_{xm} \) is not smaller than \( m \) as witnessed by the normalized counting measure on \( a_{xm} \).

Now, suppose that \( X = \bigcup_n B_n \) is a partition into Borel sets. By the Baire category theorem, there is a number \( n \in \omega \) such that \( B_n \) is not meager. In view of the first paragraph, to prove the claim it will be enough that for all but finitely many \( m \in \omega \) there is a point \( x \in X \) such that \( a_{xm} \subset B_n \). To this end, let \( t : \Gamma \to 2 \) be a finite partial function such that \( B_n \) is comeager in \( [t] \). Let \( m \in \omega \) be larger than the length of any word in \( t \); we will find a point \( x \in X \) such that \( a_{xm} \subset B_n \). Just let \( s = \bigcup \{ t_{ij} : i, j \in m \} \) where \( t_{ij} : \Gamma \to 2 \) is a finite partial function given by the demand \( t_{ij}(\beta) = t(\delta^{im}\gamma^{jm}\beta) \). By the choice of the number \( m \), \( s : \Gamma \to 2 \) is a finite function. The set \( C = \bigcap \{ \gamma^{jm}\delta^{im}B_n : i, j \in m \} \) is comeager in \( [s] \); let \( x \in C \) be an arbitrary point. Reviewing the definitions, it is clear that \( a_{xm} \subset B \) as required.

Now, consider the poset \( P \) adding an orientation of \( H \) in which every vertex in \( X \) gets an out-degree one. A condition \( p \in P \) is an orientation on countably many components of \( H \) in which every vertex in these components gets an out-degree one. The ordering is that of reverse inclusion. The poset \( P \) was analyzed in Example 9.2.15 It is not difficult to prove that the balanced conditions of \( P \) are classified by orientations of \( H \) in which every vertex gets an out-degree one, which naturally form a compact Hausdorff space. Thus, the poset \( P \) is compactly balanced, and it adds an \( H \)-coloring with countably many colors by Claim 11.5.12 while the hypergraph \( H \) does not have Borel \( \sigma \)-finite chromatic or even fractional chromatic number by Claim 11.5.12.

We conclude this section with a remark on the effect of ultrafilters on locally countable structures. It should be noted that most posets adding an ultrafilter in Chapter 7 are compactly balanced.

**Proposition 11.5.14.** (ZF+DC) Suppose that there is a nonprincipal ultrafilter on \( \omega \). Suppose that \( E \) is a hyperfinite equivalence relation on a Polish space \( X \). Then

1. there is an assignment of ultrafilters to each \( E \)-class;

2. whenever \( G \) is a hypergraph on \( X \) such that every \( G \)-hyperedge consists of pairwise \( E \)-related elements and there is \( n \in \omega \) such that \( G \) restricted to each finite set has chromatic number \( \leq m \), then the chromatic number of \( G \) on \( X \) is \( \leq m \).

For the first item, an assignment of ultrafilters to \( E \)-classes is a function \( f \) with domain \( X/E \) such that for each \( E \)-class \( c \), \( f(c) \) is an ultrafilter on \( c \). For the second item, note that it means that all analytic graphs, subsets of \( E \), with
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Borel $\sigma$-finite chromatic number receive countable chromatic number as soon as
a nonprincipal ultrafilter on $\omega$ is present.

\textbf{Proof.} Let $U$ be a nonprincipal ultrafilter on $\omega$. Let $E_n$ for $n \in \omega$ be equivalence relations
with all classes finite forming an inclusion-increasing sequence such that $E = \bigcup_n E_n$. For the first item, let $\leq$ be
any linear ordering of $X$. For each point $x \in X$ let $x(n)$ be the $\leq$-least element of $X$ which is $E_n$-related to $X$. Let
$U_x = \{ a \subseteq [x]_E : \{ n \in \omega : x(n) \in a \} \in U \}$. Thus, $U_x$ is an ultrafilter on $[x]_E$ and $x E y$ implies
that $U_x = U_y$ since $U$ is nonprincipal and for some $m \in \omega$ $x E_m y$ holds, and then for all $n > m$ $x(n) = y(n)$ holds.

For the second item, let $\leq$ be any linear ordering of the set of all finite partial functions from $X$ to $m$. For each
$x \in X$, let $c(x) = i$ just in case the set $\{ n \in \omega : \text{the } \leq \text{-least } G \text{-coloring of } [x]_E \text{ assigns } x \text{ color } i \}$ belongs to the ultrafilter $U$. It is not difficult to verify that this is a $G$-coloring with $\leq m$ colors. \hfill $\Box$

\textbf{Question 11.5.15.} (ZF+DC) Suppose that there is a nonprincipal ultrafilter on $\omega$. Let $\Gamma$ be the free group on two generators, acting on $2^\Gamma$ by shift. Let $X \subset 2^\Gamma$ be the free part of the shift, with the orbit equivalence relation $E$.

1. Is there an assignment of ultrafilters to $E$-classes?

2. Does the (acyclic) Cayley graph have chromatic number two?

3. Is there an orientation of the Cayley graph in which every vertex gets out-degree one?

Note that the affirmative answer to the first question automatically answers the next two in the positive as well. To see this, suppose that $\{ U_x : x \in X \}$ as an assignment of ultrafilters to $E$-classes. The coloring $c(x) = 0$ if the set $\{ y \in X :$ there is an even length Cayley path from $x$ to $y \}$ belongs to $U_x$ and $c(x) = 1$ otherwise is a coloring of the Cayley graph with two colors. Also, the orientation $H$ of the Cayley graph defined by $\langle [x]_E, [y]_E \rangle \in H$ if the set of $z \in X :$ the unique injective Cayley path from $x$ to $z$ includes $y$} belongs to $U_x$, is an orientation in which every vertex gets degree one.

As one limiting remark towards the affirmative answer to the first question, we mention the following. Let $P$ be the poset of infinite subsets of $\omega$ ordered by inclusion, adding a nonprincipal ultrafilter as in Section 7.1. In the $P$-extension of the Solovay model, there is no assignment of ultrafilters to $F_2$-classes; i.e. no function $f$ on the $F_2$ quotient space such that $f([x]_{F_2})$ is an ultrafilter on $\operatorname{rng}(x)$ for all $x \in \operatorname{dom}(F_2) = (2^\omega)$. To see this, note that such a function $f$ would yield a tournament $T$ on the quotient space by setting $\langle [x]_{F_2}, [y]_{F_2} \rangle \in T$ if the set $\operatorname{rng}(x) \setminus \operatorname{rng}(y)$ belongs to the ultrafilter the function $f$ assigns to the $F_2$-class consisting of all enumerations of the set $\operatorname{rng}(x) \Delta \operatorname{rng}(y)$. However, tournaments on the $F_2$ quotient space are ruled out from the $P$-extension (and many similar extensions) of the Solovay model by Theorem 13.1.2.
Chapter 12

The equi-generic divide

In this chapter, we will investigate an important dividing line among the balanced extensions of the symmetric Solovay model. We isolate the class of *equi-balanced* Suslin forcings closed under countable product. We also isolate a number of effects that these humble forcings cannot achieve. The dividing line does not seem to have any parallel in existing literature in descriptive set theory.

12.1 The equi-balanced posets

**Definition 12.1.1.** Let $V[H_0], V[H_1]$ be generic extensions of some ground model $V$. We say that the pair $V[H_0], V[H_1]$ is *equi-generic* over $V$ if for every analytic equivalence relation $E$ with code in $V$, if $\langle Q_0, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are equivalent $E$-pins, then there is an $E$-pin $\langle Q, \tau \rangle \in V$ equivalent to both.

In other words, $V[H_0], V[H_1]$ are equi-generic if every virtual $E$-class represented in both of them is represented in the ground model. It may not be immediately clear how to get useful equi-generic pairs which are in fact not mutually generic. The following simple proposition shows that equigenericity is a notion intermediate between mutual genericity and the property $V[H_0] \cap V[H_1] = V$ familiar from the treatment of the placid equivalence relations and placid posets.

**Proposition 12.1.2.** Let $V[H_0], V[H_1]$ be an equi-generic pair of extensions of $V$. Then

1. $V[H_0] \cap V[H_1] = V$;

2. if $R_0 \in V[H_0]$ and $R_1 \in V[H_1]$ are posets and $K_0 \subset R_0$ and $K_1 \subset R_1$ are filters mutually generic over $V[H_0], H_1]$, then $V[H_0][K_0]$ and $V[H_1][K_1]$ is an equi-generic pair over $V$.

**Proof.** For (1), consider the analytic equivalence relation $E$ on relations on $\omega$, connecting relations $r_0, r_1$ if either both fail to be extensional and well-founded, or they are isomorphic. The $E$-pins are classified by transitive sets by
Theorem 2.4.5. Now, suppose towards contradiction that $V[H_0] \cap V[H_1] \neq V$; there must be a set $a \subseteq V$ such that $a \in V[H_0] \cap V[H_1]$ and $a \notin V$. Let $b$ be the transitive closure of $a$, let $Q = \text{Coll}(\omega, b)$ and let $\tau$ be the $Q$-name for the pullback of the $\in$-relation on $b$ to $\omega$ under the $Q$-generic bijection of $\omega$ to $b$. Then $(Q, \tau)$ is an $E$-pin which belongs to both $V[H_0]$ and $V[H_1]$, and it is not equivalent to any $E$-pin in $V$ since $b \notin V$.

For (2), suppose that $r_0 \in R_0$ and $r_1 \in R_1$ are conditions and $(\dot{Q}_0, \tau_0)$, $(\dot{Q}_1, \tau_1)$ are $R_0$- and $R_1$-names for $E$-pins in the models $V[H_0]$ and $V[H_1]$ respectively such that $\langle r_0, r_1 \rangle \Vdash (\dot{Q}_0, \tau_0)$ and $(\dot{Q}_1, \tau_1)$ are equivalent; we must produce an $E$-pin in $V$ forced to be equivalent to both.

Work in the model $V[H_0]$ for a moment. The product forcing theorem shows that in the product $R_0 \times R_0$, the condition $(r_0, r_0)$ forced the $E$-pins $(\dot{Q}_0, \tau_0)_{\text{left}}$ and $(\dot{Q}_0, \tau_0)_{\text{right}}$ to be equivalent. It follows that $(\langle R_0 \restrict r_0 \rangle \star \dot{Q}_0, \tau_0)$ is an $E$-pin, and the product forcing theorem again shows that $r_0 \Vdash (\dot{Q}_0, \tau_0)$ is an $E$-pin equivalent to $(\langle R_0 \restrict r_0 \rangle \star \dot{Q}_0, \tau_0)$.

A similar argument on the $V[H_1]$ side shows that $(\langle R_1 \restrict r_1 \rangle \star \dot{Q}_1, \tau_1)$ is an $E$-pin in $V[H_1]$ which is forced by $r_1$ to be equivalent to $(\dot{Q}_1, \tau_1)$. By the forcing theorem, the $E$-pins $(\langle R_0 \restrict r_0 \rangle \star \dot{Q}_0, \tau_0) \in V[H_0]$ and $(\langle R_1 \restrict r_1 \rangle \star \dot{Q}_1, \tau_1) \in V[H_1]$ are equivalent. Since the pair $V[H_0], V[H_1]$ is equi-generic over $V$, there must be an $E$-pin in $V$ which is equivalent to both. This concludes the proof of (2).

Definition 12.1.3. Let $P$ be a Suslin forcing.

1. A virtual condition $\bar{p}$ in $P$ is equi-balanced if for every equi-generic pair $V[H_0], V[H_1]$ of extensions of $V$ and all conditions $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$, if $p_0, p_1 \leq \bar{p}$ then $p_0, p_1$ have a lower bound.

2. The poset $P$ is equi-balanced if for every condition $p \in P$ there is an equi-balanced virtual condition $\bar{p} \leq p$.

The fracture line between equi-balanced and balanced posets is fairly obvious from the definitions and the classifications of balanced virtual conditions proved in Chapter 6.

Example 12.1.4. The following posets are equi-balanced:

1. $P_K$ where $K$ is a modular simplicial complex;

2. the $E, F$-collapse poset of Definition 6.3.2 if $E, F$ are Borel equivalence relations and $\lambda(E) \leq \lambda(F)$;

3. the $E, F$-transversal poset of Definition 6.3.5 if $E \subseteq F$ are Borel equivalence relations and $F$ is pinned;

4. the posets defined from Fraissé classes in Definition 8.5.3.

It is also quite clear that the class of equi-balanced forcings is closed under the countable support product. On the other hand, the poset $\mathcal{P}(\omega)$ modulo finite (as well as any other poset adding a nonprincipal ultrafilter) is not equi-balanced, as will be clear from Example 12.2.5 below.
12.2 Thick graphs and hypergraphs

Numerous limitations of equi-generic extensions can be derived from certain Borel (hyper)graphs and the consideration of their chromatic numbers. The central definition:

**Definition 12.2.1.** Let $\Gamma$ be a closed hypergraph on a Polish space $X$ of a fixed finite arity $n \geq 2$. $\Gamma$ is **thick** if it is nonempty and for every $i \in n$ and every nonempty open set $O \subset \Gamma$ there are disjoint open sets $P_0, P_1 \subset X$ such that

1. the set $\Delta = \{ (y_0, y_1) \in O^2 : \forall j \neq i \ y_0(j) = y_1(j) \text{ and } y_0(i) \in P_0 \text{ and } y_1(i) \in P_1 \}$ is nonempty;
2. the maps $(y_0, y_1) \mapsto y_0, (y_0, y_1) \mapsto y_1$ and $(y_0, y_1) \mapsto (y_0(i), y_1(i))$ are continuous and open as maps from $\Delta$ to $\Gamma$ and from $\Delta$ to $X^2$ respectively.

Here, the set $\Delta$ is viewed as an open subset of the set $\{ (y_0, y_1) \in \Gamma^2 : \forall j \neq i \ y_0(j) = y_1(j) \}$ which is a closed subset of $X^{2n}$ and gets a topology inherited from $X^{2n}$. It may not be immediately clear where one can find interesting thick graphs; the elementary examples in 12.2.5 and 12.2.10 below should clarify this issue in a short order.

The following central result is stated using Convention 1.8.17.

**Theorem 12.2.2.** In equi-balanced extensions of the symmetric Solovay model, if $\Gamma$ is a thick hypergraph on a Polish space $X$ and $A \subset X$ is a $\Gamma$-anticlique, then $A$ is somewhere meager.

**Proof.** Suppose that $\Gamma$ is a thick hypergraph on a Polish space $X$, an inaccessible cardinal $\kappa$, a Suslin forcing $P$ such that in $V_\kappa$ $P$ is equi-balanced in every forcing extension, and in the symmetric Solovay model $W$ derived from $\kappa$, there is a condition $p \in P$ and a $P$-name $\tau$ such that $p \Vdash \tau \subset X$ is nowhere meager; we must find a condition $q \leq p$ and a $\Gamma$-hyperedge $a \subset X$ such that $q \Vdash \bar{a} \subset \tau$. To do this, note that the name $\tau$ as well as the condition $p$ must be definable from some $z \in 2^{<\omega}$ and some parameters in the ground model $V$. The assumptions show that there is an intermediate model $V[K]$ which is a generic extension of $V$ by a poset of size $< \kappa$, $z \in V[K]$.

Work in the model $V[K]$. Let $\bar{p} \leq p$ be an equi-balanced virtual condition in the poset $P$. Let $P_{\bar{p}}$ be the Cohen forcing on the hypergraph $\Gamma$ viewed as a closed subset of $X^n$, and let $a = \{ x_i : i \in n \}$ be a hyperedge of $\Gamma$ which is $P_{\bar{p}}$-generic over the model $V[K]$. The following claims are key:

**Claim 12.2.3.** For each $i \in n$,

1. $x_i \in X$ is a Cohen-generic point over $V[K]$;
2. there is a poset $R_i \in V[K][x_i]$ of size $< \kappa$ and an $R_i$-name $\sigma_i$ for a condition in $P$ stronger than $\bar{p}$ such that $R_i \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_i \Vdash_{P} \bar{x}_i \in \tau$. 
Proof. Return to $V[K]$. For (1), in view of Proposition 3.1.1 it is enough to show that for every nonempty open set $O \subset \Gamma$, there is a nonempty open set $O' \subset O$ such that the projection into the $i$-th coordinate is an open map on $O'$. To find $O'$, first find the open sets $P_0, P_1 \subset X$ and the associated set $\Delta$ as in Definition 12.2.1. Let $O' \subset O$ be the projection of $\Delta$ into its first coordinate and chase diagrams to show that $O'$ works as desired.

(2) immediately follows from (1). If (2) failed, then by the forcing theorem applied in $V[K]$ to the Cohen poset $P_X$, there would have to be a nonempty open set $O \subset X$ such that the failure of (2) for a Cohen generic point $\hat{x}_\text{gen} \in X$ in place of $x_i$ is forced by $O$. This means that in the model $W$, the (realization of the) condition $\check{p}$ in $P$ forces that the set $\tau$ contains no points of $O$ which are Cohen-generic over $V[K]$, in particular $\tau$ is meager in $O$. This contradicts the initial assumptions on the name $\tau$. □

Claim 12.2.4. For every $i \in n$, the pair $V[K][x_j : j \in n, j \neq i], V[K][x_i]$ is equi-generic over $V[K]$.

Proof. This is the main and only point of the definition of the thick hypergraphs. Suppose towards contradiction that this fails. Then there must be an analytic equivalence relation $E$ coded in the model $V[K]$, a nonempty open set $O \subset \Gamma$, a $P_X$-name $\chi$ for an $E$-pin and a name $\theta$ for an $E$-pin in the model $V[K][x_j : j \neq i]$ such that $O \forces_{P_X} \theta/(x_j : j \neq i)$ and $\chi/x_i$ are equivalent $E$-pins which are not equivalent to any $E$-pin in the ground model. Find the open sets $P_0, P_1 \subset X$ and consider the set $\Delta$ as in Definition 12.2.1. Let $\langle y_0, y_1 \rangle \in \Delta$ be a pair which is $P_\Delta$-generic over the model $V[K]$. By Definition 12.2.1(2) and Proposition 3.1.1, the points $y_0, y_1 \in \Gamma \cap O$ are separately $P_1$-generic over $V[K]$ and the pair $\langle y_0(i), y_1(i) \rangle \in X^2$ is mutually $P_X$-generic over $V[K]$. Note that for all $j \neq i$, $y_0(j) = y_1(j)$ holds. By the forcing theorem, the $E$-pin $\theta/(y_0(j) : j \neq i) = \theta/(y_1(j) : j \neq i)$ must be equivalent to both $\chi/y_0(i)$ and $\chi/y_1(i)$. However, the latter two $E$-pins find themselves in mutually generic extensions of $V[K]$ and so must be equivalent to an $E$-pin in $V[K]$ by Proposition 12.1.2. This contradicts the initial assumptions on the $P_X$-name $\chi$. □

Now, let $H_i \subset R_i$ for $i \in n$ be filters mutually generic over the model $V[K][x_i : i \in n]$. It follows from Proposition 12.1.2(2) and Claim 12.2.4 that for each $i \in n$, the pair $V[K][x_j, H_j : j \neq i], V[K][x_i, H_i]$ is equi-generic over $V[K]$. For each $i \in n$ let $p_i = \sigma_i/H_i$. By induction on $i \in n$ build conditions $q_i \in V[x_j, H_j : j \leq i]$ such that $q_i \leq p_j$ for all $j \leq i$. To start, let $q_0 = p_0$. For the construction of $q_{i+1}$, note that $\check{p}$ is an equi-balanced condition in $V[K]$ and the pair $V[K][x_j, H_j : j \leq i], V[K][x_{i+1}, H_{i+1}]$ is equi-generic over $V[K]$, and therefore the conditions $q_i, p_{i+1}$ must have a lower bound. By an application of Mostowski absoluteness, such a bound $q_{i+1}$ must be found in the model $V[K][x_j, H_j : j \leq i + 1]$. At the end, let $q = q_{n-1}$. By the forcing theorem applied in the model $V[K][x_i][H_i], q \forces_{P_X} \check{x}_i \in \tau$ holds for each $i \in n$. Thus, $q \forces_{P_X} \check{a} \in \Gamma$ is a hyperedge which is a subset of $\tau$ as desired. □
Figure 12.1: Finishing the proof of Theorem 12.2.2.
CHAPTER 12. THE EQUI-GENERIC DIVIDE

Example 12.2.5. In equi-balanced extensions of the symmetric Solovay model, there is no nonatomic probability measure on $\omega$.

Proof. Consider the hypergraph $\Gamma$ on $P(\omega)$ connecting three subsets of $\omega$ if they are pairwise disjoint.

Claim 12.2.6. $\Gamma$ is thick.

Proof. The argument is best started with a nearly trivial observation. Let $a$ be a finite set and $g$ a graph on $a$. Let $Y_{a,g}$ be the set of all elements $y \in P(\omega)^a$ such that for all $i,j \in a$, if $\{i,j\} \in g$ then $y(i) \cap y(j) = 0$. This is a closed subset of $P(\omega)^a$ equipped with the inherited topology. If $b \subset a$ is a nonempty subset, then the projection function $\pi: Y_{a,g} \to Y_{b,g} \upharpoonright b$ is continuous and open. This observation immediately proves that the functions of Definition 12.2.1 are continuous and open as required.

Now work in some equi-balanced extension of the symmetric Solovay model and assume towards contradiction that $\mu$ is a nonatomic probability measure on $\omega$. Let $A = \{ a \subset \omega : \mu(a) \geq 1/2 \}$. Note that $A \subset P(\omega)$ is a nowhere meager set as it is invariant under finite changes and nonmeager: writing $\pi: P(\omega) \to P(\omega)$ for the self-homeomorphism of $P(\omega)$ sending each set to its complement, $P(\omega) = A \cup \pi''A$. Now, Theorem 12.2.2 together with the claim shows that the set $A$ contains a $\Gamma$-hyperedge, which contradicts the additivity of the measure $\mu$.

Corollary 12.2.7. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the transversal poset for $E$ as in Example 6.3.8. In the $P$-extension of the symmetric Solovay model, there is no nonatomic probability measure on $P(\omega)$.

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, $E$ has a transversal, and there is no nonatomic probability measure on $P(\omega)$.

Corollary 12.2.8. Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the linearization poset for the quotient space $X/E$ as in Example 8.5.5. In the $P$-extension of the symmetric Solovay model, there is no nonatomic probability measure on $P(\omega)$.

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds, the quotient space $X/E$ is linearly ordered, and there is no nonatomic probability measure on $P(\omega)$.

Corollary 12.2.9. Let $X$ be a Polish vector space over a countable field.

1. Let $P$ be the basis poset for $X$ as in Example 6.2.16. In the $P$-extension of the symmetric Solovay model, there is no nonatomic probability measure on $P(\omega)$.
2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the space $X$ has a basis, and there is no nonatomic probability measure on $\mathcal{P}(\omega)$.

**Example 12.2.10.** Let $A \subset \mathbb{R}^+$ be a Borel set with 0 as an accumulation point. Let $\Gamma$ be the graph on $\mathbb{R}^2$ connecting two points if their Euclidean distance belongs to $A$. In equi-balanced extensions of the symmetric Solovay model, the chromatic number of $\Gamma$ is uncountable.

**Proof.** We start with a preliminary claim. Let $\epsilon > 0$ be a real number, let $O \subset \mathbb{R}^2$ be an open ball of radius $> \epsilon$, and consider the graph $\Delta$ on $O$ connecting two points if their Euclidean distance is $\epsilon$.

**Claim 12.2.11. $\Delta$ is thick.**

**Proof.** We again start with a nearly trivial observation. Let $d$ denote the Euclidean distance in $\mathbb{R}^2$. Let $O_0, O_1, O_2 \subset \mathbb{R}^2$ be open sets such that for each $x_0 \in O_0$ and $x_1 \in O_1$ there is exactly one $x_2 \in O_2$ such that $d(x_0, x_2) = d(x_1, x_2) = \epsilon$. Let $Y \subset (\mathbb{R})^2$ be the set of all triples $y$ such that $y(0) \in O_0$, $y(1) \in O_1$ and $y(2) \in O_2$, and $d(y(0), y(2)) = d(y(1), y(2)) = \epsilon$. Then the projection function from $Y$ to the 0-th and 2-nd coordinate is a continuous open map from $Y$ to $\Delta$. Similarly, the projection function from $Y$ to the 1-st and 2-nd coordinate is a continuous open map from $Y$ to $\Delta$. Finally, the projection function from $Y$ to the 0-th and 1-st coordinate is a continuous and open map, since it is the projection map from a graph of a continuous function to its domain.

The thickness of $\Delta$ now immediately follows.

Now work in some equi-balanced extension of the symmetric Solovay model and assume towards contradiction that $\mathbb{R}^2 = \bigcup_n A_n$ is a countable union of $\Gamma$-anticlques. By the Baire category theorem, there must be a number $n \in \omega$ and an open ball $O \subset \mathbb{R}^2$ such that $A_n$ is nowhere meager in $O$. Let $\epsilon > 0$ be a number in the set $A$ smaller than the radius of $O$. Then theorem 12.2.2 together with the claim now shows that $A_n$ must contain two points at distance $\epsilon$, contradicting the assumption that $A_n$ is a $\Gamma$-anticlique.

**Corollary 12.2.12.** Let $X$ be a Polish vector space over a countable field.

1. Let $P$ be the basis poset for $X$ as in Example 6.2.16. In the $P$-extension of the symmetric Solovay model, there is no transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the space $X$ has a basis, and there is no transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$.

**Proof.** Let $\Gamma$ be the graph connecting two elements of $\mathbb{R}^2$ if their Euclidean distance is rational. Use a theorem of Schmerl [67] showing that the existence of a transcendence basis implies in ZF that the chromatic number of the graph
Γ is countable. However, Example 12.2.10 shows that Γ has uncountable chromatic number in all equi-balanced extensions of the symmetric Solovay model, in particular in the P-extension.

The same proof yields

**Corollary 12.2.13.** Let E be a pinned Borel equivalence relation on a Polish space.

1. Let P be the transversal poset for E as in Example 6.3.8. In the P-extension of the symmetric Solovay model, there is no transcendence basis for \( \mathbb{R} \) over \( \mathbb{Q} \).

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, E has a transversal, and there is no transcendence basis for \( \mathbb{R} \) over \( \mathbb{Q} \).

### 12.3 OCA and similar effects

In equi-balanced extensions of the symmetric Solovay extension, there are no particularly well organized uncountable sets of reals. There are many ways of stating particular instances of this general principle, but no clear general statement implying all of them. Our first example is the well-known Open Coloring Axiom, OCA [77].

**Definition 12.3.1.** OCA is the following statement. Whenever \( X \) is a Polish space, \( A \subseteq X \) is a set, and \( \Gamma \) is a graph on \( A \) which is open in the topology on \( A \times A \) inherited from \( X \times X \), then either \( A \) is a union of countably many \( \Gamma \)-anticliques, or \( A \) contains an uncountable \( \Gamma \)-clique.

The following theorem is stated using Convention 1.8.17.

**Theorem 12.3.2.** In every equi-balanced extension of the symmetric Solovay model, OCA holds.

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that \( V_\kappa \models P \) in every forcing extension the poset \( P \) is equi-balanced. Let \( W \) be a symmetric Solovay model derived from \( \kappa \). In the model \( W \), let \( X \) be a Polish space and let \( \Gamma \subseteq X^2 \) be a symmetric open set, let \( p \in P \) be a condition, and let \( \tau \) be a \( P \)-name for a subset of \( X \) such that \( p \Vdash \tau \) cannot be covered by countably many \( \Gamma \)-anticliques. We will find a perfect set \( C \subseteq X \) such that any two points of \( C \) are \( \Gamma \)-related, and a condition \( q \leq p \) in \( P \) which forces \( C \cap \tau \) is uncountable.

To this end, choose a parameter \( z \in 2^\omega \) such that \( p, \tau, \Gamma \) are definable from \( z \). Find an intermediate generic extension \( V[K] \) of \( V \) by a poset of size \( < \kappa \) such that \( z \in V[K] \). Work in the model \( V[K] \). Let \( p \leq p \) be an equi-balanced virtual condition. Since Coll(\( \omega, \kappa \)) \( \Vdash p \Vdash \tau \) is not covered by countably many \( \Gamma \)-anticliques, and a closure of a \( \Gamma \)-anticlique is still a \( \Gamma \)-anticlique, there must be a poset \( R \) of size \( < \kappa \), and \( R \)-name \( \eta \) for an element of \( X \) which belongs
to no closed $\Gamma$-anticlique coded in $V[K]$, and an $R$-name $\sigma$ for a condition in $P$ stronger than $\bar{p}$ such that $R \vDash \text{Coll}(\omega, < \kappa) \vDash \sigma \upharpoonright P \eta \in \tau$.

Move into the model $W$. Let $\{D_n : n \in \omega\}$ enumerate all open dense subsets of finite powers of the poset $R$ in $V[K]$, with infinite repetitions. By induction on $|t|$ build conditions $r_t \in R$ for $t \in 2^{<\omega}$ so that

- $t \subset s$ implies $r_s \leq r_t$;
- whenever $D_n \subset R^m$ is an open dense set for some $m < 2^n$, then every $m$-tuple of distinct elements from the set $\{r_t : t \in 2^n\}$ belongs to $D_n$;
- for all $t \in 2^{<\omega}$ there are open sets $O_{t_0}, O_{t_1} \subset X$ such that $O_{t_0} \times O_{t_1} \subset \Gamma$ and $r_{t_0} \vDash \eta \in O_{t_0}$ and $r_{t_1} \vDash \eta \in O_{t_1}$.

The construction is routine except for the last item; we just describe how the last item is obtained. Suppose that $r_1$ has been found. Work in $V[K]$ and let $U \subset X$ be the union of all basic open sets $O \subset X$ such that $r_t \vDash \eta \notin O$. Then $r_t \vDash \eta \in X \setminus U$. Since the set $X \setminus U$ is closed, by the choice of the name $\eta$ it cannot be a $\Gamma$-anticlique and therefore there are points $x_0, x_1 \in X \setminus U$ which are $\Gamma$-connected. Since the set $\Gamma \subset X^2$ is open, there are basic open sets $O_{t_0}, O_{t_1} \subset X$ such that $x_0 \in O_{t_0}, x_1 \in O_{t_1}$, and $O_{t_0} \times O_{t_1} \subset \Gamma$. Neither of the two open sets is a subset of $U$ and therefore there must be conditions $r_{t_0}$ and $r_{t_1}$ below $r_t$ such that the former forces $\eta \in O_{t_0}$ and the latter forces $\eta \in O_{t_1}$ as desired.

In the end, for every binary sequence $y \in 2^\omega$ let $H_y \subset R$ be the filter generated by the conditions $\{r_{y^n} : n \in \omega\}$ and let $p_y = \sigma/H_y$. Note that any finite tuple of distinct filters $H_y$ for $y \in 2^\omega$ is mutually generic over the model $V[K]$ by the second item in the inductive construction above. We will show that in the poset $P$, $\bar{p} \vDash$ the set $\{y \in 2^\omega : p_y \text{ is in the generic filter}\}$ is uncountable. Towards this, let $q \not\vDash \bar{p}$ be a condition and $a \subset 2^\omega$ be a countable set; we must show that $q$ is compatible with some condition $p_y$ for $y \notin a$. To this end, find an intermediate extension $V[K][G]$ by a forcing of size $< \kappa$ such that $q, a \in V[K][G]$. Let $y \in 2^\omega$ be a point Cohen-generic over the model $V[K][G]$; we claim that the pair $V[K][H_y], V[K][G]$ is equi-generic over $V[K]$. This will show that $p_y$ is compatible with $q$, completing the proof.

Suppose towards contradiction that the pair $V[K][H_y], V[K][G]$ is not equi-generic over $V[K]$. Writing $S$ for the Cohen forcing on $2^\omega$, there must be a condition $s \in S$, an analytic equivalence relation $E$ coded in $V[K]$, a virtual $E$-class $c \in V[K][G]$ not equivalent to any virtual $E$-class in $V[K]$, and an $R$-name $\chi \in V[K]$ for a virtual $E$-class such that $r \vDash \chi/H_y$ is equivalent to $c$. Let $y_0, y_1 \in 2^\omega$ be distinct points Cohen generic over $V[G]$, both extending $r$. Consider the virtual $E$-classes $d_0 = \chi/H_{y_0}$ and $d_1 = \chi/H_{y_1}$. By the forcing theorem, both of these virtual classes are $E$-related to $c$, and so related to each other. Since they both belong to the respective models $V[K][H_{y_0}]$ and $V[K][H_{y_1}]$ and these models are mutually generic over $V[K]$, Proposition 12.1.2 says that they must be equivalent to some virtual $E$-class in $V[K]$. However, this contradicts the choice of the class $c$. \qed
Example 12.3.3. There is a balanced extension of the symmetric Solovay model in which OCA fails.

Proof. Consider the clopen graph $\Gamma$ on $2^\omega$ connecting points $x_0 \neq x_1$ if the smallest $n \in \omega$ such that $x_0(n) \neq x_1(n)$ is even. The Suslin poset $P$ adds an uncountable set $A \subset 2^\omega$ such that for every closed $\Gamma$-clique or $\Gamma$-anticlique $C \subset 2^\omega$, the set $A \cap C$ is countable. A condition in $P$ is a pair $p = \langle a_p, b_p \rangle$ where $a_p \subset 2^\omega$ is a countable set, and $b_p$ is a countable collection of closed subsets of $2^\omega$ each of which is either a $\Gamma$-clique or a $\Gamma$-anticlique. The ordering is defined by $q \leq p$ if $a_p \subset a_q$, $b_p \subset b_q$, and $\bigcup b_p \cap a_q \setminus a_p = 0$. It is not difficult to check that in fact $P$ is a Suslin $\sigma$-closed ordering. A brief argument shows that whenever $(a, b)$ is a pair such that $a \subset 2^\omega$ is a set and $b$ is the set of all closed $\Gamma$-cliques and closed $\Gamma$-anticliques, then the pair $\langle \text{Coll}(\omega, 2^\omega), \langle a, b \rangle \rangle$ is balanced, and there are no other balanced conditions up to the balanced equivalence. In the symmetric Solovay model, the poset $P$ forces the union of the first coordinates of the conditions in the generic filter to be an uncountable set such that every subset of it which is either a $\Gamma$-clique or a $\Gamma$-anticlique is countable, violating OCA.

The following theorem is stated using Convention 1.8.17.

Theorem 12.3.4. Let $I$ be an analytic $P$-ideal on $\omega$. In equi-balanced extensions of the symmetric Solovay model, if $A \subset I$ is an uncountable set, then there is a set $b \subset I$ such that the set $\{a \in A : a \subset b\}$ is uncountable.

Proof. Use the result of Solecki [72] to fix a lower-semicontinuous submeasure $\mu$ on $\omega$ such that $I = \{a \subset \omega : \lim_n \mu(a \setminus n) = 0\}$. The argument uses the following partially ordered set $Q$. The conditions in $Q$ are tuples $q = \langle s_q, \varepsilon_q, a_q \rangle$ where $s_q \subset \omega$ is finite, $\varepsilon_q > 0$ is a real number, and $a_q \subset I$ is a set such that $\mu(a_q \setminus s_q) < \varepsilon_q$. The ordering is defined by $r \leq q$ if $s_q \subset s_r$, $\varepsilon_r \leq \varepsilon_q$, and $a_q \subset s_r \cup a_r$. It is not difficult to see that $Q$ is a Suslin $\sigma$-linked poset, and if $G \subset Q$ is a generic filter, then the union of the first coordinates of conditions in $G$ is a set in $I$ which modulo finite contains every ground model element of $I$.

Now, suppose that $\kappa$ is an inaccessible cardinal and $P$ is a Suslin poset such that $V_\kappa \models P$ is equi-balanced in every forcing extension. Let $W$ be a symmetric Solovay model derived from $\kappa$ and in the model $W$, let $p \in P$ be a condition and $\tau$ be a $P$-name such that $p \forces \tau \subset I$ is an uncountable set. We need to produce a condition $q \leq p$ and a set $b \in I$ such that $q \forces \{a \in \tau : a \subset b\}$ is uncountable. The condition $p$ and the name $\tau$ must be definable from parameters in the ground model and some real parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate forcing extension by a poset of size $\kappa$ such that $z \in V[K]$, and work in the model $V[K]$.

By the assumptions on the poset $P$, there must be an equi-balanced virtual condition $\bar{p} \leq p$ in the model $V[K]$. There must be a poset $R_0$ of size $\kappa$, an $R_0$-name $\eta_0$ for an element of $I$ which is not in $V[K]$, and an $R_0$-name $\sigma_0$ for a condition in $P$ stronger than $\bar{p}$ such that $R_0 \forces \text{Coll}(\omega, \kappa) \forces \sigma_0 \forces p, \eta_0 \in \tau$. Let $H_0 \subset R_0$ be a filter generic over $V[K]$, let $a = \eta_0/H_0$, and let $p_0 = \sigma_0/H_0$. 


Let $G \subseteq Q$ be a filter generic over $V[K][H_0]$ and let $b \in I$ be the set derived from $G$. Note that $a \subseteq b$ modulo finite.

**Claim 12.3.5.** The set $b$ is $Q$-generic over $V[K]$. The pair $V[K][H_0], V[K][b]$ is equi-generic over $V[K]$.

**Proof.** This uses only the fact that $Q$ is a Suslin c.c.c. forcing. The first sentence follows from the fact that all maximal antichains in $Q$ in the model $V[K]$ are countable. Being a maximal antichain is a coanalytic property of elements of $Q^\omega$, and so all maximal antichains of $Q$ in $V[K]$ remain such in $V[K][H_0]$ by the Mostowski absoluteness, and therefore are met by the set $b$.

The equi-genericity is slightly more demanding. Suppose towards contradiction that it fails. By the forcing theorem, this means that there is an analytic equivalence relation $E$ is in the model $V[K]$, an $E$-pin $s$ in $V[K][H_0]$ which has no equivalent in $V[K]$, a $Q$-name $\chi \in V[K]$ for an $E$-pin, and in the model $V[K][H_0]$, a condition $q \in Q$ which forces $s$ to be equivalent to $\chi$. Let $b_0, b_1 \subseteq \omega$ be sets mutually $Q$-generic over the model $V[K][H_0]$, both meeting the condition $q$. By the first paragraph of the present proof and the fact that $Q \times Q$ is a Suslin c.c.c. poset, the sets $b_0, b_1 \subseteq \omega$ are mutually $Q$-generic over $V[K]$.

By the mutual genericity and Proposition 12.1.2(2) applied in $V[K]$, either the pins $\chi/b_0, \chi/b_1$ are both equivalent to the same $E$-pin in $V[K]$, or they are not equivalent at all. In any case, one or both of them are not equivalent to the pin $s$, which contradicts the initial assumption on the condition $q \in Q$. 

Move into the model $V[K][b]$. We will argue that there, $\text{Coll}(\omega, \kappa) \Vdash \bar{p} \Vdash \{ a \in \tau : a \subseteq b \}$ is uncountable; this will complete the proof. Suppose towards contradiction that this fails. Then, there must be in $V[K][b]$ a poset $R_1$ of size $< \kappa$, an $R_1$-name $\eta_1$ for an element of $P^\omega$, and an $R_1$-name $\sigma_1$ for a condition in $P$ below $\bar{p}$ such that $R_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash_P \{ c \in \tau : c \subseteq b \text{ modulo finite} \} \subseteq \text{rng}(\eta_1)$. Let $H_1 \subseteq R_1$ be a filter generic over the model $V[K][H_0][b]$, let $p_1 = \sigma_1/H_1$ and $y = \eta_1/H_1$. By Proposition 12.1.2(2) again, the models $V[K][H_0]$ and $V[K][H_1]$ are equi-generic extensions of the model $V[K]$, and by the equi-balanced assumption on the condition $\bar{p}$, $p_0, p_1$ have a lower bound $q \in P$. By the forcing theorem applied in the models $V[K][H_0]$ and $V[K][b][H_1]$, the condition $q$ forces that $\{ c \in \tau : c \subseteq b \text{ modulo finite} \} \subseteq \text{rng}(y)$ and at the same time $a \in \tau$. This is impossible, since $\text{rng}(y) \subseteq V[K][b][H_1], a \subseteq b \text{ modulo finite, and } a \notin V[K][b][H_1]$. 

**Example 12.3.6.** There is a balanced extension of the symmetric Solovay model in which the conclusion of Theorem 12.3.4 fails.

**Proof.** Let $P$ be the poset of all countable subsets of $\omega^\omega$, ordered by $q \preceq p$ if $p \subseteq q$ and no element of $p \setminus q$ is modulo finite dominated by any element of $q$. If $G \subseteq P$ is a filter generic over the symmetric Solovay model $W$, then $\bigcup G \subseteq \omega^\omega$ is a modulo finite dominating set, and for each $x \in \omega^\omega$ the set $\{ y \in \bigcup G : y \text{ is modulo finite dominated by } x \}$ is countable. Thus, the conclusion of the theorem fails with the $P$-ideal $I$ on $\omega \times \omega$ consisting of sets with all vertical sections finite.
The only thing left to check is the balance of the poset $P$. In fact, it is not difficult to classify the balanced virtual conditions in $P$. For each set $a \subset \omega$, let $\tau_a$ be the $\text{Coll}(\omega, \omega)$-name for the analytic, downward closed set of all conditions $p \in P$ such that $a = p \cap V = \{x \in p : x \text{ is modulo finite dominated by an element of } V \cap \omega\}$ and $p$ contains a function dominating each element of $V \cap \omega$.

Claim 12.3.7. 1. If $a \subset \omega$ is a set, then the pair $\langle \text{Coll}(\omega, a), \tau_a \rangle$ is a balanced pair;

2. if $\langle Q, \tau \rangle$ is a balanced pair then there is a set $a \subset \omega$ such that the balanced pairs $\langle Q, \tau \rangle$ and $\langle \text{Coll}(\omega, a), \tau_a \rangle$ are equivalent;

3. distinct subsets of $\omega$ yield inequivalent balanced pairs.

In particular, the poset $P$ is balanced. To prove the claim, start with (1) and suppose that $a \subset \omega$ is any set, and $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ are conditions in the set $\tau_a$: we must show that they are compatible, i.e. $p_0 \cup p_1$ is their lower bound. To this end, use the definition of the name $\tau_a$ to see that every element of $p_0 \setminus V$ is not bounded by any element of $V \cap \omega$, and similarly for $p_1 \setminus V$. A straightforward mutual genericity argument then shows that no element of $p_0 \setminus V$ is modulo finite comparable with any element of $p_1 \setminus V$. Since $p_0 \cap V = p_1 \cap V = a$, it follows that $p_0 \cup p_1$ is indeed a condition stronger than both $p_0, p_1$.

For (2), strengthening $\tau$ if necessary we may assume that it contains a point modulo finite dominating every element of $V \cap \omega$. Then, a balance argument shows that for each point $x \in a$, either $Q \forces \dot{x} \in \tau$ or $Q \forces \dot{x} /\in \tau$ must hold. Let $a \subset \omega$ be the set of all elements $x \in \omega$ for which the former alternative prevails, and argue that $Q \forces \tau \in \tau_a$. The routine details are left to the reader. Finally, (3) is obvious.

The poset $P$ in the previous example has been discovered by Shelah some time ago [68]. He proved that in the $P$-extension of the symmetric Solovay model, every set of reals is Lebesgue measurable, while the generic subset of $\omega$ does not have the property of Baire.

The following theorem is stated using Convention 1.8.17.

Theorem 12.3.8. Let $E$ be a Borel equivalence relation on a Polish space $X$. In equi-balanced extensions of the symmetric Solovay model, every subset of $X$ is either covered by countably many $E$-classes or contains an uncountable subset consisting of pairwise $E$-unrelated elements.

Proof. Let $\kappa$ be an inaccessible cardinal and $P$ a Suslin forcing such that $V_\kappa \models P$ is equi-balanced in every forcing extension. Let $W$ be the symmetric Solovay model derived from $\kappa$, let $p \in P$ be a condition and let $\tau$ be a $P$-name such that $p \forces \tau \subset X$ is a set which is not covered by countably many $E$-classes. We will find a perfect set $A \subset X$ consisting of pairwise $E$-unrelated elements and a condition $q \leq p$, $q \forces \tau \cap A$ is uncountable. This will prove the theorem.
The condition \( p \) and the name \( \tau \) are definable from a parameter \( z \in 2^\omega \) and some parameters in the ground model. Let \( V[K] \) be some intermediate extension of the ground model by a poset of size \(< \kappa \) and work in \( V[K] \). Let \( \bar{p} \leq p \) be an equi-balanced condition in the poset \( P \). Since the equivalence relation \( E \) is Borel, there are fewer than \( \beth_\omega \) many virtual \( E \)-classes in \( V[K] \). Since the cardinal \( \beth_\omega \) is countable in \( W \), there must be in \( V[K] \) a poset \( R \) of size \(< \kappa \) and \( R \)-names \( \sigma \) for a condition in \( P \) stronger than \( \bar{p} \) and \( \eta \) for an element of \( X \) which is not a realization of any virtual \( E \)-class in \( V[K] \) such that \( R \vDash \text{Coll}(\omega, < \kappa) \vDash \sigma \vDash_p \eta \in \tau \).

In the model \( W \), use Proposition 1.8.10 to find a perfect set \( \{ H_y \mid y \in 2^\omega \} \) of filters on \( R \) which are mutually generic over the model \( V[K] \). Write \( p_y = \sigma \rest H_y \) and \( x_y = \eta \rest H_y \). By the assumption on the name \( \eta \) and the mutual genericity, the set \( A = \{ \eta \rest H_y \mid y \in 2^\omega \} \) consists of pairwise \( E \)-unrelated elements. It will be enough to show that \( \bar{p} \vDash \) for uncountably many \( y \in 2^\omega \), \( p_y \) belongs to the generic filter. Then \( \bar{p} \vDash \tau \cap \bar{A} \) is uncountable as required.

Thus, suppose that \( q \leq \bar{p} \) be a condition and \( a \subset 2^\omega \) be countable; we need to find a point \( y \in 2^\omega \setminus a \) such that \( p_y \) is compatible with \( q \). Let \( V[K][H] \) be an intermediate generic extension obtained by a poset of size \(< \kappa \) containing \( q, a \), and the code for the continuous map \( y \mapsto H_y \). In view of the equi-balance of the condition \( \bar{p} \), it is enough to find a point \( y \in 2^\omega \setminus a \) such that \( V[K][H] \) and \( V[K][H_y] \) is an equi-generic pair of extensions of \( V[K] \). To this end, let \( y \in 2^\omega \) be a point Cohen-generic over \( V[K][H] \) and argue that \( y \) works. Suppose towards contradiction that it does not, and \( s \in 2^{<\omega} \) be a condition in Cohen forcing \( S \), let \( F \) be an analytic relation coded in \( V[K] \), \( \chi \in V[K] \) an \( R \)-name for a virtual \( F \)-class, and \( C \) a virtual \( F \)-class in the model \( V[K][H] \) which is not represented in \( V[K] \) such that \( s \vDash C \) is equivalent to \( \chi / H_y \). Let \( y_0, y_1 \in 2^\omega \) be points extending \( s \) and mutually Cohen-generic over the model \( V[K][H] \). The virtual \( F \)-classes \( \chi / H_{y_0} \) and \( \chi / H_{y_1} \) are both equivalent to \( C \) and therefore equivalent. By the mutual genericity of the filters \( H_{y_0}, H_{y_1} \subset R \) over \( V[K] \) and Proposition 12.1.2, they are both equivalent to some virtual \( F \)-class in \( V[K] \). However, \( C \) is initially assumed to be a virtual class not represented in \( V[K] \). A contradiction!

\[ \square \]

Example 12.3.9. There is a balanced extension of the symmetric Solovay model in which the conclusion of Theorem 12.3.8 fails.

\[ \text{Proof.} \] Let \( P \) be the partial order of all pairs \( p = (a_p, b_p) \) where \( a_p \subset 2^\omega \) is a countable \( \mathbb{E}_0 \)-invariant set and \( b_p \) is a countable set of closed nowhere dense subsets of \( (2^\omega)^n \) for various dimensions \( n \in \omega \), such that whenever \( C \subset (2^\omega)^{n+m} \) is a set in \( b_p \), then the meager set \( \{ x \in 2^\omega \mid C_x \text{ is somewhere dense} \} \) is covered by the closed sets in \( b_p \) of dimension \( n \). The order is defined by \( q \leq p \) if \( a_p \subset a_q \), \( b_p \subset b_q \), and if \( x \in (2^\omega)^n \) is a tuple of pairwise \( \mathbb{E}_0 \)-unrelated elements in the set \( a_q \setminus a_p \) then (i) \( x \) belongs to no set in \( b_p \) of dimension \( n \), and (ii) if \( C \in b_p \) is a set of dimension \( n + m \) then \( C_x \in b_q \). It is not difficult to check that \( P \) is a \( \sigma \)-closed Suslin partial ordering. Let \( \bar{A} \) be the \( P \)-name for the union of the first coordinates of the conditions in the generic filter. We must show that the poset
P is balanced, and that in the Solovay model W, W \models P \models \dot{A} is an E₀-invariant set containing uncountably many E₀-classes but no uncountable set of pairwise E₀-unrelated elements.

The balance of the poset P is an immediate consequence of the following claim. Write Q = \text{Coll}(\omega, 2^\omega) and let B denote the collection of all closed nowhere dense subsets of \((2^\omega)^n\) for all dimensions \(n \in \omega\).

Claim 12.3.10. Let \(p = \langle a, b \rangle \in P\) be a condition. Then the pair \(Q, \langle a, \dot{B} \rangle\) is balanced in P below p.

Proof. It is not difficult to see that \(Q \models \langle a, \dot{B} \rangle\) is a condition in P stronger than p. To see the balance, suppose that \(V[H_0], V[H_1]\) are mutually generic extensions of the ground model and \(p_0 = \langle \alpha_0, b_0 \rangle, p_1 = \langle \alpha_1, b_1 \rangle \in P\) are conditions in the respective models which are stronger than \(\langle a, B^V \rangle\); we must produce a lower bound of the conditions \(p_0, p_1\).

To do this, note that any tuple \(x \in (2^\omega)^n\) of pairwise E₀-unrelated elements of \(a_0 \setminus a\) is Cohen-generic over V. By the mutual genericity of the models \(V[H_0]\) and \(V[H_1]\), the tuple \(x\) is also Cohen-generic over the model \(V[H_1]\); in particular, it belongs to no sets in \(b_1\) of dimension \(n\). The symmetric conclusion is also true of every tuple of pairwise E₀-unrelated elements of the set \(a_1 \setminus a_0\). It follows that the condition \(\langle a_0 \cup a_1, c \rangle\) is a lower bound of \(p_0, p_1\), where \(c\) is the set containing \(b_0, b_1\) as well as all sets \(C_x\) for every tuple \(x \in (2^\omega)^{n+m}\) consisting of pairwise E₀-unrelated elements of the set \(a_0 \setminus a\) and the set \(C \in b_1\) of dimension \(n + m\), and also all sets \(C_x\) for every tuple \(x \in (2^\omega)^{n+m}\) consisting of pairwise E₀-unrelated elements of the set \(a_1 \setminus a\) and the set \(C \in b_0\) of dimension \(n + m\). \(\square\)

Now let \(\kappa\) be an inaccessible cardinal, let \(W\) be a symmetric Solovay model derived from \(\kappa\), and move to the model \(W\). Let \(p = \langle a, b \rangle \in P\) be a condition and let \(\tau\) be a P-name for an uncountable subset of \(\dot{A}\). We must find a condition \(q \leq p\) and two distinct E₀-related points \(x_0, x_1 \in 2^\omega\) such that \(q \models \dot{x}_0, \dot{x}_1 \in \tau\).

To this end, find a parameter \(z \in 2^\omega\) such that both \(p, \tau\) are definable from some parameters in the ground model and the parameter \(z\). Let \(V[K]\) be an intermediate generic extension containing the parameter \(z\), obtained as a forcing extension of the ground model by a poset of size less than \(\kappa\). Work in the model \(V[K]\).

Let \(\bar{p} \leq p\) be the balanced virtual condition isolated in Claim 12.3.10. Since \(\tau\) is forced to be an uncountable set, there must be a poset \(R\), an \(R\)-name \(\sigma\) for a condition in \(P\) stronger than \(\bar{p}\), and an \(R\)-name \(\eta\) for a point in \(a_\sigma \setminus a\) such that \(R \models \text{Coll}(\omega, \kappa) \models \sigma \models \eta \in \tau\). Note that \(\eta\) is forced to be a point Cohen-generic over the model \(V[K]\), and it is also forced that any tuple \(x \in (2^\omega)^n\) of points in the set \(a_\sigma \setminus (a \cup [\eta]_{E_0})\) is Cohen-generic over the model \(V[K][\eta]\). By the general forcing theory, we may present the poset \(R\) as a two step iteration \(S^0 \ast \dot{S}^1\), where \(S^0\) is the Cohen poset \(2^{<\omega}\) restricted to some finite binary string \(s\), adding the Cohen-generic point \(\eta\), and \(\dot{S}^1\) is the \(S^0\)-name for the remainder forcing.

Move back to the model \(W\). Choose a filter \(H_0^0 \subset S^0\) generic over the model \(V[K]\), and write \(x_0 = \eta/H_0^0 \in 2^\omega\). Flip any value of \(x_0\) past \(\text{dom}(s)\) to obtain...
another point $x_1 \in 2^\omega$. Then $x_1$ is $\mathbb{E}_0$-related to $x_0$ and also Cohen generic over the model $V[K]$, containing the string $s$ as an initial segment. Write $H^0_1 \subset S^0$ for the filter generic over $V[K]$ obtained from $x_1$; thus, $V[K][H^0_1] = V[K][H^0_0]$. Find filters $H^0_0 \subset \dot{S}^0/H^0_0$ and $H^1_1 \subset \dot{S}/H^1_1$ mutually generic over the model $V[K][H^0_0]$, and write $p_0 = \sigma/H^0_0 \ast H^1_0$ and $p_1 = \sigma/H^0_1 \ast H^1_1$. The conditions $p_0, p_1$ are compatible in the poset $P$ by a mutual genericity argument identical to Claim 12.3.10. Let $q \in P$ be any lower bound for the conditions $p_0, p_1$. The initial choices of the names $\sigma$ and $\eta$ now imply that $q \Vdash \dot{x}_0, \dot{x}_1 \in \tau$ as desired. \hfill \qed

### 12.4 Existence of generic filters

Let $P$ be a Suslin forcing. In the presence of suitable large cardinal axioms, one can replace the $P$-extension of the symmetric Solovay model with the $P$-extension of the model $L(\mathbb{R})$. A standard absoluteness argument shows that validity of $\Pi^2_1$ sentences transfers from the former to the latter, and so the latter can serve as a vehicle for most independence results presented in this book. A natural question arises: does there exist a filter $G \subset P$ which is generic over $L(\mathbb{R})$?

The answer is an easy yes if the poset $P$ in question is $\sigma$-closed an the Continuum Hypothesis holds, since then one can meet the dense subsets of the poset $P$ appearing in the model $L(\mathbb{R})$ one by one in a straightforward transfinite recursion argument. An interesting feature of equi-balance is that in a suitable variation, it guarantees the existence of generic filters over inner models such as $L(\mathbb{R})$ independently of cardinal arithmetic and assumptions such as the Continuum Hypothesis, purely as a consequence of large cardinal axioms.

**Definition 12.4.1.** Let $P$ be a Suslin forcing and let $E$ be a pinned Borel equivalence relation. A virtual condition $\bar{p}$ in $P$ is $E$-balanced if for every pair $V[H_0], V[H_1]$ of generic extensions such that every $E$-class represented in both is represented in the ground model, and any two conditions $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ such that $p_0, H_0 \leq \bar{p}$ holds, $p_0, p_1$ have a common lower bound.

**Definition 12.4.2.** Let $P$ be a Suslin forcing.

1. If $B \subset P$ is a set and $p \in P$ then write $p \leq B$ to denote that $p$ is stronger than all conditions in $B$;

2. if $B \subset P$ then $\tau_B$ is the Coll($\omega, B$)-name for the analytic set $\{ p \in P : p \leq B \}$.

3. We say that $P$ is soft if there is a pinned Borel equivalence relation $E$ on a Polish space such that for every centered set $B \subset P$ of size $< \kappa$ there is a centered set $C \supset B$ such that the pair $(\text{Coll}(\omega, C), \tau_C)$ is an $E$-balanced virtual condition.

Note that a soft poset $P$ must have the following centeredness property: if $B \subset P$ is a countable centered set, then $B$ has a lower bound. To see this, just
extend it into an $E$-balanced centered set $C \subset P$, note that in the Coll$(\omega, C)$-extension, the set $C$ has a lower bound by the definition, and by a Mostowski absoluteness argument the set $B$ must have a lower bound already in the ground model.

**Theorem 12.4.3.** Suppose that there is a weakly compact Woodin cardinal. Suppose that $P$ is a soft Suslin forcing. Then there is a filter $G \subset P$ which is generic over the model $L(\mathbb{R})$.

**Proof.** Let $B \subset P$ be a centered set of size $< \frak{c}$, and let $D \subset P$ be an open dense set in $L(\mathbb{R})$. We will show that there is a condition $p \in D$ such that $\{p\} \cup B$ is a centered set. The theorem then follows by a straightforward transfinite recursion construction, given the fact that there are only $\frak{c}$-many subsets of $P$ in the model of $L(\mathbb{R})$ as soon as a measurable cardinal exists. So, let $E$ be a pinned Borel equivalence relation on a Polish space $X$ and let $C \supset B$ be an $E$-balanced centered set witnessing the definitory properties of the soft poset $P$. The set $D$ is definable from a real parameter in the model $L(\mathbb{R}^\#)$. Below, we identify the set $D$ with its definition.

Let $\kappa$ be a weakly compact Woodin cardinal, let $\mathbb{Q}_\kappa$ be the associated countably based stationary tower, let $K \subset \mathbb{Q}_\kappa$ be a generic filter and work in the model $V[K]$. Note that $\kappa = \aleph_1$ and every element of $2^\omega$ is generic over $V$ by a poset smaller than $\kappa$. Consider the poset $R = \text{Coll}(\omega, C)$ and use Proposition 1.8.10 to find a perfect collection $\{H_y: y \in 2^\omega\}$ of filters on $R$ pairwise mutually generic over the ground model $V$.

**Claim 12.4.4.** For every finite set $a \subset P$ such that $a \cup C$ is a centered set, the set $Y_a = \{y \in 2^\omega: \exists p \in V[H_y] \ p \leq C \text{ and the set } \{p\} \cup a \text{ has no common lower bound}\}$ is countable.

**Proof.** Let $p \in P$ be a common lower bound of the countable centered set $a \cup C$. The set of $E$-classes represented in $V[p]$ and not in the ground model is countable. Now, if $y_0, y_1 \in 2^\omega$ are distinct points, then by the mutual genericity of the filters $H_{y_0}, H_{y_1} \subset R$, there are no $E$-classes which are represented in both models $V[H_{y_0}], V[H_{y_1}]$ and not in the ground model. It follows that the set $Y = \{y \in 2^\omega: \text{ there is an } E\text{-class represented in both } V[p] \text{ and } V[H_y] \text{ and not in the ground model}\}$ is countable. The $E$-balance of the virtual condition $\langle \text{Coll}(\omega, C), \tau_C \rangle$ then shows that $Y_a \subset Y$, and the claim follows. 

Let $j: V \rightarrow M$ be the generic ultrapower associated with $K$; the model $M$ is closed under $\omega$-sequences. Work in the model $M$. The closure properties of the model $M$ guarantee that the following objects belong to the model $M$: the collection $\{H_y: y \in 2^\omega\}$, the collection $\{P \cap V[H_y]: y \in Y\}$, and the map $a \mapsto Y_a$ as $a$ varies over finite subsets of $P$ such that $a \cup C$ is a centered set. Also, the model $M$ satisfies the statement that the range of this map consists of countable sets. Looking at the set $j(C)$ in the model $M$, we see that $|j(C)| < \frak{c}$, and for each finite set $a \subset j(C)$ the set $a \cup C \subset j(C)$ is centered, and therefore the set $Y_a \subset 2^\omega$ is countable. By a cardinality argument, there must be a point $y \in 2^\omega \setminus \bigcup\{Y_a: a \in [j(C)]^{<\kappa_0}\}$. 

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Work in the model $V[H_y]$. Note that the theory of $L(\mathbb{R}^\#)$ with real parameters is invariant under forcing and so the set $D \subseteq P$ is still open dense in $V[H_y]$. Thus, there must be a condition $p \in D \cap V[H_y]$, $p \leq C$. By the choice of the point $y \in Y$, the set $\{p\} \cup j(C)$ is centered. Thus $M \models$ there is a condition $p \in D$ such that $\{p\} \cup j(C)$ is centered. By the elementarity of the embedding $j$, $M \models$ there is a condition $p \in D$ such that $\{p\} \cup C$ is centered; in particular, $\{p\} \cup B$ is centered as desired.

The following examples all use the Woodin cardinal assumption which is not spelled out.

Example 12.4.5. Let $X$ be a Polish vector space over a countable field. There is a basis generic over $L(\mathbb{R})$ for the poset $P$ of Example 6.2.16. This follows from Theorem 12.4.3 and Theorem 6.2.28(1): every centered subset of $P$ yields a linearly independent set, which can be extended to a maximal linearly independent set. A maximal linearly independent set yields an $E$-balanced condition where $E$ is simply the identity on $2^\omega$.

Example 12.4.6. Let $\Gamma$ be a Borel graph on a Polish space $X$. There is a maximal acyclic subgraph $\Delta \subseteq \Gamma$ generic over $L(\mathbb{R})$ for the poset $P$ of Example 6.2.17. This follows from Theorem 12.4.3 and Theorem 6.2.28(2): every centered subset of $P$ yields an acyclic subset of $\Gamma$, which can be extended to a maximal acyclic set. A maximal acyclic set yields an $E$-balanced condition where $E$ is simply the identity on $2^\omega$.

Example 12.4.7. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$. There is a linear ordering on $X/E$ generic over $L(\mathbb{R})$ for the poset $P$ of Example 8.5.5. This follows from Theorem 12.4.3 and Theorem 8.5.4: every centered subset of $P$ yields a partial ordering on $X/E$, which can be extended to a linear order. A linear ordering on $X/E$ yields an $E$-balanced condition, by the pinned assumption on $E$ and Theorem 8.5.4.

Example 12.4.8. Let $E,F$ be pinned Borel equivalence relations on respective Polish spaces $X,Y$ with uncountably many classes each. There is an injection from $X/E$ to $Y/F$ generic over $L(\mathbb{R})$ for the poset $P$ of Definition 6.3.2. This follows from Theorem 12.4.3 and Theorem 6.3.3: every centered subset of $P$ of size $< \mathfrak{c}$ yields a partial injection from $X/E$ to $Y/F$ and also a subset of $Y/F$ of forbidden values, both of size $< \mathfrak{c}$. A counting argument provides a total injection from $X/E$ to $Y/F$ extending the given one and avoiding the given set of forbidden values. Such an injection yields a $G$-balanced condition by Theorem 6.3.3, where $G = E \times F$.

Many other examples can be produced by the observation that the class of soft forcings is closed under countable product. The nonexamples are perhaps more interesting than the examples. The equi-balance assumption cannot be removed entirely from the wording of the theorem in view of a basic observation:

Example 12.4.9. Let $P$ be the $\mathcal{P}(\omega)$ modulo finite poset. It is not equi-balanced by Example 12.2.5 above. The $P$-extension of $L(\mathbb{R})$ contains a Ramsey
ultrafilter on $\omega$, in particular a P-point; at the same time, it is consistent [69] and occurs in the product Silver model [14] that there are no P-points on $\omega$. In such circumstances, there cannot be a filter on $P$ generic over $L(\mathbb{R})$.

Another restriction on the theorem is that a soft Suslin forcing must be governed by a pinned Borel equivalence relation. While the present proof clearly needs this assumption, it is not obvious why it should be necessary. The following question addresses the simplest natural case.

**Question 12.4.10.** (In the context of ZFC+suitable large cardinal assumption.) Is there a tournament on the $F_2$-quotient space which is generic over $L(\mathbb{R})$ for the poset of Example 8.5.6?
Chapter 13

Structures on quotient spaces

13.1 Excluding tournaments

Definition 13.1.1. Let $X$ be a set. A tournament is a directed graph $\vec{\Gamma}$ on $X$ such that for any two distinct points $x_0, x_1 \in X$, exactly one of $\langle x_0, x_1 \rangle$, $\langle x_1, x_0 \rangle$ belongs to $\vec{\Gamma}$.

A straightforward Baire category proof shows that there is no Borel tournament on the $\mathbb{E}_0$-quotient space. The Glimm–Effros dichotomy then shows that no Borel tournaments exist on any non-smooth Borel quotient space. It turns out that many posets do not add tournaments on complicated quotient spaces; our most general theorem of this form is the following, stated using Convention 1.8.17.

Theorem 13.1.2. In $\sigma$-closed, $\aleph_0$-adequate extensions of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, there are no tournaments on the $\mathbb{F}_2$-quotient space.

The $\sigma$-closure demand can be relaxed to include the formally not $\sigma$-closed posets of Section 8.1; we omit the details. It is not immediately clear if the slight increase in the large cardinal hypothesis can be removed.

Proof. Let $\kappa$ be an inaccessible limit of inaccessible cardinals. Let $P$ be a Suslin forcing such that $V_\kappa \models P$ is $\aleph_0$-adequate in all generic extensions. Let $W$ be a symmetric Solovay model derived from $\kappa$ and work in $W$. Suppose towards contradiction that $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \Vdash \tau$ is a tournament on the $\mathbb{F}_2$-quotient space. Both $p, \tau$ must be definable from some real parameter $z \in 2^\omega$ and some parameters in the ground model. Let $V[K]$ is an intermediate extension by a poset of size $< \kappa$ such that $z \in V[K]$.

Work in the model $V[K]$ for a moment. Let $\bar{p} \leq p$ be a balanced virtual condition in the poset $P$ in $V[K]$. Consider the poset $Q$ of all partial countable
functions \( q : 2^\omega \to 2 \) ordered by reverse extension. Let \( \bar{q} \) be a virtual condition in the poset \( Q \) which is a \( \text{Coll}(\omega, 2^\omega) \)-name for the analytic set of all conditions in \( Q \) such that \( (2^\omega)^V \subseteq \text{dom}(q) \) and \( q \) is constantly equal to 0 on \( (2^\omega)^V \). It is immediate that \( \bar{q} \) is a balanced virtual condition in \( Q \), and moreover, \( \langle \bar{q}, \bar{q}, \bar{p} \rangle \) is a balanced condition in the product \( Q \times Q \times P \) by Theorem 6.2.2.

Let \( \lambda < \kappa \) be an inaccessible cardinal of \( V[K] \). Let \( G \subseteq \text{Coll}(\omega, \lambda) \) be a filter generic over \( V[K] \), and let \( H_0 \subseteq Q, H_1 \subseteq Q, H_2 \subseteq P \) be filters mutually generic over the model \( V[K][G] \) meeting the virtual conditions \( \bar{q}, \bar{q}, \bar{p} \) respectively. Work in the model \( V[K][H_0][H_1][H_2] \). Let \( \bar{p} \) be the virtual condition for the set of all conditions in \( P \) which are stronger than all conditions in the filter \( H_2 \). This is indeed a (nonzero) virtual condition in \( P \) by the \( \sigma \)-closedness demand on the poset \( P \); it is also a virtual balanced condition by the genericity of the filter \( H_2 \) and the \( \kappa \)-balance assumption on the poset \( P \).

Let \( f_0 = \bigcup H_0 \) and \( f_1 = \bigcup H_1 \); these are total functions from \( 2^\omega \) to 2. Let \( a_0 = f_0^{-1}\{0\} \) and \( a_1 = f_1^{-1}\{0\} \); these are subsets of \( 2^\omega \).

**Claim 13.1.3.** In \( V[K][H_0][H_1][H_2] \), either \( \text{Coll}(\omega, \kappa) \models \bar{p} \Vdash_P \langle \bar{a}_0, \bar{a}_1 \rangle \in \tau \), or \( \text{Coll}(\omega, \kappa) \models \bar{p} \Vdash_P \langle \bar{a}_1, \bar{a}_0 \rangle \in \tau \).

**Proof.** This is an elementary balance argument with \( \bar{p} \), given the assumption that \( \tau \) is forced to be a tournament on \( \mathbb{F}_2 \)-classes (or countable subsets of \( 2^\omega \)). \( \square \)

Suppose for definiteness that the first alternative of the claim prevails and denote that sentence \( \theta \). Now, return back to \( V[K] \) and record the following.

**Claim 13.1.4.** In \( V[K] \), \( \text{Coll}(\omega, \lambda) \models \langle \bar{q}, \bar{q}, \bar{p} \rangle \Vdash_{Q \times Q \times P} \theta \).

**Proof.** This is an elementary balance argument with \( \langle \bar{q}, \bar{q}, \bar{p} \rangle \), given the fact that the objects \( a_0, a_1 \), and \( \bar{q} \) above are definable in the model \( V[K][G][H_0, H_1, H_2] \) from \( H_0, H_1, H_2 \). \( \square \)

Now, come back to the model \( V[K][G][H_0, H_1, H_2] \) instead of the triple of filters \( H_0, H_1, H_2 \) on \( Q \times Q \times P \) consider the triple of filters \( H_1, H_0, H_2 \) on \( Q \times Q \times P \). It is immediate that they are generic over the model \( V[K][G] \), and they meet the condition \( \langle \bar{q}, \bar{q}, \bar{p} \rangle \) just as much as the triple \( H_0, H_1, H_2 \) does. By Claim 13.1.4 and the forcing theorem in the model \( V[K] \), it must be the case that in \( V[K][G][H_1, H_0, H_2] \), \( \text{Coll}(\omega, \kappa) \models \bar{p} \Vdash_P \langle \bar{a}_1, \bar{a}_0 \rangle \in \tau \) holds. This, however, contradicts the assumption that \( \tau \) is forced to be a tournament. \( \square \)

**Corollary 13.1.5.** 1. Let \( P = P(\omega) \) modulo finite. In the \( P \)-extension of the symmetric Solovay model, there is no tournament on the quotient space of \( \mathbb{F}_2 \);

2. it is consistent relative to an inaccessible limit of inaccessibles that \( ZF + DC \) holds, there is a nonprincipal ultrafilter on \( \omega \), and yet there is no tournament on the quotient space of \( \mathbb{F}_2 \).
There is a more careful argument for Corollary 13.1.5 which removes the need for the strengthening of the large cardinal hypothesis [51]. Theorem 13.1.2 provides many other corollaries, among which we quote the following application of Example 10.5.8:

**Corollary 13.1.6.** Let $E$ be a pinned equivalence relation on a Polish space $X$.

1. Let $P$ be the linearization poset for $E$ as in Example 8.5.5. In the $P$-extension of the Solovay model, there is no tournament on the $\mathbb{P}_2$-quotient space;

2. it is consistent relative to an inaccessible limit of inaccessibles that ZF+DC holds, the $E$-quotient space is linearly ordered, and yet there is no tournament on the $\mathbb{P}_2$-quotient space.

**Theorem 13.1.7.** Let $E$ be an orbit equivalence on a Polish space resulting from a generically turbulent action of some Polish group. In the placid extensions of the Solovay model, there is no tournament on the $E$-quotient space.

**Proof.** The argument depends on a simple abstract property of pairs of independent continuous open maps on Polish spaces of Definition 3.1.4. Let $m \in \omega$ be a number. For each natural number $n \in m$, let $X_n$ be a Polish space, $Y_n$ be Polish spaces and $f_n: X_n \to Y_n$ be continuous open maps. We define the product map $f = \prod_n f_n: \prod_n X_n \to \prod_n Y_n$ by $f(x)(n) = f_n(x(n))$. It is not difficult to check that the product map is continuous.

**Claim 13.1.8.** The set of pairs of independent maps is closed under finite product.

**Proof.** Let $m \in \omega$ be a number. For each number $n \in m$ let $X_n$ be a Polish space, $Y_{0n}, Y_{1n}$ be Polish spaces, and let $f_{0n}: X_n \to Y_{0n}$, $f_{1n}: X_n \to Y_{1n}$ be continuous open maps such that for all but finitely many $n \in \omega$ their ranges are the whole spaces $Y_{0n}, Y_{1n}$ and such that each pair $f_{0n}, f_{1n}$ is independent. Write $X = \prod_n X_n$, $Y_0 = \prod_n Y_{0n}$ and $Y_1 = \prod_n Y_{1n}$. We must prove that the product maps $f_0: X \to Y_0$ and $f_1: X \to Y_1$ are independent.

To this end, let $O \subset X$ be a nonempty open set; thinning out the set $O$ if necessary, we may find open sets $O_n \subset X_n$ for $n \in m$ such that $O = \prod_n O_n$. Let $A_n \subset Y_{0n}$ be an open set witnessing the independence of the pair $f_{0n}, f_{1n}$ for the set $O_n$. We claim that the open set $A = \prod_n A_n \subset Y_0$ witnesses the independence of the pair $f_0, f_1$ for $O$.

To see this, suppose that $B_0, B_1 \subset A$ are nonempty open sets; thinning them out if necessary, we may find open sets $B_{0n}, B_{1n} \subset A_n$ such that $B_0 = \prod_n B_{0n}$ and $B_1 = \prod_n B_{1n}$. For each $n \in m$ use the initial assumption on the set $A_n$ to find a walk $w_n$ of points in $O_n$ such that for the initial point $x_{0n} \in w_n$ we have $f_{0n}(x_{0n}) \in B_0$ and for the last point $x_{1n} \in w_n$ we have $f_{0n}(x_{1n}) \in B_1$.

Now, each walk $w_n$ is a finite string of points; shifting the indexation, we may find successive intervals $I_n$ of natural numbers for $n \in m$ such that each walk $w_n$ is indexed by numbers in $I_n$. Let $K = \bigcup_n I_n$ and consider the string
of points in the open set \( O \) indexed by \( k \in K \), defined in the following way: for each \( k \) write \( n(k) \) for that \( n \in m \) for which \( k \in I_n \), and then define point \( x_k \in O \) by \( x_k(n) = x_{i_n} \) if \( n < n(k) \), \( x_k(n) = w_k(n) \) if \( n(k) = n \), and finally \( x_k(n) = x_{0n} \) if \( n(k) < n \). It is easy to see that the sequence \( \langle x_k : k \in K \rangle \) is the desired walk in the set \( O \) from \( B_0 \) to \( B_1 \).

Now, towards the proof of Theorem 13.1.7, let \( \Gamma \) be a Polish group acting on a Polish space \( X \) in a generically turbulent way, so that \( E \) is the resulting orbit equivalence relation. Consider the space \( \Gamma \times X \times X \) by each \( x \) of points in the open set \( O \) of \( \omega \). Let \( C = \{ \langle \gamma, x_0, x_1 \rangle : \gamma \cdot x_0 = x_1 \} \) and the projection maps \( f_0 : C \to X \) into the \( x_0 \)-coordinate and \( f_1 : C \to X \) into the \( x_1 \)-coordinate. The two maps are independent by Theorem 3.2.2. Let \( g_0, g_1 \) be the product maps \( f_0 \times f_1 \) and \( f_1 \times f_0 \). Thus, \( g_0, g_1 : C \times C \to X \times X \) are continuous open maps, and they form an independent pair by the claim.

Let \( \kappa \) be an inaccessible cardinal and let \( P \) be a Suslin forcing which is placid in all generic extensions by posets of size \( < \kappa \). Let \( W \) be the Solovay model derived from \( \kappa \) and work in the model \( W \). Let \( p \in P \) be a condition and \( \tau \) be a \( P \)-name such that \( p \Vdash \tau \) is a directed graph on the \( E \)-quotient space and for each pair \( \langle c, d \rangle \) of distinct \( E \)-equivalence classes, either \( \langle c, d \rangle \) or \( \langle d, c \rangle \) is in \( \tau \). We must find a condition \( q \leq p \) and distinct \( E \)-classes \( c, d \) such that \( q \Vdash \langle c, d \rangle \in \tau \) and \( \langle d, c \rangle \notin \tau \).

Both \( p, \tau \) are definable in the model \( W \) from a parameter \( z \in 2^\omega \) and parameters in the ground model. Let \( V[K] \) be an intermediate forcing extension obtained by a poset of size \( < \kappa \) containing \( z \) and work in the model \( V[K] \). By the initial assumption on the poset \( P \), there must be a placid virtual condition \( \bar{p} \leq p \). Let \( Q \) be the Cohen forcing on \( X \times X \), adding generic points \( \bar{x}_0, \bar{x}_1 \). There must be an open set \( O \subset X \times X \) forcing the existence of a poset \( \bar{S} \) adding a condition \( \sigma \in P \) such that \( \bar{S} \Vdash \sigma \leq \bar{p} \) and \( \text{Coll}(\omega, \kappa) \Vdash \sigma \Vdash \langle \bar{x}_0, \bar{x}_1 \rangle \in \tau \) (or the whole statement repeated with \( \bar{x}_0, \bar{x}_1 \) interchanged). Since \( \Gamma \)-orbits are dense, it is possible to find open sets \( O_0, O_1 \subset X \) and \( U_0, U_1 \subset \Gamma \) such that \( O_0 \times O_1 \subset O \) and \( (U_0 \cdot O_1) \times (U_1 \cdot O_0) \subset O \).

Now, consider the Cohen poset \( R \) on the space \( C \times C \). Consider the condition \( U \in Q \) consisting of all tuples \( \langle \gamma, x_0, x_1, \delta, x_2, x_3 \rangle \in C \times C \) such that \( \gamma \in U_0 \), \( x_0 \in O_0 \), \( \delta \in U_1 \) and \( x_3 \in O_1 \). Let \( \langle \gamma, x_0, x_1, \delta, x_2, x_3 \rangle \) be an \( R \)-generic tuple over the model \( V[K] \). Note the following:

- \( \langle x_0, x_3 \rangle \) is a pair \( Q \)-generic over \( V[K] \), meeting the condition \( O \);
- similarly, \( \langle x_2, x_1 \rangle \) is a pair \( Q \)-generic over \( V[K] \), meeting the condition \( O \) by the choice of the neighborhoods \( U_0, U_1 \);
- \( x_0 E x_1 \) and \( x_3 E x_2 \);
- \( V[K][x_0, x_3] \cap V[K][x_2, x_1] = V[K] \) by the fact that \( g_0, g_1 \) are independent maps and Theorem 3.1.5.

Let \( S_0 = \bar{S}/x_0, x_3 \) and \( S_1 = \bar{S}/x_2, x_1 \). Let \( H_0 \subset S_0 \) and \( H_1 \subset S_1 \) be filters mutually generic over the model \( V[K][x_i : i \in 4] \). By the third item above and
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a mutual genericity argument, \( V[K][x_0, x_3][H_0] \cap V[K][x_2, x_1][H_1] = V[K] \). Let \( p_0 = \sigma/H_0 \) and \( p_1 = \sigma/H_1 \). By the placidity of the virtual condition \( \bar{p} \), the conditions \( p_0, p_1 \) have a lower bound \( q \in P \). The forcing theorem finally shows that in the model \( W \), \( q \vDash \) both \( \langle [x_0]_E, [x_2]_E \rangle \) and \( \langle [x_2]_E, [x_0]_E \rangle \) belong to \( \tau \). \( \square \)

Now, Example 9.3.15 together with Theorem 13.1.7 yields:

**Corollary 13.1.9.** Let \( E \) be a Borel equivalence relation on a Polish space induced by a generically turbulent Polish group action. Let \( F \) be a Borel equivalence relation on a Polish space, classifiable by countable structures.

1. Let \( P \) be the linearization poset for \( F \) as in Example 8.5.5. In the \( P \)-extension of the Solovay model, there is no tournament on the \( E \)-quotient space;
2. it is consistent relative to an inaccessible cardinal that \( ZF + DC \) holds, the \( F \)-quotient space is linearly ordered, and yet there is no tournament on the \( E \)-quotient space.

In some situations, we can exclude the existence of orientations of graphs smaller than the complete graph on the quotient space \( X/E \). An interesting example occurs when \( X = 2^\omega \), \( E \) is the modulo \( I \)-equality for some Borel ideal \( I \) on \( \omega \) containing all singletons, and \( \Gamma \) is the associated complement graph connecting (the equivalence classes of) sets \( x_0, x_1 \in 2^\omega \) if \( x_0 = 1 - x_1 \) modulo \( I \). A straightforward Baire category argument shows that \( \Gamma \) has no Borel orientation; this feature may persist to certain balanced extensions of the symmetric Solovay model as well. One reason to look at this issue is that an ultrafilter \( U \) disjoint from the ideal \( I \) yields an orientation of the complement graph: let \( \vec{x} \) be the set of all edges \( \langle x_0, x_1 \rangle \) in which \( x_0 = 1 - x_1 \) and \( x_0^{-1}\{0\} \in U \). Thus, excluding the orientation of the complement graph excludes the existence of an ultrafilter disjoint from the ideal \( I \).

**Theorem 13.1.10.** Let \( P \) be the poset of infinite subsets of \( \omega \) ordered by inclusion. In the \( P \)-extension of the symmetric Solovay model, the complement graph associated with the summable ideal on \( \omega \) cannot be oriented.

**Proof.** Write \( I \) for the summable ideal on \( \omega \); i.e. \( I \) consists of sets \( a \subset \omega \) such that the sum \( \Sigma \{ \frac{1}{n+1} : n \in a \} \) is finite. Towards contradiction, let \( \kappa \) be an inaccessible cardinal, let \( W \) be a symmetric Solovay model derived from \( \kappa \), let \( p \in P \) be a condition and \( \tau \) be a \( P \)-name such that \( p \Vdash \tau \) is an orientation of the complement graph associated with \( I \). Both \( p, \tau \) must be definable from some real parameter \( z \in 2^\omega \) and some ground model parameters. Let \( V[K] \) be an intermediate extension of the ground model by a poset of size \( < \kappa \) containing the parameter \( z \), and work in the model \( V[K] \).

We will produce a poset \( Q \) and a \( Q \)-name \( \dot{x}_{gen} \) for an element of \( 2^\omega \) such that

(I) below every condition \( q \in Q \) there are conditions \( q_0, q_1 \leq q \) and an automorphism \( \pi : Q \upharpoonright q_0 \rightarrow Q \upharpoonright q_1 \) such that \( \pi(\dot{x}_{gen}) \) is forced by \( q_1 \) to be equal to \( 1 - \dot{x}_{gen} \) modulo \( I \);
(II) the ultrafilter added by $P$ still generates an ultrafilter in the $P \times Q$-extension.

Once this is done, let $G \subset P$ and $H \subset Q$ be mutually generic filters over the model $V[K]$, with $p \in G$, and work in $V[K][G][H]$. Let $x = \check{x}_{\text{gen}} / H$. Let $U$ be the ultrafilter on $\omega$ generated by the filter $G$; this is possible by the item (II) above. By Theorem 7.1.2, $U$ is a balanced virtual condition in the poset $P$ in the model $V[K][G][H]$. The following is proved by a standard argument using the balance of $U$:

**Claim 13.1.11.** In the model $V[K][G][H]$, either $\text{Coll}(\omega, < \kappa) \vDash U \upharpoonright_P \langle x, 1-x \rangle \in \tau$, or $\text{Coll}(\omega, < \kappa) \vDash U \upharpoonright_P \langle 1-x, x \rangle \in \tau$ holds.

Suppose for definiteness that the former alternative prevails. Find a condition $\pi: Q \downarrow q_0 \to Q \downarrow q_1$ as in item (I). Find a filter $H_0 \subset Q$ generic over $V[K][G]$ meeting the condition $q_0$, and let $H_1 = \pi''H_0$. Let $x_0, x_1 \in 2^{\omega}$ be the points associated with the filters $H_0, H_1$. Then the models $V[K][G][H_0]$ and $V[K][G][H_1]$ are equal, and by the forcing theorem applied in $V[K][G]$, it must be true in $W$ that $U \upharpoonright_P \langle x_0, 1-x_0 \rangle \in \tau$, $\langle x_1, 1-x_1 \rangle \in \tau$. This is impossible though as $x_0 = 1 - x_1$ modulo $I$ and $\tau$ is forced to be a tournament.

The remainder of the proof consists of the construction of the poset $Q$ satisfying (I) and (II). This is a pure ZFC construction in which the concentration of measure phenomenon is a critical ingredient. Let $(J_n: n \in \omega)$ be a very fast sequence of successive intervals on $\omega$. The following is the concentration of measure type of demand on the intervals in question that we need. Let $\mu_n$ be the normalized counting measure on $2^{J_n}$. Let $d_n$ be the metric on $2^{J_n}$ defined by $d_n(x, y) = \Sigma\{\frac{1}{m+1}: x(m) \neq y(m)\}$. We demand that the following holds for every $n \in \omega$:

(III) for every set $a \subset 2^{J_n}$ of $\mu_n$-mass $1/n$, the $2^{-n}$-neighborhood of $a$ in $2^{J_n}$ in the sense of the metric $d_n$ has $\mu_n$-mass greater than $1/2$.

The concentration of measure computations in [60, Theorem 4.3.19] show that such a fast sequence of intervals indeed exists. Now, let $T_{\text{ini}}$ be the tree of all finite sequences $t$ such that for each $n \in \text{dom}(t)$, $t(n) \in 2^{J_n}$. Let $Q$ be the partial order of all nonempty trees $q \subset T_{\text{ini}}$ without endnodes such that for each $m \in \omega$ there is $n_m \in \omega$ such that for each $t \in Q$ of length $n > n_m$, the set $\{x \in 2^{J_n}: t \upharpoonright (x) \in q\}$ has $\mu_n$-mass $\geq m/n$. The ordering on $Q$ is that of inclusion. Let $\check{x}_{\text{gen}}$ be the $Q$-name for the concatenation of the trunks of the trees in the generic filter. We must show that items (I) and (II) above hold for $Q$ and $\check{x}_{\text{gen}}$.

Item (I) is where the concentration of measure shows up in force. We will in fact show that whenever $q_0, q_1 \in Q$ are arbitrary conditions then there are conditions $q_0' \leq q_0, q_1' \leq q_1$ and an isomorphism $\pi: Q \downarrow q_0' \to Q \downarrow q_1'$ such that $\pi(1 - \check{x}_{\text{gen}})$ is forced by $q_1'$ to be modulo $I$ equivalent to $\check{x}_{\text{gen}}$. To see this, let $t_0 \in q_0, t_1 \in q_1$ be nodes of the same length such that for every node $t \in q_0$ of
length \( n \geq |t_0| \), the set \( a_0(t) = \{ x \in 2^{J_n} : t^\prec x \in q_0 \} \) has \( \mu_n \)-mass at least \( 2/n \), and for every node \( t \in q_1 \) of length \( n \geq |t_0| \), the set \( a_1(t) = \{ x \in 2^{J_n} : t^\prec x \in q_1 \} \) has \( \mu_n \)-mass at least \( 2/n \) as well. We will produce a tree \( q'_0 \leq q_0 \upharpoonright t_0 \) in \( Q \) and a level and order preserving injection \( \pi : q'_0 \to q_0 \upharpoonright t_1 \) so that

\[
\text{(IV) for each node } t \in q'_0 \text{ and a number } n > |t_0| \text{ in } \text{dom}(t), \quad d_n(t(n), 1 - \pi(t)(n)) < 2^{-n+1}.
\]

Write \( q'_0 = \pi^* q_0 \). Since the measures \( \mu_n \) are normalized counting measures, the map \( \pi \) naturally extends to an isomorphism \( \pi : Q \upharpoonright q'_0 \to Q \upharpoonright q'_1 \). The demand (IV) then implies that \( \pi(1 - \dot{x}_{\text{gen}}) \) is forced by \( q'_1 \) to be modulo \( I \) equivalent to \( \dot{x}_{\text{gen}} \) as required.

The map \( \pi \) is obtained by a hungry algorithm. Among all level and order preserving injections from subsets of \( q_0 \upharpoonright t_0 \) to \( q_1 \upharpoonright t_1 \) which satisfy (IV), select an inclusion maximal one and call it \( \pi \). It will be enough to show that \( q'_0 = \text{dom}(\pi) \) belongs to \( Q \). To see this, first of all \( q'_0 \) is closed under initial segment by an obvious maximality argument. To verify the branching condition in the definition of \( Q \) for \( q'_0 \), let \( t \in q'_0 \) of length some \( n \geq |t_0| \). It will be enough to argue that the set \( \{ x \in 2^{J_n} : t^\prec x \in q'_0 \} \) has \( \mu_n \)-mass at least \( \min(\mu_n(a_0(t)), \mu_n(a_1(\pi(t)))) - 1/n \).

To do this, note that if both sets \( b_0 = \{ x \in 2^{J_n} : t^\prec x \in q'_0 \setminus q_0 \} \) and \( b_1 = \{ x \in 2^{J_n} : t^\prec x \in \pi(q'_0) \setminus q_1 \} \) had \( \mu_n \)-mass greater than \( 1/n \), then by (III) the \( 2^{-n} \)-neighborhood of \( b_0 \) and the set \( \{ 1 - x : x \in b_1 \} \) would both have \( \mu_n \)-mass greater than \( 1/2 \) and so they would intersect, making it possible to extend \( \pi \) while satisfying (IV) and contradicting the maximality of \( \pi \).

Item (II) follows from a density argument on \( P \) and \( Q \) given the fact that the poset \( Q \) is proper and does not add independent reals [85, Theorem 4.4.8].

**Corollary 13.1.12.** 
1. Let \( P = \mathcal{P}(\omega) \) modulo finite. In the \( P \)-extension of the symmetric Solovay model, there is no ultrafilter on \( \omega \) disjoint from the summable ideal;

2. it is consistent relative to an inaccessible cardinal that \( \text{ZF} + \text{DC} \) holds, there is a nonprincipal ultrafilter on \( \omega \), and yet there is no ultrafilter disjoint from the summable ideal.

The complicated combinatorial properties of the summable ideal were clearly key in the proof of Corollary 13.1.12. One may wonder if the same result can be achieved for other ideals \( I \) on \( \omega \).

**Question 13.1.13.** Characterize the class of Borel ideals \( I \) on \( \omega \) such that \( \text{ZF} + \text{DC} \) proves that if there is a nonprincipal ultrafilter on \( \omega \) then there is one disjoint from \( I \). Characterize the class of Borel ideals \( I \) on \( \omega \) such that in the \( \mathcal{P}(\omega) \) modulo finite extension of the symmetric Solovay model there is an ultrafilter disjoint from \( I \).
13.2 Excluding linear orders

It turns out that in many balanced extensions of the symmetric Solovay model there are no linear orderings of quotient spaces for the reason of centeredness of the balanced poset.

**Definition 13.2.1.** Let \( n < m \leq \omega \) be arbitrary. A poset \( P \) is \( m,n \)-centered if whenever \( a \subset P \) is a set of size \( m \) such that any subset of \( a \) of size \( n \) has a common lower bound, then \( a \) has a common lower bound.

Note that \( m,n \)-centeredness of Suslin posets is a \( \Pi^1_2 \) statement and therefore absolute among all forcing extensions. As usual in all balanced poset classes in this book, the class of \( m,n \)-centered posets is closed under countable product.

The following central application is stated using Convention 1.8.17.

**Theorem 13.2.2.** In 3, 2-centered, adequate forcing extensions of the symmetric Solovay model, the following are equivalent:

1. there is an \( \mathbb{E}_0 \)-transversal;

2. the \( \mathbb{E}_0 \)-quotient space is linearly ordered.

In other words, to linearly order the \( \mathbb{E}_0 \)-quotient space by a 3, 2-centered balanced Suslin forcing, it is necessary to add a transversal. Note that the transversal poset of Example 6.3.8 is indeed 3, 2-centered.

**Proof.** Clearly, (1) implies (2) in ZF: given an \( \mathbb{E}_0 \)-transversal, the \( \mathbb{E}_0 \)-quotient space can be linearly ordered by the lexicographical ordering on the representatives of the \( \mathbb{E}_0 \). The implication (2)→(1) is the heart of the matter. Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that in \( V_\kappa \), \( P \) is 3, 2-centered, adequate and balanced in every forcing extension. Let \( W \) be the symmetric Solovay model derived from \( \kappa \). Work in in the model \( W \). Suppose that there is a condition \( p \in P \) forcing the failure of (1), i.e. there is no \( \mathbb{E}_0 \)-transversal. Suppose also that there is a \( P \)-name \( \tau \) for a tournament on the \( \mathbb{E}_0 \). We will produce a condition \( q \leq p \) and an oriented 3-cycle on the \( \mathbb{E}_0 \)-quotient space such that \( q \Vdash \) the cycle is a subset of \( \tau \). Thus, \( p \Vdash \tau \) is not a name for a linear ordering, proving the failure of (2). There is a parameter \( z \in 2^\omega \) such that all of \( p, \tau \) is definable from \( p \) and parameters in the ground model in the model \( W \).

**Claim 13.2.3.** There is an intermediate model \( V[K] \) and in \( V[K] \) objects \( \bar{p}, R, \eta, \sigma \) such that

1. \( z \in V[K] \) and \( \bar{p} \leq p \) is a balanced virtual condition in \( V[K] \);

2. in \( V[K] \), \( R \) is a poset of size \( < \kappa \), \( \eta \) is an \( R \)-name for an element of \( 2^\omega \setminus V[K] \), and \( \sigma \) is an \( R \)-name for a balanced virtual condition below \( \bar{p} \);
3. the poset $R$ forces that for every condition $r \in R$ there is a filter $H' \subset R$ generic over $V[K]$ containing $r$ such that $V[K][H] = V[K][H']$, $\eta/H, \eta/H'$ are $E_0$-equivalent elements of $2^\omega$ and $\sigma/H, \sigma/H'$ are equivalent virtual conditions in $P$, where $H \subset R$ is the generic filter.

Proof. Let $G \subset P$ be a filter generic over the model $W$ containing the condition $p$. Working in $W[G]$, whenever $x \in 2^\omega$ is a point, consider the model $M_x$ of all elements of $W[G]$ which are hereditarily definable from $G, z, x$ and parameters in the ground model. Note that the construction of the model $M_x$ depends only on the $E_0$-class of $x$ and not on $x$ itself. Also, consider the model $N_x$ of all elements of $W[G]$ which are hereditarily definable from $G, z, [x]_{E_0}$ and parameters in the ground model. Clearly, $N_x \subseteq M_x$ are models of ZFC, and so by the balanced assumption and Theorem 9.1.1 both are present already in $W$.

Case 1. There is an $x \in 2^\omega$ such that $N_x \not\subseteq M_x$. This means that $x \notin M_x$. Use the adequacy assumption to argue as in Claim 10.2.3 that there are balanced virtual conditions $\bar{p}, \bar{q} \in N_x$ and $\bar{p} \in M_x$ in the filter $G$; necessarily $\bar{p} \leq \bar{q} \leq p$. The model $M_x$ is a generic extension of $N_x$, so $M_x = N_x[H]$ for some filter $H \subset R$ generic over $N_x$; let $\eta, \sigma \in N_x$ be some names such that $\eta/H = x$ and $\sigma/H = \bar{p}$. Let $D = \{r \in R : \text{there is a filter } H' \subset R \text{ containing } r, \text{ and such that } M_x[H'] = M_x, \eta/H' \text{ is } E_0\text{-related to } x \text{ and } \sigma/H' \text{ is a virtual condition equivalent to } \bar{p}\}$. The definition of the set $D$ uses only $[x]_{E_0}$ and so shows that $D \in N_x$. Clearly, $H \subset D$. By a genericity argument, it must be the case that $D$ is dense below some condition $r \in R$. A review of the definitions shows that the objects $N_x, \bar{p}, R \upharpoonright r, \eta, \sigma$ satisfy the statement of the claim.

Case 2. Case 1 fails. This means that for all $x \in 2^\omega$ it is the case that $x \in N_x$. Now, $N_x$ has a canonical HOD-type well-ordering which depends only on the equivalence class $[x]_{E_0}$. Thus, one can form the set $A = \{x \in 2^\omega : x \text{ is the first element of its } E_0\text{-class in the canonical well-ordering of } N_x\}$ and argue that $A \subset 2^\omega$ is an $E_0$-transversal. This contradicts the original assumption on the condition $p$. \hfill $\square$

Let $V[K], \bar{p}, R, \eta, \sigma$ be objects as in Claim 13.2.3.

Claim 13.2.4. In $V[K]$ the poset $R \times R \times \text{Coll}(\omega, < \kappa)$ forces the following:

1. $\sigma_{\text{right}}$ and $\sigma_{\text{left}}$ are compatible in $P$;
2. there is a lower bound of the two forcing $\langle \eta_{\text{right}}, \eta_{\text{left}} \rangle \in \tau$;
3. there is also a lower bound of the two forcing $\langle \eta_{\text{left}}, \eta_{\text{right}} \rangle \in \tau$.

Proof. (1) is immediate from the balance of the condition $\bar{p}$. The proofs of (2, 3) are symmetric; we prove (2). Suppose towards contradiction that (2) fails, as forced by some condition $\langle r_0, r_1 \rangle \in R \times R$. Let $H_0, H_1 \subset R$ be filters mutually generic over the model $V[K]$ with $r_0 \in H_0$ and $r_1 \in H_1$. The properties of $R, \eta, \sigma$ from Claim 13.2.3 show that in $V[K][H_0]$ there is a filter $H'_1 \subset R$ such that $H'_1$ is generic over $V[K]$, contains $r_1$ and such that $\eta/H_0, \eta/H'_1$ are $E_0$-related points in $2^\omega$ and $\sigma/H_0, \sigma/H'_1$ are equivalent virtual conditions in $P$. Also, in $V[K][H_1]$
there is a filter $H_0' \subset R$ such that $H_0'$ is generic over $V[K]$, contains $r_0$ and such that $\eta/H_0, \eta/H_0'$ are $\mathbb{E}_0$-related points in $2^\omega$ and $\sigma/H_1, \sigma/H_0'$ are equivalent virtual conditions in $P$. Now, the filters $H_0', H_1' \subset R$ are mutually generic over $V[K]$ by Corollary 1.8.9. The forcing theorem and the failure of (2) then show that in the model $W$ every condition below $\sigma/H_0$ and $\sigma/H_1$ simultaneously forces in $P$ that $\langle \eta/H_0, \eta/H_1 \rangle \in \tau$ and $\langle \eta/H_0', \eta/H_1' \rangle \in \tau$. This is impossible as $\eta/H_0$ is $\mathbb{E}_0$-related to $\eta/H_1'$ and $\eta/H_1$ is $\mathbb{E}_0$-related to $\eta/H_0'$.

Now, let $H_0, H_1, H_2 \subset R$ be filters mutually generic over the model $V[K]$, let $x_i = \eta/H_i$ and $p_i = \sigma/H_i$ for all $i \in 3$. The points $x_i \in 2^\omega$ for $i \in 3$ are pairwise non-$\mathbb{E}_0$-equivalent, and $p_i$ for $i \in 3$ are pairwise compatible virtual conditions in $P$. Let $\lambda < \kappa$ be a sufficiently large cardinal and let $G_0, G_1, G_2 \subset \text{Coll}(\omega, \lambda)$ be filters mutually generic over the model $V[K][H_i; i \in 3]$. In the model $V[K][H_0, H_1][G_2]$ find a lower bound $q_2 \in P$ of $p_0, p_1$ such that $q_2 \not\vdash \langle x_0, x_1 \rangle \in \tau$. In the model $V[K][H_1, H_2][G_0]$ find a lower bound $q_0 \in P$ of $p_1, p_2$ such that $q_0 \not\vdash \langle x_1, x_2 \rangle \in \tau$. In the model $V[K][H_2, H_0][G_1]$ find a lower bound $q_1 \in P$ of $p_2, p_0$ such that $q_1 \not\vdash \langle x_2, x_0 \rangle \in \tau$. This is possible by Claim 13.2.4(2, 3).

Observe that the conditions $q_2, q_0 \in P$ are compatible by the balance of $p_1$ in the model $V[K][H_1]$: they strengthen the condition $p_1$, and they belong to the respective extensions $V[K][H_1][H_0][G_2]$ and $V[K][H_1][H_2][G_0]$ which are mutually generic over $V[K][H_1]$. Similarly, the conditions $q_0, q_1$ are compatible, and the conditions $q_1, q_2$ are compatible. By the centeredness assumption on the poset $P$, the set $\{q_0, q_1, q_2\}$ has a common lower bound $q \in P$. Then $q$ forces that the ($\mathbb{E}_0$-classes of) points $x_0, x_1, x_2$ form an oriented 3-cycle in $\tau$ as desired.

\textbf{Corollary 13.2.5.} Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the tournament poset associated with $E$ as in Example 8.5.6. In the $P$-extension of a symmetric Solovay model, there is no linear ordering on the $\mathbb{E}_0$-quotient space.

2. It is consistent relative to an inaccessible cardinal that $\text{ZF} + \text{DC}$ holds, there is a tournament on the $\mathcal{E}_0$-quotient space while there is no linear ordering on the $\mathbb{E}_0$-quotient space.

\textit{Proof.} The poset $P$ is balanced by Theorem 8.5.4, adequate by Example 10.5.8, it does not add an $\mathbb{E}_0$-selector by Example 9.2.11, and it is 3, 2-centered by its definition. Thus, Theorem 13.2.2 applies.

\textbf{Corollary 13.2.6.} Let $E$ be a countable Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the poset adding an action of $\mathbb{Z}$ on $X$ inducing $E$ as in Example 6.3.10. In the $P$-extension of the symmetric Solovay model, there is no linear ordering on the $\mathbb{E}_0$-quotient space.
2. It is consistent relative to an inaccessible cardinal that \( ZF+DC \) holds, \( E \) is an orbit equivalence relation of a (discontinuous) \( \mathbb{Z} \)-action and there is no linear ordering on the \( E_0 \)-quotient space.

Proof. The poset \( P \) is balanced as proved in the initial Example 6.3.10. It is adequate by Example 10.5.6, it does not add an \( E_0 \)-selector by Corollary 11.3.12, and it is 3,2-centered by its definition. Thus, Theorem 13.2.2 applies.

The above posets are in fact \( \omega, 2 \)-centered, and the proof of Theorem 13.2.2 then with only trivial changes implies that every tournament on the \( E_0 \)-quotient space embeds every countable tournament. The following example appears not to have this feature.

**Corollary 13.2.7.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \) of coloring number \( n \in \omega \).

1. Let \( P \) be the coloring poset for \( \Gamma \) for \( 4n-3 \)-many colors. In the \( P \)-extension of the symmetric Solovay model the \( E_0 \)-quotient space cannot be linearly ordered.

2. It is consistent relative to an inaccessible cardinal that \( ZF+DC \) holds, the chromatic number of \( \Gamma \) is \( \leq 4n-3 \), and the \( E_0 \)-quotient space cannot be linearly ordered.

Proof. The poset \( P \) is balanced by Theorem 8.1.2, it is adequate by Example 10.5.4. It does not add an \( E_0 \)-selector by Example 9.2.19. The poset \( P \) is 3,2-centered and this is secured by the increase of the number of colors to \( 4n-3 \). Namely, the dense set of \( P \) isolated in the proof of Theorem 8.1.2 then has the feature that conditions \( p_0, p_1, p_2 \) have a lower bound if and only if \( p_0 \cup p_1 \cup p_2 \) is a \( \Gamma \)-coloring; the 3, 2-centeredness of \( P \) follows. Thus, Theorem 13.2.2 applies.

We conclude this section with a rather simple ZF question and observation. Which Borel quotient spaces can be linearly ordered given that a nonprincipal ultrafilter on \( \omega \) exists? In the positive direction, we have a humble ZF+DC result:

**Proposition 13.2.8.** (ZF+DC) Suppose that there is a nonprincipal ultrafilter on \( \omega \). The class of analytic equivalence relations whose quotient space can be linearly ordered is closed under the following operations:

1. Borel reducibility;
2. countable product;
3. increasing union.

Proof. The first two items do not use the ultrafilter assumption. We will argue for the third item. Let \( X \) be a Polish space and \( \langle E_n : n \in \omega \rangle \) be an increasing sequence of analytic equivalence relations on \( X \), each with linearly orderable
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quotient space. Use the DC assumption to pick linear orders $\leq_n$ for each of them. Let $U$ be a nonprincipal ultrafilter on $\omega$. If $x, y \in X$ are $E$-unrelated elements, then let $[x]_E \leq [y]_E$ if the set $a(x, y) = \{n \in \omega : [x]_{E_n} \leq_n [y]_{E_n}\}$ belongs to the ultrafilter $U$. If $x' E x$ and $y' E y$ are different representatives of the $E$-classes of $x, y$, then there is a number $m \in \omega$ such that $x E_m x'$ and $y E_m y'$; therefore, the symmetric difference $a(x, y) \Delta a(x', y')$ is a subset of $m$, and the definition of $\leq$ does not depend on the choice of the equivalence class representatives. It is easy to check that $\leq$ is a linear ordering on the $E$-classes.

In particular, the $E_0$- and $E_1$-quotient spaces can be linearly ordered given a nonprincipal ultrafilter. In the negative direction, note that no balanced poset for adding a nonprincipal ultrafilter on $\omega$ is 3.2-centered by Theorem 14.3.2 below, as any nonprincipal ultrafilter yields a discontinuous homomorphism from the Cantor group to the group on two elements. Thus, Theorem 13.2.2 does not offer any insight into the possible limitations of linear orderability of quotient spaces in models which contain a nonprincipal ultrafilter. Certain limitations can be inferred from the results of the previous section: letting $P = \mathcal{P}(\omega)$ modulo finite, then in the $P$-extension of the symmetric Solovay model there is a nonprincipal ultrafilter, yet the $F_2$- and $E_2$-quotient spaces carry no tournament so no linear order by Corollaries 13.1.5 and 13.1.12 respectively. The following remains open:

**Question 13.2.9.** In the $P$-extension of the symmetric Solovay model, are the following equivalent for a Borel equivalence relation $E$ on a Polish space?

1. the $E$-quotient space can be linearly ordered;
2. $E$ belongs to the smallest class of equivalence relations containing the identity on $2^{\omega}$ and closed under Borel reducibility, countable product, and increasing union.

### 13.3 Rigidity

The structures on quotient spaces added by the countable approximation forcings are rigid in the resulting extensions unless the automorphism is built into their definition. In absence of a general theorem in this direction, we prove a strongly suggestive and informative special case.

**Theorem 13.3.1.** Let $E$ be a Borel equivalence relation on a Polish space $X$ with uncountably many classes, and let $P$ be the linearization poset for the $E$-quotient space as isolated in Example 8.5.5. In the $P$-extension of the symmetric Solovay model, the linear order introduced by $P$ is rigid.

**Proof.** Let $\kappa$ be an inaccessible cardinal, let $W$ be an inaccessible cardinal derived from $\kappa$ and work in $W$. Suppose towards contradiction that $p \in P$ is a condition and $\tau$ is a $P$-name for a nontrivial automorphism of the generic linear ordering on $X/E$. Strengthening $p$ if necessary, we may find non-$E$-related
points \( x_0, x_1 \in X \) such that \( p \models \tau([x_0]_E) = [x_1]_E \). The objects \( p, \tau, x_0, x_1 \) are all definable from some parameter \( z \in 2^\omega \) and parameters in the ground model. Let \( V[K] \) be an intermediate generic extension of the ground model by a poset of size \( < \kappa \) such that \( z \in V[K] \). Work in the model \( V[K] \).

Let \( \bar{p} \) be a linear ordering of the virtual \( E \)-quotient space \( X^{**} \) such that \( p \subset \bar{p} \); note that \( \bar{p} \) is a virtual balanced condition in the poset \( P \). The following claim is proved via a standard balance argument.

**Claim 13.3.2.** For every virtual \( E \)-class \( c \in X^{**} \) there are virtual \( E \)-classes \( d, e \in X^{**} \) such that \( \Coll(\omega, < \kappa) \models \bar{p} \Vdash \tau(d) = c \) and \( \tau(e) = e \).

Consider the function \( g : X^{**} \to X^{**} \) which to each virtual \( E \)-class \( c \) assigns the virtual class \( e \) such that \( \Coll(\omega, < \kappa) \models \bar{p} \Vdash \tau(e) = e \). In view of the claim, this is a well-defined permutation of \( X^{**} \). Moreover, since \( \Coll(\omega, < \kappa) \models \bar{p} \Vdash g \subset \tau \), it must be the case that \( g \) is in fact an automorphism of the linear ordering \( \bar{p} \).

Now, for definiteness assume that \( [x_0]_E \) is smaller in the \( \bar{p} \)-ordering than \( [x_1]_E \). Decompose \( X^{**} \) into a union of two \( \bar{p} \)-convex sets \( C_0, C_1 \) such that 
\[ [x_0]_E \in C_0 \text{ and } [x_1]_E \in C_1. \]
Set \( D_0 = g''C_0 \) and \( D_1 = g''C_1 \); these are again \( \bar{p} \)-convex sets, and \( [x_1]_E \in D_0 \setminus C_0 \). By a density argument, there must be a poset \( R \) of size \( < \kappa \), an \( R \)-names \( \sigma \) for a countable linear ordering on the \( X/E \)-space containing the (realization of) the \( \bar{p} \)-ordering, an \( R \)-name \( \eta \) for an element of \( \dom(\sigma) \) whose \( E \)-class is in the ordering \( \sigma \) strictly between the (realizations of) the convex sets \( C_0, C_1 \), and an \( R \)-name \( \chi \) for an element of \( \dom(\sigma) \) such that \( R \Vdash \Coll(\omega, < \kappa) \models \sigma \Vdash \tau([\eta]_E) = [\chi]_E \). In \( W \), let \( H_0, H_1 \subset R \) be filters mutually generic over the model \( V[K] \), and work in the model \( V[K][H_0, H_1] \).

Let \( p_0 = \sigma/H_0 \) and \( p_1 = \sigma/H_1 \) be the resulting countable linear orders, and let \( y_0 = \eta/H_0, y_1 = \eta/H_1 \), and \( z_0 = \chi/H_0, z_1 = \chi/H_1 \). Note that the intersection of domains of \( p_0, p_1 \) consists precisely of the realizations of virtual \( E \)-classes in \( V[K] \), and on this set the linear orders \( p_0, p_1 \) agree and are equal to \( \bar{p} \).

Let \( C_0 \) be the set of realizations of virtual \( E \)-classes in the set \( C_0 \), and similarly for \( C_1, D_0, D_1 \). Let \( q \) be a linear ordering on \( \dom(p_0) \cup \dom(p_1) \) satisfying the following demands:

- \( q \upharpoonright \dom(p_0) = p_0 \) and \( q \upharpoonright \dom(p_1) = p_1 \);
- all elements of \( \dom(p_0) \) between \( C_0 \) and \( C_1 \) come before all elements of \( \dom(p_1) \) between \( C_0 \) and \( C_1 \);
- on the other hand, all elements of \( \dom(p_0) \) between \( D_0 \) and \( D_1 \) come after all elements of \( \dom(p_1) \) between \( D_0 \) and \( D_1 \).

This is possible since there are no realizations of virtual \( E \)-classes of \( V[K] \) between \( C_0 \) and \( C_1 \), and similarly there are no such classes between \( D_0 \) and \( D_1 \). Also, \([x_1]_E \in D_0 \setminus C_0 \), and so the last two items do not conflict with each other.

In the model \( W \), the forcing theorem shows that \( q \Vdash \tau([y_0]_E) = [z_0]_E \) and \( \tau([y_1]_E) = [z_1]_E \). However, the last two items above show that \([y_0]_E < [y_1]_E < [z_1]_E < [z_0]_E \) in \( q \), contradicting the assumption that \( \tau \) is forced to preserve the generic linear ordering. \( \square \)
Question 13.3.3. Is it consistent with ZF+DC to have a linear ordering of $E_0$-quotient space but no rigid linear ordering? Does it occur in the $\mathcal{P}(\omega)$ modulo finite extension of the symmetric Solovay model?
Chapter 14

Other combinatorics

The technology of balanced conditions can be applied to prove general theorems about lack of other combinatorial objects in the extensions under discourse. In this section, we include several theorems that are hopefully elegant axiomatizations and generalizations of earlier results in [51].

14.1 Maximal almost disjoint families

One rather surprising limitation of balanced extensions of the symmetric Solovay model is that they contain no maximal almost disjoint families of subsets of \( \omega \).

The following theorem is stated using Convention 1.8.17.

**Theorem 14.1.1.** In balanced extensions of the symmetric Solovay model there are no infinite maximal almost disjoint families of subsets of \( \omega \).

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that in \( V_\kappa \models P \) is balanced in every forcing extension. Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in \( W \). Suppose towards contradiction that there is a condition \( p \in P \) and a name \( \tau \) such that \( p \forces \tau \) is an infinite maximal almost disjoint family. The condition \( p \) as well as the name \( \tau \) must be definable from some parameter \( z \in 2^\omega \) and some parameters from the ground model. Use Fact 1.8.15 and the assumptions to find an intermediate generic extension \( V[K] \) such that \( z \in V[K] \) and \( P \) is balanced in \( V[K] \). Work in the model \( V[K] \).

Let \( q \leq p \) be a balanced condition. Let \( I \) be the set \( \{ a \subset \omega : \text{ for some poset } R_a \text{ of size } < \kappa \text{ and some } R_a \text{-name } \sigma_a \text{ for a condition in } P \text{ stronger than } q \text{ such that } R_a \models \text{Coll}(\omega, < \kappa) \models \sigma_a \models P, \hat{a} \in \tau \} \).

**Claim 14.1.2.** \( \omega \) cannot be covered by a finite set and finitely many elements of the set \( I \).

**Proof.** Suppose that \( J \subset I \) is a finite set such that \( \bigcup J \subset \omega \) is co-finite. Let \( H_a \subset R_a \) for \( a \in J \) be filters mutually generic over \( V[K] \) and let \( r_a = \sigma_a/H_a \in P \). By the balance of the condition \( q \), it follows that the set \( \{ r_a : a \in J \} \subset P \).
has a lower bound, denote it by $\bar{p}$. Since the model $W$ is a symmetric Solovay extension of each of the models $V[K][H_a]$, the forcing theorem applied in the model $V[K][H_a]$ shows that in $W$, for each $a \in J$, the condition $\bar{p}$ forces $J \subseteq \tau$. However, since $\bigcup J \subseteq \omega$ is cofinite, it has to be the case that $\bar{p} \not\vDash J = \tau$, contradicting the initial assumptions on the name $\tau$.  

Let $U$ be a nonprincipal ultrafilter on $\omega$ disjoint from $I$. There must be a poset $R$ of size $< \kappa$, an $R$-name $\eta$ for an infinite subset of $\omega$ which is modulo finite included in all sets in $U$, an $R$-name $\chi$ for a subset of $\omega$ and an $R$-name $\sigma$ for an element of $P$ stronger than $q$ such that $R \vDash \chi \cap \eta$ is infinite and $R \vDash \text{Coll}(\omega, < \kappa) \vDash \sigma \vDash P. \chi \in \tau$. This occurs since $\tau$ is forced to be a maximal almost disjoint family and therefore must contain an element with infinite intersection with $\eta$.

Now, let $H_0, H_1 \subset R$ be filters mutually generic over the model $V[K]$. Let $\lambda_0 = \chi/H_0, \lambda_1 = \chi/H_1 \in P(\omega)$, and $r_0 = \sigma/H_0, r_1 = \sigma/H_1 \in P$. By the balance of the condition $q \leq p$ it must be the case that $r_0, r_1 \in P$ have a lower bound, denote it by $\bar{p}$. Since $W$ is the symmetric Solovay extension of each of the models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem applied in these models shows that $W$ satisfies that $\bar{p}$ forces both $\lambda_0$ and $\lambda_1$ into $\tau$. The proof will be complete if we show that $\lambda_0 \neq \lambda_1$ and $\lambda_0, \lambda_1$ have infinite intersection.

For $\lambda_0 \neq \lambda_1$, observe that neither $\lambda_0, \lambda_1$ can belong to the model $V[K]$. If, say, $\lambda_0 \in V[K]$ then $R$ witnesses the fact that $\lambda_0 \in I$, and consequently $\lambda_0$ must have both finite and infinite intersection with $\eta/H_0$, a contradiction. Now, since $\lambda_0 \in V[K][H_0]$ and $\lambda_1 \in V[K][H_1]$, a mutual genericity argument shows that $\lambda_0 \neq \lambda_1$.

To establish that the set $\lambda_0 \cap \lambda_1$ is infinite, move back to the model $V[K]$, let $\langle \lambda_0, \lambda_1 \rangle$ be a condition in the product $R \times R$ and $n \in \omega$ be a number; we must find a number $m > n$ and conditions $t_0 \leq s_0$ and $t_1 \leq s_1$ such that $t_0 \vDash \bar{m} \in \chi$ and $t_1 \vDash \bar{m} \in \chi$. To this end, let $b_0 = \{m \in \omega: \exists t \leq s_0 \text{ such that } t \vDash \bar{m} \in \chi\}$ and $b_1 = \{m \in \omega: \exists t \leq s_1 \text{ such that } t \vDash \bar{m} \in \chi\}$. The sets $b_0, b_1 \subset \omega$ are both forced to have infinite intersection with $\eta$ and therefore must belong to the ultrafilter $U$. This means that there is a natural number $m > n$ in the intersection $b_0 \cap b_1$, and then the desired conditions $t_0 \leq s_0$ and $t_1 \leq s_1$ are found by the definition of the sets $b_0$ and $b_1$.  

Example 8.7.7 above produces a weakly balanced forcing such that in its extension of the Solovay model, a maximal almost disjoint family exists. As a result of that example and $??$, we have

**Corollary 14.1.3.** It is consistent relative to an inaccessible cardinal that $ZF + DC$ holds, there is an infinite maximal almost disjoint family of subsets of $\omega$, and there is no uncountable sequence of pairwise distinct reals.
14.2 Unbounded linear suborders

Another general obstruction for reaching ZF independence results in balanced extensions is encapsulated in the following theorem, stated using Convention 1.8.17.

**Theorem 14.2.1.** Let \( \preceq \) be an \( F_\sigma \)-preorder on a Polish space \( X \) with no maximal element such that every countable linearly ordered set has an upper bound. In balanced \( \sigma \)-closed extensions of the symmetric Solovay model, \( \preceq \) contains no unbounded linearly ordered sets.

A typical preorder satisfying the assumptions is the Turing reducibility preorder or the modulo finite domination ordering on \( \omega^\omega \).

**Proof.** We start with a simple claim which takes place in the context of ZFC. If \( R \) is a poset and \( \eta \) is an \( R \)-name for an element of \( X \), say that \( \eta \) is *unbounded* if \( R \vDash \) for every ground model element \( x \in X, \eta \preceq x \) fails.

**Claim 14.2.2.** Suppose that \( R_0, R_1 \) are posets and \( \eta_0, \eta_1 \) are respective unbounded names for elements of \( X \). Then \( R_0 \times R_1 \vDash \tau_0, \tau_1 \) are \( \preceq \)-incomparable.

**Proof.** Let \( \preceq = \bigcup_n F_n \) where for every \( n \in \omega, F_n \) is closed. If the conclusion failed, there would have to be a pair \( \langle r_0, r_1 \rangle \in R_0 \times R_1 \) forcing say \( \eta_0 \preceq \eta_1 \) and then strengthening the pair if necessary there would have to be a number \( n \in \omega \) such that \( \langle \eta_0, \eta_1 \rangle \in F_n \) is forced. Let \( M \) be a countable elementary submodel of a large structure containing \( R_1, r_1, \eta_1 \), let \( g \subset R_1 \) be a filter generic over the model \( M \) such that \( r_1 \in g \), and let \( x = \eta_1 / g \). Since \( \eta_0 \) is an unbounded name, \( R_0 \vDash \eta_0 \preceq x \) fails, and so there must be a condition \( r_0' \leq r_0 \) and basic open sets \( O_0, O_1 \subset X \) such that \( (O_0 \times O_1) \cap F_n = 0, x \in O_1, \) and \( r_0' \vDash \tau_0 \in O_0 \).

By the forcing theorem, there must be a condition \( r_1' \in g \) below \( r_1 \) such that \( r_1' \vDash \tau_1 \in O_1 \). Then the pair \( \langle r_0', r_1' \rangle \leq \langle r_0, r_1 \rangle \) forces \( \langle \eta_0, \eta_1 \rangle \in O_0 \times O_1 \) and so \( \langle \eta_0, \eta_1 \rangle \notin F_n \), in contradiction with the initial assumptions.

Now, let \( \kappa \) be an inaccessible cardinal and \( P \) be a \( \sigma \)-closed Suslin forcing such that \( V_\kappa \models P \) is balanced in every generic extension. Let \( W \) be the symmetric Solovay model derived from \( \kappa \). Suppose towards contradiction that the conclusion of the theorem fails in the \( P \)-extension of \( W \). Then, in the model \( W \) there must be a condition \( p \in P \) and a \( P \)-name \( \tau \) such that \( p \vDash \tau \subset X \) is an \( \preceq \)-unbounded linearly ordered set. The condition \( p \) as well as the name \( \tau \) must be definable from some parameter \( z \in 2^\omega \) and some parameters in the ground model. Let \( V[K] \) be a generic extension of the ground model by a poset of size \( < \kappa \) such that \( p, z \in V[K] \) and \( P \) is balanced in \( V[K] \).

Work in the model \( V[K] \). Let \( \bar{p} \leq p \) be a balanced condition in the poset \( P \). Since \( \text{Coll}(\omega, < \kappa) \vDash p \vDash \tau \) is an unbounded set in \( \leq \), for every element \( x \in X \cap V \) the set \( \tau \) is forced to contain an element which is not \( \leq x \). Since \( \text{Coll}(\omega, < \kappa) \vDash p \vDash X \cap V[K] \) is a countable set and DC holds, \( \tau \) is forced to contain a countable subset such that no element of \( X \cap V[K] \) is an upper bound of it. By the initial assumptions on the preorder \( \leq \), this countable set is forced to have an upper bound in \( \tau \), which is then an element of \( \tau \) which is not \( \leq x \)
for any element \( x \in X \cap V[K] \). In total, in \( V[K] \) there must be a poset \( R \) of size \( < \kappa \), an \( R \)-name \( \sigma \) for a condition in \( P \) such that \( \sigma \leq \bar{p} \) and an unbounded \( R \)-name \( \eta \) for an element of \( X \) such that \( R \Vdash \eta \) is not \( \leq \)-below any element of \( V[K] \) and \( R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma \). In total, in \( V[K] \) there must be a poset \( R \) of size \( < \kappa \), an \( R \)-name \( \sigma \) for a condition in \( P \) such that \( \sigma \leq \bar{p} \) and an unbounded \( R \)-name \( \eta \) for an element of \( X \) such that \( R \Vdash \eta \) is not \( \leq \)-below any element of \( V[K] \) and \( R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma \).

In the model \( W \), let \( H_0, H_1 \subset R \) be filters mutually generic over \( V[K] \). Let \( p_0 = \sigma/H_0 \in P \) and \( p_1 = \sigma/H_1 \in P \); by the balance of the condition \( \bar{p} \), the conditions \( p_0, p_1 \) are compatible with some lower bound \( q \in P \). Let \( x_0 = \eta/H_0 \in X \) and \( x_1 = \eta/H_1 \in X \); by Claim 14.2.2, these are \( \leq \)-incomparable elements of \( X \). Since the model \( W \) is a symmetric extension of both models \( V[K][H_0] \) and \( V[K][H_1] \), the forcing theorem applied in these models shows that in \( W, q \Vdash x_0, x_1 \in \tau \). This contradicts the assumption that \( \tau \) is forced to be linearly ordered by \( \leq \).

### 14.3 Homomorphisms of Polish groups

In this section, we develop a technology for ruling out discontinuous homomorphisms between Polish groups in balanced extensions of the symmetric Solovay model. One important way of doing that is to prove that in the given model of \( ZF+DC \), the chromatic number of the Hamming graph \( \mathbb{H}_2 \) is not two (say using Theorem 11.3.13 or 14.5.7), and then use a result of \([63]\) to show that in \( ZF+DC \) this rules out discontinuous homomorphisms. In this section we introduce a method which works in cases which are not treatable using the chromatic number of the Hamming graph. The following notion is central:

**Definition 14.3.1.** Let \( P \) be a Suslin poset.

1. A virtual condition \( \bar{p} \in P \) is \( m,n \)-balanced if in every generic extension, whenever \( \{H_i : i \in m\} \) are filters generic over \( V \) such that for every set \( a \subset m \) of size \( n \) the filters \( \{H_i : i \in a\} \) are mutually generic over \( V \), if \( p_i \leq \bar{p} \) are conditions in \( P \) in \( V[H_i] \), then the set \( \{p_i : i \in m\} \) has a common lower bound in \( P \).

2. The poset \( P \) is \( m,n \)-balanced if for every condition \( p \in P \) there is a \( m,n \)-balanced virtual condition \( \bar{p} \leq p \).

The following central application is stated using Convention 1.8.17.

**Theorem 14.3.2.** In 3,2-balanced extensions of the symmetric Solovay model, all homomorphisms between Polish groups are continuous.

**Proof.** Suppose that \( \kappa \) is an inaccessible cardinal. Suppose that \( P \) is a Suslin forcing such that in \( V_\kappa \models P \) is 3,2-balanced in every forcing extension. Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in \( W \). Suppose towards contradiction that the conclusion of the theorem fails: thus, there are Polish groups \( \Gamma, \Delta \), a condition \( p \in P \) and a \( P \)-name \( \tau \) such that \( p \models \tau : \Gamma \rightarrow \Delta \) is a discontinuous homomorphism. Both \( p \) and \( \tau \) must be definable in \( W \) from parameters in the ground model and some point \( z \in 2^\omega \). Let \( V[K] \) be some
intermediate extension of $V$ by a poset of size $< \kappa$ such that $z \in V[K]$ and $P$ is $3,2$-balanced in $V[K]$. Work in the model $V[K]$.

Let $\bar{p} \leq p$ be a $3,2$-balanced condition. Consider the poset $P_\Gamma$ adding a generic element $\gamma_{\text{gen}}$ of the group $\Gamma$. The following is a key claim:

**Claim 14.3.3.** $P_\Gamma$ forces that there are disjoint basic open sets $O_0, O_1 \subset \Delta$ such that $\text{Coll}(\omega, < \kappa) \forces \exists p_0 \leq \bar{p} \ p_0 \forces \tau(\gamma_{\text{gen}}) \in O_0$ and $\exists p_1 \leq \bar{p} \ p_1 \forces \tau(\gamma_{\text{gen}}) \in O_1$.

**Proof.** Suppose towards contradiction that $q \in P_\Gamma$ forces the opposite. Then, $q$ forces that in the $\text{Coll}(\omega, < \kappa)$-extension there is a unique point $\delta \in \Delta$ such that there is a condition $p' \leq \bar{p}$ in the poset $P$ forcing $\tau(\gamma_{\text{gen}}) = \delta$. Since the point $\delta$ is in the symmetric Solovay model definable from $\tau, \bar{p}$, and $\gamma_{\text{gen}}$, it must belong to the $P_\Gamma$-extension, and there is a $P_\Gamma$-name $\delta$ for it. Note that then, $q \forces \text{Coll}(\omega, < \kappa) \forces \bar{p} \forces \bar{p} \forces \tau(\gamma_{\text{gen}})$.

Now, pass to the symmetric Solovay model $W$ and consider the set $B = \{ \gamma \in \Gamma : \gamma$ is $P_\Gamma$-generic over $V[K]$ and $\gamma \in q \}$ and the function $f : B \rightarrow \Delta$ assigning to each point $\gamma \in B$ the point $\delta/\gamma$. The function $f$ is continuous on $B$, and the choice of the name $\delta$ implies that $\bar{p} \forces \bar{f} \subset \tau$ holds. The set $B \subset \Gamma$ is dense $G_\delta$ in $q$. A standard result in the theory of Polish groups [45, Theorems 9.9 and 10.10] now says that a homomorphism from $\Gamma$ to $\Delta$ which is continuous on a set comeager in a nonempty open set is in fact continuous on the whole group $\Gamma$. This contradicts the initial choice of the name $\tau$. \qed

Let $X = \{ x \in \Gamma^3 : x(0) \cdot x(1) = x(2) \}$; this is a closed subset of $\Gamma^3$, equipped with the topology inherited from $\Gamma^3$. If $i, j \in 3$ are two distinct coordinates, it is clear that the remaining coordinate of a point $x \in X$ is a continuous function of the coordinates $x(i)$ and $x(j)$, and therefore the projection from $X$ to $\Gamma^2$ given by the $i$- and $j$-th coordinate is a continuous and open surjection. Let $x \in X$ be a point $P_X$ generic over $V[K]$. Use Proposition 3.1.1 to see that whenever $i, j \in 3$ are distinct numbers then $x(i), x(j) \in \Gamma$ are points $P_\Gamma$-mutually generic over $V[K]$.

Working in the respective models $V[K][x(0)]$ and $V[K][x(1)]$, find posets $R_0, R_1$ of size $< \kappa$ and respective names $\sigma_0, \sigma_1$ for conditions in the poset $P$ stronger than $\bar{p}$ and names $\delta_0, \delta_1$ for elements of the group $\Delta$ such that $V[K][x(0)] \models R_0 \forces \text{Coll}(\omega, < \kappa) \forces \sigma_0 \forces \tau(x(0)) = \delta_0$ and similarly for $V[K][x(1)]$. In the model $V[K][x(2)]$, find the disjoint basic open sets $O_0, O_1 \subset \Delta$ as in Claim 14.3.3. Passing to a condition in the product $R_0 \times R_1$ and switching $O_0, O_1$ if necessary, we may assume that $V[K][x(0), x(1)] \models R_0 \times R_1 \forces \delta_0 \cdot \delta_1 \notin O_0$. Use the choice of the set $O_0$ to find, in the model $V[K][x(2)]$, a poset $R_2$ of size $< \kappa$, $R_2$-names $\sigma_2$ for an element of $P$ stronger than $\bar{p}$ and $\delta_2$ for an element of $O_0 \subset \Delta$ such that $R_2 \forces \text{Coll}(\omega, < \kappa) \forces \sigma_2 \forces \tau(x(2)) = \delta_2$.

Let $H_0 \subset R_0, H_1 \subset R_1, H_2 \subset R_2$ be filters mutually generic over the model $V[K][x]$. For each $i \in 3$ write $p_i = \sigma_i/H_i \in P$ and $\delta_i = \delta_i/H_i$. Note that whenever $i, j \in 3$ are distinct numbers, the models $V[K][x_i][H_i]$ and $V[K][x_j][H_j]$ are mutually generic extensions of the model $V[K]$. By the $3,2$-balanced assumption on the balanced virtual condition $\bar{p}$, the conditions $p_0, p_1, p_2 \in P$ have a lower bound, call it $q \in P$. Let $W$ be a symmetric Solovay extension
of the model $V[K][x][H_0, H_1, H_2]$ and work in $W$. The condition $q$ forces that $\tau(x(i)) = \delta_i$ for all $i \in 3$. Observe that $x(0) \cdot x(1) = x(2)$ and $\delta_0 \cdot \delta_1 \neq \delta_2$ since $\delta_0 \cdot \delta_1 \notin O_0$ while $\delta_2 \in O_0$. This contradicts the assumption that $\tau$ is forced to be a homomorphism from $\Gamma$ to $\Delta$. 

In the way of examples, numerous Suslin partial orders are 3, 2-balanced for the simplest reason: any set of pairwise compatible three conditions has a common lower bound in them. This leads to the following examples and corollaries.

**Example 14.3.4.** Let $E, F$ be Borel equivalence relations such that $\lambda(E) \leq \lambda(F)$ holds. For every number $n \in \omega$, the $E, F$-collapse poset of Definition 6.3.2 is $n, 2$-balanced and every balanced condition is $n, 2$-balanced.

**Corollary 14.3.5.** Let $E, F$ be Borel equivalence relations such that in every forcing extension, $\lambda(E) \leq \lambda(F)$ holds.

1. Let $P$ be the $E, F$-collapse poset. In the $P$-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $|E| \leq |F|$ holds, and yet there is no discontinuous homomorphism between Polish groups.

**Example 14.3.6.** Let $E \subset F$ be Borel equivalence relations on a Polish space $X$ and suppose that $F$ is pinned. For every number $n \in \omega$, the $E, F$-transversal poset of Definition 6.3.5 is $n, 2$-balanced and every balanced condition is $n, 2$-balanced.

**Corollary 14.3.7.** Let $E$ be a pinned Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the transversal poset of Example 6.3.8. In the $P$-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $E$ has a transversal, and yet there is no discontinuous homomorphism between Polish groups.

**Corollary 14.3.8.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$ with infinite classes.

1. Let $P$ be the $\mathbb{Z}$-action poset of Example 6.3.10. In the $P$-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $E$ is induced as an orbit equivalence relation of a discontinuous $\mathbb{Z}$-action, and yet there is no discontinuous homomorphism between Polish groups.
Example 14.3.9. Let $\Gamma$ be a Borel graph on a Polish space $X$ of countable coloring number. For every number $n \in \omega$, the coloring poset of Definition 8.1.1 is $n,2$-balanced and every balanced condition is $n,2$-balanced.

Corollary 14.3.10. Let $\Gamma$ be the graph on $\mathbb{R}^2$ connecting two points if they are at a rational distance from each other.

1. Let $P$ be the coloring poset of Definition 8.1.1. In the $P$-extension of the symmetric Solovay model, there are no discontinuous homomorphisms of Polish groups.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $\Gamma$ is countable, and yet there is no discontinuous homomorphism between Polish groups.

Example 14.3.11. Let $X$ be a Polish space and let $P$ be the Kurepa poset on it as in Definition 8.4.2. For every number $n \in \omega$, the poset $P$ is $n,2$-centered and every balanced condition is $n,2$-balanced. To see this, suppose that $p_i$ for $i \in n$ are conditions which are pairwise compatible; this means that for example whenever $a_0 \in p_0$ and $a_1 \in p_1$ then $a_0 \cap a_1 \in p_0 \cap p_1$ must hold. To produce the lower bound, let $q$ be the lower bound of $\bigcup_i p_i$ under intersection; we must prove that $q \leq p_i$ for each $i \in n$.

Fix a number $i \in n$. To show that $q \leq p_i$ holds, we must prove that for each $a \in q$ and each $b \in p_i$, $a \cap b \in p_i$ must hold. Suppose for definiteness that $a = \bigcap_{j \in n} a_j$ and $a_j \in p_j$. By the pairwise compatibility of the conditions $p_0,p_1,p_2$, we get that the sets $a_j \cap b$ for $j \neq i$ belong to $p_i$, and by the closure of $p_i$ under intersections, $a \cap b = (a_i \cap b) \cap \bigcap_{j \neq i} (a_j \cap b)$ belongs to $p_0$ as desired.

Corollary 14.3.12. Let $X$ be a Polish space.

1. In the extension of the symmetric Solovay model by the Kurepa poset, there are no discontinuous homomorphisms between Polish groups.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Kurepa family on $X$, and there is no discontinuous homomorphism between Polish groups.

There are posets which are 3,2-balanced for a more complicated reason, and we gather them below:

Example 14.3.13. Let $\Gamma$ be a Borel graph on a Polish space $X$. Let $P$ be the poset of all countable acyclic subsets of $\Gamma$ ordered by reverse inclusion. Whenever $n \in \omega$ is a natural number, the poset $P$ is $n,2$-balanced and every balanced condition is $n,2$-balanced.

Proof. It follows from Theorem 6.2.28 that balanced virtual conditions for $P$ are classified by maximal acyclic subgraphs of $\Gamma$. Let $\bar{p}$ be any maximal acyclic subgraph of $\Gamma$. We will check that $\bar{p}$ is $n,2$-balanced, completing the proof. To this end, let $\{V[H_i] : i \in n\}$ be pairwise mutually generic extensions of $V$, respectively containing some conditions $p_i \leq \bar{p}$ for each $i \in n$. 

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Claim 14.3.14. For each \( i \in n \) and vertices \( x_0, x_1 \in X \cap V \), the following three formulas are equivalent:

1. \( x_0, x_1 \) are connected by a path in \( \Gamma \cap V \);
2. \( x_0, x_1 \) are connected by a path in \( \bar{p} \);
3. \( x_0, x_1 \) are connected by a path in \( p_i \).

Proof. (1) \( \rightarrow \) (2) is implied by the maximality of \( \bar{p} \). (2) \( \rightarrow \) (3) follows from the fact that \( \bar{p} \subset p_i \), and the negation of (1) implies by the Mostowski absoluteness that \( x_0, x_1 \) are not connected by any path in the graph \( \Gamma \cap V[H_i] \), which is larger than \( p_i \) and therefore (3) has to fail.

The pairwise mutual genericity implies that if \( i, j \in n \) are distinct numbers and \( v_i, v_j \subset X \) are the sets of vertices mentioned in some edges in \( p_i, p_j \) respectively, then \( v_i \cap v_j \subset V \). This, together with the claim, means that the union \( \bigcup_i p_i \) cannot contain a cycle and so is the desired common lower bound of the conditions \( p_i \) for \( i \in n \).

Corollary 14.3.15. Let \( \Gamma \) be a Borel graph on a Polish space \( X \).

1. Write \( P \) for the poset adding a maximal acyclic subgraph of \( \Gamma \) as in Example 6.2.17. In the \( P \)-extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, \( \Gamma \) has a maximal acyclic subgraph and every homomorphism between Polish groups is continuous.

Example 14.3.16. Let \( \mathcal{F} \) be a Fraissé class of structures in finite relational language with strong amalgamation. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). Let \( P = P(\mathcal{F}, E) \) be the poset of Definition 8.5.3. For each number \( n \in \omega \), the poset \( P \) is \( n, 2 \)-balanced and every virtual balanced condition is \( n, 2 \)-centered.

Proof. It follows from Theorem 8.5.4 that balanced virtual conditions in \( P \) are classified by \( \mathcal{F} \)-structures on the virtual quotient space \( X^{**} \). Let \( \bar{p} \) be such a structure. Let \( V[H_i] \) for \( i \in n \) be generic extensions of the ground model which are pairwise mutually generic. Let \( p_i \in V[H_i] \) be a condition extending \( \bar{p} \), for each \( i \in n \). Note that the domains of the conditions pairwise intersect in the domain of \( \bar{p} \). Let \( b = \bigcup_i \text{dom}(p_i) \). For each finite set \( a \subset b \) use the strong amalgamation of the class \( \mathcal{F} \) \( n - 1 \)-many times to find a structure \( N_a \in \mathcal{F} \) on \( a \) so that for each \( i \in n \), \( N_a \upharpoonright (\text{dom}(p_i) \cap a) = p_i \upharpoonright (\text{dom}(p_i) \cap a) \). Let \( U \) be an ultrafilter on \( [b]^{<\aleph_0} \) such that for every finite set \( c \subset b \), the set \( \{a \in [b]^{<\aleph_0} : c \subset a\} \) belongs to \( U \). Let \( N \) be the structure on \( b \) which is the \( U \)-integral of the structures \( N_a \). It is immediate that for all \( N \upharpoonright \text{dom}(p_i) = p_i \) for each \( i \in 3 \), and the closure of the class \( \mathcal{F} \) under substructures shows that \( N \) is a \( \mathcal{F} \)-structure.

It is the desired lower bound of the conditions \( p_i \) for \( i \in n \).
Corollary 14.3.17. Let $E$ be a Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the $E$-linearization as in Example 8.5.5. In the $P$-generic extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the $E$-quotient space carries a linear ordering and every homomorphism between Polish groups is continuous.

As usual, the independence results obtained in this section can be combined. For natural numbers $n \leq m$, the class of $m,n$-balanced posets is closed under countable, full support product. Thus for a given Borel graph $G$ and a Borel equivalence relation $E$, it is possible to get a model of ZF+DC in which $G$ has a maximal acyclic subgraph and the $E$-quotient space is linearly ordered, yet every homomorphism between Polish groups is continuous.

Question 14.3.18. Are there $\Sigma^2_1$ sentences $\phi_0, \phi_1$ such that each of them is consistent with ZF+(DC+) there are no discontinuous homomorphisms between Polish groups but their conjunction implies in ZF(+DC) an existence of a discontinuous homomorphism?

14.4 Perfect balanced forcing

The purpose of this section is to isolate the class of perfect balanced forcings, and to show that certain dichotomies survive the passage from the symmetric Solovay model to its perfect balanced extensions.

Definition 14.4.1. A Suslin forcing $P$ is perfect if for every Borel function $f : 2^{\omega} \to P$, either there is a finite set $a \subset 2^{\omega}$ such that the set $f''a \subset P$ has no lower bound, or else there is a perfect set $B \subset 2^{\omega}$ such that $f''B$ has a lower bound in the separative quotient.

If the nonexistence of lower bounds for finite subsets of $P$ is an analytic relation (which is universally true for all classes of partial orders discussed in this paper) then perfectness is a $\Pi^1_2$ condition and therefore absolute to all forcing extensions. As usual with the classes of partial orders isolated in this book, the class of perfect partial orders is closed under countable product. It turns out that perfect balanced extensions of the symmetric Solovay model satisfy stronger versions of certain regularity properties of the equi-balanced extensions.

Definition 14.4.2. The full Silver dichotomy is the following statement: If $E$ is a Borel equivalence relation on a Polish space $X$ and $A \subset X$ is an $E$-invariant set, then either $A$ contains only countably many $E$-classes or $A$ contains a perfect set consisting of pairwise $E$-unrelated elements.

As a consequence, the full Silver dichotomy implies that among all sets whose cardinality is smaller than a Borel quotient space, $2^{\omega}$ is the set with the smallest
CHAPTER 14. OTHER COMBINATORICS

uncountable cardinality. The terminology refers to the classical Silver dichotomy [39, Theorem 5.7.1], a theorem of ZF+DC which says in particular that the dichotomy holds for analytic sets \( A \subseteq X \). The Solovay model satisfies the full Silver dichotomy. The following result is stated using Convention 1.8.17.

**Theorem 14.4.3.** In the perfect balanced extensions of the symmetric Solovay model, the full Silver dichotomy holds.

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a perfect Suslin forcing such that \( V_\kappa \models P \) is balanced in every forcing extension. Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in the model \( W \). Let \( p \in P \) be a condition and \( \tau \) be a \( P \)-name such that \( p \Vdash \tau \subseteq X \) is an \( E \)-invariant set containing uncountably many \( E \)-classes. The condition \( p \) as well as the name \( \tau \) have to be definable from parameters in \( V \) as well as some parameter \( z \in 2^\omega \).

Use the assumptions to find an intermediate model \( V[K] \) obtained as a generic extension of \( V \) by a poset of size \( \kappa \) such that \( z \in V[K] \) and \( P \) is balanced in \( V[K] \).

Work in the model \( V[K] \). Find a virtual balanced condition \( \bar{p} \preceq p \). Note that the equivalence relation \( E \) is Borel and therefore it has fewer than \( 2^{\omega_1}[K] \) many virtual classes. The cardinality \( 2^{\omega_1}[K] \) is countable in the model \( W \), while the set \( \tau \) is forced to contain uncountably many \( E \)-classes. It follows that in the model \( V[K] \) there has to be a poset \( R \) of size \( \kappa \), an \( R \)-name \( \sigma \) for a condition in \( P \) stronger than \( \bar{p} \) and an \( R \)-name \( \eta \) for an element of \( X \) which is forced not to be a realization of any virtual \( E \)-class in the model \( V[K] \), and \( R \Vdash \text{Coll}(\omega, \kappa) \Vdash \sigma \Vdash \eta \in \tau \).

Back in the model \( W \), use Proposition 1.8.10 to find a function continuous function \( y \mapsto H_y \) which to each element \( y \in 2^\omega \) assigns a filter \( H_y \subseteq R \) generic over \( V[K] \) so that for every finite set \( a \subseteq 2^\omega \), the filters \( \{H_y : y \in a\} \) are mutually generic over \( V[K] \). Let \( p_y = \sigma/H_y \in P \) and \( x_y = \eta/H_y \) for all \( y \in 2^\omega \). The map \( y \mapsto p_y \) is Borel, and the map \( y \mapsto x_y \) is a continuous injection from \( 2^\omega \) to \( X \). By the assumption on the name \( \eta \), the points \( x_y \) for \( y \in 2^\omega \) are pairwise \( E \)-unrelated.

**Claim 14.4.4.** For every finite set \( a \subseteq 2^\omega \), the set \( \{p_y : y \in a\} \subseteq P \) has a common lower bound.

**Proof.** Let \( \langle y_j : j \in i \rangle \) enumerate the set \( a \) without repetitions. By induction on \( j \in i \), construct a descending sequence \( \langle q_j : j \in i \rangle \) of conditions in \( P \) such that for each \( j \in i \), \( \bar{p}_j \in V[K][H_k : k \leq j] \) and \( q_j \preceq p_{y_j} \). This is easy to do using the balance of the condition \( \bar{p} \) at every stage of the induction, noting that \( q_j, p_{y_{j+1}} \) are both strengthenings of \( \bar{p} \) in mutually generic extensions of the model \( V[K] \).

In the end, the condition \( q_{i-1} \) is a lower bound of the set \( \{p_y : y \in a\} \subseteq P \).

Since the poset \( P \) is perfect, there is a perfect set \( C \subseteq 2^\omega \) such that the set \( \{p_y : y \in C\} \) has a lower bound, say \( q \in P \) in the separative quotient of \( P \). By the forcing theorem applied in every model \( V[K][H_y] \) for \( y \in B \) it is the case that in the model \( W \), the condition \( q \) forces the perfect set \( \{x_y : y \in C\} \)
consisting of pairwise $E$-unrelated elements to be a subset of $\tau$. The proof is complete. \hfill $\Box$

**Definition 14.4.5.** OCA+ is the following statement. Whenever $X$ is a Polish space, $A \subset X$ is a set, and $\Gamma$ is a graph on $A$ which is open in the topology on $A \times A$ inherited from $X \times X$, then either $A$ is a union of countably many $\Gamma$-anticliques, or $A$ contains a perfect $\Gamma$-clique.

The following result is stated using Convention 1.8.17.

**Theorem 14.4.6.** In every balanced, perfect extension of the symmetric Solovay model, OCA+ holds.

**Proof.** Let $X$ be a Polish space and let $P$ be a perfect Suslin forcing. Let $\kappa$ be an inaccessible cardinal such that in $V_\kappa$, in every forcing extension the poset $P$ is balanced. Let $W$ be a symmetric Solovay model derived from $\kappa$. In the model $W$, let $A \subset X^2$ be a symmetric open set, let $p \in P$ be a condition, and let $\tau$ be a $P$-name for a subset of $X$ such that $p \Vdash \tau$ cannot be covered by countably many $\Gamma$-anticliques. We must find a perfect set $C \subset X$ such that any two points of $C$ are $\Gamma$-related, and a condition $q \leq p$ in $P$ which forces $\check{C} \subset \tau$.

To this end, choose a parameter $z \in 2^\omega$ such that $p, \tau, \Gamma$ are definable from $z$. Find an intermediate generic extension $V[K]$ of $V$ by a poset of size $< \kappa$ such that $z \in V[K]$. Work in the model $V[K]$. Let $\bar{p} \leq p$ be a balanced virtual condition in $P$. Since $\Coll(\omega, < \kappa) \Vdash p \Vdash \tau$ is not covered by countably many $\Gamma$-anticliques, and a closure of a $\Gamma$-anticlique is still a $\Gamma$-anticlique, there must be a poset $R$ of size $< \kappa$, and $R$-name $\eta$ for an element of $X$ which belongs to no closed $\Gamma$-anticlique coded in $V[K]$, and an $R$-name $\sigma$ for a condition in $P$ stronger than $\bar{p}$ such that $\check{R} \Vdash \Coll(\omega, < \kappa) \Vdash \sigma \Vdash R \check{\eta} \in \tau$.

Move into the model $W$. Let $\{D_n : n \in \omega\}$ enumerate all open dense subsets of finite powers of the poset $R$ in $V[K]$, with infinite repetitions. As in the proof of Theorem 12.3.2, by induction on $|t|$ build conditions $r_t \in R$ for $t \in 2^{<\omega}$ so that

- $t \subset s$ implies $r_s \leq r_t$;
- whenever $D_n \subset R^m$ is an open dense set for some $m < 2^n$, then every $m$-tuple of distinct elements from the set $\{r_t : t \in 2^n\}$ belongs to $D_n$;
- for all $t \in 2^{<\omega}$ there are open sets $O_{t_0}, O_{t_1} \subset X$ such that $O_{t_0} \times O_{t_1} \subset \Gamma$ and $r_{t_0} \Vdash \eta \in O_{t_0}$ and $r_{t_1} \Vdash \eta \in O_{t_1}$.

In the end, for every binary sequence $y \in 2^\omega$ let $H_y \subset R$ be the filter generated by the conditions $\{r_{y|n} : n \in \omega\}$ and let $p_y = \check{\sigma}/H_y$. Note that any finite tuple of distinct filters $H_y$ for $y \in 2^\omega$ is mutually generic over the model $V[K]$ by the second item in the inductive construction above. By the balance of the condition $\bar{p}$, any finite tuple of the conditions $p_y$ for $y \in 2^\omega$ has a lower bound. By the perfectness of the poset $P$, there is a perfect set $B \subset 2^\omega$ such that the conditions $\{p_y : y \in B\}$ have a lower bound $q$ in the separative quotient of the
poset $P$. Let $C = \{\eta/y : y \in B\}$; this is a perfect subset of $X$, and it is a
$\Gamma$-clique by the last item of the inductive construction above. Let $y \in B$ be
any point. Since $W$ is a symmetric Solovay extension of the model $V[K][H_y]$, it
follows by the choice of the name $\eta$ and the forcing theorem applied in the
model $V[K][H_y]$ that $q \vDash \eta/y \in \tau$. To recapitulate, $C$ is a perfect $\Gamma$-clique and
$q \vDash \dot{C} \subset \tau$ as required. \hfill \square

**Example 14.4.7.** Let $I$ be an $F_\sigma$-ideal on $\omega$. The poset $P$ of all $I$-positive
subsets of $\omega$ ordered by inclusion is perfect.

As a special case, this includes the poset of infinite subsets of $\omega$ ordered by
inclusion of Section 7.1 and the posets of Section 7.2, both adding ultrafilters
on $\omega$ with various Ramsey properties.

**Proof.** Recall that $P(I)$ is the poset of all $I$-positive subsets of $\omega$, ordered by
inclusion. Write $I = \bigcup_n I_n$ as a countable union of closed sets, each of which is
closed under taking subset. Let $f : 2^\omega \to \mathcal{P}(\omega)$ be a Borel function such that for
any finite set $a \subset 2^\omega$, $\bigcap f^{-1}(a) \notin I$ holds; we must find a perfect set $B \subset 2^\omega$
such that the set $f''B \subset P$ has a lower bound. Thinning the domain of $f$ if necessary
we may assume that the function $f$ is in fact continuous. By induction on $n \in \omega$
build nodes $u_t \in 2^{<\omega}$ for all $t \in 2^n$ and finite sets $b_n \subset \omega$ such that

- $b_n \notin I_n$;
- $s \subset t$ implies that $u_s \subset u_t$ and $s$ is incompatible with $t$ implies $u_s$ is
  incompatible with $u_t$;
- for each $t \in 2^{n+1}$ and every $y \in [u_t]$ it is the case that $b_n \subset f(y)$.

Once the induction is performed, let $b = \bigcup_n b_n$, let $B \subset 2^\omega$ be the perfect set
of all points $y \in 2^\omega$ such that $\forall n \exists t \in 2^n u_t \subset y$ and use the continuity of the
function $f$ to prove that $b$ is the lower bound of the set $f''B$.

To start the induction, let $u_0 = 0$. Now suppose that the nodes $u_t \in 2^{<\omega}$ for
t $\in 2^n$ as well as sets $b_m$ for $m \in n$ have been constructed. For each $t \in 2^n$ choose
distinct points $y_{t0}, y_{t1} \in [u_t]$ and use the initial assumption on the function $f$
to observe that $c = \bigcap_{t \in 2^n} f(y_{t0}) \cap \bigcap_{t \in 2^n} f(y_{t1})$ is an $I$-positive set. Since the
set $I_n \subset \mathcal{P}(\omega)$ is closed, there must be a finite subset $b_n \subset c$ which is not in
$I_n$. Use the continuity of the function $f$ to find initial segments $u_{t^0} \subset y_{t0}$
and $u_{t^1} \subset y_{t1}$ satisfying the second item of the induction hypothesis. This
concludes the inductive step. \hfill \square

**Corollary 14.4.8.** [16] Let $P = \mathcal{P}(\omega)$ modulo finite.

1. In the $P$-extension of the symmetric Solovay model, the full Silver di-
chotomy holds, and OCA+ holds.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there
is a nonprincipal ultrafilter on $\omega$, the full Silver dichotomy holds, and
OCA+ holds.
Example 14.4.9. Let \( \langle \Gamma, \cdot \rangle \) be a countable semigroup. The poset \( P = P(\Gamma) \) of Subsection 7.4 designed to add a union ultrafilter is perfect.

Proof. Recall that elements of \( P \) are just sequences in \( \Gamma^\omega \) and the ordering is defined by \( q \leq p \) if there are pairwise disjoint nonempty finite sets \( a_n \subset \omega \) such that \( q(n) = \prod_{m \in a_n} p(m) \). The proof of the perfect property of the poset \( P \) is another fusion argument. Let \( f : 2^\omega \to P \) be a Borel function such that for every finite set \( b \subset 2^\omega \), the conditions in the set \( f^{-1}[b] \) have a common lower bound in \( P \); we must find a perfect set \( B \subset 2^\omega \) such that the set \( f^{-1}[B] \subset P \) has a lower bound.

Thinning the domain of \( f \) if necessary we may assume that the function \( f \) is continuous. By induction on \( n \in \omega \) build nodes \( u_t \in 2^{<\omega} \) for all \( t \in 2^n \), finite sets \( a_t \subset \omega \) and elements \( \gamma_n \) so that

- \( \min(a_{n+1}) > \max(a_n) \);
- \( s \subset t \) implies that \( u_s \subset u_t \) and \( s \) is incompatible with \( t \) implies \( u_s \) is incompatible with \( u_t \);
- \( a_0 = 0 \) and if \( s \subset t \) then \( a_s \cap a_t = 0 \);
- for each \( t \in 2^{n+1} \) and each \( y \in [u_t] \), \( \gamma_n = \prod_{m \in a_t} f(y)(m) \).

Once the induction is performed, let \( p = \langle \gamma_n : n \in \omega \rangle \), let \( B \subset 2^\omega \) be the perfect set of all points \( y \in 2^\omega \) such that \( \forall n \exists t \in 2^n \ u_t \subset y \) and observe that \( p \) is the lower bound of the set \( f^{-1}[B] \).

To start the induction, let \( u_0 = 0 \) and \( a_0 = 0 \). Now suppose that the nodes \( u_t \in 2^{<\omega} \) for \( t \in 2^n \) as well as sets \( a_s \) for \( s \in 2^{<n} \) and elements \( \gamma_m \) for \( m \in n \) have been constructed. For each \( t \in 2^n \) choose distinct points \( y_{t0}, y_{t1} \in [u_t] \) and use the initial assumption on the function \( f \) to observe that the set \( c = \{ f(y_{t0}), f(y_{t1}) : t \in 2^n \} \) has a lower bound in the poset \( P \). This means that there are nonempty finite sets \( a_{t-0}, a_{t-1} \subset \omega \) disjoint from \( \bigcup_{s \in 2^{<n}} a_s \) and a semigroup element \( \gamma_n \in \Gamma \) such that \( \prod_{m \in a_{t-0}} f(y_{t0})(m) = \gamma_n \) and \( \prod_{m \in a_{t-1}} f(y_{t1})(m) = \gamma_n \) holds for all \( t \in 2^n \). Use the continuity of the function \( f \) to find initial segments \( u_{t-0} \subset y_{t0} \) and \( u_{t-1} \subset y_{t1} \) such that \( f(y_{t0})(m) = f(y)(m) \) holds for all \( m \in a_{t-0} \) and all \( y \in [u_{t-0}] \) and \( f(y_{t1})(m) = f(y)(m) \) holds for all \( m \in a_{t-1} \) and all \( y \in [u_{t-1}] \). This concludes the inductive step.

Corollary 14.4.10. Let \( P \) be the poset of Subsection 7.4 designed to add an ordered union ultrafilter.

1. In the \( P \)-extension of the symmetric Solovay model, the full Silver dichotomy holds, and OCA+ holds.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is an ordered union ultrafilter on \( \omega \), the full Silver dichotomy holds, and OCA+ holds.
Example 14.4.11. The poset $P$ of subsets of $\omega \times \omega$ with infinitely many infinite vertical sections, ordered by inclusion, is not perfect.

Proof. For each $n \in \omega$ and each binary string $t \in 2^n$, it is easy to find a perfect set $A_t \subset \mathcal{P}(\omega)$ such that

- $A_t$ consists of pairwise almost disjoint infinite sets;
- whenever $\{x_t : t \in 2^n\}$ is a collection such that $x_t \in A_t$, the set $\bigcap_t x_t \subset \omega$ is infinite.

Then, let $f : [t] \to \mathcal{P}(\omega)$ be a continuous injection into $A_t$ for every $t \in 2^{<\omega}$. Now, for every $y \in 2^\omega$ let $g(y) \subset \omega \times \omega$ be the set of all pairs $\langle n, m \rangle$ such that $m \in f_{y|n}(y)$. It is clear that the set $g(y) \subset \omega \times \omega$ has all vertical sections infinite and so is a condition in $P$. We will show that any finite subset of $g''2^\omega$ has a lower bound in $P$, yet no condition in $P$ is below more than countably many elements of $g''2^\omega$.

For the first assertion, let $a \subset 2^\omega$ be a finite set. Let $m \in \omega$ be such that the strings $y \upharpoonright m$ for $y \in a$ are pairwise distinct. Then, for all $n \geq m$ the second item implies that the set $\bigcap_{y \in a} f_{y|n}(y)$ is infinite. In other words, the set $\bigcap_{y \in a} g(y)$ has all vertical sections beyond $m$ infinite, and so is a lower bound for $g''a$. For the second assertion, suppose that $p \in P$ is a condition, and assume towards contradiction that the set $\{y \in 2^\omega : p \leq g(y)\}$ is uncountable. By a counting argument, there must be a number $n \in \omega$ such that the set $\{y \in 2^\omega : p_n \text{ is infinite and modulo finite} \}$ is uncountable. Find two distinct elements $y_0, y_1$ of the latter set such that $y_0 \upharpoonright n = y_1 \upharpoonright n$; denote the common value by $t$. Then, the intersection $f_t(y_0) \cap f_t(y_1)$ should be finite by the choice of the function $f_t$, and at the same time it should modulo finite contain the infinite set $p_n$. This is a contradiction. \hfill \Box

To conclude this section, we include an example showing that in a non-perfect balanced extension of the Solovay model, the cardinalities below $|2^\omega|$ can exhibit chaotic structure. This concern was addressed in [83].

Example 14.4.12. Let $P$ be the poset of all countable functions from $2^\omega$ to 2, ordered by reverse extension. Let $\kappa$ be an inaccessible cardinal, let $W$ be the symmetric Solovay model derived from $\kappa$, and let $G \subset P$ be a filter generic over $W$. In $W[G]$, let $a_0 = \{y \in 2^\omega : \exists p \in G \ p(y) = 0\}$ and $a_1 = \{y \in 2^\omega : \exists p \in G \ p(y) = 1\}$. In $W[G]$, $|a_0| \neq |a_1|$.

Thus, $2^\omega$ can be decomposed into two uncountable sets of distinct cardinalities, contradicting the full Silver dichotomy. In fact, one can prove all kinds of pathologies regarding the cardinalities of $a_0$ and $a_1$ respectively; for example $|a_0^\omega| \nleq |a_0|$ holds in $W[G]$.

Proof. Work in $W$. Suppose towards contradiction that $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \Vdash \tau : a_0 \rightarrow a_1$ is a bijection. The condition $p$ as well as the name $\tau$ have to be definable from some ground model parameters.
and some parameter $z \in 2^\omega$. Find an intermediate model $V[K]$ obtained as a generic extension of the ground model by a poset of size $< \kappa$ such that $z \in V[K]$. 

Work in the model $V[K]$. Let $\bar{p}$ be the $\text{Coll}(\omega, \kappa)$-name for the set \{\(q \in P: p \subset q \wedge \forall x \in V[K] \cap 2^\omega \setminus \text{dom}(p) \ q(x) = 0\}\}. This is a balanced virtual condition in $P$ for the model $V[K]$. For every point $x \in 2^\omega \setminus \text{dom}(p)$ let $y_x \in 2^\omega$ be a point such that $\text{Coll}(\omega, \kappa) \not\models \bar{p} \Vdash_P \tau(\bar{x}) = y_x$, if it exists. As $\tau$ is forced to be an injection from $\check{\dot{a}}_0$ to $\check{\dot{a}}_1$, for each $x$ there can be at most one $y_x$ of this kind, and the function $g: x \mapsto y_x$ must be an injection from $2^\omega \setminus \text{dom}(p)$ to $\text{dom}(p)$. Since the former set is uncountable and the latter is countable, there must be a point $x \in 2^\omega \setminus \text{dom}(p)$ which is not in the domain of $g$.

Still in the model $V[K]$, the statement that $x \not\in \text{dom}(y)$ means that $\text{Coll}(\omega, < \kappa)$ forces that either there is a condition below $\bar{p}$ in $P$ which forces $\tau(\bar{x})$ out of $V[K]$, or there are two distinct conditions in $P$ stronger than $\bar{p}$ which force the value $\tau(\bar{x})$ to be two distinct points in $V[K]$. Suppose for definiteness that the former is the case. Then there must be a poset $R$ of size $< \kappa$, an $R$-name $\sigma$ for a condition in $P$ stronger than $\bar{p}$ and an $R$-name $\eta$ for an element of $2^\omega \setminus V[K]$ such that $R \models \text{Coll}(\omega, < \kappa) \not\models \sigma \Vdash_P \tau(\bar{x}) = \eta$.

In the model $W$, pick filters $H_0, H_1 \subset R$ mutually generic over the model $V[K]$. Let $p_0 = \sigma/H_0 \in P$ and $p_1 = \sigma/H_1 \in P$; by the balance of the condition $\bar{p}$, $p_0$ and $p_1$ are conditions compatible in $P$, with some lower bound $q \in P$. Let $y_0 = \eta/H_0 \in 2^\omega$ and $y_1 = \eta/H_1 \in 2^\omega$; by the product forcing theorem and the choice of the name $\eta$, $y_0 \neq y_1$ holds. But then, $q \Vdash \tau(\bar{x}) = y_0$ and $\tau(\bar{x}) = y_1$, an impossibility. \qed

### 14.5 Definably balanced forcing

This section is devoted to the class of apparently the softest balanced Suslin extensions one can find: those in which the balanced virtual conditions can be found in an easily definable way.

**Definition 14.5.1.** Let $P$ be a Suslin forcing. We say that $P$ is *definably balanced* if for each condition $p \in P$ there is a balanced virtual condition $\bar{p} \leq p$ which is definable from a real parameter.

**Example 14.5.2.** Let $E$ be a Borel equivalence relation on a Polish space $X$ the poset $P$ of Example 6.3.9 adding a countable complete section of $E$ is definably balanced. Namely, let $Y = X^\omega$, let $B \subset Y$ be the set of those elements $y \in Y$ whose range consists of pairwise $E$-related points, and let $P$ consist of all countable sets $p \subset B$ such that if $y_0, y_1 \in p$ satisfy $[\text{rng}(y_0)]_E = [\text{rng}(y_1)]_E$ then in fact $\text{rng}(y_0) = \text{rng}(y_1)$ holds. The ordering on $P$ is that of inclusion. For every condition $p \in P$ one can look at the definable balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \text{Coll}(\omega, X), \tau \rangle$ where $\tau$ is the name for the set of all conditions $q \in Q$ such that $p \subset q$ and for every point $x \in V \setminus \bigcup_{y \in p}[\text{rng}(y)]_E$, $q$ contains an enumeration of the set $[x]_E \cap V$.

**Example 14.5.3.** Let $P$ be the poset of Section 8.4 adding a cofinal Kurepa family on a fixed Polish space $X$. For every condition $p \in P$ (a countable set
of countable subsets of $X$ closed under intersection) one can find a balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \text{Coll}(\omega, X), \tau \rangle$ where $\tau$ is the name for the condition obtained from $p$ by adding the set $X \cap V$. This is indeed a balanced pair as proved in Theorem 8.4.3, and it is definable from a real.

**Example 14.5.4.** Let $P$ be the partial ordering introducing a nontrivial automorphism of the algebra $\mathcal{P}(\omega)$ modulo finite as in Section 8.3. As proved in Theorem 8.3.3, the balanced virtual conditions are classified precisely by automorphisms of the algebra. At the same time, every condition (an automorphism of a countable subalgebra) can be extended into a total automorphism which is trivial, i.e. generated by a bijection between cofinite subsets of $\omega$ [7, Theorem 2.3]. Such a total automorphism is clearly definable from a real.

**Example 14.5.5.** Let $\Gamma$ be the clopen graph on $2^\omega$ connecting points $x_0 \neq x_1$ if the smallest number $n$ such that $x_0(n) \neq x_1(n)$ is even. Let $P$ be the partial order of Example 12.3.3 forcing a failure of OCA by adding an uncountable subset of $2^\omega$ containing no uncountable $\Gamma$-clique or $\Gamma$-anticlique. For every condition $p \in P$ (which is a pair $\langle a_p, b_p \rangle$ where $a_p \subset 2^\omega$ is a countable set and $b_p$ is a countable collection of closed $\Gamma$-cliques and closed $\Gamma$-anticliques) one can find a balanced virtual condition $\bar{p} \leq p$ represented by the pair $\langle \text{Coll}(\omega, 2^\omega), \tau \rangle$ where $\tau$ is the name for the pair $\langle a_p, \bar{b}_p \rangle$ where $\bar{b}_p$ is the set of all closed $\Gamma$-cliques and all closed $\Gamma$-anticliques coded in the ground model. This is indeed a balanced pair below $p$ as shown in Example 12.3.3 and it is definable from a real.

The preservation theorems we prove are best organized with a well-known factorization fact and its corollary.

**Fact 14.5.6.** Let $Q$ be the finite support product of $\omega_1$-many copies of the Cohen forcing. Let $G \subset Q$ be a generic filter.

1. Whenever $x \in V[G]$ is an element of $2^\omega$, $V[G]$ is a $Q$-extension of $V[x]$.

2. Suppose that $X, Y$ are Polish spaces in $V$, $E$ is a pinned Borel equivalence relation on $Y$ coded in $V$. In $V[G]$, if $x \in X$ is a point and $c \subset Y$ is an $E$-class definable from $x$ and some elements of the ground model, then $c$ has a representative in $V[x]$.

The following results are stated using Convention 1.8.17.

**Theorem 14.5.7.** In definably balanced extensions of the symmetric Solovay model, whenever $\Gamma$ is an analytic hypergraph of countable, possibly infinite arity on a Polish space $X$ of uncountable Borel chromatic number, then $\Gamma$ has uncountable chromatic number.

**Proof.** In view of the dichotomy theorem for uncountable Borel chromatic number of analytic hypergraphs of countable arity [53] it is enough to verify that the conclusion of the theorem holds for a specific analytic hypergraph $\Gamma$. Let $X$ be
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the dense $G_δ$ subspace of $ω^ω$ consisting of all functions $x ∈ ω^ω$ such that for infinitely many $n ∈ ω$, $x(n) > x(m)$ for all $m ∈ n$. Let $\{s_n : n ∈ ω\}$ be a collection of finite strings of natural numbers such that $s_n ∈ ω^n$ and $\{s_n : n ∈ ω\} ⊆ ω^ω$ is dense. Define the hypergraph $Γ$ to consist of all tuples $⟨x_m : m ∈ ω⟩$ such that there is $n ∈ ω$ such that $s_n$ is an initial segment of all points $x_m$, $x_m(n) = m$ holds for all $m$, and the tails $x_m \upharpoonright (n, ω)$ are the same for all $m ∈ ω$. [53] shows that the graph $Γ$ has uncountable Borel chromatic numbers and moreover, it can be continuously homomorphically mapped to any analytic hypergraph of uncountable Borel chromatic number. Thus, it is enough to prove the conclusion of the theorem for $Γ$.

Suppose that $κ$ is an inaccessible cardinal. Suppose that $P$ is a Suslin forcing such that $V_κ ⪰ P$ is definably balanced in all forcing extensions. Let $W$ be the symmetric Solovay model derived from $κ$ and work in $W$. Let $p ∈ P$ be a condition and let $τ$ be a $P$-name for a function from $X$ to $ω$; we must find a $Γ$-edge $⟨x_m : m ∈ ω⟩$, a number $k ∈ ω$ and some strengthening of the condition $p$ which forces $τ(⟨x_m⟩) = k$ for all $m ∈ ω$. The condition $p$ as well as the name $τ$ are defined from some parameters in the ground model and perhaps an additional parameter $z ∈ 2^ω$. Find an intermediate extension $V[K]$ of the ground model using a poset of size $< κ$ such that $z ∈ V[K]$, and work in the model $V[K]$.

Let $Q$ be the product of uncountably many copies of Cohen forcing. By the definable balance assumption, there must be a $Q$-name $η$ for a real number and a condition $q ∈ Q$ which identifies a definition of a balanced virtual condition $\check{p} ≤ p$ which uses only $η$ as a parameter. By Fact 14.5.6, passing to an intermediate extension if necessary we may assume that $η$ is in fact a name for a specific element of the ground model. For the sake of brevity, we ignore $η$ and $q$ below. By a balance argument with $\check{p}, Q ⪰ ∀x ∈ X \exists k ∈ Coll(ω, < κ) \models p \Vdash R(\check{x}) = k$. Let $\check{h}$ be the $Q$-name for the map $x → k$; thus, $\check{h}$ is forced to be a definable $Γ$-coloring.

Consider the Cohen poset $P_X$ with its name $\check{x}_gen$ for a generic point of the space $X$. Find a condition $O ∈ P_X$ and a number $k ∈ ω$ such that $O ⪰ Q ⪰ \check{h}(\check{x}_gen) = k$. Now, let $x ∈ O$ be a point $P_X$-generic over $V[K]$. Use a genericity argument to find a number $m$ large enough that $s_m$ is an initial segment of $x$ and $[s_m] \cap X ⊆ O$. For each number $n ∈ ω$ let $x_n ∈ X$ be the point obtained from $x$ by overwriting its $m$-th entry with $n$. Thus, each point $x_n$ is $Q_0$-generic over $V[K]$ and belongs to the set $O$. Now, let $G ⊆ Q$ be a filter generic over the model $V[K]$ such that $x ∈ V[K][G]$, and consider the definable balanced condition $\check{p} ≤ p$ in $V[K][G]$. By the forcing theorem applied in the model $V[K]$, in the model $W$ $\check{p} \Vdash Q(\check{x}_n) = k$ must hold for every number $n$. In view of the fact that the sequence $⟨x_n : n ∈ ω⟩$ is a $Γ$-hyperedge, the proof is complete.

Corollary 14.5.8. 1. Let $P$ be the poset for adding a nontrivial automorphism of the algebra $P(ω)$ modulo finite of Section 8.3. In the $P$-extension of the Solovay model, there is no discontinuous homomorphism between Polish groups.

2. It is consistent relative to an inaccessible cardinal that $ZF+DC$ holds,
there is a nontrivial automorphism of \( P(\omega) \) modulo finite, and there is no discontinuous homomorphism between Polish groups.

**Proof.** The poset \( P \) is definably balanced by Example 14.5.4. It is clearly \( \sigma \)-closed, so in the \( P \)-extension of the Solovay model, \( \text{ZF}+\text{DC} \) holds. By Theorem 14.5.7, in the \( P \)-extension the chromatic number of the Hamming graph \( H_2 \) is uncountable. By a \( \text{ZF}+\text{DC} \) result of [63], this abstractly implies that there are no discontinuous homomorphisms between Polish groups. \( \square \)

**Theorem 14.5.9.** In a definably balanced extension of the symmetric Solovay model, \( |E_1| \not\leq |F| \) holds for every pinned Borel orbit equivalence relation \( F \).

**Proof.** Let \( \Gamma \) be a Polish group continuously acting on a Polish space \( X \), inducing an equivalence relation \( F \). Suppose that \( F \) is pinned. Suppose that \( P \) is a Suslin forcing. Suppose further that \( \kappa \) is an inaccessible cardinal such that \( V_\kappa \models P \) is definably balanced in every forcing extension. Let \( W \) be the symmetric Solovay model derived from \( \kappa \) and work in \( W \). Suppose that \( p \in P \) is a condition and \( \tau \) is a \( P \)-name such that \( p \Vdash \tau \) is a function from the \( E_1 \)-quotient space to the \( F \)-quotient space. We must find a strengthening of the condition \( p \) and two distinct \( E_1 \)-classes such that their \( \tau \)-images are forced to be the same.

The condition \( p \) as well as the name \( \tau \) are definable from a parameter \( z \in 2^\omega \) as well as some parameters from the ground model. Let \( V[K] \) be an intermediate extension obtained from a poset of size \( \kappa \) such that \( z \in V[K] \). Work in the model \( V[K] \). Let \( Q \) be the finite support product of uncountably many copies of the Cohen forcing. Use the definable balance assumption to find a \( Q \)-name \( \eta \) for a real and a condition \( q \in Q \) which identifies a definition of a virtual balanced condition \( \bar{p} \leq p \) with a parameter \( \eta \). By Fact 14.5.6, passing into a larger intermediate extension, we may assume that \( \eta \) is in fact a name for a specific element of \( V[K] \). For the brevity of notation, we neglect \( \eta \) and \( q \) below.

Let \( H \subset Q \) be a filter generic over \( V[K] \) and work in the model \( V[K][H] \) for the moment. Note that the equivalence relation \( F \) is pinned there by Theorem 2.6.1. By a balance argument with \( \bar{p} \), for every \( E_1 \)-class \( c \) in the model \( V[K][H] \) there is an \( F \)-class \( d \) such that \( \text{Coll}(\omega, \kappa) \Vdash \bar{p} \Vdash \tau(c) = d \). Go back to \( V[K] \). Fix a \( Q \)-name \( \bar{h} \) for the map \( c \mapsto d \). Note that \( \bar{h} \) is forced to be a definable injection from the \( E_1 \)-quotient space to the \( F \)-quotient space.

Let \( y \in (2^\omega)^\omega \) be a sequence of points in \( 2^\omega \) generic over \( V[K] \) for the finite support product of the usual Cohen forcing. Working in the model \( V[K][y] \), by Fact 14.5.6, there is a point \( x \in X \) such that \( Q \Vdash \bar{h}([y]_{E_1}) = [x]_F \). Let \( \gamma \in \Gamma \) be a point generic over \( V[K][y] \) for the poset \( P_1 \). As in the proof of Theorem 4.1.1, \( V[K][\gamma \cdot x] \) contains no entries of the sequence \( y \), and therefore no representatives of the \( E_1 \)-class of \( y \). Since the class \( [y]_{E_1} \) is definable from \( \gamma \cdot x \) as the class which is mapped by \( h \) to \( [\gamma \cdot x]_F \), this contradicts Fact 14.5.6(2). \( \square \)

Using Example 14.5.2, we now get

**Corollary 14.5.10.** 1. Let \( P \) be the partial ordering adding a countable complete section to \( E_1 \). In the \( P \)-extension of the symmetric Solovay model, \( |E_1| \not\leq |F| \) holds for every pinned orbit equivalence relation \( F \).
2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, 
\(|E_1| \leq |F_2|\) and yet \(|E_1| \not\leq |F|\) for any pinned Borel orbit equivalence relation \(F\).
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