Geometric set theory

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Preface

We wish to present to the reader a fresh and exciting new area of mathematics: geometric set theory. The purpose of this research direction is to compare transitive models of set theory with respect to their extensional agreement and definability. It turns out that many fracture lines in descriptive set theory, analysis, and model theory can be efficiently isolated and treated from this point of view. A particular success is the comparison of various $\Sigma^1_2$ consequences of the Axiom of Choice in unparalleled detail and depth.

The subject matter of the book was rather slow in coming. The initial work, restating Hjorth’s turbulence in geometric terms and isolating the notion of a virtual quotient space of an analytic equivalence relation, existed in rudimentary versions since about 2013 in unpublished manuscripts of the second author. The joint effort [25] contained some independence results in choiceless set theory similar to those of the present book, but in decidedly suboptimal framework. It was not until January 2018 discovery of balanced Suslin forcing that the flexibility and power of the geometric method fully asserted itself. The period from that discovery was filled with intense wonder–passing from one configuration of models of set theory to another and testing how they separate various well-known concepts in descriptive set theory, analysis, and model theory. At the time of writing, geometric set theory seems to be an area wide open for innumerable applications.

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Chapter 1

Introduction

1.1 Main results

This paper provides a thoroughly modern look at independence results in choiceless ZF set theory. We look at $\Sigma^2_1$ consequences of Axiom of Choice which have immediate interplay with the concerns of modern descriptive set theory, and provide a uniform methodology for independence proofs among them. This should lead to improved comparative understanding of the methods used in descriptive set theory and model theory.

The models of ZF set theory under investigation are all similar. We start with the classical symmetric Solovay model [16, Theorem 26.14] originally constructed to show that it is consistent with choiceless set theory that all sets of reals are Lebesgue measurable and have the Baire property. Over the years, this model became one of the most thoroughly studied objects in set theory. All of our models are generic extensions of the Solovay model by $\sigma$-closed Suslin partial orders. This means that all of them use as an underlying assumption the existence of an inaccessible cardinal since the Solovay model is constructed from one. For the sake of brevity, we drop this technical assumption from the consistency results stated in this section. Also, all of our models satisfy the Axiom of Dependent Choices (DC) since DC holds in the Solovay model and it is preserved under $\sigma$-closed extensions.

It turns out that for most independence tasks within a wide syntactical class there is a nearly canonical choice of the Suslin partial order to look at. The Suslin partial order is then studied in the familiar context of set theory with choice to obtain a full analysis of the choiceless generic extension of the Solovay model in question. To support the general framework, we develop the concept of balanced Suslin forcing and balanced conditions in the said forcing. From one point of view, these are conditions closely reminiscent to master conditions in proper forcing; a similar scope of variations and preservation theorems appears. From another point of view, these are conditions naturally obtained by a straightforward generalization of the notion of a virtual equivalence class
to Suslin partial orders. The balanced conditions are obtained in ZFC as a consequence of various combinatorial applications of the axiom of choice, and typically are classified by the combinatorial objects obtained. Coming back full circle, they are used to analyze the choiceless extensions of the Solovay model by the Suslin partial orders.

The independence results and test problems addressed in this book can be divided into several broad categories, which correspond to various branches of algebra, analysis, and descriptive set theory.

1.1a By topic

Cardinalities of quotient spaces. The part of the book most intimately tied with traditional concerns of descriptive set theory is the work on cardinalities of quotient spaces. Given Polish spaces $X, Y$ and Borel equivalence relations $E, F$ on each, the descriptive set theorists study the question when there can be a Borel function $h : X \to Y$ which reduces $E$ to $F$. This line of work has been very successful in the last two decades.

In the ZF+DC context, the quotient spaces $X/E$ and $Y/F$ can have distinct cardinalities, and the existence of a Borel reduction implies the inequality $|E| \leq |F|$, where we abuse the notation to write $|E|$ for the cardinality of the quotient space $X/E$. On the other hand, the nonexistence of a Borel reduction is often connected with the possibility that ZF+DC cannot prove the cardinal inequality $|E| \leq |F|$. A number of our results deal with the possible consistent combinations of various cardinal inequalities in the context of ZF+DC. The methodological import of such results is that the various nonreducibility proof methods from descriptive set theory are independent from each other. The following consistency results use various benchmark equivalence relations identified in Section 1.3.

**Theorem 1.1.1.** (Theorem 10.1.28) It is consistent with ZF+DC that $|E_0| \leq |2^\omega|$ and yet there is no $E_0$-transversal.

To illustrate the degree of control we achieve, in the model for Theorem 1.1.1 we can also verify that $|E_1| \not\leq |F|$ holds for any orbit equivalence relation $F$, and that $|E| \not\leq |F|$ holds for any orbit equivalence relation $E$ of a turbulent group action and an equivalence $F$ of isomorphism of countable structures.

**Theorem 1.1.2.** It is consistent with ZF+DC that $E_1$ has a complete countable section and yet $|E_1| \not\leq |F|$ for any pinned orbit equivalence relation $F$.

Among the cardinal features of the model for Theorem 1.1.2, we can verify that $|E_0| \not\leq |2^\omega|$ holds and $|E| \not\leq |F|$ holds for any orbit equivalence relation $E$ of a turbulent group action and an equivalence $F$ of isomorphism of countable structures.

**Theorem 1.1.3.** (Theorem 8.5.7) Let $E$ be a pinned Borel equivalence relation on a Polish space. It is consistent with ZF+DC that $|E| \leq |E_0|$ and yet $|E_0| \not\leq |2^\omega|$.
1.1. MAIN RESULTS

The most interesting open question in this area is whether the cardinalities of countable Borel equivalence relations can be manipulated in the context of ZF+DC. For example, are there countable Borel equivalence relations $E, F$ such that both inequalities $|E| < |F|$ and $|F| < |E|$ are consistent with ZF+DC?

Model theory. Given a pre-geometry on a Polish structure (for example, the pre-geometry of linear span on a Polish vector space), the axiom of choice yields a maximal independent set (a basis for the vector space). This broad stroke erases many fine combinatorial distinctions between the various pre-geometries. These distinctions come back to light when independence results in choiceless set theory. A sample result:

**Theorem 1.1.4.** (Corollary 8.3.9 and 10.3.13) It is consistent with ZF+DC that there is a Hamel basis but no transcendence basis for the reals.

The proof highlights the difference between the modular pre-geometry of linear span and the non-modular geometry of algebraic closure. As such, it can handle any uncountable Polish vector space and any uncountable Polish field in a similar fashion, and preservation theorems for balanced forcing also allow simultaneous treatment of many cases simultaneously. Many questions remain open. To state the most egregious example, we do not know if the existence of a transcendence basis for the reals implies the existence of a well-ordering of the real line.

In order to handle the models resulting from maximal sets in pre-geometries, we have to develop the abstract notion of modular simplicial complex, which greatly generalizes the notion of modular pre-geometry. It turns out that such complexes as the acyclic complex in a given Borel graph or the acyclic complex in a directed Borel graph are modular, while they typically do not arise from any pre-geometry. Further fine model theoretic distinctions yield independence results of the following kind:

**Theorem 1.1.5.** It is consistent with ZF+DC that every Borel graph contains a maximal acyclic subgraph and there is no Hamel basis for the reals.

In another direction connecting model theory with choiceless ZF+DC arguments, we study theories which can have models on the various quotient spaces. For example, call a set $X$ a tournament cardinal if there is a tournament on it, or a linearly ordered cardinal if there is a linear ordering on it. Both of these notions are clearly inherited by sets of smaller cardinality, and every linearly ordered cardinal is a tournament cardinal. If all sets of reals have the Baire property, these notions are trivial on Borel quotients: they simply coincide with the smooth quotients. In the wider ZF+DC context, they take a life on their own:

**Theorem 1.1.6.** (Corollary 11.3.18) Let $E$ be a Borel equivalence relation on a Polish space $X$. It is consistent with ZF+DC that $X/E$ is a tournament cardinal while $2^\omega/E_0$ is not a linearly ordered cardinal.

**Theorem 1.1.7.** (Corollary 8.3.9 and 10.3.15) Let $E$ be a Borel equivalence relation on a Polish space $X$ classifiable by countable structures and let $F$ be an
orbit equivalence relation of some turbulent Polish group action on some Polish space $Y$. It is consistent with ZF+DC that $X/E$ is a linearly ordered cardinal yet $Y/F$ is not even a tournament cardinal.

**Ultrafilters.** The methods of the book can separate various types of ultrafilters on $\omega$ and other combinatorial objects in the context of ZF+DC.

**Theorem 1.1.8.** (Corollary 10.3.7, Corollary ?? and Theorem 13.4.4) In ZF+DC, there are no provable implications between the existence of a Hamel basis and existence of a nonprincipal ultrafilter on natural numbers.

**Theorem 1.1.9.** (Corollary 10.2.10) In ZF+DC, the existence of a nonprincipal ultrafilter on natural numbers does not imply the existence of one which is disjoint from the summable ideal.

Many combinations of implications remain unresolved. For example, we do not know how to construct a model of ZF+DC in which there is a nonprincipal ultrafilter on $\omega$, $|E_1| \leq |E_0|$ and yet $|E_0| \nleq |2^\omega|$.

**Uniformization.** Uniformization problems belong to the guiding lights of descriptive set theory. One of the most notorious versions of it is the countable to one uniformization, the statement that every subset of the plane with nonempty countable vertical sections contains the graph of a total function. In Chapter 9, we develop a satisfactory criterion which guarantees that various strong uniformization principles (including the countable-to-one uniformization as the humblest special case) hold in nearly all models under study. Among the consistency results stated so far, only the first one (Theorem 1.1.1) necessitates the failure of the countable to one uniformization, all others are consistent with it. Separating the various uniformization principles seems to be a difficult task beyond the techniques of the present book.

**Limitations of the method.** The balanced forcing offers a very flexible and powerful method for obtaining independence results in ZF+DC set theory. However, certain desirable types of independence results are outside of its reach. Certain basic limitations are encapsulated in the following theorems.

**Theorem 1.1.10.** (Corollary 8.1.2) In every balanced extension of the symmetric Solovay model, there is no $\omega_1$ sequence of distinct Borel sets of bounded Borel rank.

**Theorem 1.1.11.** (Theorem 11.1.1) In every balanced extension of the symmetric Solovay model, there is no maximal almost disjoint family of subsets of $\omega$.

**Theorem 1.1.12.** (Theorem 11.2.1 simplified) In every balanced extension of the symmetric Solovay model, there is no linearly ordered, unbounded set of elements of $\omega^\omega$ in the modulo finite domination ordering.

Thus, the balanced forcing cannot be used to prove for example the consistency of the possibility that $\aleph_1 \leq |E_1|$ but $\aleph_1 \nleq |E_0|$. For consistency results of this type, one has to reach to weakly balanced forcing ???
1.1. MAIN RESULTS

1.1b By model

One appealing way to present the work in this book is to consider several more or less canonical generic extensions in the Solovay model and list the statements that we know hold in them. There are also test problems that we do not know how to resolve. The reader needs to keep in mind that all the models are balanced extensions of the Solovay model $W$ and as such obey the general limitations described in the previous section.

**The Ramsey ultrafilter model.** Consider the poset $P(\omega)$ modulo finite and the associated extension of the Solovay model $W$. The generic filter is identified with a Ramsey ultrafilter, and well-known pre-existing results show that in a suitable sense, every Ramsey ultrafilter is generic over $W$ for the poset $P(\omega)$ modulo finite.

**Theorem 1.1.13.** In the model $W[U]$, the following statements hold:

1. (Corollary ??) $|E_0| \not\leq |2^\omega|$;
2. (Corollary 8.4.7) $|E_1| \not\leq |F|$ for any orbit equivalence relation $F$;
3. the quotient spaces of $E_0$ and $E_1$ are linearly orderable (Theorem 13.4.2), while the quotient spaces of $E_2$ (Corollary 10.2.10) and $F_2$ (Corollary 10.2.7) do not carry even any tournament;
4. (Example 8.1.9) if $E$ is a Borel equivalence relation and $A$ is a subset of the $E$-quotient space then either $|A| \leq \aleph_0$ or $|2^\omega| \leq |A|$;
5. (Corollary 10.2.10) every nonprincipal ultrafilter on $\omega$ has nonempty intersection with the summable ideal;
6. (Proposition 9.5.4) countable-to-one uniformization.

One can consider many variations, adding an ultrafilter which is not Ramsey. We study the cases of a union ultrafilter and a generic ultrafilter which is disjoint from a given $F_\sigma$-ideal. It is challenging to discern between the resulting models by sentences which do not mention ultrafilters.

There are many open questions about the Ramsey ultrafilter model. We do not know how to classify ultrafilters on $\omega$ in it. We do not know if there can be Borel equivalence relations $E, F$ such that $|E| \not\leq |F|$ holds in $W$ and $|E| \leq |F|$ holds in $W[U]$. In particular, we do not know if this can occur for countable Borel equivalence relations $E, F$ or in the situation where $E$ is the orbit equivalence relation of a turbulent action of a Polish group and an equivalence $F$ classifiable by countable structures.

**The Hamel basis model.** Consider the poset $P$ of countable sets of reals which are linearly independent over the rationals. The generic filter yields a Hamel basis $B \subset \mathbb{R}$. It does not seem to be easy to provide a simple criterion which would guarantee that a given Hamel basis is $P$-generic over the Solovay model. Still, we understand the theory of the model $W[B]$ fairly well:
Theorem 1.1.14. In the model $W[B]$, the following statements hold:

1. (Theorem 13.4.4) $|E_0| \leq |2^\omega|$;
2. (Theorem 13.4.7) $|E_1| \leq |F_2|$;
3. (Corollary 10.3.13) there is no transcendental basis for any uncountable Polish field;
4. (Corollary 10.3.7) there is no nonprincipal ultrafilter on $\omega$;
5. (Proposition 9.5.2) countable-to-one uniformization.

One can consider a different Polish vector space in place of $\mathbb{R}$ and add a basis to it; a natural example is $\mathcal{P}(\omega)$ viewed with the symmetric difference operation as a vector space over the binary field. The conclusions of the above theorem except for (2) remain in force. We do not know how to discern between the resulting models in general.

If suitable large cardinals exist, one can find a Hamel basis generic over the model $L(\mathbb{R})$, independently of the size or structure of the continuum. The model $L(\mathbb{R})[B]$ inherits all features quoted in Theorem 1.1.14.

The transversal models. Consider the poset $P$ of countable partial $E_0$-independent sets. The generic filter yields an $E_0$-transversal $T \subset 2^\omega$. The study of the resulting model is fairly straightforward with our methods:

Theorem 1.1.15. In the model $W[T]$, the following statements hold:

1. (Corollary 8.4.12) $|E_1| \not\leq |F|$ for any orbit equivalence relation $F$;
2. $|E| \not\leq |F|$ for any equivalence relation $E$ generated by a turbulent group action and equivalence relation $F$ classifiable by countable structures;
3. there is no Hamel basis;
4. (Corollary 10.3.7) there is no nonprincipal ultrafilter on $\omega$;
5. (Proposition 9.5.2) countable-to-one uniformization.

It is interesting to compare the model $W[T]$ with the model $W[I]$ obtained by adding a generic injection from the $E_0$-quotient space to $2^\omega$ studied in Theorem 10.1.28. In that model, there is no $E_0$-transversal, therefore the countable-to-one uniformization fails and the chromatic number of $E_0$ is uncountable.

One can add generic transversals to any Borel pinned equivalence relation $P$ by a balanced $\sigma$-closed Suslin forcings. The resulting models may obviously differ on the inequalities between various pinned quotient spaces, such as the items (1) and (2) of the above theorem. However, the impact on the combinatorial objects unrelated to cardinalities appears to be mostly identical.

If suitable large cardinals exist, one can find a transversal generic over the model $L(\mathbb{R})$, independently of the size or structure of the continuum. The model $L(\mathbb{R})[B]$ inherits all features quoted in Theorem 1.1.15.
1.1c Supporting results

While the ZF independence results are the raison d’etre of the present book, we do not wish to discount the importance of the supporting results that we had to prove in order to support the weight of the balanced forcing technology.

In Chapter 2, we develop the notion of the virtual quotient space for a given analytic equivalence relation on a Polish space. To size up the virtual quotient space, in Definition 2.5.1 we isolate cardinal invariants $\kappa(E)$ and $\lambda(E)$ of an equivalence relation $E$: $\lambda(E)$ is the cardinality of the virtual space, while $\kappa(E)$ is its forcing-theoretic complexity. The most important piece of information is that for Borel equivalence relations, these cardinals are well-defined and obey an explicit upper bound:

**Theorem 1.1.16.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Then $\kappa(E), \lambda(E) < \beth_1$.

As a result, the well-known Friedman–Stanley theorem [9] on nonreducibility of a jump $E^+$ of a Borel equivalence relation $E$ to $E$ is immediately translated to the Cantor theorem on the cardinality of the powerset: $\lambda(E^+) = 2^{\lambda(E)} > \lambda(E)$—Example 2.5.5. Other similar conceptual and brief nonreducibility results appear as well.

As is usual in the realm of equivalence relations, we aim to prove various dichotomies for the cardinals $\kappa(E)$ and $\lambda(E)$. The most attractive and informative of them characterizes the equivalence relations for which the cardinals are well-defined:

**Theorem 1.1.17.** Suppose that there is a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. Exactly one of the following occurs:

1. $\kappa(E), \lambda(E) < \infty$;
2. $E^\omega_1$ is almost Borel reducible to $E$.

In other results, we show that for orbit equivalence relations, the forcing-theoretic complexity of every element of the virtual quotient space can be characterized by a cardinal number (Theorem 2.8.4), and that the various notions defined are suitably absolute.

In Chapter 3, we develop a restatement of Hjorth’s method of turbulence in terms of comparison of forcing extensions. This turns out to be the most practical approach to turbulence for anyone familiar with the forcing relation:

**Theorem 1.1.18.** (Theorem 3.2.2) Let $\Gamma$ be a Polish group acting on a Polish space $X$ with all orbits meager and dense. The following are equivalent:

1. the action is generically meager;
2. $V[x] \cap V[\gamma \cdot x]$ whenever $\gamma \in \Gamma$ and $x \in X$ are mutually generic points.
To support a broad extension of ergodicity results due to Hjorth and Kechris [20, Theorem 12.5], we develop the classes of trim and virtually trim equivalence relations (the latter including all equivalence relations classifiable by countable structures) and prove:

**Corollary 1.1.19.** (Theorem 3.3.5) Let $E$ be the orbit equivalence relation on a Polish space $X$ resulting from a turbulent Polish group action. Let $F$ be an analytic, virtually trim equivalence relation on a Polish space $Y$ and let $h : X \to Y$ be a Borel homomorphism of $E$ to $F$. Then there is an $F$-class whose $h$-preimage is comeager in $X$.

It turns out that there is a version of turbulence for measure which leads to many of the same ergodicity results. It uses the notion of concentration of measure quite close to the abstract whirly actions on measure algebras due to [11].

**Theorem 1.1.20.** (Theorem 3.6.2) Let $\Gamma$ be a Polish group continuously acting on a Polish space $X$ with an invariant Borel probability measure and an invariant ultrametric. Suppose that the action has concentration of measure. Then $V[x] \cap V[\gamma \cdot x]$ whenever $x \in X$ is a random point and $\gamma \in \Gamma$ is a generic point mutually generic with $x$.

As usual with the forcing reconceptualizations of various notions in descriptive set theory, we have to show that the resulting notions are suitably absolute and evaluate their complexity–Theorem 3.5.6. There is also a convenient characterization of pairs of continuous functions which yield generic extensions with trivial intersections, entirely divorced from any group action context and still touchingly close to the original turbulence idea–Theorem 3.1.5.

1.2 Navigation

The structure of the book is quite complicated and some navigation advice will greatly enhance the reader’s experience.

The first two chapters are preparatory and do not contain essentially any choiceless independence results. They are interesting in their own right, and the work on balanced extensions of the symmetric Solovay model is probably incomprehensible without some appreciation for the concepts developed in the first two chapters.

**Chapter 2** defines and explores the notion of the virtual quotient space for an analytic equivalence relation $E$ on a Polish space $X$. On an intuitive level, a virtual equivalence class is one which may only exist in some forcing extension but still we already have a sensible calculus to speak about it.

Section 2.1 provides the basic definitions. If $P$ is a partial order and $\tau$ is a $P$-name for an element of $X$, the name is called $E$-pinned if its $E$-class does not depend on the generic filter on $P$. There is a natural equivalence $\bar{E}$ on pinned names, and the $\bar{E}$ equivalence classes are referred to as the virtual $E$-classes. Section 2.2 shows that if one has a definable structure on the $E$-quotient space, it is possible to consider the virtual version of that structure.
1.2. NAVIGATION

on the virtual quotient space. This is a structure \( \Pi_1 \)-elementarily equivalent to the original one. Structures such as virtual versions of partially ordered sets of Borel simplicial complexes will come handy later in the book.

Sections 2.3 and 2.4 deal with the most immediate concern: the classification problem for the virtual quotient spaces. The virtual equivalence classes should correspond to some immediately recognizable combinatorial objects, and in many important cases this hope is fulfilled. In a broad class of equivalence relations (the pinned equivalence relations of [17, Section 17]) the virtual quotient space is simply identical to the quotient space. Many jump-type operations on equivalence relations have a natural translation to operations on virtual quotient spaces; for example, the Friedman–Stanley jump [9] is translated to a powerset operation–Theorem 2.3.4. Virtual classes of isomorphism relations on countable structures naturally correspond to uncountable structures of the same type (Theorem 2.4.5), even though in some cases there may be mysterious virtual classes which are not classified in this way \( \ast \). Still, there are many wide open questions. For example, we do not know how to classify the virtual quotient space for the measure equivalence. The virtual space of homomorphism of second countable compact Hausdorff spaces seems to be naturally classified by compact Hausdorff spaces, but there is no theorem to that effect.

Section 2.5 deals with the next most natural concern: the attempt to size up the virtual quotient space in terms of cardinality. The effort is surprisingly successful, generating cardinal invariants of analytic equivalence relations which can take all kinds of exotic and informative values. Given an analytic equivalence relation \( E \), one defines the \textit{pinned cardinal} \( \kappa(E) \) as the minimal cardinal such that all virtual classes of \( E \) are represented by names on posets of size \( < \kappa \). One also defines \( \lambda(E) \) as the cardinality of the virtual quotient space, and \( \lambda(E, P) \) as the cardinality of the set of virtual classes which are represented by names on the poset \( P \). These cardinals respect the Borel reducibility order and so serve as viable tools for the Borel nonreducibility results. In addition, they interact well with a number of operations on equivalence relations. In a jarring development, these cardinals can connect Borel nonreducibility results to hard questions about singular cardinal arithmetic or similar concerns of transfinite set theory.

Chapter 3 restates and greatly generalizes Hjorth’s notion of turbulence in forcing terms. This development shows that nonturbulent equivalence relations are in fact parallel to pinned equivalence relations in a very precise sense. The forcing relation encapsulates many distracting estimates needed in the traditional treatment of turbulence, resulting in a clean and efficient general calculus.

Section 3.1 provides the motivating insight—namely, that building pairs of models \( V[y_0], V[y_1] \) where \( y_0 \in Y_0 \) and \( y_1 \in Y_1 \) are separately Cohen generic elements of their respective Polish spaces such that \( V[y_0] \cap V[y_1] = V \) depends on a suitable notion of a walk–Theorem 3.1.5. The comparison of generic extensions vis-a-vis their intersection is an important concern in the later parts of this book. Section 3.2 provides two basic examples. In the first, Hjorth’s notion of turbulence of group actions is restated in geometric terms–Theorem 3.2.2. The
second example provides a complete characterization of those analytic ideals on \( \omega \) such that it is possible to find points \( x_0, x_1 \in 2^{\omega} \) separately generic over \( V \) such that they are equal modulo \( I \) and \( V[x_0] \cap V[x_1] = V \)—Theorem 3.2.4.

Section 3.3 defines two classes of equivalence relations that are simple from the turbulent point of view. An equivalence relation \( E \) on a Polish space \( X \) is trim if any \( E \)-class represented in generic extensions with trivial intersection is represented in the ground model. \( E \) is virtually trim if any virtual \( E \)-class represented in such generic extensions is represented as a virtual class in the ground model. These definitions are preserved under Borel reducibility and under many natural operations on equivalence relations, as shown in Section 3.4. The main application is the generalization of an ergodicity theorem by Hjorth and Kechris [20, Theorem 12.5] (Theorem 3.3.5): any Borel homomorphism from a turbulent equivalence relation to a virtually trim one stabilizes on a comeager set.

Section 3.5 provides a necessary complement to the forcing development in this chapter: a theorem asserting that the notions introduced are absolute among all generic extensions and evaluating their descriptive complexity—Theorem 3.5.6. Finally, Section 3.6 develops turbulence of group actions for measure. It turns out that the appropriate concept uses the familiar notion of concentration of measure of \([30]\).

**Chapter 5** develops the basic general theory of balanced Suslin forcing, and develops several broad classes of examples. The treatment of Suslin forcing is quite different from that in [2] or similar treatments on the structure of the real line. Our partial orders are typically (but not necessarily) \( \sigma \)-closed, and serve to add combinatorial objects by explicit countable approximations in the context of the symmetric Solovay model. It is important to understand that the study of these partial orders takes place exclusively within set theory with choice.

Section 5.1 defines the notion of a virtual condition in a Suslin forcing \( P \). A virtual condition is an element of the completion of the poset \( P \) which only exists in some generic extension, but for which one can develop a sensible calculus of comparison already in the ground model, quite in parallel to virtual equivalence classes developed in Chapter 2. There are natural equivalence and ordering relations on virtual conditions in \( P \).

Section 5.2 defines a notion of a balanced condition in a Suslin forcing \( P \). A balanced condition is an element of the completion of the forcing \( P \) which exists in some generic extension and behaves very predictably in mutually generic extensions. There is a natural notion of equivalence on balanced conditions in \( P \). The motherload feature exploited throughout the rest of the book is the simple Theorem 5.2.6: every balanced class contains exactly one virtual condition. Thus, one can start the study of balanced virtual conditions in a given Suslin forcing \( P \).

**Chapter 6** analyzes a class of balanced forcings arising from simplicial complexes of various algebraic forms. Namely, let \( \mathcal{K} \) be a Borel simplicial complex on a Polish space \( X \). Let \( P_\mathcal{K} \) be the poset of countable subsets \( p \subset X \) such that \([p]^{<\aleph_0} \subset \mathcal{K}\), ordered by reverse inclusion. Many useful posets in this book arise in this fashion, and the balance properties of the posets depend on model
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theoretic properties of $K$. Section 6.2 introduces the most interesting notion: that of a modular simplicial complex (Definition 6.2.1), parallel to the complex of all free sets in a modular pre-geometry, with a number of examples and related notions. Section 6.4 investigates the quotient simplicial complexes on quotient spaces, and shows that the balanced conditions are naturally found in the related virtual quotient spaces.

**Chapter 7** analyzes various ways of adding a nonprincipal ultrafilter on $\omega$ to a choiceless model. The basic method, the poset of all infinite subsets of $\omega$ ordered by inclusion, is investigated in Section 7.1. This poset adds a Ramsey ultrafilter, and its balanced conditions are classified simply by ultrafilters. The resulting model has been investigated from many directions previously [6, 35]. Section 7.3 shows how to add a union ultrafilter; the most interesting insight—the balanced conditions are classified by idempotent ultrafilters (Theorem 7.3.7). Section 7.2 shows how to add an ultrafilter disjoint from a given $F_{\sigma}$-ideal.

**Chapter 8** compares cardinalities of quotient spaces in the balanced extensions of the symmetric Solovay model $W$. On the surface, this is an enterprise very similar to the traditional forcing concerns in the context of the axiom of choice. However, it is important to understand that the non-well-ordered quotient cardinals offer many more opportunities for meaningful independence work and also for surprising ZF results preventing various patterns of cardinal collapses.

Section 8.1 shows that no new inequalities between well-ordered cardinalities and quotient cardinalities are added by balanced forcing. It also provides a practicable criterion (Theorem 8.1.7) to check that $|2^\omega|$ remains the smallest non-well-ordered cardinality among the quotient spaces and their subsets—this is a strong version of the perfect set property. The criterion is easily verified for most of the partial orders adding ultrafilters; unlike the earlier proofs that the perfect set property holds in models like $W[U]$ where $U$ is a Ramsey ultrafilter, the verification needs no sophisticated Ramsey theoretic technology.

Section 8.2 shows that the nonreducibility results between analytic equivalence relations obtained by the comparisons of their cardinal invariants $\kappa$ and $\lambda$ mostly turn into cardinal inequalities in the Solovay model $W$, and these inequalities survive unharmed into the balanced extension.

Section 8.3 shows that the nonreducibility results obtained by Hjorth’s turbulence method turn into cardinal inequalities in certain classes of balanced extensions. The main contribution here is Definition 8.3.1, isolating the relevant class of trim balanced Suslin forcings. The trim balance is later used to rule out combinatorial objects from the generic extensions which have nothing to do with any group actions, such as nonprincipal ultrafilters.

Section 8.4 provides practical criteria for showing that the classical nonreducibility of $\mathbb{E}_1$ to any orbit equivalence relation translates into cardinal inequalities in certain classes of balanced extensions of the Solovay model. This turns out to require close study of nested sequences of models of ZFC ordered in the reverse $\omega$ type. The resulting tricky combinatorial work can be used to separate for example the existence of a Hamel basis from existence of maximal acyclic subgraphs in Borel graphs.
Chapter 9 deals with the question of whether countable-to-one uniformization and similar uniformization principles hold in the balanced forcing extensions. Section 9.1 provides the a practical criterion on a Suslin forcing $P$ which implies that many strong uniformization principles hold in the forcing extension of the Solovay model by $P$. It turns out that the criterion is satisfied in a great number of interesting cases, and its verification mostly follows from the careful classification of balanced conditions for $P$. A number of examples appears in Section 9.5. Sections 9.2, 9.3, and 9.4 study various uniformization principles in turn and show that they in fact hold in adequate extensions of the symmetric Solovay model. Lastly, Section 9.5 provides a great number of examples and non-examples.

Chapter 10 deals with the evaluation of chromatic numbers of Borel hypergraphs in balanced extensions. This makes it possible to rule out a number of other, seemingly unrelated combinatorial objects in these extensions via ZF+DC theorems proved elsewhere.

Section 10.1 develops the critical notion of a hypergraph charm (Definition 10.1.1) and shows that it serves well for the purposes of showing that the chromatic number of the hypergraph is large (Theorem 10.1.3). There are many examples, in particular showing that $|E_0| \leq |2^\omega|$ is possible in conjunction with no $E_0$-transversal (Corollary 10.1.30), and that it is possible to linearize any Borel quotient space while keeping $|E_0| > |2^\omega|$ (Corollary 10.1.25). Section 10.2 adapts these concepts to a situation where the hypergraph in question is really a quotient hypergraph, showing that in the Ramsey ultrafilter model there are no tournaments on the $E_2$ and $F_2$-quotient spaces; this was proved already in [25] without the general technology of the current book.

Section 10.3 develops a notion of a turbulent hypergraph (Definition 10.3.2) parallel to the turbulent group actions of Greg Hjorth, and uses it to guarantee that chromatic numbers of such hypergraphs remain high in trim balanced extensions of the symmetric Solovay model. One particularly interesting turbulent hypergraph is the set of quadruples $(r_i; i \in 4)$ of real numbers such that $r_0r_2 + r_1r_3 = 1$; this is used to show that in trim balanced extensions the reals have no transcendence basis over the rationals (Corollary 10.3.13). Another turbulent hypergraph is the set of all pairs $(x_0, x_1)$ of subsets of $\omega$ such that $x_0 \cap x_1$ is finite; this is used to show that in trim balanced extensions there are no nonprincipal ultrafilters on $\omega$ (Corollary 10.3.7).

The appendix 13 contains results which are in some sense tangential to the main developments in this book, all the while being necessary at various stages in the earlier chapters.

Section 13.1 contains a number of complexity calculations regarding generic extensions of countable models of ZFC. For example, if $\langle M_x : x \in 2^\omega \rangle$ is a Borel family of well-founded countable models of ZFC, how complex a task it is to find generic filters $\langle G_x : x \in 2^\omega \rangle$ for each so that the resulting extensions satisfy certain properties, etc. The resulting theorems may be more difficult to state properly than to prove, but they are really indispensable at several places in the book.

Section 13.2 deals with the finite support iterations of c.c.c. very Suslin
forcing notions, proving in particular several preservation theorems stated in descriptive terms: Corollary 13.2.13 (the iterations are very Suslin again), Theorem 13.2.16 (preservation of definable $\sigma$-centeredness) and Theorem 13.2.21 (preservation of definable Ramsey-centeredness). These theorems seem to be entirely unrelated to the rest of the book; however, it is apparently difficult to prove certain results of Section 10.1 without them. Interestingly, the iteration theorems are applied only to posets which from the standpoint of ZFC independence proofs are essentially worthless—in the ZFC context, they are isomorphic to the product of continuum many Cohen reals.

Section 13.3 deals with several aspects of product forcing. This may seem to be a nearly trivial subject, but there is a useful and surprising characterization of mutual genericity in Theorem 13.3.2. It has the advantage of speaking of the resulting models and not any specific forcings. This feature is significantly used in the deep and useful Theorem 4.3.7, where mutual genericity is verified without calculating any partial orders. It also allows a definition of mutual genericity for non-generic extensions, and gradations of mutual genericity indexed by ordinals, both novel concepts.

Section 13.4 contains ZF or ZF+DC results which connect various combinatorial objects, chromatic numbers, and orbit cardinalities. Most of them are quite simple, but there is a potential for deep results here. Theorem 13.4.7 in particular seems hard to believe at first: in ZF, if there is a basis for a separable Banach space over the rationals then $E_1$ has a countable complete section. The proof uses a nested sequence of models of ZFC and is in a precise sense complementary to the results of Section 8.4. Recent results of Christian Rosendal regarding discontinuous homomorphisms of Polish groups also indicate that the theory ZF+DC proves highly nontrivial and surprising implications between $\Sigma^2_1$ sentences. Perhaps, it will turn out that this one little section in comparison to the mass of the independence results proved in the previous chapters does not do the subject justice at all.

### 1.3 Notation and terminology

**Analytic equivalence relations.** A number of concepts and results in this book is stated in terms of Borel equivalence relations on a Polish space. The following definition records several benchmark relations which are used throughout the book.

**Definition 1.3.1.**

1. $E_0$ is the *Vitali equivalence* on $2^\omega$, connecting $x, y \in 2^\omega$ if they differ at only finite set of entries;

2. $E_1$ is the equivalence relation on $(2^\omega)\omega$ connecting $x, y$ if they differ at only finite number of entries;

3. $E_2$ is the relation on $2^\omega$ connecting $x, y$ if the sum $\Sigma\left\{\frac{1}{n+1} : x(n) \neq y(n)\right\}$ is finite;

4. $F_2$ is the equivalence relation on $(2^\omega)\omega$ connecting $x, y$ if $\text{rng}(x) = \text{rng}(y)$;
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5. \( H \subset C \) is the equivalence relation on \( \mathcal{P}(\omega \times \omega) \) connecting relations \( x, y \) if either both are illfounded or fail the axiom of extensionality or fail to have a maximal element, or they are isomorphic.

6. \( E_{\omega_1} \) is the equivalence relation on \( \mathcal{P}(\omega \times \omega) \) connecting relations \( x, y \) if either both are illfounded or are not linear orders, or they are isomorphic.

7. if \( \Gamma \) is a coanalytic class of structures on \( \omega \) invariant under isomorphism, \( E_{\Gamma} \) is the equivalence relation on countable structures on \( \omega \) connecting two such structures if they are both fail to belong to \( \Gamma \) or they are isomorphic.

8. if \( I \) is an ideal on \( \omega \) then \( = I \) on \( 2^{\omega} \) is the equivalence relation connecting \( x, y \) if \( \{ n \in \omega : x(n) \neq y(n) \} \in I \). There is an identically defined equivalence relation on \( (2^{\omega})^{\omega} \).

Borel equivalence relations are naturally ordered by Borel reducibility. In the case of analytic equivalence relations, the Borel reducibility relation exhibits certain pathologies, and it is best replaced by some of its strengthenings. This is the contents of the following definition.

Definition 1.3.2. Let \( E \) and \( F \) be analytic equivalence relations on respective Polish spaces \( X \) and \( Y \).

1. \( E \) is Borel reducible to \( F \), in symbols \( E \leq B F \), if there is a Borel function \( h : X \to Y \) such that \( \forall x_0, x_1 \in X \ x_0 E x_1 \iff h(x_0) F h(x_1) \).

2. \( E \) is almost Borel reducible to \( F \), in symbols \( E \leq a F \), if there is a Borel function \( h : X \to Y \) and a set \( Z \subset X \) consisting of countably many \( E \)-classes such that \( \forall x_0, x_1 \in X \setminus Z \ x_0 E x_1 \iff h(x_0) F h(x_1) \).

The most permissive comparison of equivalence relations is the one which compares just the cardinalities of the quotient spaces. The following abuse of notation is used throughout:

Definition 1.3.3. If \( E \) is an equivalence relation on a Polish space \( X \) then \( |E| \) denotes the cardinality of the quotient space \( X/E \).

The Silver dichotomy shows that for a Borel equivalence relation \( E \), \( |E| \) is either countable or \( |2^{\omega}| \leq |E| \). In the context of the axiom of choice, the latter disjunct implies that \( |E| = |2^{\omega}| \) since the quotient space \( X/E \) is a surjective image of \( 2^{\omega} \).

In choiceless context though, the dichotomies satisfying the latter disjunct may represent many different cardinalities, and this is the subject of study of several sections in this book. It is clear that \( E \leq F \) implies that \( |E| \leq |F| \), and in the context of the axiom of dependent choices, \( E \leq a F \) and \( |F| \) is uncountable implies that \( |E| \leq |F| \).

Analytic hypergraphs. A hypergraph on a set \( X \) is an arbitrary subset \( \Gamma \subset \mathcal{P}(X) \) consisting of nonempty sets. The elements of \( \Gamma \) will be called edges while the elements of \( X \) will be called vertices. We will be interested in analytic hypergraphs of finite arity. Say that \( \Gamma \subset [X]^{<\aleph_0} \) is an analytic hypergraph if \( X \) is Polish and the set \( \{ y \in X^\omega : \text{rng}(y) \in \Gamma \} \) is analytic. There is a number of standard hypergraphs that will follow us throughout.
Definition 1.3.4.  1. $G_0$ is the minimal analytic graph of uncountable Borel chromatic number. To obtain it, choose binary strings $s_n \in 2^n$ for each $n \in \omega$ so that $\{s_n : n \in \omega\} \subset 2^\omega$ is dense, and let $x \in G_0$ if there is a unique number $n$ such that $x(n) \neq y(n)$ and moreover for this number $n$, $s_n \subset x$;

2. $\mathbb{H}$, the Hamming graph on $2^\omega$, connects $x, y \in 2^\omega$ if there is exactly one $n \in \omega$ such that $x(n) \neq y(n)$;

3. if $G$ is an analytic graph on a Polish space $X$ without cycles of odd length, the bipartization graph of $G$ connects $x, y \in X$ if there is a $G$-path of odd length connecting $x, y$;

4. if $E$ is an equivalence relation on a Polish space $X$ then the polarization graph of $E$ is a graph on $X^2$ connecting pairs $(x_0, x_1)$ and $(y_0, y_1)$ if $x_0 E y_1$ and $x_1 E y_0$;

5. if $I$ is an analytic ideal on $\omega$ then the complement graph of $I$ is a graph on $2^\omega$ connecting sets $x, y$ if $x =_{I} 1 - y$.

Recall that the chromatic number $\chi(\Gamma)$ of the hypergraph $\Gamma$ is the smallest cardinal number $\kappa$ such that $X$ can be partitioned into pieces $X = \bigcup_{\alpha \in \kappa} X_\alpha$ in such a way that there is no edge $e \in \Gamma$ whose vertices all come from one and the same piece of the partition. This definition makes sense only in the context of the Axiom of Choice in which cardinals are well-ordered and therefore $\chi(\Gamma)$ must exist. We will study the chromatic number of hypergraphs on Polish spaces $X$ in the choiceless context, where only a limited version is available. We say that the chromatic number of the hypergraph $\Gamma$ is countable if $X$ can be partitioned into pieces $X = \bigcup_{n \in \omega} X_n$ in such a way that there is no edge $e \in \Gamma$ whose vertices all come from one and the same piece of the partition; otherwise, the chromatic number of $\Gamma$ will be termed uncountable. The Borel chromatic number of $\Gamma$ is countable if $X$ can be decomposed into countably many Borel sets $X = \bigcup_n X_n$ neither of which contains all vertices of a $\Gamma$-edge.

Forcing. A great part of this book is devoted to forcing. We use the Boolean notation: $q \leq p$ means that the condition $q$ is stronger, more informative than $p$. The formula $P \models \phi$ denotes the statement that every condition $p \in P$, $p \models \phi$. Every Polish space $X$ and every analytic set $A \subset X$ have a canonical interpretation in every generic extension, which commutes with all usual descriptive set theoretic operations on Polish spaces. For the detailed theory of interpretations, see [39]; we will use it without explicit mention as is customary in the current forcing practice. The interpretations obey two basic absoluteness rules:

Fact 1.3.5. (Mostowski absoluteness) If $M \subset N$ are transitive models of ZF+DC and $\phi(\vec{p})$ is a $\Sigma^1_1$-formula with parameters in $M$, then $M \models \phi(\vec{p})$ if and only if $N \models \phi(i\vec{p})$ where $i$ is the interpretation operation.

Fact 1.3.6. (Shoenfield absoluteness) If $M \subset N$ are transitive models of ZF+DC such that $\omega_1 \subset M$ holds, and $\phi(\vec{p})$ is a $\Pi^1_2$-formula with parameters in $M$, then $M \models \phi(\vec{p})$ if and only if $N \models \phi(i\vec{p})$ where $i$ is the interpretation operation.
In particular, interpretations of analytic equivalence relations are equivalence relations again, and interpretations of Polish groups and continuous Polish group actions on Polish spaces are Polish groups and continuous Polish group actions again.

Mutual relationships between forcing extensions of ZFC are captured in the following fact:

**Fact 1.3.7.** If $V$ is a model of ZFC, $V[G]$ its forcing extension, and $M$ is a model of ZFC such that $V \subseteq M \subseteq V[G]$, then $M$ is a forcing extension of $V$ and $V[G]$ is a forcing extension of $M$.

There are several standard forcing notions used throughout:

**Definition 1.3.8.**
1. If $X$ is a set then $\text{Coll}(\omega, X)$ is the poset of finite partial functions from $\omega$ to $X$ ordered by reverse extension;
2. if $\kappa$ is a cardinal then $\text{Coll}(\omega, < \kappa)$ is the finite support product of the posets $\text{Coll}(\omega, \alpha)$ for all $\alpha \in \kappa$;
3. if $X$ is a topological space then $P_X$ is the poset of all nonempty open subsets of $X$ ordered by inclusion. If $X$ is in addition Polish, then $\dot{x}_{\text{gen}}$ denotes the $P_X$-name for the generic point of $P_X$—the unique element of the interpretation of the Polish space $X$ in the $P_X$-extension which belongs to all open sets in the generic filter.
4. the Vitali forcing or $\mathbb{E}_0$ forcing is obtained as a quotient algebra. Let $I$ be the $\sigma$-ideal on $2^\omega$ generated by Borel partial $E_0$ transversals; the Vitali forcing is the poset of all Borel $I$-positive subsets of $2^\omega$ ordered by inclusion.

There is a number of basic properties of these partial orders used in the text below.

**Fact 1.3.9.** ([16, Corollary 26.11]) Let $\lambda$ be a cardinal and $P$ is a poset of size $< \lambda$. Suppose that $G \subseteq \text{Coll}(\omega, \lambda)$ is a generic filter and in $V[G]$, $H \subseteq P$ is a filter generic over $V$. Then there is a filter $K \subseteq \text{Coll}(\omega, \lambda)$ generic over $V[H]$ such that $V[G] = V[H][K]$.

**Fact 1.3.10.** ([16, Lemma 26.7]) Let $P$ be a forcing of size $\lambda > \aleph_0$ such that $P \models |\lambda| = \aleph_0$. Then $P$ is in the forcing sense isomorphic to $\text{Coll}(\omega, \lambda)$.

Let $\kappa$ be an inaccessible cardinal. The symmetric Solovay model derived from $\kappa$ is obtained as follows. Let $G \subseteq \text{Coll}(\omega, < \kappa)$ be a generic filter and in $V[G]$, form the model $W$ of all sets hereditarily definable from reals and elements of the ground model. The theory of the Solovay model has been thoroughly investigated throughout the years. We note the following:

**Fact 1.3.11.** [34] In $W$, $ZF+DC$ holds, every set has the Baire property and is Lebesgue measurable, and there is no uncountable well-ordered sequence of distinct Borel sets of bounded Borel rank.
During the investigation of the symmetric Solovay model, the following technical fact and its corollary about the symmetric Solovay model will be used without mention.

**Fact 1.3.12.** ([16, Section 26]) Let $\kappa$ be an inaccessible cardinal and let $W$ be the symmetric Solovay extension of $V$ associated with $\kappa$. Then

1. every set in $W$ is definable from parameters in $V$ and in $2^\omega$;
2. in $W$, whenever $M$ is a generic extension of $V$ then $W$ is a symmetric Solovay extension of $M$.

**Corollary 1.3.13.** Suppose that $\kappa$ is an inaccessible cardinal, $X$ is a Polish space, $\phi(x,\vec{y})$ is a formula of set theory with all free variables displayed, and $\vec{p}$ is a sequence of sets of the same length as $\vec{y}$. The following are equivalent:

1. $\operatorname{Coll}(\omega, < \kappa) \models \exists x \in X \phi(x, \vec{p})$;
2. there is a poset $R$ of size $< \kappa$ and an $R$-name $\sigma$ for an element of $X$ such that $R \models \operatorname{Coll}(\omega, < \kappa) \models \phi(\sigma, \vec{p})$.

In several cases, we will need the basics of the stationary tower forcing.

**Definition 1.3.14.** [26] Let $\kappa$ be an inaccessible cardinal. Let $Q_\kappa$ denote the stationary tower up to $\kappa$. That is, $Q_\kappa$ consists of sets $S$ such that $S \subset [\operatorname{dom}(S)]^{\aleph_0}$ is stationary and $\operatorname{dom}(S) \in V_\kappa$. The ordering is defined by $T \leq S$ if $\operatorname{dom}(S) \subset \operatorname{dom}(T)$ and $\{x \cap \operatorname{dom}(S) : x \in T\} \subset S$. If $G \subset Q_\kappa$ is a generic filter, $j : V \to M$ denotes the generic ultrapower derived from $G$.

**Fact 1.3.15.** [26] Let $\kappa$ be an inaccessible cardinal, $G \subset Q_\kappa$ be a generic filter, and $j : V \to M$ be the generic embedding.

1. if $\kappa$ is a Woodin cardinal in the ground model, then $\kappa = \omega_1^V[G]$ and $M^{\omega_1} \subset M$ in $V[G]$. In particular, $M$ is well-founded and identified with its transitive collapse;
2. if $\kappa$ is a weakly compact Woodin cardinal in the ground model, then for every $z \in 2^\omega$ in $V[G]$ there is a Woodin cardinal $\lambda < \kappa$ such that $G \cap Q_\lambda$ is generic over $V$ and $z \in V[G \cap Q_\lambda]$. In particular, the model $W = V(\mathbb{R}^V[G])$ is a symmetric Solovay extension of $V$ derived from $\kappa$.

On several occasions, certain finer properties of the Vitali forcing will be used, as recorded in the following fact. To make sense of the second item, consider the natural action of the rational Cantor group on $2^\omega$ and observe that it naturally extends to an action by automorphisms on the poset $P$.

**Fact 1.3.16.** Let $P$ be the Vitali forcing.

1. $P$ is a proper bounding forcing not adding any independent reals;
2. for every condition $B \in P$ there is a nonzero element $\gamma$ of the rational Cantor group such that $B \cap \gamma \cdot B \in P$. 
Chapter 2

The virtual realm

There are many quotient structures in mathematics. It turns out that a typical quotient structure allows a useful canonical extension to its virtual version. The purpose of this chapter is to lay the foundations of the theory of virtual structures.

2.1 Virtual equivalence classes

The basis of any quotient structure of interest in the present book is a Polish space $X$. A quotient structure worth its salt will also use an equivalence relation on the underlying Polish space $E$. In this section we indicate how to extend the quotient space $X/E$ in a canonical way to a potentially much larger set or class.

Definition 2.1.1. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P$ be a poset and $\tau$ a $P$-name for an element of $X$.

1. The name $\tau$ is $E$-pinned if $P \times P \models \tau_{\text{left}} E \tau_{\text{right}}$;
2. if $\tau$ is $E$-pinned, then the pair $\langle P, \tau \rangle$ will be called an $E$-pin.

The definition may puzzle a novice reader. Its meaning is best illustrated by the following proposition. A pinned $\tau$-name is one which in all forcing extensions points at the same $E$-class, even though that $E$-class may not have any representative in the ground model.

Proposition 2.1.2. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P$ be a poset and $\tau$ a $P$-name for an element of $X$. The following are equivalent:

1. $\tau$ is $E$-pinned;
2. in every forcing extension, if $G_0, G_1 \subset P$ are filters separately generic over the ground model, then $\tau/G_0 E \tau/G_1$ holds.
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Proof. (2) immediately implies (1) by considering the $P \times P$ extension. To see how (1) implies (2), suppose that $V[H]$ is a forcing extension and in $V[H]$ there are filters $G_0, G_1 \subset P$ separately generic over $V$. Let $G_2 \subset P$ be a filter generic over $V[H]$. By the product forcing theorem, $G_0, G_2 \subset P$ are mutually generic filters, and so are $G_1, G_2 \subset P$. Applying the assumption (1), we see that $\tau/G_0 E \tau/G_2 E \tau/G_1$, and so $\tau/G_0 E \tau/G_1$ by the transitivity of the equivalence relation $E$. □

The following is the archetypal example of a non-trivial pinned name.

Example 2.1.3. Consider the poset $P = \text{Coll}(\omega, 2^\omega)$ and its name $\tau$ for the generic surjection from $\omega$ to $(2^\omega)^V$. The name $\tau$ is $F_2$-pinned since no matter which generic filter $G \subset P$ one selects, the range of $\tau/G$ is the same: it is the set $(2^\omega)^V$. Clearly, it is a nontrivial name since the set $2^\omega$ is uncountable and so there is no ground model element of $(2^\omega)^\omega$ that can enumerate it and be equivalent to $\tau$.

Given an analytic equivalence relation $E$ on a Polish space $X$, the $E$-pinned names form a seemingly inexhaustible and complicated class. However, this class admits a natural equivalence relation which usually greatly clarifies matters:

Definition 2.1.4. Let $E$ be an analytic equivalence relation on a Polish space $X$. Suppose that $(P, \tau)$ and $(Q, \sigma)$ are $E$-pinned names on their respective posets. Define $(P, \tau) \bar{E} (Q, \sigma)$ if $P \times Q \Vdash \tau E \sigma$.

Proposition 2.1.5. The relation $\bar{E}$ is an equivalence. If $P, Q$ are posets with $E$-pinned names $\tau, \sigma$ on them, the following are equivalent:

1. $(P, \tau) \bar{E} (Q, \sigma)$;

2. in every forcing extension, if $G \subset P$ and $H \subset Q$ are filters separately generic over the ground model, then $\tau/G E \sigma/H$ holds.

Proof. $\bar{E}$ is clearly symmetric and reflexive by its definition. To see the transitivity, suppose that $(P_0, \tau_0) \bar{E} (P_1, \tau_1) \bar{E} (P_2, \tau_2)$. This means that $P_0 \times P_1 \times P_2 \Vdash \tau_0 E \tau_1 E \tau_2$, and by the transitivity of the equivalence relation $E$, $P_0 \times P_1 \times P_2 \Vdash \tau_0 E \tau_2$. By the Mostowski absoluteness between the $P_0 \times P_1 \times P_2$ extension and $P_0 \times P_2$ extension, it is the case that $P_0 \times P_2 \Vdash \tau_0 E \tau_2$ and consequently $(P_0, \tau_0) \bar{E} (P_2, \tau_2)$.

Now, (2) immediately implies (1) by considering the $P \times Q$ extension. To see how (1) implies (2), suppose that $V[K]$ is a forcing extension and in $V[K]$ there are filters $G \subset P, H \subset Q$ separately generic over $V$. Let $H' \subset Q$ be a filter generic over $V[K]$. By the product forcing theorem, $G, H'$ are mutually generic filters, and so are $H, H'$. Applying the assumption (1), we see that $\tau/G E \sigma/H' E \sigma/H$, and so $\tau/G E \sigma/H$ by the transitivity of the equivalence relation $E$. □

Definition 2.1.6. Let $E$ be an analytic equivalence relation on a Polish space $X$. 


1. The $E$-classes are referred to as the virtual $E$-classes;

2. if $z$ is a virtual $E$-class and in some generic extension $V[G]$ $y$ is an $E$-class, we say that $y$ is a realization of $z$ if for some (equivalently, all) representatives $\langle P, \tau \rangle \in z$ and $x \in y$, $V[G] \models P \models \tau E \bar{x}$ holds.

The following proposition is used throughout this book. It says that $E$-classes represented in mutually generic extensions must be realizations of a virtual $E$-class in the ground model.

**Proposition 2.1.7.** Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $P_0, P_1$ be partial orders and $G_0 \subset P_0$ and $G_1 \subset P_1$ be mutually generic filters. If $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ are $E$-equivalent points then $[x_0]_E$ is the realization of some virtual $E$-class from the ground model.

**Proof.** Suppose that $p_0 \in P_0$, $p_1 \in P_1$, $\tau_0$ is a $P_0$-name and $\tau_1$ is a $P_1$-name such that $\langle p_0, p_1 \rangle \models \tau_0 \models \tau_1$. It immediately follows that $\tau_0$ must be an $E$-pinned name on the poset $P \upharpoonright p_0$ and so $p_0 \models [\tau_0]_E$ is the realization of the virtual $E$-class represented by the pair $\langle P \upharpoonright p_0, \tau_0 \rangle$.

The main question surrounding the virtual $E$-classes is whether they can be classified in some informative way. Is there a proper class of virtual $E$-classes or just a set? If there is just a set, what is its cardinality? Do virtual $E$-classes correspond to some more tangible combinatorial objects? This chapter contains many good answers to similar questions, even though many problems remain unsolved.

### 2.2 Virtual structures

It is now possible to define virtual versions of quotient structures on Polish spaces.

**Definition 2.2.1.** An analytic quotient structure is a tuple $\mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle$ where

1. $X$ is a Polish space;

2. $E$ is an analytic equivalence relation on $X$;

3. for every $i \in \omega$, $R_i \subset X^n$ is an analytic relation which is invariant under $E$;

4. for every $j \in \omega$, $f_j \subset X^{m_j+1}$ is an analytic relation which is invariant under $E$, and in the quotient $X/E$ it is a graph of a function.

The structure $\mathcal{M}$ is Borel based if the equivalence relation $E$ is Borel.
There are many popular examples of analytic quotient structures. If $\langle G, \cdot \rangle$ is a Polish group and $H \subset G$ is an analytic normal subgroup, one can form the quotient group $G/H$. If $\langle X, \leq \rangle$ is an analytic partial ordering, one can form the separative quotient under the assumption that the quotient equivalence relation is analytic. Embeddability of countable structures forms an ordering on the quotients space of all countable structures modulo the equivalence relation of biembeddability etc.

If $\mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle$ is an analytic quotient structure, then we write $\mathcal{M}^* = \langle X^*, R_i^* : i \in \omega, f_j^* : j \in \omega \rangle$ for its associated structure on the actual quotient space $X/E$ of all $E$-classes.

**Definition 2.2.2.** Let $\mathcal{M} = \langle X, E, R_i : i \in \omega, f_j : j \in \omega \rangle$ be an analytic quotient structure. The *virtual version* of $\mathcal{M}$ is the tuple $\mathcal{M}^{**} = \langle X^{**}, R_i^{**} : i \in \omega, f_j^{**} : j \in \omega \rangle$ where

1. $X^{**}$ is the set or class of all virtual $E$-classes;
2. for each $i \in \omega$, $R_i^{**}$ is the relation on $X^{**}$ of arity $n_i$ given by $\langle \langle Q_k, \tau_k \rangle : k \in n_i \rangle \in R_i^{**}$ if $\prod_k Q_k \Vdash \langle \tau_k : k \in n_i \rangle \in \hat{R}_i$;
3. for each $j \in \omega$, $f_j^{**}$ is the relation on $X^{**}$ of arity $m_j + 1$ given by $\langle \langle Q_k, \tau_k \rangle : k \in m_j + 1 \rangle \in f_j^{**}$ if $\prod_k Q_k \Vdash \langle \tau_k : k \in m_j + 1 \rangle \in \hat{f}_j$.

The first proposition says that Definition 2.2.2 is sound and that we receive a structure with the same signature as the original one:

**Proposition 2.2.3.** The definition of $R_i^{**}$ and $f_j^{**}$ does not depend on the choice of representatives of the virtual classes. Moreover, $f_j^{**}$ is a graph of a (total) function.

**Proof.** The statements “$R_i, f_j$ are $E$-invariant relations” and “$f_j^{**}$ is a graph of a total function in the quotient” are $\Pi^1_1$ and therefore absolute between $V$ and all forcing extensions. The first sentence of the proposition immediately follows. For the second sentence, suppose that the function $f_j$ has arity $m_j$ and $\langle \langle Q_k, \tau_k \rangle : k \in m_j \rangle$ is a tuple of $E$-pins. Let $Q = \prod_k Q_k$ and let $\tau$ be a $Q$-name for an element of $X$ such that $f_j(\tau_k : k \in m_j, \tau)$ is forced to hold. Since in the $Q \times Q$-extension, $f_j$ is a graph of a function on the quotient and the names $\tau_k$ for $k \in m_j$ are $E$-pinned, it follows that the name $\tau$ is $E$-pinned as well. Clearly, the tuple $\langle \langle Q_k, \tau_k \rangle : k \in m_j, \langle Q, \tau \rangle \rangle$ belongs to $f_j^{**}$ and the second sentence of the proposition follows. \[\square\]

In the case of an analytic quotient structure, it is possible that its virtual version is a proper class. In particular, it is possible to express the whole ordinal axis as an isomorph of a virtual version of an analytic quotient structure—Example 2.4.6. However, if the equivalence relation $E$ is Borel then the virtual version is a set of size $< \beth_1$ by Theorem 2.5.6. In any case, it is possible to stratify $\mathcal{M}^{**}$ into set sized pieces:
2.3. CLASSIFICATION: GENERAL THEOREMS

**Definition 2.2.4.** Let $\mathcal{M} = (X, E, R_i : i \in \omega, f_j : j \in \omega)$ be an analytic quotient structure. $\mathcal{M}^*\kappa$ is the substructure of $\mathcal{M}^{**}$ consisting of the virtual $E$-classes represented by names on $\text{Coll}(\omega, \kappa)$.

A proof identical to that of Proposition 2.2.3 shows that $\mathcal{M}^*\kappa$ is closed under all functions of $\mathcal{M}^{**}$ and so it is truly a substructure of $\mathcal{M}^{**}$. An elementary name-counting argument shows that for each infinite $\kappa$, the structure $\mathcal{M}^*\kappa$ has size at most $2^{\kappa}$. If $\kappa \leq \lambda$ are cardinals then $\text{Coll}(\omega, \kappa)$ is regularly embedded in $\text{Coll}(\omega, \lambda)$ and so $\mathcal{M}^*\kappa \subseteq \mathcal{M}^*\lambda$. Every poset is regularly embedded in $\text{Coll}(\omega, \kappa)$ for some $\kappa$ and so $\mathcal{M}^{**}$ decomposes into a monotone union $\bigcup \mathcal{M}^*\kappa$.

For each analytic quotient structure $\mathcal{M}$, there is a canonical embedding $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}$ which maps each class $[x]_E$ to the virtual $E$-class of $\langle Q, \tilde{x} \rangle$ for a trivial poset $Q$. The most important fact about the virtual structures is that there is some degree of elementaricity:

**Proposition 2.2.5.** The canonical embedding $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}$ is $\Pi_1$-elementary in $L_{\omega_1}\omega$ logic.

**Proof.** Let $\kappa$ be a cardinal, and let $G \subset \text{Coll}(\omega, \kappa)$ be a generic filter. In the model $V[G]$, let $\chi : (\mathcal{M}^{**})^V \rightarrow (\mathcal{M}^*)^{V[G]}$ be the map sending a virtual $E$-class in $(\mathcal{M}^{**})^V$ to its realization. Thus, we have maps $(\mathcal{M}^*)^V \xrightarrow{\chi} (\mathcal{M}^{**})^V \xrightarrow{\pi} (\mathcal{M}^*)^{V[G]}$. The composition $\chi \circ \pi$ sends each equivalence class in $V$ to its interpretation in $V[G]$.

Let $\phi$ be a $\Pi_1$ formula with possible parameters such that $\mathcal{M}^* \models \phi$. The statement $\mathcal{M}^* \models \phi$ is a $\Pi_1^3$ sentence about the structure $\mathcal{M}$, and by Shoenfield absoluteness it transfers from the ground model $V$ to the generic extension $V[G]$. Thus, the map $\chi \circ \pi$ is a $\Pi_1$ elementary embedding from $(\mathcal{M}^*)^V$ to $(\mathcal{M}^*)^{V[G]}$. It follows that the map $\pi : \mathcal{M}^* \rightarrow \mathcal{M}^{**}$ in $V$ must be a $\Pi_1$-elementary embedding. Since $\mathcal{M}^{**}$ is an increasing union $\bigcup \mathcal{M}^*\kappa$, the proposition follows. \qed

In particular, if the original quotient structure was a group, a partial order, or an acyclic graph, its virtual version maintains these properties. However, it is important to understand that the embedding does not have to be $\Sigma_2$-elementary, and so virtual versions of connected graphs may become disconnected, virtual versions of divisible groups may not be divisible anymore, and virtual versions of nonatomic partial orders may have atoms. The study of these apparent pathologies is the fuel and fire of this book.

2.3 Classification: general theorems

In this section, we provide a number of general classification theorems for virtual equivalence classes. The theorems are all of the same type: if $F$ is an analytic equivalence relation which is obtained from another equivalence relation $E$ using a certain operation, then all virtual $F$-classes are obtained from virtual $E$-classes using a similar operation. This will provide a suitable background to the investigation of specific cases in Section 2.4.
To start with, great many analytic equivalence relations yield only utterly uninteresting virtual classes: only those already realized in the ground model. This phenomenon is isolated in the following definition.

**Definition 2.3.1.** Let $E$ be an analytic equivalence relation on a Polish space $X$. A virtual $E$-class represented by $\langle P, \tau \rangle$ is trivial if there is $x \in X$ such that $P \vdash \tau E \check{x}$. The equivalence relation $E$ is pinned if it has only trivial virtual classes.

The class of pinned equivalence relations has been investigated for a number of years. The basic pre-existing knowledge about this class is subsumed in the following fact.

**Fact 2.3.2.** [17, Theorem 17.1.3] The analytic equivalence relations in the following classes are pinned:

1. orbit equivalence relations generated by actions of Polish cli groups;
2. Borel equivalence relations with all classes $\Sigma^0_3$;
3. equivalence relations Borel reducible to pinned ones.

The operation on equivalence relations which has the most informative translation into virtual classes is that of the Friedman–Stanley jump.

**Definition 2.3.3.** Let $E$ be an analytic equivalence relation on a Polish space $X$. The Friedman–Stanley jump of $E$ is the equivalence relation $E^+$ on the space $Y = X^\omega$ defined by $y_0 E^+ y_1$ if $[\text{rng}(y_0)]_E = [\text{rng}(y_1)]_E$.

Here, the classification of pinned names is right at hand: a pinned name for the jump is essentially just a set of pinned names for the original equivalence relation. For a nonempty set $S = \{\langle P_i, \tau_i \rangle : i \in I\}$ of representatives of virtual $E$-classes, let $\tau_S$ be the name on the poset $Q_S = \prod_i P_i \times \text{Coll}(\omega, I)$ for an element of $X^\omega$ enumerating the set $\{\tau_i : i \in I\}$.

**Theorem 2.3.4.** Let $E$ be an analytic equivalence relation on a Polish space $X$.

1. If $S$ is a set of $E$-pinned names then $\langle Q_S, \tau_S \rangle$ is an $E^+$-pinned name;
2. $\langle Q_S, \tau_S \rangle$ $E^+$ $\langle Q_T, \tau_T \rangle$ iff the sets $S, T$ represent the same set of virtual $E$-classes;
3. whenever $\langle P, \tau \rangle$ is an $E^+$-pinned name, there is a set $S$ of $E$-pinned names such that $\langle P, \tau \rangle$ $E^+$ $\langle Q_S, \tau_S \rangle$.

**Proof.** Items (1) and (2) are immediate. To prove (3), suppose that $\tau$ is an $E^+$-pinned name on a poset $P$. For every virtual $E$-class $y$, the statement $\phi(y) = "\text{rng}(\tau) contains a realization of the class } y^\gamma"$ must be decided in the same way by every condition in $P$. Let $S$ be any set of $E$-pinned names which
collects representatives from all virtual \( E \)-classes \( y \) such that \( \forall p \in P \; p \Vdash \phi(y) \).

We claim that the set \( S \) works.

Indeed, suppose that \( G_0, G_1 \subset P \) are mutually generic filters. The sets \( \text{rng}(\tau/G_0) \) and \( \text{rng}(\tau/G_1) \) are equal, and by Proposition 2.1.7 they contain only realizations of ground model virtual \( E \)-classes. By the choice of the set \( S \), these sets contain exactly realizations of virtual \( E \)-classes represented by names in \( S \). Since the generic ultrafilter \( G_0 \) was arbitrary, \( \langle P, \tau \rangle \overset{E}{\Vdash} \langle Q_S, \tau_S \rangle \) as desired.

Example 2.3.5. The relation \( \mathbb{F}_2 \) is the Friedman–Stanley jump of the identity on \( X = 2^\omega \). The identity is pinned by Fact 2.3.2 and so its virtual classes can be identified with elements of \( X \). The \( \mathbb{F}_2 \)-classes then correspond to subsets of \( 2^\omega \).

The countable product of equivalence relations translates to the virtual realm without change.

Definition 2.3.6. For each \( n \in \omega \), let \( E_n \) be an analytic equivalence relation on a Polish space \( X_n \). The product \( \Pi_n E_n \) is the equivalence relation \( E \) on \( Y = \prod_n X_n \) defined by \( y_0 E y_1 \) if for every \( n \in \omega \), \( y_0(n) E_n y_1(n) \).

For every function \( g \) such that \( \text{dom}(g) = \omega \) and for all \( n \in \omega \) \( g(n) \) is some \( E_n \)-pinned name \( \langle Q_n, \tau_n \rangle \), let \( \tau_g \) be the name on the poset \( Q_g = \prod_n Q_n \) (the support applied in the product is irrelevant for the equivalence class of the resulting \( \Pi_n E_n \)-pin) for the sequence \( \langle \tau_n : n \in \omega \rangle \), which is forced to be an element of \( Y \). The following is nearly immediate.

Theorem 2.3.7. For each \( n \in \omega \), let \( E_n \) be analytic equivalence relations on respective Polish spaces \( X_n \) and let \( E = \Pi_n E_n \).

1. If \( g \) is a sequence of \( E_n \)-pinned names, then \( \langle Q_g, \tau_g \rangle \) is an \( \Pi_n E_n \)-pin;
2. \( \langle Q_g, \tau_g \rangle \overset{E}{\Vdash} \langle Q_h, \tau_h \rangle \) iff for each \( n \in \omega \), \( g(n) \overset{E_n}{\Vdash} h(n) \);
3. whenever \( \langle P, \tau \rangle \) is an \( E \)-pinned name, there is a function \( g \) such that \( \langle P, \tau \rangle \overset{E}{\Vdash} \langle Q_g, \tau_g \rangle \).

The countable increasing union of equivalence relations translates to the virtual realm without change as well.

Theorem 2.3.8. Let \( \{ E_n : n \in \omega \} \) be an increasing sequence of analytic equivalence relations on a Polish space \( X \), and let \( E = \bigcup_n E_n \).

1. Whenever \( \langle Q, \sigma \rangle \) is an \( E_n \)-pinned name for some \( n \in \omega \) then it is also \( E \)-pinned;
2. whenever \( \langle P, \tau \rangle \) is an \( E \)-pinned name, there is a number \( n \in \omega \) and an \( E_n \)-pinned name \( \langle Q, \sigma \rangle \) such that \( \langle P, \tau \rangle \overset{E}{\Vdash} \langle Q, \sigma \rangle \).
Proof. (1) is immediate as $E_n \subset E$. For (2), by the forcing theorem there must be conditions $p_0, p_1 \in P$ and a number $n$ such that $\langle p_0, p_1 \rangle \Vdash_{P \times P} \exists \tau \left( E_n \tau \right)$. The transitivity of the relation $E_n$ then shows that $\tau$ on the poset $P \upharpoonright p_0$ is an $E_n$-pinned name. The initial assumptions show that $\langle P, \tau \rangle \bar{E} \langle P \upharpoonright p_0, \tau \rangle$, as desired.

Example 2.3.9. The Louveau jump survives into the virtual realm without change. The Louveau jump of an analytic equivalence relation $E$ on a Polish space $X$ is the equivalence relation $E^{+L}$ on $Y = X^\omega$ connecting $y_0, y_1 \in Y$ if for all but finitely many $n \in \omega$, $y_0(n) E y_1(n)$. The Louveau jump can be written as a countable increasing union of countable products of $E$, which by Theorems 2.3.7 and 2.3.8 yields a complete analysis of its virtual classes in terms of virtual $E$-classes. In particular, if $E$ is pinned then its Louveau jump is pinned.

The virtual realm also correctly reflects the situation in which the equivalence classes of one relation consist of countably many equivalence classes of another one.

Definition 2.3.10. Let $E, F$ be analytic equivalence relations on a Polish space $X$. We say that $F$ is countable over $E$ if $E \subset F$ and every $F$-class consists of countably many $E$-classes.

Theorem 2.3.11. Let $E, F$ be analytic equivalence relations on a Polish space $X$, with $F$ countable over $E$.

1. If $\langle Q, \sigma \rangle$ is an $E$-pinned name then it is $F$-pinned as well;

2. if $\langle P, \tau \rangle$ is an $F$-pinned name then there is an $E$-pinned name $\langle Q, \sigma \rangle$ such that $\langle P, \tau \rangle \bar{F} \langle Q, \sigma \rangle$.

Proof. (1) is immediate. For (2), it will be enough to show that there is a condition $p \in P$ such that $\tau$ is an $E$-pinned name on $P \upharpoonright p$, for then $\langle P, \tau \rangle \bar{F} \langle P \upharpoonright p, \tau \rangle$ as desired.

Suppose towards contradiction that there is no condition $p \in P$ such that $\tau$ is $E$-pinned on the poset $P \upharpoonright p$. It follows by the forcing theorem that $P \times P \Vdash \neg \exists \tau \left( E \tau \right)$ holds. Let $M$ be a countable elementary submodel of a large structure containing $P, \tau$ and the codes for $E, F$. Use Theorem 13.3.2 to find an uncountable collection $\{ g_i : i \in I \}$ of filters on $M \cap P$ pairwise mutually generic over the model $M$. By the Mostowski absoluteness between the models $M[g_i, g_j]$ and $V$ for $i \neq j \in I$, the elements $\tau/g_i \in X$ for $i \in I$ are pairwise $F$-related, but pairwise $E$-unrelated, contradicting the initial assumptions on the relations $E, F$.

Example 2.3.12. The Clemens jump survives into the virtual realm without change. Here the Clemens jump of an analytic equivalence relation $E$ on a Polish space $X$ is the equivalence relation $E^{+C}$ on $Y = X^Z$ connecting $y_0, y_1 \in Y$ if there is $n \in Z$ such that for every $m \in Z$ $y_0(m) E y_1 (m + n)$.
2.4 Classification: specific examples

There are many analytic equivalence relations for which the virtual space can be classified by more tangible combinatorial objects, but which do not fit into the context of the theorems of Section 2.3. The purpose of this section is to investigate these more difficult, but also more informative, possibilities.

The most interesting issues arise in equivalence relations classifiable by countable structures. Among these, the Borel equivalence relations are Borel reducible to an iterated jump of the identity [17, Theorem 12.5.2], and so can be handled by Theorems 2.3.4 and 2.3.8. For example, for all equivalence relations \( E \) Borel reducible to \( \mathbb{F}_2 \), the virtual \( E \)-classes are classifiable by subsets of \( 2^\omega \) by Example 2.3.5. More interesting issues arise with analytic equivalence relations. Let \( E \) be the equivalence relation of isomorphism of structures on \( \omega \). It is natural to attempt to classify virtual classes of \( E \) by uncountable structures as in the following definition.

**Definition 2.4.1.** Let \( M \) be a (possibly uncountable) structure of a countable signature. \( \tau_M \) is a \( \text{Coll}(\omega, M) \)-name for some structure on \( \omega \) isomorphic to \( M \).

It is immediate that the pair \( \langle \text{Coll}(\omega, M), \tau_M \rangle \) is an \( E \)-pin, and its \( \bar{E} \)-equivalence relation does not depend on the choice of the name \( \tau_M \). It turns out that the \( \bar{E} \)-equivalence relation on the \( E \)-pins obtained in this way coincides with a familiar concept of model theory:

**Definition 2.4.2.** [28, Section 2.4] Let \( M, N \) be structures with the same signature. Say that \( M, N \) are Ehrenfeucht–Fraïssé-equivalent if Player II has a winning strategy in the Ehrenfeucht–Fraïssé game. In the EF-game, the two players take turns, at round \( i \in \omega \) Player I starting with an element \( n_i \in N \) or \( m_i \in M \) and Player II responding with an element \( m_i \in M \) or \( n_1 \in N \) respectively. After all rounds indexed by \( i \in \omega \) have been completed, Player II wins if the map \( n_i \mapsto m_i \) for \( i \in \omega \) preserves all relations and functions of \( N, M \) in the signature.

**Theorem 2.4.3.** Let \( M, N \) be models with the same countable signature. The following are equivalent:

1. \( M, N \) are Ehrenfeucht–Fraïssé equivalent;
2. \( \langle \text{Coll}(\omega, M), \tau_M \rangle \bar{E} \langle \text{Coll}(\omega, N), \tau_N \rangle \).

**Proof.** For the (1)\( \rightarrow \) (2) direction, if \( M \) is Ehrenfeucht–Fraïssé equivalent to \( N \) as witnessed by a winning strategy \( \sigma \) for Player I in the EF-game, then \( \text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models \tau_M \bar{E} \tau_N \), since a generic run of the EF-game in which Player II
follows the strategy $\sigma$ will generate an isomorphism between $M$ and $N$ in the extension. For the $(2) \rightarrow (1)$ direction, suppose that $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models \tau_M E \tau_N$ and let $\pi : M \rightarrow N$ be a product name for the isomorphism. The winning strategy for Player II can be described as follows: as the game develops, Player II also maintains on the side conditions $q_i \in \text{Coll}(\omega, M) \times \text{Coll}(\omega, N)$ such that $q_0 \geq q_1 \geq \ldots$ and $q_i \models \pi(\check{m}_i) = \check{n}_i$. It is immediate that this is possible and Player II must win in the end.

However, in general, there are virtual $E$-classes which are not represented by a straightforward collapse name as in Definition 2.4.1. In order to extract more information, we have to restrict to more specific classes of structures.

**Definition 2.4.4.** Let $\Gamma$ be a coanalytic set of structures on $\omega$, closed under isomorphism.

1. $\bar{E}_\Gamma$ is the equivalence relation on structure on $\omega$ connecting $M, N$ if either both $M, N$ fail to belong to $\Gamma$ or else $M, N$ are isomorphic;

2. a (possibly uncountable) structure $M$ is a $\Gamma^{**}$-structure if $\text{Coll}(\omega, M) \models \tau_M \in \Gamma$.

We proceed to show that for some interesting coanalytic classes $\Gamma$, every $\bar{E}_\Gamma$-class is represented by a collapse name of a $\Gamma^{**}$-structure as in Definition 2.4.1. In the classification results, we always ignore the trivial class of structures which do not belong to $\Gamma$.

**Theorem 2.4.5.** Let $\Gamma$ be a coanalytic class of countable structures on $\omega$, invariant under isomorphism, consisting of rigid structures only.

1. For $\Gamma^{**}$-structures, the Ehrenfeucht–Fraissé equivalence and isomorphism coincide;

2. for every $\bar{E}_\Gamma$-pin $\langle P, \sigma \rangle$ there is a $\Gamma^{**}$-structure $M$ such that $\langle P, \sigma \rangle \bar{E} \langle \text{Coll}(\omega, M), \tau_M \rangle$.

**Proof.** Before we begin the argument, note that the statement that every structure in the set $A$ is rigid is $\Pi^1_2$ and so holds also in all generic extensions by the Shoenfield absoluteness.

For (1), it is clear that isomorphic structures are Ehrenfeucht–Fraissé equivalent. For the opposite implication, suppose that $M, N$ are $\Gamma^{**}$ structures which are EF-equivalent. By Theorem 2.4.3, $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models \tau_M E \tau_N$ must hold. As $\Gamma$ consists of rigid structures even in the collapse extension, $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N) \models$ there is a unique isomorphism $\pi : M \rightarrow N$. Since $\text{Coll}(\omega, M) \times \text{Coll}(\omega, N)$ is a homogeneous notion of forcing, for each $m \in M$ the value of $\pi(\check{m})$ is decided by the largest condition to be some $h(m) \in N$. The function $h : M \rightarrow N$ is an isomorphism of $M$ to $N$ present already in $V$.

For (2), let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over $V$. In the model $V[G_0, G_1]$, let $N_0 = \tau/G_0$ and $N_1 = \tau/G_1$. To define the model $M \in V$, let $x_0 =$
\{s: s is the Scott sentence of the model \langle N_0, a \rangle \text{ for some } a \in N_0 \} \in V[G_0] \text{ and } x_1 = \{s: s \text{ is the Scott sentence of the model } \langle N_1, a \rangle \text{ for some } a \in N_1 \} \in V[G_1].

Since \( N_0 \) is isomorphic to \( N_1 \), it follows from Karp’s theorem [10, Lemma 12.1.6] that \( x_0 = x_1 \), so \( x_0 = x_1 \in V[G_0] \cap V[G_1] \in V \). The set \( x_0 \) will be the universe of the model \( M \). Note that since the model \( N_0 \) is rigid, the elements of \( N_0 \) are in one-to-one correspondence with \( x_0 \) by Karp’s theorem again and the unique isomorphism between \( N_0 \) and \( N_1 \) factors through the identity on the set \( x_0 = x_1 \).

To construct the realizations of relational and functional symbols of the model \( M \), for every relational symbol \( R \) (say binary) of the language of the models and \( s, t \in x_0 \), let \( s \overset{R}{\leftrightarrow} t \) if for the unique \( a, b \in N_0 \) such that \( s \) is the Scott sentence of \( a \) and \( t \) is a Scott sentence of \( b \), \( N_0 \models s \overset{R}{\leftrightarrow} t \). The same definition using the model \( N_1 \) yields the same relation, and so \( R^M \in V[G_0] \cap V[G_1] = V \). Define the realizations of all functional and relational symbols of the model \( M \) in this way. As a result, \( M \) is a model in \( V \) and the map \( a \mapsto \text{the Scott sentence of } \langle M, a \rangle \) is an isomorphism of \( N_0 \) and \( M \) in the model \( V[G_0] \). Thus, (2) follows. \( \square \)

Theorem 2.4.5 makes it possible to describe some class-sized virtual spaces explicitly:

**Example 2.4.6.** The virtual \( \mathbb{E}_{\omega^1} \)-classes are precisely classified by ordinals, since \( \mathbb{E}_{\omega^1} = \mathbb{E}_\Gamma \) where \( \Gamma \) is the class of all well-orderings on \( \omega \). Well-orderings are rigid, and up to isomorphism are classified by ordinals.

**Example 2.4.7.** The virtual \( \mathbb{H}C \)-classes are classified by transitive sets with the \( \varepsilon \)-relation.

In the case of non-rigid structures, the classification may become more complicated. We will treat the case of acyclic graphs, which has the virtue of being Borel-complete among the equivalence relations classifiable by countable structures. Note that the Ehrenfeucht–Fraïssé equivalence on uncountable acyclic graphs is distinct from isomorphism, as the case of an empty graph on \( \aleph_0 \) shows.

**Theorem 2.4.8.** Let \( \Gamma \) be the class of all acyclic graphs on \( \omega \). For every \( \mathbb{E}_\Gamma \)-pin \( \langle P, \sigma \rangle \) there is an acyclic graph \( H \) such that \( \langle P, \sigma \rangle \cong \langle \text{Coll}(\omega, H), \tau_H \rangle \).

**Proof.** The point of the proof is that an acyclic graph can be explicitly built from automorphism orbits of its elements. This procedure is captured in the following observation. Suppose \( x \) is a set, \( f: x^2 \to \omega + 1 \) is a function such that \( f(s, t) > 0 \iff f(t, s) > 0 \), and \( g: x^2 \to \omega + 1 \) is a function such that \( f(s, t) > 0 \) implies \( g(s) = g(t) \). Then there is, up to an isomorphism unique, acyclic graph \( H(x, f, g) \) together with an onto map \( h: y \to x \), where \( y \) is the set of vertices of \( H(x, f, g) \), such that

- for every \( s, t \in x \) and every vertex \( v \in y \), if \( h(v) = s \) then the set of all neighbors of \( v \) mapped to \( t \) has size \( f(s, t) \);

- for every \( s \in x \) there are \( g(s) \) many connected components of the graph \( H \) containing a vertex \( v \) with \( h(v) = s \).
The construction of the graph $H(x, f, g)$ is straightforward. Note that whenever $u, v \in y$ are two vertices such that $h(u) = h(v)$ then there is an automorphism of the graph $H(x, f, g)$ sending $u$ to $v$.

Now, suppose that $\sigma$ is an $E$-pinned name on a poset $P$ and let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over $V$. In the model $V[G_0, G_1]$, let $H_0 = \sigma/G_0$ and $H_1 = \sigma/G_1$. To define the graph $H \in V$, let $x_0 = \{s: s$ is the Scott sentence of the model $\langle H_0, v \rangle$ for some vertex $v$ of $H_0 \} \in V[G_0]$ and $x_1 = \{s: s$ is the Scott sentence of the model $\langle H_1, v \rangle$ for some vertex $v$ in $H_1 \} \in V[G_1]$. Since $H_0$ is isomorphic to $H_1$, it follows from Karp’s theorem ([10, Lemma 12.1.6]) that $x_0 = x_1$, so $x = x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. Let $f: x^2 \to \omega + 1$ be the function defined by $f(s, t) = i$ if every vertex of $H_0$ of type $s$ has $i$-many neighbors of type $t$ when $i \in \omega$, and $f(s, t) = \omega$ if every vertex of $H_0$ of type $s$ has infinitely many neighbors of type $t$. Let $g: x \to \omega + 1$ be the function defined by $g(s) = i$ if there are $i$-many connected components of $H_0$ containing a node of type $s$ when $i \in \omega$, and $g(s) = \omega$ if there are infinitely many connected components of $H_0$ containing a node of type $s$. Note that these functions are well-defined and the graph $H_0$ is isomorphic to $H(x, f, g)$ in the model $V[G_0]$.

Virtual spaces for equivalence relations which are not classifiable by countable structures are often quite difficult to understand. To conclude this section, we state another classification theorem and a couple of open questions.

**Fact 2.4.9.** [19, Section 4] Let $E$ be the equivalence relation of isomorphism of structures on $\omega$.

1. If $P$ is a poset forcing $|N_2^P| > \aleph_0$ then every $E$-pin $\langle P, \sigma \rangle$ is $\bar{E}$-equivalent to $\langle \text{Coll}(\omega, M), \tau_M \rangle$ for some structure $M$ of size $\leq \aleph_1$;

2. if $P$ forces $|N_2^P| = \aleph_0$, then there is an $E$-pin $\langle P, \sigma \rangle$ which is not $\bar{E}$-equivalent to $\langle \text{Coll}(\omega, M), \tau_M \rangle$ for any structure $M$.

Virtual spaces for equivalence relations which are not classifiable by countable structures are often quite difficult to understand. To conclude this section, we state another classification theorem and a couple of open questions.

**Theorem 2.4.10.** Let $E$ be the equivalence relation on $X = (P(\omega))^{\omega}$ connecting $x_0, x_1$ if $\text{rng}(x_0)$ and $\text{rng}(x_1)$ generate the same filter on $\omega$. The virtual $E$-classes are classified by filters on $\omega$.

**Proof.** On one hand, if $F$ is a filter on $\omega$, one can consider the $\text{Coll}(\omega, F)$-name $\tau_F$ for a generic enumeration of the filter $F$. It is immediate that the name $\tau_F$ is $E$-pinned, and distinct filters yield inequivalent names.

For the more difficult part, suppose that $P$ is a partial order and $\tau$ is an $E$-pinned name on $P$; we must find a filter $F$ on $\omega$ such that $\tau$ is equivalent to $\tau_F$. To do this, let $F = \{a \subset \omega: \exists p \vdash \bar{a} \text{ belongs to the filter generated by } \text{rng}(\tau)\}$. By the pinned property of the name $\tau$, the existential quantifier in
the definition of $F$ can be replaced by universal without changing the resulting set $F$. It follows immediately that the set $F$ is a filter; we must show that $\tau$ is equivalent to $\tau_F$.

Suppose towards contradiction that this fails; then it must be the case that for some condition $p \in P$ and some number $n \in \omega$, $p \Vdash \tau(\check{n})$ has no subset in the filter $F$. Since the name $\tau$ is $E$-pinned, there must exist conditions $p_0, p_1 \leq p$ and a finite set $a \subseteq \omega$ such that $\langle p_0, p_1 \rangle \Vdash \bigcap_{m \in a} \tau(m) \subseteq \tau(\check{n})$. Let $b = \{ k \in \omega : \exists r \leq p_1 \ r \Vdash \hat{k} \in \bigcap_{m \in a} \tau(m) \}$ and let $c = \{ k \in \omega : p_0 \Vdash \hat{k} \in \tau(\check{n}) \}$. Observe that $b \in F$ (since $p_1 \Vdash \bigcap_{m \in a} \tau(m) \subseteq \check{b}$) and $c \notin F$ (since $p_0 \Vdash \check{c} \subseteq \tau$). Thus, there has to be a number $k \in b \setminus c$, and for this number $k$ there are conditions $r_0 \leq p_0$ and $r_1 \leq p_1$ such that $r_1 \Vdash \hat{k} \in \bigcap_{m \in a} \tau(m)$ and $r_0 \Vdash \hat{k} \notin \tau(\check{n})$. This, however, contradicts the choice of the conditions $p_0, p_1$.

Consider the equivalence relation $E$ of homeomorphism of compact metrizable spaces. This is known to be a largest equivalence relation reducible to an orbit equivalence in the sense of Borel reducibility [40]. In an attempt to describe its virtual space, consider any compact Hausdorff space $X$ with a topology basis of size $\kappa$, and consider the $\text{Coll}(\omega, \kappa)$-name $\tau_X$ for the interpretation of the space $X$ in the extension in the sense of [39]. The interpretation of $X$ will have a countable basis, and so the interpretation will be a compact metrizable space. The basic theory of interpretations shows that $\langle \text{Coll}(\omega, \kappa), \tau_X \rangle$ is an $E$-pin. The most natural question is open:

**Question 2.4.11.** Is every virtual $E$-class represented by a compact Hausdorff space?

The measure equivalence $E$ is one of the hardest equivalence relations to understand. It connects two Borel probability measures $\mu, \nu$ on the Cantor space if they share the same ideal of null sets. The relation $\mathbb{F}_2$ Borel reduces to $E$, and so $E$ is not pinned.

**Question 2.4.12.** Classify the virtual space for the measure equivalence.

### 2.5 Cardinal invariants

There are several cardinal invariants of equivalence relations which are associated with the concept of virtual equivalence classes. They respect the Borel reducibility order and so they are useful as tools for nonreducibility results. They also come conceptually handy in several places in this book.

#### 2.5a Basic definitions

The following definition records the most natural cardinal invariants associated with the virtual spaces.

**Definition 2.5.1.** Let $E$ be an analytic equivalence relation on a Polish space $X$. 
1. \( \kappa(E) \), the pinned cardinal of \( E \), is the smallest \( \kappa \) (if it exists) such that every virtual \( E \)-class has a representative on a poset of size \( < \kappa \). If \( E \) is pinned, we let \( \kappa(E) = \aleph_1 \). If \( \kappa(E) \) does not exist, we write \( \kappa(E) = \infty \);

2. \( \lambda(E) \) is the cardinality of the set of all virtual \( E \)-classes. If the virtual \( E \)-classes form a proper class, we let \( \lambda(E) = \infty \);

3. \( \lambda(E, P) \) is the cardinality of the set of all virtual \( E \)-classes represented on the poset \( P \).

The rather mysterious demand that \( \kappa(E) = \aleph_1 \) for all pinned equivalence relations \( E \) is explained by a reference to Theorem 2.8.2: there are no nontrivial pinned names on countable posets for any analytic equivalence relation. It turns out that the cardinals \( \kappa(E) \) and \( \lambda(E, P) \) can attain all kinds of exotic and informative values. We will start with the two archetypal and somewhat boring computations.

**Example 2.5.2.** \( \kappa(F_2) = c^+ \) and \( \lambda(F_2) = 2^c \).

*Proof.* This follows immediately from the classification of pinned names for the Friedman–Stanley jump. Every virtual \( F_2 \) class is represented by a subset of \( 2^\omega \), and distinct subsets of \( 2^\omega \) give rise to distinct virtual \( F_2 \) classes. \( \square \)

**Example 2.5.3.** \( \kappa(E_{\omega_1}) = \infty \). To see this, let \( \Gamma \) be the class of all countable well-orders, so \( E_{\omega_1} = E_\Gamma \). Note that all elements of \( \Gamma \) are rigid. By Theorem 2.4.5, the virtual \( E_{\omega_1} \)-classes are classified by ordinals, and so there is a proper class of them.

Theorem 2.7.1 shows that in fact \( E_{\omega_1} \) is a minimal example of an equivalence relation \( E \) with \( \kappa(E) = \infty \).

The most appealing fact about the cardinal invariants \( \kappa(E) \) and \( \lambda(E, P) \) is that they respect the Borel reducibility order and therefore can be used to prove Borel nonreducibility results. In a good number of instances, the comparison of the cardinal invariants is the fastest and most intuitive way of proving nonreducibility. One of the main features of this style of argumentation is that it automatically survives the transfer to nonreducibility by functions more complicated than Borel.

**Theorem 2.5.4.** Let \( E, F \) be analytic equivalence relations on Polish spaces \( X, Y \) respectively. If \( E \leq_a F \) then \( \kappa(E) \leq \kappa(F) \) and \( \lambda(E, P) \leq \lambda(F, P) \) holds for every partial order \( P \).

*Proof.* Suppose that \( h: X \to Y \) is a Borel function witnessing the reduction of \( E \) to \( F \) everywhere except for a set \( Z \subset X \) consisting of countably many \( E \)-classes. By a Shoenfield absoluteness argument, these properties of the function \( h \) transfer to all generic extensions. If \( P \) is a partial ordering and \( \tau \) is an \( E \)-pinned name on \( P \) which is not a name for one of the classes in the set \( Z \), then \( h(\tau) \) is an \( F \)-pinned name on \( P \), and the map \( \tau \mapsto h(\tau) \) respects virtual \( E \)- and
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F-classes, and it is an injection from the former to the latter. It follows that \( \lambda(E, P) \leq \lambda(F, P) \).

Now, suppose that \( Q \) is another poset and that \( h(\tau) \) has a \( \bar{F} \)-equivalent name \( \sigma \) on \( Q \). By the Shoenfield absoluteness, \( Q \models \exists x \in X \setminus Z \ h(x) = \tau \); any \( Q \)-name for such an \( x \) is \( E \)-pinned and the name for \( h(x) \) is \( \bar{F} \)-equivalent to \( h(\tau) \). A brief bout of diagram chasing now shows that \( \kappa(E) \leq \kappa(F) \) and the theorem follows.

The following neat application has been observed by [37] for the case of Borel equivalence relations classified by countable structures.

Example 2.5.5. Friedman and Stanley proved that the Friedman–Stanley jump of a Borel equivalence relation \( E \) is not Borel reducible to \( E \). The most appealing proof of this fact uses the cardinal invariant \( \lambda(E) \). Theorem 2.5.6 below shows that for Borel equivalence relations, the value of \( \lambda(E) \) is an actual cardinal as opposed to \( \infty \). Theorem 2.3.4 shows that virtual \( E^+ \)-classes are classified by sets of virtual \( E \)-classes, in other words \( \lambda(E^+) = 2^{\lambda(E)} \). Theorem 2.5.4 then completes the argument.

2.5b Estimates

The key fact about the pinned cardinal is the following theorem. It shows that there are a priori bounds on the size of the pinned cardinal; in particular, if the equivalence relation \( E \) is Borel, then \( \kappa(E) \leq \beth_\omega \) and the virtual space of \( E \) is a set.

Theorem 2.5.6. Let \( E \) be a Borel equivalence relation on a Polish space \( X \) of rank \( \Pi^0_\alpha \). Then \( \kappa(E) \leq (\beth_\alpha)^+ \).

Proof. Let \( \tau \) be an \( E \)-pinned name on a poset \( P \); we must produce a Coll\((\omega, \beth_\alpha)\)-name \( \sigma \) which is \( E \)-related to \( \tau \). Note that \( [\tau]_E \) is a \( P \)-name for a Borel set of rank \( \leq \alpha \). As is the case for every name for a Borel set, [34, Corollary 2.9] shows that in the Coll\((\omega, \beth_\alpha)\) extension \( V[G] \) there is a Borel code for a Borel set \( B \subset X \) such that in every further forcing extension \( V[G][H] \) and every \( x \in X \cap V[G][H] \) in that extension, \( x \in B \) if and only if \( V[x] \models P \models \bar{x} \in [\tau]_E \).

Note that if \( H \subset P \) is generic over \( V[G] \), then the set \( B \) is nonempty in \( V[G][H] \), containing the point \( \tau/H \); this follows from the fact that \( \tau \) is \( E \)-pinned. Thus, the set \( B \) is nonempty already in \( V[G] \) by the Mostowski absoluteness between \( V[G] \) and \( V[G][H] \). Back in \( V \), let \( \sigma \) be any Coll\((\omega, \beth_\alpha)\)-name for an element of the set \( B \). This clearly works.

Example 2.5.7. For any given countable ordinal \( \alpha \), let \( \Gamma_\alpha \) be the class of binary relations on \( \omega \) which are extensional and wellfounded of rank \( < \alpha \); let \( E_\alpha \) be the isomorphism relation. It is not difficult to check (Theorem 2.4.5) that for each countable ordinal \( \alpha \), the relation \( E_\alpha \) is Borel, and its pinned names are collapse names for isomorphs of the membership relation on sets in \( V_\alpha \). Thus, the cardinals \( \kappa(E_\alpha) \) converge to \( \beth_{\omega_1} \).
Theorem 2.5.8. Let $E$ be an analytic equivalence relation almost reducible to an orbit equivalence relation of a continuous Polish group action. If $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first $\omega_1$-Erdős cardinal.

Proof. Let $\kappa$ be the first $\omega_1$-Erdős cardinal, and suppose that $\kappa(E) > \kappa$; we must show that $\kappa(E) = \infty$. Since the cardinal $\kappa$ is the Hanf number for the class of wellfounded models of first order sentences, for every cardinal $\lambda$ there is a wellfounded model $M$ such that $M \models \kappa(E) > \lambda$. Now, since $E$ is almost reducible to an orbit equivalence relation, Corollary 2.6.4 shows that the wellfounded model $M$ is correct about $\kappa(E)$ to the extent that $|\kappa(E)^M| \leq \kappa(E)$. It follows that $\kappa(E) > \lambda$, and since $\lambda$ was arbitrary, $\kappa(E) = \infty$. \hfill \Box

Example 2.5.9. For every countable ordinal $\alpha$ there is a coanalytic class $\Gamma$ of structures on $\omega$, invariant under isomorphism, such that $\kappa(\mathbb{E}_\Gamma) =$the first $\alpha$-Erdős cardinal.

Proof. Let $\Gamma$ be the class of all binary relations on $\omega$ which are extensional, wellfounded, and do not admit a sequence of indiscernibles of ordertype $\alpha$; we claim that this class works.

Clearly, $\Gamma$ is a coanalytic set of rigid structures invariant under isomorphism. Write $E = \mathbb{E}_\Gamma$ and $\kappa =$the first $\alpha$-Erdős cardinal. By Theorem 2.4.5, every virtual $E$-class is represented by a transitive set $A$ without indiscernibles of ordertype $\alpha$. It must be the case that $|A| < \kappa$ and so $\kappa(E) \leq \kappa$. On the other hand, whenever $\lambda < \kappa$ is an ordinal, then the structure $(\mathbb{V}_{\lambda}, \in)$ has no indiscernibles of ordertype $\alpha$, and it remains such in every forcing extension by a wellfoundedness argument. Thus, the $\text{Coll}(\omega, \mathbb{V}_{\lambda})$-name for the generic isomorph of this structure is $E$-pinned, and it is not equivalent to any $E$-pinned name on a poset of size $|\mathbb{V}_{\lambda}|$ since it entails the collapse of $|\mathbb{V}_{\lambda}|$ to $\aleph_0$. Thus, $\kappa(E) = \kappa$ as desired. \hfill \Box

Theorem 2.5.10. Let $E$ be an analytic equivalence relation on a Polish space $X$. Let $\kappa$ be a measurable cardinal. If $\kappa(E) < \infty$ then $\kappa(E) < \kappa$.

Proof. Suppose that there is a poset $P$ and an $E$-pinned name $\tau$ on $P$ which is not $E$-related to any name on a poset of size $< \kappa$. We will produce a proper class of pairwise non-$E$-related $E$-pinned names.

First note that the poset $P$ and the name $\tau$ can be selected so that $|P| = \kappa$. Simply take an elementary submodel $M$ of size $\kappa$ of large structure with $\mathbb{V}_\kappa \subset M$ and consider $Q = P \cap M$ and $\sigma = \tau \cap M$; so $|Q| = \kappa$. As $M$ is correct about pinned names and the equivalence $\tilde{E}$ by a Shoenfield absoluteness argument, $\sigma$ is an $E$-pinned name on $Q$ and it is not $\tilde{E}$-equivalent to any pinned names on posets of size $< \kappa$.

Thus, assume that the poset $P$ has size $\kappa$. Let $j: V \rightarrow N$ be an elementary embedding into a transitive model with critical point equal to $\kappa$. Note that $H(\kappa) \subset N$ and so both $P, \tau$ are (isomorphic to) elements of $N$. Let $\{N_\alpha, j_\beta: \beta \in \alpha\}$ be the usual system of iteration of the elementary embedding $j$ along the ordinal axis. Let $P_\alpha = j_\alpha(P)$ and $\tau_\alpha = j_\alpha(\tau)$. It will be enough to show that the pairs $\langle P_\alpha, \tau_\alpha \rangle$ for $\alpha \in \text{Ord}$ are pairwise $\tilde{E}$-unrelated. To see this,
pick ordinals \( \alpha \in \beta \). As the original poset had size \( \kappa \), it is the case that
\( P_\alpha, \tau_\alpha, P_\beta, \tau_\beta \) are in the model \( N_\beta \). By the elementarity of the embedding \( j_{0, \beta} \),
\( N_\beta \models \lnot \langle P_\alpha, \tau_\alpha \rangle \ E \langle P_\beta, \tau_\beta \rangle \). The wellfounded model \( N_\beta \) is correct about \( E \) by a
Shoenfield absoluteness argument and so \( \langle P_\alpha, \tau_\alpha \rangle \ E \langle P_\beta, \tau_\beta \rangle \) fails also in \( V \) as
required.

Unlike the previous theorems in this section, we do not have a complementary
example showing that the measurable cardinal bound is, at least to some extent,
onimal.

The last theorem in this section provides an estimate of the \( \lambda \) cardinal for
unpinned equivalence relations. The well-known Silver dichotomy says that
every Borel equivalence relation with uncountably many classes has in fact \( 2^{\aleph_0} \)
many classes. We would like to show that every unpinned Borel equivalence
relation has at least \( 2^{\aleph_1} \) many virtual classes. However, in ZFC this is still
open. The best we can do is the following.

**Theorem 2.5.11.** Let \( E \) be an unpinned Borel equivalence relation. Let \( \kappa \) be
an inaccessible cardinal. Then \( \text{Coll}(\omega, \kappa) \models \lambda(E, \text{Coll}(\omega, \omega_1)) = 2^{\aleph_1} \).

**Proof.** Let \( A \subset P(\kappa) \) be a set of size \( 2^\kappa \) such that any two distinct elements
of \( A \) have intersection of size \( < \kappa \). Recall that the poset \( P = \text{Coll}(\omega, \kappa) \) is
a finite support product of posets \( \text{Coll}(\omega, \alpha) \) for \( \alpha \in \kappa \). For every set \( a \subset \kappa \)
write \( P_a = \Pi_{\alpha \in a} \text{Coll}(\omega, \alpha) \subset P \). Let \( G \subset \text{Coll}(\omega, \kappa) \) be a generic filter. Use
Theorem 2.6.1 to argue that for every \( a \in A \), \( V[G \cap P_a] \models E \) is unpinned, and
let \( \langle Q_a, \tau_a \rangle \in V[G \cap P_a] \) be a nontrivial \( E \)-pinned name with its attendant poset.
It will be enough to show that the pairs \( \langle Q_a, \tau_a \rangle \) for \( a \in A \) represent pairwise
distinct virtual classes.

Suppose towards contradiction that for some \( a \neq b \) in the set \( A \) the \( E \)-pinned
names \( \tau_a, \tau_b \) are equivalent. Let \( c = a \cap b \) and work in the model \( V[G \cap P_c] \). The
assumptions imply that the poset \( (P_a \restriction c \ast \bar{Q}_a) \times (P_b \restriction c \ast \bar{Q}_b) \) forces (at least below
some condition) that \( \tau_a \models E \tau_b \). It follows that the name \( \tau_a \) on the poset \( P_a \restriction c \ast \bar{Q}_a \)
is \( E \)-pinned. Since the equivalence relation \( E \) is Borel, it follows that that the
virtual class of \( \langle P_a \restriction c \ast \bar{Q}_a, \tau_a \rangle \) is also represented by some pair \( \langle R, \sigma \rangle \) by some
poset of size \( < \aleph_1 \) by Theorem 2.5.6. However, in the model \( V[G \cap P_a] \),
there is a filter \( H \subset R \) generic over the model \( V[G \cap P_a] \). Thus, it would have
to be the case that in the model \( V[G \cap P_a] \), \( Q_a \models \tau_a \models E \sigma/H, \) contradicting the
non triviality of the name \( \tau_a \).

**Example 2.5.12.** Similarly to the Silver dichotomy, the conclusion fails for
analytic equivalence relations. Consider the case of the equivalence relation
\( E_{\omega_1} \) and an inaccessible cardinal \( \kappa \) such that \( 2^\kappa > \kappa^+ \). In the \( \text{Coll}(\omega, \kappa) \) extension,
\( \aleph_2 < 2^{\aleph_1} \) will hold; at the same time, \( E_{\omega_1} \)-names realized on \( \text{Coll}(\omega, \omega_1) \) are
classified by ordinals of cardinality \( \aleph_1 \) and so there are only \( \aleph_2 \)-many of them.

### 2.5c Cardinal arithmetic examples

The cardinal invariants \( \kappa(E) \) and \( \lambda(E) \) provide a basis on which equivalence
relations can be compared with uncountable cardinals of all sorts. In this section,
we will introduce several jump operations which have direct translations into cardinal arithmetic operations. Using this approach, one can formally encode statements such as the failure of the singular cardinal hypothesis into reducibility results between Borel or analytic equivalence relations. For brevity, given a coanalytic class $\Gamma$ of structures on $\omega$ invariant under isomorphism, we write $\kappa(\Gamma)$ for $\kappa(\mathcal{E}_\Gamma)$ in this section. A possibly uncountable structure $M$ is a $\Gamma^{**}$-structure if $\text{Coll}(\omega, |M|)$ forces that there is a structure in $\Gamma$ isomorphic to $M$.

The constructions in this section depend on certain types of jump operators on structures and equivalence relation. They are all provisionally denoted by the $+$ sign, not to be confused with the Friedman–Stanley jump. The first jump operation on equivalence relations we will consider translates into the powerset operation using the pinned cardinal:

**Definition 2.5.13.** Let $\Gamma$ be a coanalytic class of structures on $\omega$, invariant under isomorphism. $\Gamma^+$ is the class of structures on $\omega$ of the following description: there is a partition $\omega = a \cup b$ into two infinite sets, there is a $\Gamma$-structure on $a$, and there is an extra relation $R$ on $b \times a$ such that the vertical sections $R_m$ for $m \in b$ are pairwise distinct subsets of $a$.

**Proposition 2.5.14.** Let $\Gamma$ be a coanalytic class of structures on $\omega$, invariant under isomorphism.

1. $\Gamma^+$ is coanalytic, and if $\Gamma$ is Borel then so is $\Gamma^+$;
2. if $\Gamma$ consists of rigid structures, then so does $\Gamma^+$;
3. if $\Gamma$ consists of rigid structures and $\kappa(\Gamma) = \lambda^+$ then $\kappa(\Gamma^+) = (2^\lambda)^+$. 

**Proof.** The first two items are obvious. For (3), suppose that $\kappa(\Gamma) = \lambda^+$. Then there must be a $\Gamma^{**}$ structure $M$ of size $\lambda$ and no $\Gamma^{**}$ structures of larger size. Every $\Gamma^{**}$-structure consists of a $\Gamma^{**}$-structure and some family of its pairwise distinct subsets; the largest such structure then is of size $2^\lambda$ exactly. By Theorem 2.4.5 and (2), every virtual $\mathcal{E}_\Gamma$-class is associated with a $\Gamma^{**}$ structure. This completes the proof. 

**Example 2.5.15.** For every countable ordinal $\alpha$ there is a Borel class $\Gamma$ of rigid structures such that $\kappa(\mathcal{E}_\Gamma) = \beth^{+}_\alpha$. 

**Proof.** By transfinite recursion on $\alpha$ define Borel classes $\Gamma_\alpha$ consisting of rigid structures as follows. Let $\Gamma_0$ be the class of structures isomorphic to $(\omega, \in)$. Let $\Gamma_{\alpha+1} = (\Gamma_\alpha)^+$. For a limit cardinal $\alpha$ let $\Gamma_\alpha$ be the class of structures which consist of exactly one copy of a structure in class $\Gamma_\beta$ for each $\beta \in \alpha$. It is not difficult to prove by induction on $\alpha$ using the Proposition 2.5.14 at the successor stage and Theorem 2.4.5 at the limit stage that $\kappa(\mathcal{E}_\Gamma) = \beth^{+}_\alpha$ as desired.

**Definition 2.5.16.** Let $\Gamma$ be a coanalytic class consisting of structures on $\omega$ invariant under isomorphism. $\Gamma^+$ is the class of structures on $\omega$ of the following description. A structure $M \in \Gamma^+$ has a linear order $\leq$ and on cofinally many initial segments $\{m : m \leq n\}$ it has a $\Gamma$-structure $M_n$, so that for distinct elements $n_0, n_1 \in M$, the structures $(M_{n_0}, \leq)$ and $(M_{n_1}, \leq)$ are nonisomorphic.
Proposition 2.5.17. Let \( \Gamma \) be a coanalytic class of structures on \( \omega \), invariant under isomorphism.

1. \( \Gamma^+ \) is coanalytic;

2. if \( \Gamma \) is Borel and consists of rigid structures, then \( \Gamma^+ \) is Borel and consists of rigid structures;

3. if \( \Gamma \) consists of rigid structures, then \( \kappa(\Gamma^+) = \kappa(\Gamma) + \).

Proof. (1) is clear. For (2), the rigidity conclusion is clear. To see the Borelness of the class \( \Gamma^+ \) note that the statement “\( \langle M, \leq_M \rangle \) is isomorphic to \( \langle N, \leq_N \rangle \)” for structures \( M, N \in \Gamma \) is Borel by the rigidity of the structures in \( \Gamma \) and the Lusin–Suslin theorem [22, Theorem 15.1]. For (3), write \( \kappa = \kappa(\Gamma) \). To show that \( \kappa(\Gamma^+) \leq \kappa^+ \), use Theorem 2.4.5 to argue that every virtual \( \Gamma^+ \) class is classified by a generic \( \Gamma^+ \)-structure. Such a structure contains a linear ordering, and on cofinally many initial segments of the ordering there is a generic \( \Gamma \)-structure. It follows that every initial segment of the ordering is of size \( < \kappa \), and so the underlying set has size at most \( \kappa \).

To show that \( \kappa(\Gamma^+) \geq \kappa^+ \), treat first the case that \( \kappa \) is a limit cardinal. Let \( M \) be the structure on \( \kappa \) which includes the usual \( \in \)-ordering on \( \kappa \) and on each cardinal \( \lambda < \kappa \) on which there is a generic \( \Gamma \)-structure, \( M \) contains one. This is a generic \( \Gamma^+ \)-structure of size \( \kappa \) and so \( \kappa(\Gamma^+) \geq \kappa^+ \).

Suppose now that \( \kappa \) is a successor cardinal, \( \kappa = \lambda^+ \). This means that there is a generic \( \Gamma \)-structure \( N \) of size \( \lambda \). Let \( M \) be the structure on \( \kappa \) which includes the usual \( \in \)-ordering of \( \kappa \) and on each ordinal \( \mu < \kappa \) of cardinality \( \lambda \), it contains a copy of \( N \). This is a generic \( \Gamma^+ \)-structure of size \( \kappa \) and so \( \kappa(\Gamma^+) \geq \kappa^+ \) in this case as well. The proof is complete.

Example 2.5.18. For every countable ordinal \( \alpha > 0 \) there is a Borel equivalence relation \( E \) such that (provably) \( \kappa(E) = \aleph_\alpha \).

Proof. Now, by recursion on a countable ordinal \( \alpha \) construct a Borel class \( \Gamma_\alpha \) of rigid structures as follows. \( \Gamma_0 \) is the set of structures isomorphic to \( \langle \omega, \in \rangle \), containing just one isomorphism class. Then let \( \Gamma_{\alpha+1} = \Gamma_\alpha^+ \) and \( \Gamma_\alpha = \bigcup_{\beta < \alpha} \Gamma_\beta \) if \( \alpha \) is limit. Proposition 2.5.17 can be used to show by transfinite induction that \( \kappa(\Gamma_\alpha) = \aleph_{1+\alpha} \) as desired.

Note that an equivalence relation \( E \) as above for \( \alpha \geq 2 \) cannot be reducible to \( \mathcal{F}_2 \) and \( \mathcal{F}_2 \) cannot be reducible to it. This answers a question of Kechris [17, Question 17.6.1] in the negative as well as some related questions of Simon Thomas. To see that \( \mathcal{F}_2 \) cannot be Borel reducible to any \( E \), suppose for contradiction that \( h : \text{dom}(E) \to \text{dom}(\mathcal{F}_2) \) is a Borel reduction. Pass to a generic extension in which \( c > \aleph_{\omega_1} \). There, \( h \) is still a reduction of \( E \) to \( F \), while \( \kappa(E) > \kappa(F) \). This contradicts Theorem 2.5.4. To see that \( E \) cannot be reducible to \( \mathcal{F}_2 \) for any \( \alpha > 2 \), pass to a generic extension in which the Continuum Hypothesis holds instead.
The following two examples deal with jump operations designed to mimic cardinal exponentiation. They lead to nonreducibility results which, at least on the face of it, use the failure of singular cardinal hypothesis in various situations. This means that the proofs presented use large cardinal assumptions, as they are needed to get the failure of the singular cardinal hypothesis. We make no claim as to whether the large cardinal assumptions are necessary for the conclusion.

**Definition 2.5.19.** Let \( \Gamma, \Delta \) be coanalytic classes of structures on \( \omega \), invariant under isomorphism. The symbol \( \Gamma^\Delta \) stands for the coanalytic class of structures \( M \) on \( \omega \) of the following form: \( \omega \) is partitioned into infinite sets \( \omega = a \cup b \cup c \), on \( M \upharpoonright a \) is a structure in class \( \Gamma \), \( M \upharpoonright b \) is a structure in class \( \Delta \), and there is an extra relation \( R \subset c \times b \times a \) such that for every \( m \in c \), the vertical section \( R_m \) is a function from \( b \) to \( a \), and for \( m_0 \neq m_1 \in c \) the vertical sections \( R_{m_0} \) and \( R_{m_1} \) are distinct.

**Proposition 2.5.20.** Suppose that the classes \( \Gamma, \Delta \) consist of rigid structures only. Then

1. \( \Gamma^\Delta \) consists of rigid structures only;
2. \( \kappa(\Gamma^\Delta) = \sup\{ \kappa^\lambda : \kappa < \kappa(\Gamma), \lambda < \kappa(\Delta) \} \).

**Proof.** The first item is nearly trivial. For the second item, use Theorem 2.4.5 to note that a \((\Gamma^\Delta)^{**}\) structure is represented by a \(\Gamma^{**}\)-structure, a \(\Delta^{**}\)-structure, and an infinite set of functions from the latter to the former. \(\square\)

**Example 2.5.21.** Let \( \Delta \) be the class of structures isomorphic to \( \langle \omega, \in \rangle \) (consisting of just one equivalence class). For every countable ordinal \( \alpha \), let \( \Gamma_\alpha \) be the class derived in Example 2.5.18. Let \( E \) be the equivalence relation of isomorphism on the class \((\Gamma_\omega)^\Delta\) and \( F \) be the isomorphism equivalence relation on the class \( E_{\Gamma_\alpha} \). Then \( E \) is not Borel reducible to \( F \times F \).

**Proof.** Move to a model of ZFC where \( \varepsilon = \aleph_1 \) and \( \aleph_\omega^\varepsilon > \aleph_\alpha \). Proposition 2.5.20 shows that \( \kappa(E) = \aleph_\omega^\varepsilon > \varepsilon \cdot \aleph_\alpha = \kappa(F \times F) \). The conclusion of the example follows from Theorem 2.5.4. \(\square\)

**Definition 2.5.22.** Let \( E \) be an analytic equivalence relation on a Polish space \( X \) and \( F \) be a Borel equivalence relation on a Polish space \( Y \). \( E^F \) is the equivalence relation \((F \times E)^+\) on the space \((Y \times X)^\omega\) restricted to the Borel set \( \{ z \in (Y \times X)^\omega : \text{rng}(z) \text{ is a partial function from } Y \text{ to } X \text{ whose domain consists of pairwise } F\text{-unrelated elements} \} \).

It seems to be impossible to formulate this concept in an analytic form without the additional demand that \( F \) be Borel.

**Proposition 2.5.23.** Let \( E \) be an analytic equivalence relation on a Polish space \( X \) and \( F \) be a Borel equivalence relation on a Polish space \( Y \). Then

1. \( \lambda(E^F) = \lambda(E)^{\lambda(F)} \);
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2. if $P$ is a partial order such that $P \vdash |\lambda(F)| = |\kappa(F)| = \aleph_0$ then $\lambda(E^F, P) = \lambda(E, P)^{\lambda(F)}$.

Note that as $F$ is assumed to be Borel, the value of $\lambda(F)$ is not $\infty$.

Proof. Both statements follow from the classification of virtual classes for product and Friedman–Stanley jump in Theorem 2.3.4. An $E^F$-virtual class is represented by a function from $F$-virtual classes to $E$-virtual classes.

Example 2.5.24. Let $F_0$ be any Borel equivalence relation with countably many classes and let $F_1$ be the identity on $2^{\omega}$. For every Borel equivalence relation $E$, $E_{\omega_1}F_0$ is not Borel reducible to $E \times E_{\omega_1}^F$.

Proof. Move to a model of set theory where $\mathfrak{c} = \aleph_1$ and there is a cardinal $\kappa > \beth_1$ such that $(\kappa^+)^{\beth_1} > (\kappa^+)^{\aleph_0}$. The first such a cardinal must be in violation of the singular cardinal hypothesis at cofinality $\omega_1$. Let $P = \text{Coll}(\omega, \kappa)$. Clearly, $\lambda(E_{\omega_1}, P) = \kappa^+$, since the pinned names on $P$ correspond to ordinals below $\kappa^+$. Since $E$ is Borel, $\lambda(E), \kappa(E) < \beth_1, < \kappa$ by Theorem 2.5.6 and so Proposition 2.5.23 shows that $\lambda(E \times E_{\omega_1}^F) = \lambda(E) \cdot (\kappa^+)^{\aleph_0} = (\kappa^+)^{\aleph_0} < (\kappa^+)^{\beth_1} = \lambda(E_{\omega_1}^F)$. The argument is concluded by a reference to Theorem 2.5.4. \qed

A similar argument with a failure of the singular cardinal hypothesis at larger cofinalities yields

Example 2.5.25. Let $F$ be any Borel equivalence relation. For every Borel equivalence relation $E$, $E_{\omega_1}^F$ is not Borel reducible to $E \times E_{\omega_1}^F$.

2.5d Hypergraph examples

The previous examples were to some extent artificial in the sense that the values of the pinned cardinal were directly built into them. The following examples, all Borel reducible to $\mathbb{F}_2$, are connected with combinatorics of small uncountable cardinals via the pinned cardinal, even though the connection is not at first sight obvious.

Definition 2.5.26. A hypergraph on a set $X$ is a subset of $X^{\leq \omega}$. If $G$ is a hypergraph on $X$, a $G$-anticlique is a set $A \subset X$ such that $A^{\leq \omega} \cap G = 0$. If $G$ is an analytic hypergraph on a Polish space $X$, write $E_G$ for the equivalence relation on $X^\omega$ connecting $y, z$ if the sets $\text{rng}(y), \text{rng}(z)$ either both fail to be $G$-anticliques or they are equal.

It is clear that the equivalence relations of the form $E_G$ are all analytic and almost Borel reducible to $\mathbb{F}_2$. In the common case when $G$ is Borel and contains only finite edges, the equivalence relation is in fact Borel. A simple Mostowski absoluteness argument shows that every (even uncountable) $G$-anticlique remains such in every forcing extension. By Example 2.3.5, the virtual $E_G$-classes are classified by $G$-anticliques, and so $\kappa(E_G) = \sup\{|A|^{\uparrow}: A \subset X\}$ is
a $G$-anticlique}. The values of cardinals defined in this way are subject to forcing manipulations and transfinite combinatorics. The interesting examples are always connected with a universality feature of the hypergraph in question.

Our first two examples use Borel hypergraphs of finite arity.

**Definition 2.5.27.** Let $X$ be a Polish space. A Borel relation $R \subset [X]^{<\aleph_0} \times X$ is **combinatorially universal** if it has countable vertical sections and for every cardinal $\kappa$ and every relation $T \subset \kappa^{<\aleph_0} \times \kappa$ with countable vertical sections, there is a c.c.c. poset $P$ such that $P \models$ there is an injective function $\pi: \kappa \to X$ which is a homomorphism of $T$ to $R$.

It is not clear whether combinatorial universality of this sort is actually a property absolute among transitive models of ZFC, but all universal examples found in this section are absolutely universal.

**Theorem 2.5.28.** Let $X$ be a Polish space. There is a combinatorially universal Borel relation on $[X]^{<\aleph_0} \times X$ with countable vertical sections.

**Proof.** Let $X = \mathcal{P}(\omega)$. We will show that the relation $R \subset [X]^{<\aleph_0} \times X$ defined by $(a, x) \in R$ if $x$ is computable from $a$ is universal. Let $\kappa$ be a cardinal and let $T \subset [\kappa]^n \times \kappa$ be a relation with countable vertical sections. Let $(k_m: m \in \omega)$ be a recursive sequence of increasing functions in $\omega^\omega$ with disjoint ranges. For a finite set $b \subset \mathcal{P}(\omega)$ let $e_b$ be the increasing enumeration of the set $\bigcap b$, for every $m \in \omega$ let $h_m(b) \subset \mathcal{P}(\omega)$ be the set of all $l \in \omega$ such that $e_b \circ k_m(l)$ is an odd number. We will produce a forcing which adds an injection $\pi: \kappa \to X$ such that for every finite set $a \subset \kappa$ and every $\alpha \in T_a$, there is a number $m \in \omega$ such that $\pi(\alpha)$ is modulo finite equal to $h_m(\pi'' a)$.

Let $P$ be the poset of all tuples $p = (n_p, \pi_p, \nu_p)$ so that

- $n_p \in \omega$, $\pi_p$ is a partial function from $\kappa$ to $\mathcal{P}(n_p)$ with finite domain $\text{dom}(p)$;
- $\nu_p$ is a finite partial function from $\mathcal{P}(\text{dom}(p)) \times \omega$ to $\text{dom}(p)$ such that $(a, \nu_p(a, m)) \in T$ whenever $(a, m) \in \text{dom}(\nu_p)$.

The ordering on $P$ is defined by $q \leq p$ if $n_q \leq n_p$, $\text{dom}(p) \subset \text{dom}(q)$, $\forall \alpha \in \text{dom}(p)$ $\pi_p(\alpha) = \pi_q(\alpha) \cap n_p$, $\nu_p \subset \nu_q$, and for every $(a, m) \in \text{dom}(\nu_p)$, whenever $l$ is a number in the domain of $(e_{\pi_p''a} \setminus e_{\pi_q''a}) \circ k_m$ then $e_{\pi_q''a} \circ k_m(l)$ is odd if and only if $l \in \pi_q(\nu_p(a, m))$. It is not difficult to see that $P$ is indeed an ordering.

**Claim 2.5.29.** The poset $P$ is c.c.c.

**Proof.** Let $\langle p_\alpha: \alpha \in \omega_1 \rangle$ be conditions in $P$. The usual $\Delta$-system and counting arguments can be used to thin down the collection if necessary so that the sets $\text{dom}(p_\alpha)$ for $\alpha \in \omega_1$ form a $\Delta$-system with root $b$ and for all $a \in |b|^\omega$ and all $\alpha \in \omega_1$, $T_a \cap \text{dom}(p_\alpha) \subset b$. Moreover, we can require that the increasing bijection between $\text{dom}(p_\alpha)$ and $\text{dom}(p_\beta)$ extends to an isomorphism of $p_\alpha$ and $p_\beta$ for every $\alpha, \beta \in \omega_1$. 


Consider the condition $q$.

Proof. Assume that $\beta \neq \alpha$.

Recall that if $q$ is $\text{R}$-free, then the condition $q$ defined by $n_q = n_{q_\alpha}$, $\pi_q = \pi_p, \cup \pi_{p_\beta}$ and $\nu_q = \nu_{p_\alpha} \cup \nu_{p_\beta}$ is easily checked to be a common lower bound of the conditions $p_{\alpha}, p_{\beta}$.

Claim 2.5.30. Whenever $a \in [\kappa]^n$ and $\beta \in T_a$, the set $D_{a, \beta} = \{p \in P : a \cup \{\beta\} \subset \text{dom}(p), \exists m \nu_p(a, m) = \beta\}$ is dense in $P$.

Proof. Let $p \in P$; we must find a condition $q \leq p$ in the set $D_{a, \beta}$. For definiteness assume that $\beta \notin \text{dom}(p)$. Choose $m \in \omega$ such that $\langle a, m \rangle \notin \text{dom}(\nu_p)$.

Consider the condition $q \leq p$ defined by $n_q = n_p$, $\pi_q = \pi_p \cup \{\langle \alpha, 0 \rangle : \alpha \in a \setminus \text{dom}(p), \langle \beta, 0 \rangle\}$, $\nu_q = \nu_p \cup \{\langle a, m, \beta \rangle\}$. The condition $q \leq p$ is in the set $D_{a, \beta}$ as required.

Claim 2.5.31. For every finite set $a \subset \kappa$ and every $k \in \omega$, the set $D_{a, k} = \{p \in P : a \subset \text{dom}(p) \text{ and } \text{the set } \bigcap \pi''_p a \text{ has at least } k \text{ elements}\}$ is dense in $P$.

Proof. Fix $a, k$ and let $p \in P$ be an arbitrary condition. We must find a condition $q \leq p$ in the set $D_{a, k}$. First of all, the previous claim shows that one can strengthen $p$ to include all ordinals in $a$. Increasing $n_p$, if necessary, we may also assume that $k < n_p$.

Consider the set $b = \pi''_p a$ and the function $e_b$; write $k' = \text{dom}(e_b)$. If $k \leq k'$ then $q \doteq p$ will work. Otherwise, it is easy to find an increasing sequence $d = \langle m_i : k' \leq i < k \rangle$ of numbers larger than $n_q$ such that, writing $e = e_b \cup d$, for every natural number $m$ such that $\langle a, m \rangle \in \text{dom}(\nu_p)$ and every $l$ such that $k' \leq k_m(l) < k$, $m_{k_m(l)}$ is odd if and only if $l \in p(\nu_p(a, m))$. The condition $q \leq p$ defined by $n_q = m_{k-1} + 1$, $\text{dom}(\pi_q) = \text{dom}(\pi_p)$, $\forall \beta \in a \pi_q(\beta) = \pi_p(a) \cup \{m_i : k' \leq i < k\}$, $\forall \beta \in \text{dom}(\pi_p) \setminus a \pi_q(\beta) = \pi_p(\beta)$, and $\nu_q = \nu_p$, is in the set $D_{a, k}$ as desired.

The last two claims show that the function $\pi : \kappa \to X$ defined as $\pi(\alpha) = \bigcup\{\pi_p : p \in \text{the generic filter}\}$ is forced by $P$ to be the desired homomorphism.

Recall that if $R \subset [X]^{|\omega_1} \times X$ is a relation then $a \subset X$ is $R$-free if for every $x \in a \langle a \setminus \{x\}, x \rangle \notin R$.

Example 2.5.32. Let $X$ be a Polish space and let $R \subset [X]^{|\omega_0} \times X$ be a combinatorially universal Borel relation with countable vertical sections. Let $n \geq 1$ be a number. Let $G_n$ be the hypergraph of all sets $a \in [X]^n$ which are $R$-free. Then

1. $\kappa(E_{G_n}) \leq \aleph_{n+1}$;
2. if Martin’s Axiom for $\aleph_n$ holds then equality is attained.
Proof. Fix the number \( n \in \omega \). The argument depends on an old theorem of Sierpiński [12]: \( \mathbb{N}_{n+1} \) is the smallest cardinal \( \kappa \) such that every relation \( T \subset [\kappa]^n \times \kappa \) with countable vertical sections has a \( T \)-free \( n+1 \)-tuple. To prove (1), apply the Sierpiński theorem to show that there is no \( \mathcal{G}_n \)-anticlique of size \( \geq \mathbb{N}_{n+1} \). To prove (2), let \( \kappa = \mathbb{N}_n \) and use the Sierpiński theorem again to find a relation \( T \subset \kappa^{<\mathbb{N}_0} \times \kappa \) with countable vertical sections and no \( T \)-free \( n \)-tuple. Then use the universality assumption to find an injective homomorphism \( \pi: \kappa \to X \) which is a homomorphism of \( T \) to \( R \). Observe that \( \text{rng}(\pi) \) is a \( \mathcal{G}_n \)-anticlique of size \( \kappa \), representing a \( \kappa(E_{\mathcal{G}_n}) \)-pin which is realized only by posets collapsing \( \kappa \) to \( \mathbb{N}_0 \).

Definition 2.5.33. Let \( X \) be a Polish space. A Borel equivalence relation \( R \) on \( [X]^{<\mathbb{N}_0} \) with countably many classes is combinatorially universal if for every cardinal \( \kappa \) and every equivalence relation \( T \) on \( [\kappa]^{<\mathbb{N}_0} \) with countably many classes, there is a c.c.c. poset \( P \) such that \( P \models \text{there is an injection \( \pi: \kappa \to X \) which is a homomorphism of \( \neg T \) to \( \neg R \).} \)

Theorem 2.5.34. Let \( X \) be a Polish space. There is a universal Borel equivalence relation on \( [X]^{<\mathbb{N}_0} \) with countably many classes.

Proof. Without loss of generality assume \( X = [\omega]^{\mathbb{N}_0} \). Consider the following relation \( R \) on \( [X]^{<\mathbb{N}_0} \). Define a Borel function \( g: [X]^{<\mathbb{N}_0} \to [\omega]^{<\mathbb{N}_0} \) by \( g(a) = \{ \min(x \setminus m+1): x \in a \} \) if \( a \) is a set of size at least two and consists of pairwise almost disjoint sets and \( m \) is the largest number which appears in at least two of them; \( g(a) = \min(x) \) if \( a = \{ x \} \) is a singleton; and otherwise \( g(a) = 0 \). Let \( R \) be the equivalence relation induced by the function \( g \). We will show that \( R \) is universal.

Let \( \kappa \) be a cardinal, \( T \) an equivalence relation on \( [\kappa]^{<\mathbb{N}_0} \) with countably many classes, and let \( f: [\kappa]^{<\mathbb{N}_0} \to [\omega]^{<\mathbb{N}_0} \) be a map inducing the equivalence relation \( T \). Let \( \nu: [\omega]^{<\mathbb{N}_0} \to [\omega] \) be a sufficiently generic map such that \( \nu(g(0)) = f(0) \). Let \( P \) be the poset of all maps \( p \) such that

- \( \text{dom}(p) \subset \kappa \) is a finite set;
- for every \( \alpha \in \text{dom}(p) \) the value \( p(\alpha) \) is a nonempty subset of \( \omega \);
- for every \( \alpha \in \text{dom}(p), \nu(\min(p(\alpha))) = f(\alpha) \);
- for every set \( a \subset \text{dom}(p) \) of size at least 2 there is a number which belongs to at least two sets \( p(\alpha), p(\beta) \) for \( \alpha \neq \beta \in a \), and writing \( m \) for the largest such number, \( p(\alpha) \setminus m + 1 \neq 0 \) holds for every \( \alpha \in a \), and \( \nu(\{ \min(p(\alpha)) \setminus m + 1: \alpha \in a \}) = f(a) \).

The ordering is defined by \( q \leq p \) if for every \( \alpha \in \text{dom}(p), q(\alpha) \) end-extends \( p(\alpha) \), and the sets \( q(\alpha) \setminus p(\alpha) \) are pairwise disjoint for \( \alpha \in \text{dom}(p) \).

Claim 2.5.35. \( P \) is c.c.c.
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Proof. In fact, $P$ is semi-Cohen in the sense of [1], but we will not need that fact here. By the usual $\Delta$-system arguments, it is enough to show that any two conditions $p, q \in P$ such that $p \upharpoonright \text{dom}(p) \cap \text{dom}(q) = q \upharpoonright \text{dom}(p) \cap \text{dom}(q)$, are compatible. To find the lower bound, enumerate $\text{dom}(p) \cup \text{dom}(q)$ as $\beta_i$ for $i \in k$, enumerate $(\text{dom}(p) \setminus \text{dom}(q)) \times (\text{dom}(q) \setminus \text{dom}(p))$ as $\nu_j$ for $j \in l$. Use the genericity of the function $\nu$ to build numbers $m_0 < m_1 < \cdots < m_{l-1}$ and pairwise distinct numbers $n_i^j$ for $i \in k$ and $j \in l$ so that

- $m_0 > \max(\bigcup \text{rng}(p) \cup \text{rng}(q))$;
- $m_j < n_i^j < m_{j-1}$ for every $j \in l$;
- for every set $a \subset \text{dom}(p) \cup \text{dom}(q)$, $\nu(\{n_i^j : \beta_i \in a\}) = f(a)$.

The lower bound is then a function $r$ defined by $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$, for $\alpha \in \text{dom}(p)$, $\alpha = \beta_i$ set $r(\alpha) = p(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha \text{ appears in the pair } \nu_j\}$. Similarly, for $\alpha \in \text{dom}(q)$, $\alpha = \beta_i$ set $r(\alpha) = q(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha \text{ appears in the pair } \nu_j\}$. It is not difficult to check that $r \leq p, q$ as required. \hfill $\square$

Claim 2.5.36. The set $D_\alpha = \{p \in P : \alpha \in \text{dom}(p)\}$ is dense in $P$ for every $\alpha \in \kappa$.

Proof. Let $\alpha \in \kappa$ and $p \in P$; we must produce $q \leq p$ such that $\alpha \in \text{dom}(q)$. Enumerate $\text{dom}(p)$ as $\beta_i$ for $i \in k$ and write $\alpha = \beta_k$. Use the genericity of the function $\nu$ to find numbers $m < m_0 < m_1 < \cdots m_{k-1}$ and pairwise distinct numbers $n_i^j$ for $i \in k, j \in k + 1$ so that

- $\nu(m) = f(\alpha)$ and $m_0 > \max(\bigcup \text{rng}(p))$;
- $m_j < n_i^j$ for every $j \in k + 1$;
- for every nonempty set $a \subset \text{dom}(p) \cup \{\alpha\}$ and every $i \in k$, $\nu(\{n_i^j : \beta_j \in a\}) = f(a)$.

Once this is done, just consider the function $q$ defined by $\text{dom}(q) = \text{dom}(p) \cup \{\alpha\}$, $s(\alpha) = \{m, m_i : i \in k, n_i^i : i \in k\}$, and for every $i \in k$, $q(\beta_i) = p(\beta_i) \cup \{m_i, n_i^j : j \in k\}$. It is not difficult to observe that $q \in P$ and $q \leq p$ as desired. \hfill $\square$

Now, if $G \subset P$ is a generic filter, then in $V[G]$ let $\pi: \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{p \in G} p(\alpha)$. The claims show that $f = \nu \circ g \circ \pi$, in particular $\pi$ is a homomorphism of $\neg T$ to $\neg R$. \hfill $\square$

Example 2.5.37. Let $X$ be a Polish space and $R$ a universal equivalence relation on $[X]^{<\aleph_0}$ with countably many classes. Let $n > 0$ be a number. Let $\mathcal{G}_n$ be the Borel hypergraph on $X$ consisting of all finite sets $a \subset X$ which can be written in more than $2^n - 1$ ways as a union of two distinct $R$-equivalent sets. Then

1. $\kappa(E_{\mathcal{G}_n}) \leq \aleph_{n+1}$:
2. if Martin’s Axiom for $\aleph_n$ holds, then the equality is attained.

\textit{Proof.} Fix the number $n$. The computations depend on and are motivated by a theorem of Komjáth and Shelah [23]. Let $\kappa = \aleph_n$. For every equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes, there is a finite set $a \subseteq \kappa$ which can be written in at least $2^n - 1$ ways as a union of two distinct $T$-related sets. In addition, if Martin’s Axiom for $\kappa$ holds then there is an equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes such that every finite set $a \subseteq \kappa$ can be written in at most $2^n - 1$ ways as a union of two distinct $T$-related sets.

For (1), the first part of this result shows that there is no $G_n$-anticlique of size $\kappa^+$. For (2), use Martin’s Axiom and the second part of the Komjáth–Shelah result to find the equivalence relation $T$ on $[\kappa]^{<\aleph_0}$ with countably many classes as above, and use the universality of the relation $R$ to find an injective homomorphism $\pi: \kappa \to X$ of $\neg T$ to $\neg R$. It is immediate that $\text{rng}(\pi) \subseteq X$ is a $G_n$-anticlique of size $\kappa$, and therefore associate with a virtual $E_{\mu_n}$-class realized only on posets which collapse $\kappa$ to $\aleph_0$.

Much more complicated effects can be realized if the hypergraph $G$ is allowed to have infinite edges. We conclude this section with one more example of this type.

\textbf{Definition 2.5.38.} Let $X$ be a Polish space. A Borel equivalence relation $R$ on $X^2$ with countably many classes is \textit{combinatorially universal} of for every ordinal $\kappa$ and every equivalence relation $T$ on $\kappa^2$ with countably many classes, there is a c.c.c. poset $P$ such that $P \Vdash$ there is an injective function $\pi: \kappa \to X$ which is a homomorphism of $\neg T$ to $\neg R$.

\textbf{Theorem 2.5.39.} There is a combinatorially universal Borel equivalence relation with countably many classes on $X^2$ for every Polish space $X$.

\textit{Proof.} Without loss of generality, assume $X = P(\omega)$. Let $\omega = \bigcup_{n,m \in \omega} a_{n,m}$ be a partition of $\omega$ into infinite sets. For almost disjoint sets $b, c \subseteq \omega$ such that $b$ is lexicographically less than $c$ define $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$, in other cases define $f(b, c) = 0$. Let $R$ be the equivalence relation induced by the function $f$. We will show that $R$ is combinatorially universal.

Fix a cardinal $\kappa$ and a function $g: \kappa^2 \to \omega$ which induces an equivalence relation $T$ with countably many classes. Define the poset $P$ as the collection of all functions $p$ such that

- $\text{dom}(p) \subseteq \kappa$ is a finite set;
- $\text{rng}(p)$ consists of finite subsets of $\omega$ such that neither of them is an initial segment of another;
- for every $\alpha \neq \beta$ such that $p(\alpha)$ is lexicographically smaller than $p(\beta)$, the set $p(\alpha) \cap p(\beta)$ is nonempty, and its maximum belongs to the set $a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$. 

The ordering on $P$ is defined by $q \leq p$ if $\text{dom}(p) \subseteq \text{dom}(q)$, for every $\alpha \in \text{dom}(p)$ the set $p(\alpha)$ is an initial segment of $q(\alpha)$, and the sets $\{q(\alpha) \setminus p(\alpha) : \alpha \in \text{dom}(p)\}$ are pairwise disjoint. The following routine claims complete the proof of the theorem.

**Claim 2.5.40.** The poset $P$ is c.c.c.

*Proof.* By the usual $\Delta$-system arguments it is only necessary to show that any two conditions $p,q \in P$ such that $p \upharpoonright \text{dom}(p) \cap \text{dom}(q) = q \upharpoonright \text{dom}(p) \cap \text{dom}(q)$ are compatible in the poset $P$. Strengthening the conditions $p,q$ on $\text{dom}(p) \setminus \text{dom}(q)$ and $\text{dom}(p) \setminus \text{dom}(q)$ respectively if necessary, we may assume that no set in $\text{rng}(p) \cup \text{rng}(q)$ is an initial segment of another. Enumerate $(\text{dom}(p) \setminus \text{dom}(q)) \times (\text{dom}(q) \setminus \text{dom}(p))$ as $u_i$ for $i \in j$ and find pairwise distinct numbers $m_i$ for $i \in j$ such that

- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically smaller than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$;
- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically greater than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = n$ and $g(\beta, \alpha) = m$;
- all numbers $m_i$ are greater than $\max(\bigcup \text{rng}(p) \cup \bigcup \text{rng}(q))$.

In the end, let $r$ be the function defined by $\text{dom}(r) = \text{dom}(p) \cup \text{dom}(q)$, for all $\alpha \in \text{dom}(p)$ let $r(\alpha) = p(\alpha) \cup \{m_i : \alpha \text{ appears in } u_i\}$, and for all $\beta \in \text{dom}(q)$ let $r(\beta) = q(\beta) \cup \{m_i : \beta \text{ appears in } u_i\}$. It is not difficult to check that $r$ is a common lower bound of the conditions $p,q$ as desired.

**Claim 2.5.41.** For every $\alpha \in \kappa$ the set $D_\alpha = \{p \in P : \alpha \in \text{dom}(r)\}$ is dense in $P$.

*Proof.* Let $\alpha \in \kappa$ be an ordinal and $p \in P$ be a condition; we must produce a condition $q \leq p$ such that $\alpha \in \text{dom}(q)$. It will be the case that $q(\alpha) \cap \max(\bigcup \text{rng}(p)) + 1 = 0$; this way, $q(\alpha)$ will be lexicographically smaller than all $q(\beta)$ for $\beta \in \text{dom}(p)$. List $\text{dom}(p)$ as $\beta_i$ for $i \in j$, and find pairwise distinct numbers $m_i$ for $i \in j$ so that

- $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\alpha, \beta) = n$;
- each $m_i$ is greater than $\max(\bigcup \text{rng}(p))$.

Then, let $q$ be the function defined by $\text{dom}(q) = \text{dom}(p) \cup \{\alpha\}$ and $q(\beta_i) = p(\beta_i) \cup \{m_i\}$ and $q(\alpha) = \{m_i : i \in j\}$. It is immediate that the condition $q$ works.

Now it is easy to see that if $G \subseteq P$ is a generic filter, the function $\pi : \kappa \to \mathcal{P}(\omega)$ defined by $\pi(\alpha) = \bigcup \{p(\alpha) : p \in G\}$ induces a homomorphism of $\neg T$ to $\neg R$ as desired.
Example 2.5.42. Let $X$ be a Polish space and let $R$ be a combinatorially universal Borel equivalence relation on $X^2$ with countably many classes. Let $\mathcal{G}$ be the hypergraph on $X$ consisting of all unions $b_0 \cup b_1$ where $b_0, b_1$ are infinite sets such that $b_0 \times b_1$ is a subset of a single $R$-class.

1. if Chang’s conjecture holds, then $\kappa(E_\mathcal{G}) \leq \aleph_2$;
2. if Martin’s Axiom for $\aleph_2$ holds then Chang’s conjecture is equivalent to $\kappa(E_\mathcal{G}) \leq \aleph_2$.

Proof. The argument is based on and motivated by two results of Todorcevic [36]. Namely, if Chang’s conjecture holds, then for every partition of $\omega^2$ into countably many pieces, one piece of the partition contains a product of infinite sets. In addition, if Martin’s Axiom for $\aleph_2$ holds and Chang’s conjecture fails, then there is a partition of $\omega^2$ into countably many pieces such that no piece of the partition contains a product of infinite sets.

Now, for (1) use the first result of Todorcevic to argue that under Chang’s conjecture there is no $\mathcal{G}$-anticlique of size $\geq \aleph_2$. For (2), suppose that Martin’s Axiom holds and Chang’s conjecture fails. By the second result of Todorcevic, find an equivalence relation $T$ on $\omega^2$ with countably many classes without any monochromatic infinite rectangles. Use the universality of the equivalence relation $R$ to find an injection $\pi: \omega_2 \to X$ which is a homomorphism of $\neg T$ to $\neg R$. Observe that $\text{rng}(\pi)$ is a $\mathcal{G}$-anticlique of size $\aleph_2$ which therefore identifies a virtual $E_\mathcal{G}$-class realized only on posets which collapse $\aleph_2$ to $\aleph_0$. \hfill \Box

2.6 Absoluteness

This section compiles the available information regarding the absoluteness of various notions regarding the virtual equivalence classes. The most substantial and useful statement is Corollary 2.6.3: if $E$ is a Borel equivalence relation then its pinned status is absolute among all generic extensions.

Theorem 2.6.1. Suppose that $E$ is a Borel equivalence relation on a Polish space $X$. The following are equivalent:

1. $E$ is pinned;
2. For every $\omega$-model $M$ of ZFC containing the code for $E$, $M \models E$ is pinned.

Proof. For simplicity assume $X = \omega^\omega$. The implication (2)→(1) is trivial. If (1) fails, then $E$ is not pinned, and the failure of (2) is witnessed by $M = V$.

The implication (1)→(2) is more difficult. Suppose that (2) fails. Fix a $\omega$-model $M$ of ZFC containing the code for $E$ such that $M \models E$ is unpinned; taking an elementary submodel if necessary we may assume that $M$ is countable. In the model $M$, find a nontrivial $E$-pinned name $\tau_0$ on some poset $Q_0$. Form a transfinite sequence of models and a commuting system of elementary embeddings $\langle M_\alpha, Q_\alpha, \tau_\alpha, j_{\beta\alpha} : \beta < \alpha \leq \omega_1 \rangle$ so that
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1. $M_0 = M$, for every countable $\alpha$, $M_\alpha$ is a countable $\omega$-model of ZFC, and $j_{\beta\alpha}(q_\beta, r_\beta) = q_\alpha, r_\alpha$;

2. for limit $\alpha$ the model $M_\alpha$ is obtained as a direct limit of the earlier models;

3. if there is $x \in X$ such that $Q_\alpha \models \tau_\alpha E \bar{x}$ then there is such an $x$ in the model $M_{\alpha+1}$.

After the induction is performed, we will show that $\tau_{\omega_1}$ is a nontrivial $E$-pinned name on $Q_{\omega_1}$, and therefore $E$ is indeed unpinned and (1) fails.

The successor step of the induction is arranged through the following theorem of ZFC applied in the model $M_\alpha$:

**Claim 2.6.2.** Whenever $P$ is a poset, then in some generic extension there is an elementary embedding $j : V \to W$ into a possibly illfounded $\omega$-model $W$ such that $W$ contains $j''P$ as well as some subset of $j''P$ whose $j$-preimage is a $P$-generic filter over $V$.

**Proof.** Consider the set $Y = |P \cup \mathcal{P}(P)|^{|\omega_0}$ and functions $f, g$ with domain $Y$ such that $f(a) = a \cap P$ and $g(a)$ is some filter on $a \cap P$ which meets all open dense subsets of $a \cap P$ in the set $a$. Let $I$ be the $\sigma$-ideal of nonstationary subsets of the set $Y$ and consider the poset $R = \mathcal{P}(Y)$ modulo $I$ and the associated generic ultrapower $j : V \to W$. It is easy to see that the functions $f, g$ represent the desired elements in the generic ultrapower: $[f] = j''P$ and $[g]$ is a filter on $j''P$ meeting $j(D)$ for every open dense subset $D \subset P$ in the ground model.

Now working in $M_\alpha$, find a poset $R$ forcing the existence of the elementary embedding as above for $P = Q_\alpha$. Let $R'$ be the disjoint union of $R$ and $Q_\alpha$, with the ordering defined by $r \leq q$ if $r \Vdash j(q) \in \dot{g}$. Then $Q_\alpha$ is a regular subposet of $R'$ and $R$ is a dense subset of $R'$. Now suppose that $Q_\alpha \models \tau_\alpha E \bar{x}$; perhaps the point $x$ is not in the model $M_\alpha$. Let $N$ be a countable elementary submodel of a large enough structure and let $h \subset R'$ be generic over $N$. Then $h \subset R'$ is also generic over $M$. Let $j = j_{\omega+1} : M_\alpha \to M_{\omega+1}$ be the generic embedding obtained by an application of the claim in $M_\alpha$. Let $g = h \cap Q_\alpha$. Then $\tau/g \in X$ by the forcing theorem applied in the model $N$ and the Mostowski absoluteness between $N[h]$ and $V$. Also, $\tau/g \in M_{\omega+1}$ since $j''g \in M_{\omega+1}$ and $\tau/g$ is reconstructed as the unique point $y \in \omega^\omega$ such that for every $n \in \omega$, $y(n) = m$ if there is $p \in j''g$ which forces in the poset $jQ_\alpha$ that $j(\tau)(n) = m$.

Once the induction is performed, consider the poset $Q = Q_{\omega_1}$, as well as the name $\tau = \tau_{\omega_1}$ from the point of view of $V$ as opposed to the model $M_{\omega_1}$. By the absoluteness of the Borel equivalence relation $E$ between the $Q \times Q$-extensions of $M_{\omega_1}$ and $V$, $\tau$ is an $E$-pinned name on $Q$ in $V$. We will complete the proof by showing that $\tau$ is nontrivial—again as viewed in $V$ as opposed to $M_{\omega_1}$.

Suppose that there is a point $x \in X$ such that $Q \models \tau E \bar{x}$. Let $N$ be a countable elementary submodel of a large enough structure containing the iteration and the point $x$ and write $a = \omega_1 \cap N$. Since the model $M_{\omega_1}$ is a direct limit of the tower of earlier models, the transitive collapse $\pi$ of $N \cap M_{\omega_1}$ is an isomorphism of $Q_\alpha, \tau_\alpha$ with $Q \cap N, \tau \cap N$. Whenever $g \subset Q \cap N$ is a filter generic...
over $V$, it must be the case that $N[g] \models x \in V \tau / g$ by the forcing theorem applied in $N$, and $V[g] \models x \in V \tau / \pi'g = \tau / g$ by the Mostowski absoluteness between $V[g]$ and $N[g]$. Thus, $Q_{\alpha} \models \tau_\alpha \in x \in \bar{\tau} / g$. By the inductive assumption there is a point $y \in X \cap M_{\alpha+1}$ which is $E$-related to $x$. Then, $M_{\alpha+1} \models \tau_\alpha \in x$. By the Borel absoluteness between the $Q$-extension of $V$ and $M_{\omega_1}$, it must be the case that $M_{\omega_1} \models \tau_\alpha \in x$. This contradicts the fact that $M_{\omega_1} \models \tau_0$ is a nontrivial $E$-pinned name together with the elementarity of the embedding $j_{\omega_1}$. □

**Corollary 2.6.3.** Let $E$ be a Borel equivalence relation on a Polish space $X$. The statement “$E$ is pinned” is absolute between all generic extensions.

**Proof.** The validity of item (2) of Theorem 2.6.1 does not change if one only considers countable models. This follows from an immediate downward Löwenheim-Skolem argument. The countable model version of (2) is a coanalytic statement, and as such it is absolute among all forcing extensions by the Mostowski absoluteness. □

The absoluteness of the pinned status of analytic equivalence relations is a more difficult question. A rather primitive example shows that in the constructible universe it fails in general; on the other hand, in the presence of sufficiently large cardinals the pinned status of every analytic equivalence relation is absolute. We refrain from giving the rather uninformative proofs.

Another natural question concerns the computation of the various cardinal invariants of equivalence relations in inner models. In particular, we would like to see that if $M \subseteq N$ are transitive models of ZFC containing the code for an analytic equivalence relation $E$, then $(\kappa(E))^M \leq (\kappa(E))^N$. Interestingly enough, this is not clear, and we only have a positive answer in the case of orbit equivalence relations and their relatives.

**Theorem 2.6.4.** Let $E$ be an analytic equivalence relation almost reducible to an orbit equivalence relation. Let $M$ be a transitive model of large portion of set theory containing the codes for $E$, the group action, and the almost reduction. Then $|\kappa(E)| \leq (\kappa(E))^M$ holds.

**Proof.** The proof uses the transitivity of the model $M$ in one fairly tricky abstract point. Let $M \models B$ be a complete Boolean algebra, completely generated by a set $A \subseteq B$. Then (viewed from $V$!) $RO(B)$ is a complete Boolean algebra completely generated by $A$ again. To see this, observe that every element of $B$ is obtained by a transfinite series of Boolean operations from the set $A$ in the model $M$, which is also a transfinite series of operations in the algebra $RO(B)$ in $V$, and these operations are evaluated in the same way in $B$ as in $RO(B)$ as $B$ is dense in $RO(B)$. It follows that $A$ generates a dense subset of $RO(B)$ in $V$, and therefore it generates $RO(B)$ as well.

Let $\Gamma$ be a Polish group continuously acting on a Polish space $Y$, $F$ its orbit equivalence relation, and $h: X \to Y$ a Borel almost reduction of $E$ to $F$ such that the codes of these objects belong to the model $M$. Suppose that $\lambda < |\kappa(E)|^M$ is a cardinal; we have to find an $E$-pinned name which has no equivalent on a poset of size $< \lambda$. 
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Work in the model $M$. Let $\tau$ be an $E$-pinned name on a poset $P$ which does not have an equivalent on a poset of size $< \lambda$. It will be enough to show that $\tau$ remains such also in $V$. To this end, let $\sigma = h(\tau)$. This is an $F$-pinned name which has no $F$-equivalent on a poset of size $< \lambda$. Consider the poset $\hat{P}$. It follows by Theorem 2.8.3 that the poset $\hat{P}$ collapses the cardinal $\lambda$. The initial remark of the proof shows that $\hat{P}$ as evaluated in $M$ is dense in $\hat{P}$ as evaluated in $V$.

Now, step out of the model $M$ and argue that $\langle P, \tau \rangle$ does not have an $\bar{E}$-equivalent on a poset of size $< \lambda$ even in $V$. Suppose for contradiction that $\langle Q, \eta \rangle$ is such an equivalent. Let $\chi = \dot{h}(\eta)$; this is a name $\bar{F}$-equivalent to $\sigma$. It follows from Claim 2.8.6 that $\hat{Q} \times P_G$ adds a point which is $\hat{P}$-generic over $V$ and therefore also over the smaller model $M$. Therefore, $\lambda$ is collapsed in the $\hat{Q} \times P_G$-extension. This contradicts the fact that the density of $\hat{Q} \times P_G$ is smaller than $\lambda$. \qed

2.7 Dichotomies

This section contains three theorems which characterize various complex features of the virtual realm in terms of dichotomies.

**Theorem 2.7.1.** Assume that there is a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. The following are equivalent:

1. $\kappa(E) = \infty$;
2. $E \leq_{\omega_1} \kappa$.

**Proof.** (2) implies (1) by Example 2.4.6 and Theorem 2.5.4. The large cardinal assumption is not needed for this direction. For the (1)$\to$(2) implication, suppose that $\kappa(E) = \infty$. Let $\kappa$ be a measurable cardinal.

**Claim 2.7.2.** There is a poset $P$ of size $\kappa$ and an $E$-pinned $P$-name $\tau$ such that $\tau$ is not $\bar{E}$-related to any name on a poset of size $< \kappa$.

**Proof.** Since $\kappa(E) = \infty$, there is a poset $Q$ and an $E$-pinned $P$-name $\sigma$ which is not $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Choose an elementary submodel $M$ of a large enough structure such that $|M| = \kappa$, $V_\kappa \subset M$ and $Q, \sigma \in M$, and let $P = Q \cap M$ and let $\tau = \sigma \cap M$. We claim that $P, \tau$ works as required.

Indeed, if $G \times H \subset P \times P$ is a filter generic over $V$, then it is also generic over $M$, by the elementarity of $M$ and the forcing theorem in $M$. $M[G, H] \models \tau / G E \tau / H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models \tau / G E \tau / H$. This proves that the name $\tau$ is $E$-pinned. If $R$ is a poset in $V$ of size $< \kappa$ and $\nu$ is an $E$-pinned $R$-name and $G \times H \subset P \times R$ is a generic filter over $V$, then $R, \nu \in M$, the filter $G \times H$ also generic over $M$, by the elementarity of $M$ and the forcing theorem in $M$. $M[G, H] \models \neg \tau / G E \nu / H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models$
\[\neg \tau \cup E \sigma / H.\] This proves that \(\nu \bar{E}\tau\) fails and completes the proof of the claim. \(\square\)

Choose a poset \(P\) of size \(\kappa\) and an \(E\)-pinned name \(\tau\) as in the claim. Let \(M\) be a countable elementary submodel of a large enough structure. Let \(Y\) be the space of binary relations on \(\omega\), so \(Y = \text{dom}(E_\omega)\). By Lemma \ref{lemma} and \ref{lemma2}, there are Borel functions \(f: Y \to Y\), \(g: Y \times M \to \omega\), \(h: Y \to \mathcal{P}(\omega)\) and \(k: Y \to X\) such that whenever \(y \in Y\) is a wellorder then \(f(y)\) is an isomorph of the iteration of the model \(M\) of length \(y\), \(g(y)\) is the iteration elementary embedding of \(M\) into \(f(y)\), \(h(y)\) is a filter on \(g(y)(P)\) generic over \(f(y)\), and \(k(y) = g(y)(\tau)/h(y)\). It will be enough to show that \(k\) is a Borel reduction of \(E_\omega\) to \(E\) on the set of \(y \in Y\) which code well-orders.

Suppose first that \(y, z \in Y\) are well-orders of the same length. Then \(f(y), f(z)\) are wellfounded and isomorphic. Write \(N\) for their common transitive isomorph, \(j: M \to N\) for the iteration map, and let \(Q = j(P)\) and \(\sigma = j(\tau)\). By the elementarity of the embedding \(j\), \(N \models \sigma\) is an \(E\)-pinned \(Q\)-name. Identify \(h(y), h(z)\) with filters on \(Q\) separately generic over \(N\), so \(k(y) = \sigma/h(y)\) and \(k(z) = \sigma/h(z)\). Let \(h' \subset Q\) be a filter generic over both countable models \(N[h(y)]\) and \(N[h(z)]\). By the product forcing theorem, the filters \(h' \times h(y)\) and \(h' \times h(z)\) are both \(Q \times Q\)-generic over \(N\). As \(N \models \tau\) is an \(E\)-pinned name, it follows that \(N[h', h(y)] \models k(y) = \sigma/h(y)\) in \(E\sigma/h'\) and \(N[h', h(z)] \models k(z) = \sigma/h(z)\) in \(E\sigma/h'\). By the Mostowski absoluteness between these two models and \(V\), and the transitivity of the relation \(E\), \(k(y) E k(z)\) follows.

Suppose now that \(y, z \in Y\) are well-orders of different lengths; say that \(y\) is shorter than \(z\). Let \(N_y\) be the transitive isomorph of \(f(y)\), \(j_y: M \to N_y\) the iteration map, \(Q_y = j_y(P), \sigma_y = j_y(\tau)\); similarly for \(N_z, j_z, Q_z, \sigma_z\). Identify \(h(y), h(z)\) with filters on \(Q_y, Q_z\) separately generic over \(N_y, N_z\), so \(k(y) = \sigma/h(y)\) and \(k(z) = \sigma/h(z)\). Now, as \(P, \tau \in H_{\omega_1}\), it is the case that \(Q_y, \sigma_y \in N_y\) and \(N_z \models [Q_y] < j_z(\kappa)\). Find a filter \(h' \subset Q_y\) generic over both the countable models \(N_y[h(y)]\) and \(N_z\), and let \(h' \subset Q_z\) be a filter generic over both the countable models \(N_z[h'][h(z)]\) and \(N_z[h(z)]\). Let \(x' = \sigma_y/h'\) and \(x'' = \sigma_z/h''\), both elements of the space \(X\). Since \(N_y \models \sigma_y\) is \(E\)-pinned, the Mostowski absoluteness between \(V\) and \(N_y[h(y), h']\) implies that \(k(y) E x'\) holds. Since \(N_z \models \sigma_z\) is \(E\)-pinned, the Mostowski absoluteness between \(V\) and \(N_y[h(z), h'']\) implies that \(k(z) E x''\) holds. Finally, since \(N_z \models [Q_z, \tau_z] E [Q_y, \tau_0]\) fails by the choice of \(P\) and the elementarity of the embedding \(j_z\), the Mostowski absoluteness between \(V\) and \(N_z[h', h'']\) implies that \(x' E x''\) fails. In conclusion \(k(y) E k(z)\) fails as required. \(\square\)

The status of a Borel equivalence relation as pinned/unpinned may be absolute between models of ZFC by Corollary \ref{corollary}. However, if one dares to look at choiceless models, much more colorful picture comes to sight. The simplest description of the pinned status occurs in the Solovay model, where it actually obeys a former conjecture of Kechris [17, Question 17.6.1].
2.7. DICHOTOMIES

**Theorem 2.7.3.** The following holds in the Solovay model derived from a measurable cardinal. Let $E$ be an analytic equivalence relation on a Polish space $X$. The following are equivalent:

1. $E$ is unpinned;
2. $\mathcal{F}_2 \leq E$ or $E_{\omega_1} \leq_\alpha E$.

**Proof.** Let $\kappa$ be a measurable cardinal and let $W$ be the derived Solovay model. In $W$, (2) certainly implies (1) as the proofs that $\mathcal{F}_2, E_{\omega_1}$ are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the reducibility orderings works in ZF+DC.

For the implication (1)$\rightarrow$ (2), assume that $W \models E$ is unpinned. There must be a poset $P$ and a nontrivial $E$-pinned name $\tau$ on the poset $P$, both in $W$. Both $P$ and $\tau$ must be definable in $W$ from a ground model parameter $z \in 2^\omega$. For simplicity of the notation assume that $z \in V$. Return to $V$. Let $Q$ be the two-step iteration $\text{Coll}(\omega, < \kappa) \ast \hat{P}$, and write $\sigma$ for the $Q$-name obtained from the $\hat{P}$-name $\tau$. There are two cases.

**Case 1.** There is a condition $q \in Q$ such that the $Q \upharpoonright q$-name $\sigma$ is $E$-pinned. In this case, we will conclude that $E_{\omega_1} \leq_\alpha E$ and use the Shoenfield absoluteness to transfer the almost reducibility to the Solovay model. To simplify the notation assume that $q$ is the largest element of the poset $Q$.

Observe that the name $\sigma$ cannot be $\check{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that $R$ is a poset of size $< \kappa$ and $\chi$ an $E$-pinned $R$-name such that $\langle R, \chi \rangle \check{E} \langle Q, \sigma \rangle$. In $W$, let $H \subset R$ be a filter generic over $V$. By Proposition 2.1.5 applied over $V$, $W \models P \Vdash \tau E \chi/H$. This contradicts the assumption that $V[G] \models P \Vdash \tau$ is not $E$-related to any point in $V[G]$.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 2.5.6(2), since $\kappa$ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 2.7.1, $E_{\omega_1} \leq_\alpha E$ as desired.

**Case 2.** For every condition $q \in Q$, the $Q \upharpoonright q$-name $\sigma$ is not $E$-pinned. In this case, we will conclude that $\mathcal{F}_2$ is reducible to $E$. Then, a Shoenfield absoluteness argument shows that the Borel reduction of $\mathcal{F}_2$ to $E$ remains a Borel reduction also in the Solovay model.

Fix a countable elementary submodel $M$ of a large structure containing the code for $E$, the posets $P, Q$ and the name $\tau$. We will start with an auxiliary lemma. A collection $\langle g_i : i \in I \rangle$ of filters on $\text{Coll}(\omega, < \kappa) \cap M$ is called **mutually generic over** $M$ if for every finite set $a \subset I$ the filter $\prod_{i \in a} g_i \subset \text{Coll}(\omega, < \kappa)^{[a]}$ is generic over $M$. For every set $a \subset I$ write $2_a^\omega = \bigcup \{2^\omega \cap M[\prod_{i \in b} g_i] : b \subset a \text{ finite}\}$, $M_a = M(2_a^\omega)$, $P_a$ and $\tau_a$ for the poset and name in $M_a$ defined in the model $M(2_a^\omega)$ by the formulas $\phi_P$ and $\phi_\tau$. Similar usage will prevail for functions $y : \omega \rightarrow I$, writing $P_y = P_{\text{rng}(y)}$ etc.

**Lemma 2.7.4.** Suppose that $\langle g_i : i \in I \rangle$ is a mutually generic collection of filters on $\text{Coll}(\omega, < \kappa)$.

1. whenever $a \subset I$ is a nonempty set then in there is a filter $h \text{Coll}(\omega, < \kappa)$-generic over $M$ such that $2^\omega \cap M[h] = 2_a^\omega$.
2. whenever \(a, b, c \subseteq I\) are pairwise disjoint countable nonempty sets then there is a filter \(h_a \times h_b \times h_c \subseteq \text{Coll}(\omega, < \kappa)^3\) generic over \(M\) such that \(2^\omega_a = 2^\omega \cap V[h_a]\) and similarly for \(2^\omega_b\) and \(2^\omega_c\).

3. whenever \(a\) and \(b\) are distinct countable subsets of \(I\) then \(P_a \times P_b \not\vDash \neg \tau_a \land \tau_b\).

**Proof.** For (1), let \(R = \{k : \exists \alpha < \kappa \exists \beta < a \text{ finite } k \subseteq \text{Coll}(\omega, < \alpha)\) is a filter generic over \(M\) and \(k \in M[g_i : i \in b]\}\) and order \(R\) by inclusion. Let \(K \subseteq R\) be a sufficiently generic filter; we claim that \(h = \bigcup K\) works as desired. Indeed, a simple density argument shows that \(h \subseteq \text{Coll}(\omega, < \kappa)\) is an ultrafilter all of whose proper initial segments are generic over \(M\). By the \(\kappa\)-c.c. of \(\text{Coll}(\omega, < \kappa)\), the filter \(h\) is in fact generic over \(M\) itself. A straightforward genericity argument then shows that \(2^\omega_a = 2^\omega \cap M[h]\) as desired.

(2) follows easily from (1). Let \(h_a, h_b, h_c \subseteq \text{Coll}(\omega, < \kappa)\) be any filters obtained from (1); we will show that these filters are in fact mutually generic over the model \(V\). Since \(\text{Coll}(\omega, < \kappa)^3\) has \(\kappa\)-c.c.c., it is enough to show that for every ordinal \(\alpha < \kappa\), the filters \(h^\alpha_a = h_a \cap \text{Coll}(\omega, < \alpha)\), \(h^\alpha_b = h_b \cap \text{Coll}(\omega, < \alpha)\), and \(h^\alpha_c = h_c \cap \text{Coll}(\omega, < \alpha)\) are mutually generic over \(V\). Since the filters \(h^\alpha_a\), \(h^\alpha_b\) and \(h^\alpha_c\) are coded by reals in the models \(M[h_a], M[h_b],\) and \(M[h_c]\), there are finite subsets \(a', b', c'\) of \(a, b, c\) respectively such that \(h^\alpha_i \in M[\prod_{i \in a} g_i]\) etc. The mutual genericity now follows from the general Corollary 13.3.3 about product forcing.

(3) is proved in several parallel cases depending on the mutual position of the sets \(a, b\) vis-a-vis inclusion. We will treat the case in which all three sets \(a \cap b, a \setminus b, b \setminus a\) are nonempty. Suppose for contradiction that \(P_a \times P_b \not\vDash \tau_a \land \tau_b\). From (2), it follows that in \(V\), the triple product \(\text{Coll}(\omega, < \kappa)^3\) forces \(\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \not\vDash \tau_{\{0,1\}} \land \tau_{\{1,2\}}\). Then, the quadruple product \(\text{Coll}(\omega, < \kappa)^4\) forces in \(V\) that \(\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \times \dot{P}_{\{2,3\}} \not\vDash \tau_{\{0,1\}} \land \tau_{\{1,2\}} \land \tau_{\{2,3\}}\), in particular \(\dot{P}_{\{0,1\}} \not\vDash \tau_{\{0,1\}} \land \tau_{\{2,3\}}\). In view of (2) again, this means that the product \(\text{Coll}(\omega, < \kappa) \times \text{Coll}(\omega, < \kappa)\) forces \(\dot{P}_{\leftarrow} \times \dot{P}_{\rightarrow} \not\vDash \tau_{\leftarrow} \land \tau_{\rightarrow}\). In other words, \((\text{Coll}(\omega, < \kappa) \times \dot{P}) \times (\text{Coll}(\omega, < \kappa) \times \dot{P})\) forces \(\tau_{\leftarrow} \land \tau_{\rightarrow}\), contradicting the case assumption.

Use Theorem 13.3.2 to find a continuous map \(f : 2^\omega \to \mathcal{P}(\text{Coll}(\omega, < \kappa) \cap M)\) such that its range consists of mutually generic filters over \(M\). Write \(Y = (2^\omega)^{\omega} = \text{dom}(\mathbb{F}_2)\). It is easy to find a Borel map \(g : Y \to (2^\omega)^{\omega}\) such that for every \(y \in Y\), \(g(y)\) enumerates the set \(2^\omega_y\). Use Lemmas 2.7.4(1) and 13.1.4 to find a Borel map \(h : Y \to \mathcal{P}(Q \cap M)\) such that for every \(y \in Y\), \(h(y) \subseteq Q\) is a filter generic over \(M\) and \(\text{rng}(g(y)) = 2^\omega \cap V[h_0(y)]\), where \(h_0(y) \subseteq \text{Coll}(\omega, < \kappa)\) is the filter generic over \(M\) obtained from \(h(y)\). Let \(k : Y \to X\) be given by \(k(y) = \tau/h(y)\); this is a Borel map by Lemma 13.1.3. We will show that \(k\) is a reduction of \(\mathbb{F}_2\) to \(E\).

First, assume that \(y_0, y_1 \in Y\) are \(\mathbb{F}_2\)-related. Then \(\text{rng}(y_0) = \text{rng}(y_1)\), \(\text{rng}(g(y_0)) = \text{rng}(g(y_1))\), and so \(M_{y_0} = M_{y_1}, P_{y_0} = P_{y_1}\) and \(\tau_{y_0} = \tau_{y_1}\). Let \(H \subseteq P_{y_0}\) be a filter generic over both countable models \(M_{y_0}[k(y_0)]\) and \(M_{y_1}[k(y_1)]\) and let \(x = \tau_{y_0}/H\). By the forcing theorem applied in the model \(M_{y_0} = M_{y_1}\) and
the fact that \( \tau_{y_0} \) is an \( E \)-pinned name, conclude that \( x \in E \ k(y_0) \) and \( x \in E \ k(y_1) \) and so \( k(x_0) \in E \ k(x_1) \) as desired.

Second, assume that \( y_0, y_1 \in Y \) are not \( \mathbb{F}_2 \)-related. Choose a sufficiently generic filter \( H_0 \times H_1 \subset P_{y_0} \times P_{y_1} \), so that \( H_0 \) is generic over \( M_{y_0}[k(y_0)] \) and \( H_1 \) is generic over \( M_{y_1}[k(y_1)] \). As the names \( \tau_{y_0} \) and \( \tau_{y_1} \) are \( E \)-pinned, the forcing theorem in the models \( M_{y_0} \) and \( M_{y_1} \) implies that \( k(y_0) \in E \ \tau_{y_0}/H_0 \) and \( k(y_1) \in E \ \tau_{y_1}/H_1 \). Now, \( \tau_{y_0}/H_0 \in E \ \tau_{y_1}/H_1 \) fails by Lemma 2.7.4(3), and so \( k(y_0) \in E \ k(y_1) \) must fail as well. This completes the proof.

**Corollary 2.7.5.** The following holds in the symmetric Solovay model derived from a measurable cardinal. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). \( E \) is unpinned if and only if \( \mathbb{F}_2 \leq E \).

**Proof.** This follows from Theorem 2.7.3 once we show that the option \( \mathbb{E}_{\omega_1} \leq_\alpha E \) is not available for any Borel equivalence relation \( E \). This in turn follows easily from results on the pinned cardinal \( \kappa(E) \) obtained in Section 2.5: \( \kappa(E) < \infty \) by Theorem 2.5.6(1), \( \kappa(\mathbb{E}_{\omega_1}) = \infty \) by Example 2.4.6, and the pinned cardinal is monotone with respect to the reducibility ordering \( \leq_\alpha \). Theorem 2.5.4.

Among the equivalence relations Borel reducible to \( \mathbb{F}_2 \), \( \mathbb{F}_2 \) is the only one whose pinned cardinal can be proved in \( \text{ZFC} \) to be equal to \( \varepsilon^+ \). This is the contents of the following theorem:

**Theorem 2.7.6.** Let \( \kappa > \beth_{\omega_1} \) be a regular cardinal such that \( \kappa^{\beth_\omega} = \kappa \). Let \( P \) be the usual poset for adding \( \kappa \) Cohen reals. In the \( P \)-extension, exactly one of the following holds for every Borel equivalence relation \( E \) Borel-reducible to \( \mathbb{F}_2 \):

1. \( E \) is bireducible with \( \mathbb{F}_2 \);
2. \( \kappa(E) \leq \varepsilon \).

**Proof.** It is clear that (1) implies the negation of (2). If \( \mathbb{F}_2 \) is Borel reducible to \( E \) then \( \kappa(E) \geq \kappa(\mathbb{F}_2) = \varepsilon^+ \) holds in every generic extension, and so (2) fails. We must show that the negation of (2) implies that \( \mathbb{F}_2 \) is Borel-reducible to \( \mathbb{F}_2 \).

For simplicity assume that \( E \) is a Borel equivalence relation on \( X = 2^\omega \) with a code in the ground model, and fix a Borel reduction \( h : 2^\omega \to (2^\omega)^\omega \) of \( E \) to \( \mathbb{F}_2 \).

Let \( G \subset P \) be a filter generic over the ground model and work in \( V[G] \). Since \( V[G] \models \kappa(E) = \varepsilon^+ \), there must be a poset \( Q \) and an \( E \)-pinned name \( \sigma \) on the poset \( Q \) such that \( \sigma \) is not \( E \)-equivalent to any name on a poset of size \( < \varepsilon \). The \( Q \)-name \( \dot{h}(\sigma) \) is \( \mathbb{F}_2 \)-pinned and so there is a set \( A \subset 2^\omega \) such that \( Q \models \dot{A} = \text{rng}(\dot{h}(\sigma)) \) by Example 2.3.5. By the Mostowski absoluteness, there is a \( \text{Coll}(\omega, A) \)-name \( \chi \) for an element of \( X \) such that \( \text{rng}(\dot{h}(\chi)) = A \). The name \( \chi \) is \( E \)-related to \( \sigma \). Since \( \sigma \) was chosen to have no equivalents on posets of size \( < \varepsilon \), it follows that \( |A| = \varepsilon \).

Back to the ground model. The previous paragraph shows that there must be a \( P \)-name \( \dot{A} \) for a subset of \( 2^\omega \) of size \( \varepsilon = \kappa \) such that in some further forcing extension (by Mostowski absoluteness it is enough to take \( \text{Coll}(\alpha, \kappa) \)-extension) there is an element \( x \in X \) such that \( \text{rng}(\dot{h}(x)) = A \).
Since $\kappa > \beth_\omega$, essentially by Theorem 2.5.8 there is a countable model $M$ elementarily equivalent to $\mathbb{R}$ such that the integers of $M$ are isomorphic to $\omega$ and $M$ has an infinite collection of ordered indiscernibles below $\kappa^M$. Let $N$ be a model obtained as a Skolem hull of indiscernibles of ordertype $2^\omega$ with the usual lexicographic ordering from $M$. More precisely, let $Y = \{ (t, \vec{a}) \mid n \in \omega, t$ is an $n$-ary Skolem term and $\vec{a} \in (2^\omega)^n \}$. Let $F$ be the equivalence relation on $Y$ defined by $(t, \vec{a}) F (s, \vec{b})$ if for some sequences $\vec{a}' \vec{b}'$ of indiscernibles in the model $M$ ordered in the same way as $\vec{a}, \vec{b}$, $M \models t(\vec{a}') = t(\vec{b}')$. Build the model $N$ on the collection of $F$-classes in the usual way: $\{ \}$

The following piece of notation will be useful below. For a pair $y = (t, \vec{a}) \in Y$, write $\text{supp}(y)$ for the intersection of all sets $b \subset 2^\omega$ such that for some Skolem term $s$ and some sequence $\vec{c}$ with $\text{rng}(\vec{c}) \subseteq b$, $(s, \vec{c}) F (t, \vec{a})$. An easy indiscernibility argument shows that $y \mapsto \text{supp}(y)$ is a Borel function in a natural sense.

**Claim 2.7.7.** $N$ is (can be presented as) a Borel model. The natural numbers of $N$ are isomorphic to $\omega$.

**Proof.** The equivalence relation $F$ is clearly Borel; we will show that it is smooth. $\Box$

**Claim 2.7.8.** There is a filter $G \subset P^N$ generic over the model $N$ which is moreover a Borel subset of $N$.

Let $G \subset P^N$ be a filter generic over the model $N$. For any point $z \in (2^\omega)^\omega$, let $N_z$ be the Skolem hull of the $N$-indiscernibles in the set $\text{rng}(z)$, naturally presented as a structure on $\omega$. Note that $G_z = G \cap N_z$ is a filter generic over the model $N_z$; this follows from the fact that $N_z$ is an elementary submodel of $N$, $N \models P^N$ is c.c.c., and all natural numbers of $N$ belong to $N_z$. Let $a_z = \bar{A}^N_z/G_z$; this is a subset of $2^\omega$.

**Claim 2.7.9.** For $z_0, z_1 \in (2^\omega)^\omega$, $z_0 \mathbb{F}_2 z_1 \Leftrightarrow a_{z_0} = a_{z_1}$ holds.

Use Proposition 13.1.2 to find a Borel map $z \mapsto H_z$ assigning to every point $z \in (2^\omega)^\omega$ a filter $H_z \subset \text{Coll}(\omega, \kappa)^{N_z}$ generic over the model $N_z[G_z]$. Use Proposition 13.1.3 to find a Borel map $z \mapsto x_z$ such that $x_z \in X^{N_z[G_z][H_z]}$ such that $N_z[G_z[H_z]] \models \text{rng}(h(x_z)) = a_z$. By the Borel absoluteness between $V$ and the model $N_z[G_z][H_z]$, the statement $\text{rng}(h(x_z)) = a_z$ holds even in $V$. Finally, Claim 2.7.9 shows that the map $z \mapsto x_z$ is the desired Borel reduction of $\mathbb{F}_2$ to $E$. $\Box$

### 2.8 Restrictions on partial orders

Given an analytic equivalence relation $E$ on a Polish space $X$, it may be informative to investigate which posets can carry $E$-pinned names. After all, essentially all pinned names discussed in this chapter naturally live on collapse posets, and so one may easily (and wrongly) assume that no forcing sophistication is needed.
when it comes to the investigation of the virtual realm. This section contains several theorems on this topic.

First of all, there are some partial orders which can never carry a nontrivial pinned name.

**Definition 2.8.1.** [8] A poset $P$ is reasonable if for every ordinal $\lambda$ and for every function $f: \lambda^{<\omega} \to \lambda$ in the $P$-extension there is a set $a \subset \lambda$ which is closed under $f$, belongs to the ground model, and it is countable in the ground model.

In particular, all c.c.c. and all proper forcings are reasonable. Good examples of unreasonable forcings are posets which collapse $\aleph_1$, Namba forcing and Prikry forcing.

**Theorem 2.8.2.** Let $E$ be an analytic equivalence relation on a Polish space $X$. If $P$ is a reasonable forcing and $\tau$ is an $E$-pinned name on $P$, then $\tau$ is trivial.

**Proof.** Suppose that $P$ is a reasonable poset and $\tau$ is an $E$-pinned name on $P$. We will produce a condition $p \in P$ and a point $x \in X$ such that $p \Vdash \tau E \bar{x}$.

Towards this end, choose a large structure and use the reasonability of $P$ to find a countable elementary submodel $M$ of it containing $P, E$ and $\tau$ and a condition $p \in P$ such that $p \Vdash G \cap \bar{M}$ is generic over $\bar{M}$, where $\bar{G}$ is the canonical $P$-name for its generic ultrafilter. As $M$ is countable, there is a filter $H \subset P \cap M$ generic over $M$ in the ground model $V$. Let $x = \tau/H \in X$. Proposition 2.1.5 applied to the model $M$ and the filters $H$ and $\bar{G} \cap M$ now says that $p \Vdash \bar{x} E \tau$, completing the proof.

The key feature of partial orders from the point of view of existence of pinned names is collapsing $\aleph_1$ as the following theorem shows:

**Theorem 2.8.3.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Exactly one of the following occurs:

1. $E$ is pinned;
2. for every poset $P$ collapsing $\aleph_1$, $P$ carries a nontrivial $E$-pinned name.

**Proof.** Clearly (1) implies the negation of (2). For the difficult direction, suppose that (1) fails and work to confirm (2). Let $\tau$ be a nontrivial $E$-pinned name on some poset $P$. Let $\langle M_\alpha: \alpha \in \omega_1 \rangle$ be a continuous $\bar{\epsilon}$-tower of countable elementary submodels of a large structure containing $X$ and $E$. Let $M = \bigcup_\alpha M_\alpha$, let $Q = P \cap M$ and let $\sigma = \tau \cap M$.

Observe that in any generic extension, whenever $G, H \subset Q$ are generic filters over $M$, then $\sigma/G E \sigma/H$ and their equivalence class contains no ground model elements. The first part of this statement follows from Proposition 2.1.5 applied in the transitive collapse of the model $M$. For the second part of the statement, it is enough to show that in $V$, $Q \Vdash \sigma$ is not $E$-equivalent to any element of the ground model. Suppose for contradiction that there is a point $x \in X$ such that
$Q \Vdash \sigma \in \check{x}$. We will show that there then must be $y \in M \cap X$ which is $E$-related to $x$. Then $Q \Vdash \sigma \in \check{y}$, by the Mostowski absoluteness between the $Q$-extensions of $M$ and $V$. We have just proved that $\nu$ is a nontrivial $E$-pinned name on $\check{x}$. This does not occur for orbit equivalence relations. In them, every pinned name is inescapably connected with a poset carry nontrivial $E$-pinned names. This does not occur for orbit equivalence relations. In them, every pinned name is inescapably connected with a poset collapsing its own density character to $\aleph_1$.

To find the point $y \in M \cap X$, let $N$ be a countable elementary submodel of a large structure containing $\langle M_\alpha : \alpha \in \omega_1 \rangle, Q, x$. Since the tower of models $\langle M_\alpha : \alpha \in \omega_1 \rangle$ is continuous, there is a limit ordinal $\alpha \in \omega_1$ such that $M_\alpha = N \cap M$. Let $Q_\alpha = Q \cap M_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \sigma \cap M_\alpha = \tau \cap M_\alpha$. By elementarity of the model $N$ and analytic absoluteness between the models $M_\alpha$-extension of $N$ and $V$, $Q_\alpha \Vdash \sigma_\alpha \in \check{x}$. Since $Q_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \tau \cap M_\alpha$, both $Q_\alpha, \tau_\alpha$ belong to the model $M_{\alpha+1}$. By the elementarity of the model $M_{\alpha+1}$, there must be a point $y \in X \cap M_{\alpha+1}$ such that $Q_\alpha \Vdash \sigma_\alpha \in \check{y}$. By the transitivity of $E$, it follows that $x \in E \check{y}$. The point $y \in M_{\alpha+1} \subset M$ works.

Now, suppose that $R$ is a poset collapsing $\aleph_1$. Since $|M| = \aleph_1$, in the $R$-extension there is a filter $Q$-generic over $M$. Let $\check{H}$ be an $R$-name for such a filter and let $\nu$ be the $R$-name for $\sigma/\check{H}$. We have just proved that $\nu$ is a nontrivial $E$-pinned name on $R$. \hfill \Box

Theorem 2.8.3 does not rule out the possibility that some $\aleph_1$ preserving posets carry nontrivial $E$-pinned names. This does not occur for orbit equivalence relations. In them, every pinned name is inescapably connected with a poset collapsing a certain cardinal to $\aleph_0$. This is the contents of the following:

**Theorem 2.8.4.** Let $E$ be an analytic equivalence relation on a Polish space $X$, Borel reducible to an orbit equivalence relation. Let $\langle P, \tau \rangle$ be an $E$-pinned name. There is a cardinal $\kappa$ such that in every forcing extension, the following are equivalent:

1. $|\kappa| = \aleph_0$;
2. the virtual $E$-class of $\langle P, \tau \rangle$ is realized.

**Proof.** We will start with a preliminary general claim.

**Claim 2.8.5.** Suppose that $P$ is a poset such that $P \Vdash \check{P}$ has a countable dense subset. Then $P$ is in the forcing sense equivalent to $\text{Coll}(\omega, \kappa)$ where $\kappa$ is the minimum size of a dense subset of $P$.

**Proof.** Let $\check{x}$ be a $P$-name for an enumeration of a countable dense subset of $P$. Then $D = \{ q \in P : \exists p \in P \exists n \in \omega \ p \Vdash \check{q} = \check{x}(n) \}$ must be a dense subset of $P$ and as such has size at least $\kappa$. In the $P$-extension, the set $D$ is equal to $\{ q \in P : \exists p \in \text{rng}(x) \exists n \in \omega \ p \Vdash \check{q} = \check{x}(n) \}$ which is clearly countable. Thus, $P \Vdash |\kappa| = \aleph_0$. Now, every poset collapsing its own density character to $\aleph_0$ is in the forcing sense equivalent to the collapse poset by a classical theorem of McAloon–Fact 1.3.10. \hfill \Box

Now, some notation. Let $\Gamma$ be a Polish group acting on a Polish space $X$ inducing $E$ as its orbit equivalence relation. Write $P_{\Gamma}$ for the Cohen poset on
the acting group $\Gamma$, adding a generic group element $\dot{\gamma}$. For an $E$-pinned name $\tau$ on a poset $P$, write $\dot{\tau}$ for the $P \times P_\Gamma$-name $\dot{\gamma} \cdot \tau$ and $\dot{P}$ for the complete subalgebra of $RO(P \times P_\Gamma)$ generated by the name $\dot{\tau}$. Note that $\langle P, \dot{\tau} \rangle E \langle P, \tau \rangle$.

**Claim 2.8.6.** Suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \models \tau E \dot{x}$. In $V[H]$, $P_\Gamma \models \dot{\gamma} \cdot \dot{x}$ is $P$-generic over $V[H]$. Moreover, for every condition $\dot{p} \in \dot{P}$ there is a condition $q \in P_\Gamma$ forcing the filter generated by $\dot{\gamma} \cdot \dot{x}$ to contain $\dot{p}$.

**Proof.** Let $p \in P$ and $q \in P_\Gamma$ be arbitrary conditions. Let $K \subseteq P$ be a generic filter over $V[H]$ and write $\gamma = \tau/K$. It will be enough to find points $g, h \in \Gamma$ $P_\Gamma$-generic over the model $V[H][K]$, such that $h \in q$ and $gx = hy$. For then, $g$ is $P_\Gamma$-generic over $V[H]$ and $gx$ is a $\bar{\dot{P}}$-generic point over $V[H]$ meeting the projection of the condition $\langle p, q \rangle$ into $\dot{P}$.

By the assumption on the point $x$, it is the case that $V[H][K] \models x E y$, and there is an element $\beta \in \Gamma \cap V[H][K]$ such that $\beta \cdot x = y$. Let $\delta \in q$ be a $P_\Gamma$-generic point over $V[H][K]$ and let $\gamma = \delta \beta$. As the topology on the group $G$ is invariant under translations, $\gamma$ is also $P_\Gamma$-generic over $V[H][K]$, and $\gamma \cdot x = \delta \beta \cdot x = \delta \cdot y$ as required. \hfill $\square$

**Claim 2.8.7.** $\dot{P}$ is in the forcing sense equivalent to $Coll(\omega, \kappa)$ for some cardinal $\kappa$.

**Proof.** Since $\tau$ is an $E$-pinned name, $\dot{P} \models P \models \tau E \dot{\tau}$. Thus, by Claim 2.8.6, in the $\dot{P}$-extension the algebra $RO(\dot{P})$ is completely embedded into $RO(P_\Gamma)$ and therefore $\dot{P}$ contains a countable dense set. The proof is completed by a reference to Claim 2.8.5. \hfill $\square$

We will now show that $\kappa$ works as desired. On one hand, suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \models \tau E \dot{x}$. By Claim 2.8.6, in $V[H]$ the poset $P_\Gamma$ adds a $\bar{\dot{P}}$-generic and therefore by Claim 2.8.7, it collapses $\kappa$ to $\aleph_0$. Since $P_\Gamma$ is just the Cohen poset, this can happen only if $|\kappa| = \aleph_0$ already in $V[H]$.

On the other hand, suppose that $V[H]$ is a generic extension in which $|\kappa| = \aleph_0$. Back in the ground model, find an elementary submodel $M$ of a large enough structure containing $P, \tau$ such that $|M| = \kappa$ and $M \cap P$ is dense in $\dot{P}$. In the model $V[H]$, there is a $\bar{\dot{P}}$-generic filter $K$ over $M$. Let $x = \tau/K$. It follows that $\dot{P} \models \dot{\tau} E \dot{x}$ and so $P \models \tau E \dot{x}$. \hfill $\square$

**Corollary 2.8.8.** Let $E$ be a Borel equivalence relation on a Polish space $X$, Borel reducible to an orbit equivalence relation. If $E$ is not pinned then the following are equivalent:

1. $P$ carries a nontrivial $E$-pinned name;
2. $P$ collapses $\aleph_1$ to $\aleph_0$. 


Example 2.8.9. Consider the equivalence relation $\mathbb{F}_2$ on $(2^\omega)^\omega$. This is clearly an orbit equivalence relation. Every $E$-pinned name is represented by a set $A \subset 2^\omega$ by Example 2.3.5. The cardinal $\kappa$ of Theorem 2.8.4 is equal to $|A|$.

Example 2.8.10. Let $E$ be the equivalence relation on $X = (\mathcal{P}(\omega))^\omega$ defined by $x_0 E x_1$ if $\text{rng}(x_0)$ and $\text{rng}(x_1)$ generate the same filter on $\omega$. Suppose that in $V$, there is a modulo finite strictly decreasing sequence $a = \langle a_\alpha : \alpha \in \omega_2 \rangle$ of subsets of $\omega$. Let $P$ be the Namba forcing. It is well-known that $P$ preserves $\aleph_1$ and adds a cofinal sequence $\sigma : \omega \to \omega_2^V$ to $\omega_2$. Let $\tau$ be the $P$-name for $a \circ \sigma$. It is immediate that $\tau$ is a nontrivial $E$-pinned name on $P$. Thus, the assumption that $E$ be reducible to an orbit equivalence in Corollary 2.8.8 cannot be omitted.
Chapter 3

Turbulence

Hjorth and Kechris isolated the notion of turbulent actions of Polish groups and showed [20, Theorem 12.5] that orbit equivalence relations of turbulent actions are not Borel reducible to equivalence relations classifiable by countable structures. This non-reducibility result transfers to cardinal inequalities in many models studied in this paper.

3.1 Independent functions

The purpose of this section is to isolate a practical criterion which would imply that generic extensions $V[G_0]$ and $V[G_1]$ have trivial intersection, i.e. $V[G_0] \cap V[G_1] = V$. We need a folkloric observation.

**Proposition 3.1.1.** Let $X, Y$ be Polish spaces and $f : X \to Y$ be a continuous open function. Then $P_X \Vdash f(x_{gen})$ is a $P_Y$-generic element of $Y$.

**Proof.** It is just necessary to show that $P_X \Vdash f(x_{gen}) \in D$ for every open dense set $D \subset Y$. To this end, let $O \in P_X$ be a condition. The set $f''O \subset Y$ is open, and therefore it has nonempty intersection with $D$. Consider the nonempty open set $O' = (f^{-1}(f''O \cap D)) \cap O \subset O$. For every point $x \in O'$, $f(x) \in D$ holds, and so $O' \Vdash f(x_{gen}) \in D$ as required.

Now suppose that $X, Y_0, Y_1$ are Polish spaces and $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ are continuous open maps. We want to find a criterion implying that $P_X \Vdash V[f_0(x_{gen})] \cap V[f_1(x_{gen})] = V$. The following turbulence–like definition is central.

**Definition 3.1.2.** Let $X, Y_0, Y_1$ be Polish spaces and $f_0 : X \to Y_0$ and $f_1 : X \to Y_1$ be continuous open maps. If $O \subset X$ is an open set, a walk in $O$ is a sequence $\langle x_i : i \in 2k + 1 \rangle$ of points in $O$ such that for all $i \in k$, $f_0(x_{2i}) = f_0(x_{2i+1})$ and $f_1(x_{2i+1}) = f_1(x_{2i+2})$.

First, a simple complexity observation.
Proposition 3.1.3. Let $X, Y_0, Y_1$ be Polish spaces and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ be continuous open maps. Let $O \subseteq X$ and $D \subseteq O$ be nonempty open sets and $k \in \omega$ be a number. The set $C(k, D) \subseteq O$ of all points $x \in O$ such that there is a walk $\langle x_i : i \in 2k + 1 \rangle$ in $O$ with $x_0 = x$ and $x_{2k} \in D$ is open.

Proof. By induction on $k$ simultaneously for all open sets $D \subseteq O$. The case $k = 0$ is trivially satisfied, since $C(k, O) = D$. The case $k = 1$ is satisfied with $f_0^{-1}f_0''(f_1^{-1}(f_0''D) \cap O) \cap O$. Now suppose that the proposition is verified for $k$ and all open sets $D \subseteq Y_0$. To check it for $k + 1$ and a fixed $D$, apply the induction hypothesis to conclude that the set $C(k, D) \subseteq O$ is open. Apply the case $k = 1$ to show that the set $C(1, C(k, D)) \subseteq O$ is open. The definitions show that $C(1, C(k, D)) = C(k + 1, D)$, completing the induction step.

Definition 3.1.4. Let $X, Y_0, Y_1$ be Polish spaces and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ be continuous open maps. The functions $f_0, f_1$ are independent if for every nonempty open set $O \subseteq X$ there is a nonempty open set $A \subseteq X$ such that for all nonempty open subsets $B_0, B_1 \subseteq A$ there is a walk $\langle x_i : i \in 2k \rangle$ in $O$ such that $f_0(x_0) \in B_0$ and $f_0(x_{2k}) \in B_1$.

It may appear that the definition is not symmetric with respect to the maps $f_0, f_1$. In fact, the definition is symmetric, and this follows as a small corollary from the following central theorem.

Theorem 3.1.5. Suppose that $X, Y_0, Y_1$ are Polish spaces and $f_0: X \to Y_0$ and $f_1: X \to Y_1$ are continuous open maps. The following are equivalent:

1. $f_0, f_1$ are independent;
2. $P_X \models V[f_0(x_{\text{gen}})] \cap V[f_1(x_{\text{gen}})] = V$.

Proof. For (1)$\to$(2) implication, since the models in question are models of set theory with choice, it is enough to show that every set of ordinals in the intersection is actually in $V$. To this end, suppose that $\tau_0, \tau_1$ are $P_{Y_0}$ and $P_{Y_1}$-names for sets of ordinals respectively and $O \subseteq X$ is a nonempty open set forcing $\tau_0/f_0(x_{\text{gen}}) = \tau_1/f_1(x_{\text{gen}})$. We need to find an open set $O' \subseteq O$ deciding the statement $\dot{\alpha} \in \tau_0/f_0(x_{\text{gen}})$ for every ordinal $\alpha$.

Let $A \subseteq Y_0$ be a nonempty open set standing witness to the independence of the functions $f_0, f_1$. We claim that for every ordinal $\alpha$, in the poset $P_{\dot{\alpha}}$, $A \models \dot{\alpha} \in \tau_0$ or $A \models \dot{\alpha} \notin \tau_0$. Once this is proved, let $O' = O \cap f_0^{-1}A$ and note that for every ordinal $\alpha$, in the poset $P_{\dot{\alpha}}$, $O' \models \dot{\alpha} \in \tau_0/f_0(x_{\text{gen}})$ or $A \models \dot{\alpha} \notin \tau_0/f_1(x_{\text{gen}})$. This will complete the proof.

Suppose towards contradiction that there is an ordinal $\alpha$ and nonempty open sets $B_0, B_1 \subseteq A$ such that $B_0 \models \dot{\alpha} \in \tau_0$ and $B_1 \models \dot{\alpha} \in \tau_1$. Let $k \in \omega$ be the minimal number such that there is a walk $\langle x_i : i \leq 2k \rangle$ in $O$ with $f_0(x_0) \in B_0$ and $f_0(x_{2k}) \in B_1$. The following is the central walk claim:

Claim 3.1.6. There are nonempty open sets $O_i \subseteq O$ for $i \leq 2k$ such that

1. $f_0''O_0 \subseteq B_0$ and $f_0''O_{2k} \subseteq B_1$;
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2. for each \( i \leq k \), \( f_i^0 O_{2i} \) in \( P_{Y_0} \) decides the statement \( \bar{\alpha} \in \tau_0 \);

3. for each \( i < k \), \( f_i^1 O_{2i+1} \) in \( P_{Y_1} \) decides the statement \( \bar{\alpha} \in \tau_1 \);

4. for each \( i < k \), the sets \( f_i^0 O_{2i+1} \subset f_i^1 O_{2i} \) and \( f_i^0 O_{2i+2} \subset f_i^1 O_{2i+1} \).

Once this is proved, note that by the additional assumptions on the sets \( B_0, B_1 \) there has to be \( i \leq k \) such that in the poset \( O_{2i} \models \bar{\alpha} \notin \tau_0 / f_0(\hat{x}_{\text{gen}}) \) and \( O_{2i+2} \models \bar{\alpha} \in \tau_0 / f_0(\hat{x}_{\text{gen}}) \). For such \( i \), no matter how \( O_{2i+1} \) decides the statement \( \bar{\alpha} \in \tau_1 \), it will be in disagreement with the decision made by either \( O_{2i} \) or \( O_{2i+2} \). Appropriately, one of the conditions \( O_{2i+1} \) or \( O_{2i+2} \) in the poset \( P_X \) will force \( \bar{\alpha} \in \tau_0 / f_0(\hat{x}_{\text{gen}}) \Delta \tau_1 / f_1(\hat{x}_{\text{gen}}) \), contradicting the initial choice of the open set \( O \subset X \).

For every number \( i \leq k \) let \( P_i = \{ x \in O : \) there is a walk \( \langle x_j : j \in 2k+1-2i \rangle \) in \( O \) such that \( x = x_{2i} \), and for all \( j \) \( x_{2j} \in O_{2j} \) and \( f_0(x_{2k}) \in B_1 \} \). The sets \( P_i \) are open by Proposition 3.1.3. The construction of the sets \( O_{2i}, O_{2i+1} \) proceeds by recursion on \( i \) with the additional hypothesis that \( O_{2i} \subset P_i \). To start the recursion, just let \( O_0 = O \cap f_0^{-1} B_0 \cap P_0 \); this is a nonempty set by the initial assumptions. Now suppose that the set \( O_{2i} \subset X \) has been found.

Let \( Q = j_1^{-1}(j_1^0 f_0^1(j_1^1 O_{2i}) \cap O) \cap O \). The set \( Q \subset O \) is open, and must have nonempty intersection with \( P_{i+1} \). Find an open set \( O_{2i+2} \subset Q \cap P_{i+1} \) such that \( f_i^0 O_{2i+2} \) in \( P_{Y_0} \) decides the statement \( \bar{\alpha} \in \tau_0 \) and \( f_i^0 O_{2i+2} \) in \( P_{Y_1} \) decides the statement \( \bar{\alpha} \in \tau_1 \), and let \( O_{2i+1} = j_1^{-1}(f_i^1 O_{2i+2} \cap f_0^1 f_i^0 O_{2i} \cap O \). A review of the demands of the claim shows that the sets \( O_{2i+1} \) and \( O_{2i+2} \) work as required.

To see why the negation of (1) implies the negation of (2), let \( O \subset X \) be a nonempty open set witnessing the failure of (1). For each \( y \in Y_0 \), define the orbit of \( y \) to be the set of all elements \( z \in Y_0 \) such that there is a walk \( \langle x_i : i \leq 2k \rangle \) in \( O \) such that \( f_0(x_0) = y \) and \( f_0(x_{2k}) = z \). Let \( x \in O \) be a \( P_X \)-generic and work in the model \( V[x] \).

**Claim 3.1.7.** The orbit of \( f_0(x) \) is nowhere dense in \( Y_0 \).

**Proof.** If the orbit were dense in some nonempty open set \( B \subset Y_0 \), then in the ground model, the set \( B \) would show that \( O \) is not a witness to the failure of (1). To see this, suppose that \( B_0, B_1 \subset B \) are nonempty open subsets of \( B \). Working in \( V[x] \), find walks \( w_0, w_1 \) in \( O \) which lead from \( f_0(x) \) to \( B_0 \) and \( B_1 \), invert \( w_0 \) and concatenate with \( w_1 \) and get a walk from \( B_0 \) to \( B_1 \). By the Mostowski absoluteness between \( V[x] \) and \( V \), there must be such a walk in \( V \) as well. \( \square \)

Now look at the set \( A = \{ B \subset Y_0 : B \) is a basic open subset of \( Y_0 \) and \( B \) contains no element of the orbit of \( f_0(x) \} \).

**Claim 3.1.8.** \( A \subset V[f_0(x)] \cap V[f_1(x)] \).

**Proof.** In \( V[x] \), the set \( A \) is defined as the set of all basic open sets \( B \subset Y_0 \) such that there is no walk \( \langle x_i : i \leq 2k \rangle \) in \( O \) such that \( f_0(x_0) = f_0(x) \) and \( f_0(x_{2k}) \in B \). This is a \( \Pi_1^1 \) definition with parameter \( f_0(x) \) which therefore yields the same set in the model \( V[f_0(x)] \).
In $V[x]$, the same set $A$ also has an alternative definition: it is the set of all basic open sets $B \subset Y_0$ such that there is no walk $\langle x_i : i \leq 2k \rangle$ in $O$ such that $f_1(x_0) = f_1(x)$ and $f_0(x_{2k}) \in B$. To see this, note that a walk with $f_0(x_0) = f_0(x)$ can be transformed into a walk with $f_1(x_0) = f_1(x)$ (and also vice versa) by simply adding the point $x$ twice to the beginning of the walk. This alternative definition of the set $A$ is again $\Pi^1_1$ with parameter $f_1(x)$, and so by the Mostowski absoluteness yields the same set in the model $V[f_1(x)]$.

To show that (2) fails, it is enough to argue that $A \notin V$. However, if $A \in V$ then $\bigcup A \in V$ is an open dense subset of $Y_0$ by Claim 3.1.7, and $f_0(x) \notin \bigcup(A)$. This contradicts Proposition 3.1.1: the point $f_0(x) \in Y$ is forced to be $P_Y$-generic over the ground model and therefore to belong to every open dense set in the ground model.

The independent pairs of open maps are closed under a product operation of a certain type. The easiest proof uses Theorem 3.1.5.

**Definition 3.1.9.** Let $X_0, Y_0, Z_0$ and $X_1, Y_1, Z_1$ be Polish spaces and $f_0: X_0 \to Y_0, g_0: X_0 \to Z_0, f_1: X_1 \to Y_1$ and $g_1: X_1 \to Z_1$ be continuous open maps. The product of the triples $\langle X_0, f_0, g_0 \rangle$ and $\langle X_1, f_1, g_1 \rangle$ is the triple $\langle X_0 \times X_1, f_0 \times f_1, g_0 \times g_1 \rangle$ where the maps $f_0 \times f_1: X_0 \times X_1 \to Y_0 \times Y_1$ and $g_0 \times g_1: X_0 \times X_1 \to Z_0 \times Z_1$ are defined in the natural way: $(f_0 \times f_1)(x_0, x_1) = (f_0(x_0), f_1(x_1))$, and similarly for $g_0 \times g_1$.

**Theorem 3.1.10.** The product of two pairs of independent maps is a pair of independent maps.

**Proof.** Use the notation of Definition 3.1.9. Suppose that $f_0, g_0$ and $f_1, g_1$ are pairs of independent open maps. It is clear that the product maps are again continuous and open. To verify the independence, let $\dot{x}_0, \dot{x}_1$ be the respective $P_{X_0}$- and $P_{X_1}$-names for the generic points of $X_0, X_1$. Let $y_0, \dot{z}_0$ be the $P_{X_0}$-names for $f_0(\dot{x}_0)$ and $g_0(\dot{x}_0)$ and similarly for subscript 1. In view of Theorem 3.1.5, it is enough to show that $P_{X_0} \times P_{X_1} \Vdash V[y_0, \dot{y}_1] \cap V[\dot{z}_0, z_1] = V$.

To this end, suppose that $p \in P_{X_0} \times P_{X_1}$ is a condition and $\tau$ is a $P_{X_0} \times P_{X_1}$-name such that $p \Vdash \tau \in V[\dot{z}_0, \dot{z}_1]$; we must show that (strengthening the condition $p$ if necessary), $p \Vdash \tau \in V$. Towards this end, first note that $p \Vdash \tau \in V[y_0, \dot{z}_0][\dot{y}_1] \cap V[y_0, \dot{z}_0][z_1]$. Since the independence of the pair of the continuous open maps $f_1, g_1$ is an analytic statement, it transfers from $V$ to $V[y_0, \dot{z}_0]$, showing that the intersection of the two models is equal to $V[y_0, \dot{z}_0]$. Thus, $p \Vdash \tau \in V[y_0, \dot{z}_0]$. By the product forcing theorem, the intersections $V[y_0, \dot{z}_0] \cap V[\dot{y}_0, \dot{y}_1]$ and $V[y_0, \dot{z}_0] \cap V[\dot{z}_0, \dot{z}_1]$ are equal to $V[y_0]$ and $V[z_0]$ respectively, and so $p \Vdash \tau \in V[y_0] \cap V[z_0]$. By the independence of the maps $f_0, g_0$, the intersection $V[y_0] \cap V[z_0]$ is equal to $V$ and so $p \Vdash \tau \in V$ as desired.

### 3.2 Examples

Several groups of results in this book depend on an identification of suitable pair of independent maps. The most important example comes from Hjorth’s
3.2. EXAMPLES

The notion of turbulence of group actions. Recall the standard definition:

**Definition 3.2.1.** [17, Section 13.1] Let $\Gamma$ be a Polish group continuously acting on a Polish space $Y$.

1. If $U \subset \Gamma, O \subset Y$ are sets then a $U, O$-walk is a sequence $\langle y_i: i \leq k, \gamma_i: i < k \rangle$ such that for each $i \leq k \gamma_i \in O$ holds, and for each $i < k \gamma_i \in U$ and $y_{i+1} = \gamma_i \cdot y_i$ both hold;

2. if $y \in O$ then $U, O$-orbit of $y$ is the set of all $z \in O$ such that there is a $U, O$-walk starting at $y$ and ending at $z$;

3. The action is turbulent at $y \in Y$ if for all open sets $U \subset \Gamma$ and $O \subset X$ with $1 \in U$ and $y \in O$ the $U, O$-orbit of $y$ is somewhere dense.

4. The action is generically turbulent if its orbits are meager and dense and the set of points of turbulence is comeager.

Now, suppose that $\Gamma$ is a Polish group acting continuously on a Polish space $Y$. Let $X = \{\langle \gamma, y_0, y_1 \rangle \in \Gamma \times Y \times Y: \gamma \cdot y_0 = y_1 \}$; this is a closed subset of $\Gamma \times Y \times Y$ and therefore Polish in the inherited topology. Let $f_0: X \to Y$ be the projection into the second coordinate, $f_1: X \to Y$ be the projection into the third coordinate, and let $f_2: X \to \Gamma \times Y$ be the projection into the first two coordinates. It is not difficult to check that these mappings are continuous and open.

**Theorem 3.2.2.** Let $\Gamma$ be a Polish group continuously acting on a Polish space $Y$ such that all orbits are meager and dense. The following are equivalent:

1. the action is generically turbulent;

2. $f_0, f_1: X \to Y$ is an independent pair of functions;

3. $P_X \times P_Y$ forces $V[x_{\text{gen}}] \cap V[\gamma_{\text{gen}} \cdot x_{\text{gen}}] = V$.

**Proof.** For the implication (1)$\to$(2), suppose that the action is generically turbulent. Let $O \subset X$ be an open set. Find a point $\delta \in \Gamma$ and open sets $U \subset \Gamma$ and $O_0, O_1 \subset Y$ such that $1 \in U$, $U = U^{-1}$, and $\delta \cdot U \times O_0 \times O_1 \cap X \subset O$. Find a point $y \in O_0$ such that the $U, O_0$-orbit of $y$ is somewhere dense; let $A \subset Y$ be a nonempty open set such that the $U, O_0$-orbit of $y$ is dense in $A$. We claim that $A \subset Y$ is an open set witnessing the independence of functions $f_0$ and $f_1$.

To see this, let $B_0, B_1 \subset A$ be nonempty open sets. Concatenating $U, O_0$-walks from $y$ to $B_0$ and $B_1$, we can find a $U, O$-walk $\langle y_i: i \leq k, \gamma_i: i < k \rangle$ which starts in $B_0$ and ends in $B_1$. Consider the sequence $\langle x_i: i \leq 2k \rangle$ of points in the open set $O \subset X$ given by $x_2i = \langle \delta, y_i, \delta \cdot y_i \rangle$ and $x_{2i+1} = \langle \gamma_i, y_i, \delta y_{i+1} \rangle$ if $i < k$. It is immediate that this is an $O$-walk in the sense of Definition 3.1.2, confirming the independence of functions $f_0, f_1$.

For the implication (2)$\to$(1), suppose that the functions $f_0, f_1$ are independent. Suppose that $U \subset \Gamma$ and $O_0 \subset Y$ are nonempty open sets and let
Definition 3.2.3. [3] A set $I$ is called $\omega$-hitting if for every countable collection of infinite subsets of $\omega$ there is a set $b \in I$ which has nonempty intersection with each set in the collection.

A typical analytic $\omega$-hitting ideal is $I_p$, where $p$ is a partition of $\omega$ into finite sets and $I_p$ is the collection of all sets $a \subset \omega$ such that for some number $n \in \omega$, for every set $b \in p$, $|a \cap b| \leq n$. In fact, [14] shows that every Borel $\omega$-hitting ideal contains one of the ideals $I_p$ as a subset in a suitable sense. $\omega$-hitting ideals give rise to pairs of independent functions in the proof of the following theorem:

Theorem 3.2.4. Let $I$ be an analytic ideal on $\omega$. The following are equivalent:

1. $I$ is $\omega$-hitting;

2. in some forcing extension, there are points $y_0, y_1 \in 2^\omega$ separately Cohen-generic over $V$ such that $y_0 =_I y_1$ and $V[y_0] \cap V[y_1] = V$.

Proof. For the direction (1)$\Rightarrow$(2), assume that the ideal $I$ is $\omega$-hitting. Let $h : \omega^\omega \rightarrow \mathcal{P}(\omega)$ be a continuous map such that $I = \text{rng}(h)$. Let $T = \{ t \in \omega^\omega : h''[t] \text{ is } \omega\text{-hitting} \}$. The set $T \subset \omega^\omega$ is clearly closed under initial segment.

Claim 3.2.5. For all $t \in T$ there is $n \in \omega$ such that for all $m > n$ there is $z \in [T \restriction t]$ such that $m \in h(z)$.

Proof. Suppose towards contradiction that the conclusion fails for some node $t \in T$. Let $a = \omega \setminus \bigcup h''[T \restriction t]$; the set $a \subset \omega$ is infinite. For each $s \in \omega^\omega \setminus T$ let $c_s$ be a countable collection of infinite subsets of $\omega$ such that for all $z \in [s]$, $h(z)$ has finite intersection with one element of $c_s$. Let $c = \bigcup_{s \in T} c_s \cup \{ a \}$; it is not difficult to see that for every $z \in [t]$, $h(z)$ has finite intersection with an element of $c$, contradicting the assumption that $t \in T$. \qed
3.3. TRIM EQUIVALENCE RELATIONS

Now, let $X$ be the closed set of all triples $\langle y_0, y_1, z \rangle \subset 2^\omega \times 2^\omega \times [T]$ such that \{ $n \in \omega: y_0(n) \neq y_1(n)$ \} $\subset h(z)$. Let $f_0, f_1 : X \rightarrow 2^\omega$ be the projections into the first and second coordinate. It is not difficult to see that both of these maps are continuous and open. In view of Theorem 3.1.5, the following claim concludes the proof of (2).

Claim 3.2.6. The maps $f_0, f_1$ are independent.

Proof. Let $O \subset X$ be an open set. Find $u_0, u_1 \in 2^{<\omega}$ and $t \in T$ such that $([u_0] \times [u_1] \times [t]) \cap X$ is a nonempty subset of $O$. Thinning out further, we may assume that the binary strings $u_0, u_1$ have the same length and for all $z \in [T \upharpoonright t]$ and every $l$ such that $u_0(l) \neq u_1(l)$, $n \in h(z)$ holds. Let $n \in \omega$ be such that for all $m > n$ there is $z \in [T \upharpoonright t]$ such that $m \in h(z)$. Let $v \in 2^{<\omega}$ be any binary string extending $u_0$ of length $> n$ and $A = [v]$. We claim that the open set $A$ witnesses the definition of independence for $f_0, f_1$ and $O$.

Indeed, suppose that $w_0, w_1$ are two binary strings of the same length extending $v$. To produce a walk from $[w_0]$ to $[w_1]$, list all entries on which the strings $w_0, w_1$ differ by $\{ m_j: j \in i \}$. Let $y \in [w_0]$ be any point, and let $y_j \in 2^\omega$ be the point obtained from $y$ by flipping the outputs at all $m_k$ for $k \in j$. Thus, $y_j$ differs from $y_{j+1}$ only at entry $m_j$ and $y_j \in [w_1]$. Let $y_{j}' \in 2^\omega$ be the point obtained by replacing the initial segment $u_0 \subset y_j$ with $u_1$. Also, let $z_j \in [T \upharpoonright t]$ be any point such that $m_j \in h(z_j)$. Now, define points $x_{2j+1}, x_{2j+2} \in X$ for $j \in i$ as follows:

- $x_0 = \langle y_0, y'_0, z \rangle$ for any $z \in [T \upharpoonright t]$;
- $x_{2j+1} = \langle y_j, y'_{j+1}, z_j \rangle$;
- $x_{2j+2} = \langle y_{j+1}, y'_{j+1}, z \rangle$ for any $z \in [T \upharpoonright t]$.

Clearly, the sequence $\langle x_k: k \leq 2i \rangle$ is a walk from $[w_0]$ to $[w_1]$ as required. □

To prove that (2) implies (1), assume that (1) fails, as witnessed by a countable collection $\{ a_n: n \in \omega \}$ of infinite subsets of $\omega$. Suppose that in some generic extension, $y_0, y_1 \in 2^\omega$ are $=_I$-related points which are separately Cohen-generic over $V$. The set $b = \{ n \in \omega: y_0(n) \neq y_1(n) \}$ belongs to the ideal $I$, and therefore there is a number $n \in \omega$ such that $b \cap a_n$ is finite. It follows that the functions $y_0 \upharpoonright a_n$ and $y_1 \upharpoonright a_n$ are modulo finite equal and so they belong to $V[y_0] \cap V[y_1]$. By a genericity argument, $y_0 \upharpoonright a_n \notin V$ holds, and so (2) fails. □

3.3 Trim equivalence relations

The characterization of turbulence in Theorem 3.2.2 leads to many Borel nonreducibility results and cardinal preservation results in the generic extensions of the Solovay model. To state these results succinctly, the following definitions are helpful.
Definition 3.3.1. Let $E$ be an analytic equivalence relation on a Polish space $X$. We say that $E$ is trim if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of $V$ (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$ are $E$-related points in the space $X$, then they are $E$-related to some point in $V$.

Definition 3.3.2. Let $E$ be an analytic equivalence relation on a Polish space $X$. We say that $E$ is virtually trim if, whenever $V[H_0]$ and $V[H_1]$ are separately generic extensions of $V$ (inside some ambient generic extension) such that $V[H_0] \cap V[H_1] = V$ and $\langle Q_0, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are $E$-related $E$-pins, then they are related to some $E$-pin in $V$.

In other words, an equivalence relation $E$ is trim if disjoint generic extensions do not share any $E$-classes which are not represented in $V$. An equivalence relation $E$ is virtually trim if disjoint generic extensions do not share any virtual $E$-classes which are not represented in $V$. The following propositions encapsulate the basic properties of the two classes of equivalence relations.

Proposition 3.3.3. Let $E$ be an analytic equivalence relation on a Polish space $X$. $E$ is virtually trim if and only if for any separately generic extensions $V[H_0], V[H_1]$ such that $V[H_0] \cap V[H_1] = V$ and $E$-related points $x_0 \in V[H_0]$ and $x_1 \in V[H_1]$, $x_0, x_1$ are realizations of some virtual $E$-class in $V$.

Proof. The left-to-right implication is immediate from the definitions. For the right-to-left implication, suppose that $V[H_0]$ and $V[H_1]$ are separately generic extensions such that $V[H_0] \cap V[H_1] = V$ and $\langle Q_0, \tau_0 \rangle \in V[H_0]$ and $\langle Q_1, \tau_1 \rangle \in V[H_1]$ are $E$-related pins. Let $K_0 \subset Q_0$ and $K_1 \subset Q_1$ be filters mutually generic over the model $V[H_0, H_1]$ and let $x_0 = \tau_0/K_0$ and $x_1 = \tau_1/K_1$. By the mutual genericity, $V[H_0][K_0] \cap V[H_1][K_1] = V$, and the assumption on $E$ gives that $x_0, x_1$ are realizations of some virtual $E$-class from the ground model. It follows that both $E$-pins are $\bar{E}$-related to some $E$-pin from $V$.

Proposition 3.3.4. An analytic equivalence relation $E$ on a Polish space $X$ is trim if and only if it is pinned and virtually trim.

Proof. The right-to-left implication is immediate. If $E$ is a pinned and virtually trim equivalence relation and $V[H_0], V[H_1]$ are separately generic extensions such that $V[H_0] \cap V[H_1] = V$, then any $E$-class represented in both must be represented as a virtual class in $V$ by the virtual trim assumption, and this virtual class must be trivial by the pinned assumption. This means that $E$ is trim. For the left-to-right implication, if $E$ is trim then it must be pinned because if $V[H_0], V[H_1]$ are mutually generic extensions then $V[H_0] \cap V[H_1] = V$ by the product forcing theorem. To show that $E$ must be virtually trim, suppose that $V[H_0], V[H_1]$ are separately generic extensions. By Proposition 3.3.3, it is enough to verify that every $E$-class represented in both extensions is a realization of some virtual $E$-class in $V$. However, the trim assumption even implies that it is a realization of a trivial virtual $E$-class in $V$. 


A good example of an equivalence relation which is virtually trim but not trim is \( \mathbb{P}_2 \). It is not trim since it is not pinned.

The following ergodicity theorem is a great generalization of the ergodicity theorems of Hjorth and Kechris [20, Theorem 12.5], and it is the main motivation behind the definition of trim and virtually trim classes of equivalence relations.

**Theorem 3.3.5.** Suppose that a Polish group \( \Gamma \) acts continuously and turbulently on a Polish space \( X \) such that the resulting orbit equivalence relation \( E \) is analytic, with all orbits dense. Suppose that \( F \) is a virtually trim equivalence relation on a Polish space \( Y \). Suppose that \( h : X \to Y \) is a homomorphism of \( E \) to \( F \). Then there is a comeager set \( B \subset X \) which is mapped into a single \( F \)-class.

**Proof.** Let \( \gamma \in \Gamma, x \in X \) be a mutually generic pair of points for the posets \( P_1 \) and \( P_X \). Theorem 3.2.2 implies the instrumental equality: \( V[\gamma] \cap V[\gamma \cdot x] = V \). Since \( x \in E \gamma \cdot x \) holds, it must be the case that \( h(x) F h(\gamma \cdot x) \) must hold. By the virtual trimness of the equivalence relation \( F \), it must be the case that \( h(x) \) is a realization of a ground model virtual \( F \)-class. However, the poset \( P_X \) is countable, so all virtual classes realized in its extension are already represented in the ground model by Theorem 2.8.2. Thus, there must be a point \( y \in Y \cap V \) such that \( h(x) F y \). Since the generic point \( x \in X \) avoids all ground model coded meager sets, it must be the case that the set \( B = h^{-1}(\{y\}|_E) \subset X \) is nonmeager. Since this is an analytic set which is invariant under the continuous group action all of whose orbits are dense, it follows that the set \( B \) is comeager.

**Theorem 3.3.6.** Let \( P \) be the poset for adding \( \aleph_1 \) many Cohen reals with finite support. Suppose that a Polish group \( \Gamma \) acts continuously and turbulently on a Polish space \( X \) such that the resulting orbit equivalence relation \( E \) is analytic, with all orbits dense. Suppose that \( F \) is a virtually trim equivalence relation on a Polish space \( Y \). Then \( P \) forces that there is no injection from \( X/E \) to \( Y/F \) definable from the parameters in the ground model.

**Proof.** Suppose toward contradiction that there is a condition \( p \in P \) which forces that some function, definable by a specific formula with ground model parameters, is an injection from \( X/E \) to \( Y/F \). We will denote the function by \( h \) everywhere below. By the homogeneity of the poset \( P \), the condition \( p \) can then be chosen as the largest condition of \( P \). The poset \( P \) is in the forcing sense equivalent to the product of \( P_X \) and \( P \) where \( P_X \) is the Cohen poset on the space \( X \), adding a point \( \dot{x}_{gen} \in X \). One can ask whether in the \( P_X \times P \)-extension, there is a representative of the \( F \)-equivalence class \( h(\dot{x}_{gen}) \) in the ground model, in the \( P_X \)-extension or outside of the \( P_X \)-extension. The theorem will be proved by deriving contradiction in all three cases.

**Case 1.** There is a condition \( p \in P_X \) and a point \( y \in Y \) in the ground model such that \( p \vdash X \vdash \dot{y} \in h(\dot{x}_{gen}) \). Choose points \( x_0, x_1 \in X \) which are mutually \( P_X \)-generic over \( V \) and both contained in the condition \( p \), and choose a filter \( \mathcal{H} \subset P \) generic over the model \( V[x_0, x_1] \). Observe that \( x_0 E x_1 \) must fail since the set \( \{x_0\}_E \subset X \) is meager and the point \( x_1 \in X \) is generic over \( V[x_0] \), in
particular avoiding the meager set \([x_0]_E\). Also, the model \(V[x_0, x_1][H]\) is a \(P\)-extension of both \(V[x_0]\) and \(V[x_1]\) and by the forcing theorem applied in both of these models, \(h([x_0]_E) = h([x_1]_E) = [y]_F\) for both of the distinct classes \([x_0]_E\) and \([x_1]_E\). This contradicts the assumption that \(h\) is forced to be an injection.

**Case 2.** Case 1 fails and there is a condition \(p \in P_X\) such that \(p \vdash P \vdash h([\check{x}_{\text{gen}}]_E)\) is represented in the \(P_X\)-extension. Let \(x_0 \in X\) and \(\gamma \in \Gamma\) be points mutually \(P_X\)- and \(P\)-generic over the ground model such that \(x_0 \in p\) and \(x_1 = \gamma \cdot x_0 \in p\), and let \(H \subset P\) be a filter generic over the model \(V[x_0, \gamma, H]\).

The model \(V[x_0, \gamma, H]\) is a \(P\)-extension of both models \(V[x_0]\) and \(V[x_1]\), and by the forcing theorem applied in the two models and the case assumption, \(h([x_0]_E) = h([x_1]_E)\) is an \(F\)-class which is represented in both models \(V[x_0]\) and \(V[x_1]\) by some \(y_0, y_1 \in Y\). Theorem 3.2.2 shows that \(V[x_0] \cap V[x_1] = V\). The virtual trimness assumption on the relation \(F\) shows that the points \(y_0, y_1\) must be realizations of a single ground model virtual \(F\)-class. Theorem 2.8.2 shows that there are no nontrivial virtual \(F\)-classes on Cohen forcing, and so there is a ground model element \(y \in Y\) equivalent to both \(y_0, y_1\). This contradicts the case assumption.

**Case 3.** Cases 1 and 2 both fail. Let \(x \in X\) be a point Cohen-generic over the ground model and move to the model \(V[x]\). Let \(Q\) be the Cohen forcing. By a chain condition argument, there must be a \(Q\)-name \(\sigma\) for an element of \(Y\) such that \(Q \times P \vdash [\sigma]_F = h([x]_E)\). We will show that \(\sigma\) is an \(F\)-pinned name.

By Theorem 2.8.2, the Cohen forcing \(Q\) carries only trivial \(F\)-pinned names, and so it is forced that \(\sigma\) is \(F\)-related to some element of the model \(V[x]\). This contradicts the case assumption.

To show that \(\sigma\) is an \(F\)-pinned name, let \(K_0, K_1 \subset Q\) be filters mutually generic over the model \(V[x]\), and let \(y_0 = \sigma/K_0\) and \(y_1 = \sigma/K_1\). We must prove that \(y_0 F y_1\) holds. To this end, let \(H \subset P\) be a filter generic over the model \(V[x][K_0][K_1]\). Since the model \(V[x][K_0][K_1][H]\) is a \(P\)-generic extension of both models \(V[x][K_0]\) and \(V[x][K_1]\), the forcing theorem applied in the two models shows that \(h([x]_E) = [y_0]_F = [y_1]_F\) and therefore \(y_0 F y_1\) holds as desired. \(\square\)

### 3.4 Examples and operations

**Theorem 3.4.1.** The class of trim equivalence relations is closed under the following operations:

1. Borel almost reduction;
2. countable product;
3. countable increasing union;
4. countable factor;

The class of virtually trim equivalence relations is closed under the same operations and the Friedman–Stanley jump.
Proof. In view of Proposition 3.3.4, it is enough to show the closure of virtual trimness under these operations, since the class of pinned equivalence relations is closed under (1–4) by the work of Chapter 2. For virtual trimness, we will prove (1) and the closure under the Friedman–Stanley jump.

For (1), suppose that \( E,F \) are analytic equivalence relations on Polish spaces \( X,Y \) and \( h: X \to Y \) is a Borel function which is a reduction of \( E \to Y \) everywhere except for a set \( Z \subset X \) consisting of countably many \( E \)-classes. Suppose that \( F \) is virtually trim and work towards the conclusion that \( E \) is virtually trim. Let \( V[H_0], V[H_1] \) be generic extensions such that \( V[H_0] \cap V[H_1] = V \) and let \( x_0 \in V[H_0] \) and \( x_1 \in V[H_1] \) be \( E \)-related points in \( X \); we need to show that they are realizations of some virtual \( E \)-class. If \( x_0 \in Z \) then this certainly occurs as \( V \) contains a countable set of points whose \( E \)-classes cover \( Z \), and this feature of the countable set persists to \( V[H_0] \) by a Shoenfield absoluteness argument. If \( x_0 \notin Z \) then \( x_1 \notin Z \); look at the points \( h(x_0) \in V[H_0] \) and \( h(x_1) \in V[H_1] \). These are \( F \)-related points; since \( F \) is trim they realize some virtual \( F \)-class represented by some \( F \)-pin \( \langle Q, \sigma \rangle \). By a Shoenfield absoluteness argument, \( Q \models \exists x \in X \setminus Z \ h(x) E \sigma \); let \( \tau \) be a \( Q \)-name for such a point \( x \in X \). It is not difficult to see that \( \langle Q, \tau \rangle \) is an \( E \)-pin and \( x_0, x_1 \) realize the virtual class of \( \langle Q, \sigma \rangle \).

For (5), suppose that \( E \) is a virtually trim equivalence relation on a Polish space \( X \), \( V[H_0] \) and \( V[H_1] \) are separately generic extensions of \( V \) such that \( V[H_0] \cap V[H_1] = V \) and \( y_0, y_1 \in X^\omega \) are elements in the respective models such that \( \left[ \text{rng}(y_0) \right]_E = \left[ \text{rng}(y_1) \right]_E \). Since \( E \) is virtually trim, every element of \( \text{rng}(y_0) \) is a realization of a virtual \( E \)-class from \( V \). Let \( B \) be the set of all virtual \( E \)-classes in \( V \) which have realizations in \( \text{rng}(y_0) \). Since \( B \) is also the set of all virtual \( E \)-classes in \( V \) which have realizations in \( \text{rng}(y_1) \), it is clear that \( B \in V[H_0] \cap V[H_1] = V \). Thus, the points \( y_0, y_1 \in X^\omega \) are realizations of the virtual \( E^+ \)-class represented by the set \( B \). By Proposition 3.3.3, we conclude that \( E^+ \) is virtually trim. \( \square \)

**Theorem 3.4.2.** Suppose that \( \Gamma \) is a Polish group continuously acting on a Polish space \( X \) and \( E \) is the resulting orbit equivalence relation. The following are equivalent:

1. \( E \) is virtually trim;

2. for every Borel set \( B \subset X \) such that \( E \upharpoonright B \) is Borel, \( E \upharpoonright B \) is virtually trim.

Proof. (1) immediately implies (2) since \( E \upharpoonright B \) is reducible to \( E \) by the identity map on \( B \). Now suppose that (1) fails, and let \( P \) be a poset and \( \tau \) a name for an element of \( X \) such that \( P \times P \models \tau \text{left} \ E \tau \text{right} \) fails, and let \( Q \) be a poset with names \( \hat{H}_0, \hat{H}_1 \) for filters generic over the ground model such that \( Q \models V[\hat{H}_0] \cap V[\hat{H}_1] = V \) and \( \tau/\hat{H}_0 = \tau/\hat{H}_1 \).

Let \( M \) be a countable elementary submodel of a large structure containing all objects mentioned so far. Let \( M \) be the transitive collapse of \( M \), and \( P, Q \) etc. be the images of \( P, Q \) etc. under the transitive collapse. By Proposition 3.5.5, \( Q \models \ldots \)
write $= \text{trim. To this end, consider the transfinite sequence }\langle a \rangle \text{ with ideals on countable sets. If }$

\[\text{Theorem 3.4.1. The following examples deal with equivalence relations associated from the identity by repeated application of the operations indicated in Theorem 3.4.2, it is only necessary to show that all Borel fragments of } F \text{ is virtually trim.}
\]

Proof. In view of Theorem 3.4.1(1), it is only necessary to show that every orbit equivalence relation $E$ obtained from an action of $S_\omega$ is virtually trim. By Theorem 3.4.2, it is only necessary to show that all Borel fragments of $E$ are virtually trim. To this end, consider the transfinite sequence $\langle F_\alpha : \alpha \in \omega_1 \rangle$ obtained from $F_1 =$identity on $2^\omega$ by repeated application of Friedman–Stanley jump, at limit stages taking disjoint unions. It is well-known cite[Theorem 12.5.2]kanovei:book that every Borel equivalence relation classifiable by countable structures is Borel reducible to $F_\alpha$ for some countable ordinal $\alpha$. Theorem 3.4.1 iterated trans-finitely shows that each $F_\alpha$ is virtually trim. Thus, every Borel fragment of $E$ is virtually trim, and so is $E$.

Theorem allows the transfer of the virtual trim property from Borel fragments of a given orbit equivalence relation to the whole equivalence relation. Consider the following attractive corollary:

**Corollary 3.4.3.** Every equivalence relation classifiable by countable structures is virtually trim.

Proof. In view of Theorem 3.4.1(1), it is only necessary to show that every orbit equivalence relation $E$ obtained from an action of $S_\omega$ is virtually trim. By Theorem 3.4.2, it is only necessary to show that all Borel fragments of $E$ are virtually trim. To this end, consider the transfinite sequence $\langle F_\alpha : \alpha \in \omega_1 \rangle$ obtained from $F_1 =$identity on $2^\omega$ by repeated application of Friedman–Stanley jump, at limit stages taking disjoint unions. It is well-known cite[Theorem 12.5.2]kanovei:book that every Borel equivalence relation classifiable by countable structures is Borel reducible to $F_\alpha$ for some countable ordinal $\alpha$. Theorem 3.4.1 iterated trans-finitely shows that each $F_\alpha$ is virtually trim. Thus, every Borel fragment of $E$ is virtually trim, and so is $E$.

There are many Borel trim equivalence relations which cannot be obtained from the identity by repeated application of the operations indicated in Theorem 3.4.1. The following examples deal with equivalence relations associated with ideals on countable sets. If $a$ is a countable set and $J$ is an ideal on $a$, write $=_J$ for the equivalence relation on the space $X = (2^\omega)^a$ connecting points $x_0, x_1$ if the set $\{n \in a : x_0(n) \neq x_1(n)\}$ belongs to $J$.

**Example 3.4.4.** Let $a = 2^{<\omega}$ and let $J$ be the branch ideal: the ideal generated by the subsets of $a$ linearly ordered by inclusion. The equivalence relation $=_J$ is trim.

Proof. Write $X = (2^\omega)^{2^{<\omega}}$. For every node $t \in \omega^{<\omega}$ write $[t]$ for the set of all nodes in $2^{<\omega}$ extending $t$. Let $V[G_0], V[G_1]$ be two generic extensions containing respective $=_J$-related points $x_0, x_1 \in X$. Assume that $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to both $x_0, x_1$.

Let $T = \{t \in 2^{<\omega} : x_0 \upharpoonright [t] \text{ is not } =_J\text{-equivalent to any point in the ground model}\}$. This is a subtree of $\omega^{<\omega}$. If $0 \notin T$ then the proof is complete; thus, it is only necessary to derive a contradiction from the assumption $0 \in T$. First, observe that the tree $T$ cannot have any terminal nodes: if $t$ was a terminal node of $T$ then one could combine the ground model witnesses for $t^\frown 0 \notin T$ and $t^\frown 1 \notin T$ to find a witness for $t \notin T$. Second, observe that the definition of the tree $T$ depends only on the $=_J$-class of $x_0$ and so $T \in V[G_0] \cap V[G_1] = V$. Since $T$ is a nonempty ground model tree without endnodes, it must have an infinite branch $y \in 2^{<\omega}$ in the ground model. Since $x_0 =_J x_1$, there is a number $n \in \omega$ such that for every $t \in 2^{<\omega}$ such that $x_0(t) \neq x_1(t)$, either $t$ is incompatible
with \( y \upharpoonright n \) or else \( t \) is an initial segment of \( y \). Let \( e = [y \upharpoonright n] \setminus [y \upharpoonright m : m \geq n] \) and observe that \( e \in V \) and \( x_0 \) and \( x_1 \) coincide on \( e \), therefore \( x_0 \upharpoonright e \in V \). If \( z \in V \) is any function in \( 2^{[y \upharpoonright n]} \) extending \( x_0 \upharpoonright e \), then \( z =_{f} x_0 \upharpoonright [y] \), contradicting the assumption that \( y \upharpoonright n \in T \). \hfill \( \square \)

**Theorem 3.4.5.** The equivalence relation \( =_{I} \) is trim for every countably separated Borel ideal \( I \) on \( \omega \).

**Proof.** Let \( \{a_n : n \in \omega\} \) be a countable collection of subsets of \( \omega \) witnessing the countable separation of the ideal \( I \). Write \( X = (2^{\omega})^{\omega} \). Let \( V[G_0], V[G_1] \) be generic extensions containing respective \( =_{I} \)-related points \( x_0, x_1 \in X \). Assume \( V[G_0] \cap V[G_1] = V \) and work to find a ground model point \( x \in X \) which is \( =_{I} \)-related to both \( x_0, x_1 \).

Consider the set \( b = \{n \in \omega : x_0 \upharpoonright a_n \text{ is } I\text{-equivalent to some element of the ground model}\} \). The definition of the set \( b \) depends only on the \( =_{I} \)-equivalence class of \( x_0 \), therefore the set \( b \) belongs to both \( V[G_0] \) and \( V[G_1] \), and by the initial assumptions, to \( V \). Let \( f \) be the map with domain \( b \) which to each \( n \in b \) identifies the \( =_{I} \)-class in \( V \) which contains \( x_0 \upharpoonright a_n \). Again, the definition of \( f \) depends only on the \( =_{I} \)-class of \( x_0 \) and so \( f \in V[G_0] \) and \( f \in V[G_1] \), therefore \( f \in V \). By the Mostowski absoluteness between \( V \) and \( V[G_0] \), there is a point \( x \in X \) such that for all \( n \in b \), \( x \upharpoonright a_n \in f(n) \). We will show that \( x =_{f} x_0 \) holds.

Suppose towards contradiction that this fails. Consider the set \( c = \{i \in \omega : x(i) \neq x_0(i)\} \notin I \) and the set \( d = \{i \in \omega : x_0(i) \neq x_1(i)\} \in I \). By the countable separation of the ideal \( I \), there must be a number \( n \in \omega \) such that \( c \cap a_n \notin I \) and \( d \cap a_n = 0 \). However, the latter equality shows that \( x_0 \upharpoonright a_n = x_1 \upharpoonright a_n \) and so \( x_0 \upharpoonright a_n \in V \) and therefore \( n \in b \). But then \( \{i \in a_n : x(i) \neq x_0(i)\} \in I \) by the choice of the point \( x \). This contradicts the fact that \( c \cap a_n \notin I \). The proof of the theorem is complete. \hfill \( \square \)

There is a natural operation on analytic ideals which seeks a countably separated closure. For the discussion below, let \( \{a_n : n \in \omega\} \) be a fixed countable collection of subsets of \( \omega \).

**Definition 3.4.6.** Let \( I \) be an ideal on \( \omega \). \( I^+ \) is the ideal on \( \omega \) consisting of all sets \( b \subset \omega \) such that there is \( c \in I \) such that \( \forall n \in \omega \ b \cap a_n \notin I \rightarrow c \cap a_n \neq 0 \).

We will start with some simple observations. It is clear that \( I^+ \) is an ideal containing \( I \) as a subset. If \( I \) is analytic, then so is \( I^+ \). Moreover, \( I \) is separated by the sequence \( \{a_n : n \in \omega\} \) just in case \( I = I^+ \). We will show that under suitable additional assumption, the jump preserves trimness of the equivalence relation \( =_{J} \).

**Definition 3.4.7.** Let \( I \) be an ideal on \( \omega \). \( I \) is skew if for every set \( c \subset \omega \) there is a subset \( c' \subset c \) such that for every finite set \( d \subset \omega \), if \( c \cap \bigcap_{n \in d} a_n \notin I \) then \( c' \cap \bigcap_{n \in d} a_n \notin I \) and if \( c \cap \bigcap_{n \in d} a_n \in I \) then \( c' \cap \bigcap_{n \in d} a_n \) is finite.

It is not difficult to see that every \( \mathbb{F}_\alpha \) ideal is skew and every analytic \( P \)-ideal is skew.
**Theorem 3.4.8.** If \( I \) is a skew analytic ideal on \( \omega \) then \( I^+ \) is a skew analytic ideal as well. If in addition the equivalence relation \( =_I \) is trim then the equivalence relation \( =_{I^+} \) is trim as well.

**Proof.** For the first sentence, if \( c \subseteq \omega \) is a set and \( c' \subset c \) witnesses the skew property for \( I \) and \( c \), then \( c' \) also witnesses the skew property for \( I^+ \) and \( c \). The proof of the second sentence is more involved. Write \( X = (2^\omega)^\omega \). Let \( V[G_0], V[G_1] \) be generic extensions containing respective \( =_{I^+} \)-related points \( x_0, x_1 \in X \). Assume \( V[G_0] \cap V[G_1] = V \) and work to find a ground model point \( x \in X \) which is \( =_{I^+} \)-related to both \( x_0, x_1 \).

Consider the set \( b = \{ n \in \omega : x_0 \upharpoonright a_n \text{ is } I^+ \text{-equivalent to some element of the ground model} \} \). The definition of the set \( b \) depends only on the \( =_{I^+} \)-equivalence class of \( x_0 \), therefore the set \( b \) belongs to both \( V[G_0] \) and \( V[G_1] \), and by the initial assumptions, to \( V \). Let \( f \) be the map with domain \( b \) which to each \( n \in b \) identifies the \( =_{I^+} \)-class in \( V \) which contains \( x_0 \upharpoonright a_n \). Again, the definition of \( f \) depends only on the \( =_{I^+} \)-class of \( x_0 \) and so \( f \in V[G_0] \) and \( f \in V[G_1] \), therefore \( f \in V \). By the Mostowski absoluteness between \( V \) and \( V[G_0] \), there is a point \( x \in X \) such that for all \( n \in b, x \upharpoonright a_n \in f(n) \). We will show that \( x =_{I^+} x_0 \) holds.

Suppose towards contradiction that this fails. Consider the set \( c = \{ i \in \omega : x(i) \neq x_0(i) \} \notin I^+ \) and the set \( d = \{ i \in \omega : x_0(i) \neq x_1(i) \} \in I^+ \). It is now time to use the skew assumption on the ideal \( I \). Find a set \( c' \subset c \) witnessing the skew property of \( I^+ \). Use the definition of the jump to find a set \( e \in I \) such that for all \( n \in \omega, d \cap a_n \notin I \) implies \( e \cap a_n \neq 0 \). Since the set \( c' \) is positive in the jump, there must be a number \( n \in \omega \) such that \( c' \cap a_n \notin I \) and \( e \cap a_n = 0 \). The choice of the set \( c' \) shows that \( c' \cap a_n \) is in fact \( I^+ \)-positive. The choice of the set \( e \) shows that \( d \cap a_n \in I \). This means that \( x_0 \upharpoonright a_n =_I x_1 \upharpoonright a_n \), and by the trim assumption on the equivalence relation \( =_I \), \( n \in b \) must hold. This stands in contradiction with the fact that \( c' \cap a_n \) and so \( c \cap a_n \) is \( I^+ \)-positive. The proof of the theorem is complete. \( \Box \)

**Example 3.4.9.** Let \( I \) be the branch ideal on \( 2^\omega \), and let \( \{ a_n : n \in \omega \} \) be an enumeration of the basic open subsets of \( 2^{<\omega} \). The ideal \( I \) is skew. One can start iterating the jump for countable ordinals \( \alpha \), at limit stages taking unions. The resulting ideals consist of subsets of \( \omega^{<\omega} \) whose closure in \( 2^\omega \) is countable with Cantor–Bendixson rank \( \leq \alpha \). All the resulting equivalence relations are trim.

### 3.5 Absoluteness

The trim and virtually trim classes of equivalence relations are defined in such a way that it is not clear whether the membership in them is absolute, and what the actual complexity is. This section provides a satisfactory resolution to these questions.

To reach the absoluteness result, one has to perform a computation of intersections of forcing extensions of independent interest. The computation starts with several definitions:
Definition 3.5.1. Let $B$ be a Boolean algebra. A subalgebra $A \subset B$ is projective if the projection function $\pi: B \to A$ assigning to each $b \in B$ the smallest element of $A$ which is $\geq b$, is defined for every $b$.

A good example of a projection is any complete subalgebra of a complete Boolean algebra. The notion of a projective pair is no longer a first order statement about $A$, but it transfers from model to model without damage. Note that if $A \subset B$ is a projective subalgebra, then every maximal antichain of $A$ is also a maximal antichain of $B$, and therefore the intersection of a generic filter on $B$ will be a generic filter on $A$. To see this, let $D \subset A$ be a maximal antichain of $A$ and $b \in B$ be a condition. Let $d \in D$ be an element of $A$ compatible with $\pi(b)$. Then $\pi(b) - d \in A$ is strictly smaller element of $A$ than $\pi(b)$, and so it is not greater than $b$ anymore. This means that $d$ and $b$ must be compatible.

Definition 3.5.2. Let $B$ be a Boolean algebra and $A_0, A_1 \subset B$ be projective subalgebras with projection functions $\pi_0, \pi_1$.

1. The projection sequence starting at $b \in B$ is the sequence $\langle b_n : n \in \omega \rangle$ defined by $b_0 = b$, $b_{2n+1} = \pi_0(b_{2n})$ and $b_{2n+2} = \pi_1(b_{2n+1})$.

2. the pair $\{A_0, A_1\}$ is projective if for each $b \in B$, the supremum of the projection sequence starting at $b$ exists in $B$ and belongs to $A_0 \cap A_1$.

Again, a good example of a projective pair is a pair of complete subalgebras of a complete Boolean algebra. The notion of a projective pair is no longer a first order statement about $A_0, A_1, B$, but it still transfers between $\omega$-models of ZFC without damage.

Proposition 3.5.3. Let $B$ be a Boolean algebra and $\{A_0, A_1\}$ be a projective pair of subalgebras. Then

1. $A_0 \cap A_1 \subset B$ is a projective subalgebra of $B$;

2. if $\tau_0, \tau_1$ and $A_0$- and $A_1$-names for sets of ordinals and $B \models \tau_0 = \tau_1$, then there is a $A_0 \cap A_1$-name $\tau_2$ such that $B \models \tau_0 = \tau_1 = \tau_2$.

Proof. Let $\pi_0, \pi_1$ be the projections of $B$ to $A_0, A_1$. Let $\pi: B \to A_0 \cap A_1$ be the function which assigns to each $b \in B$ the supremum of the projection sequence starting from $b$. It is clear from the definitions that $\pi(b)$ is the smallest element of $A_0 \cap A_1$ above $b$ and (1) follows.

For (2), first note that for every $b \in B$ and every ordinal $\alpha$, $b \models \bar{\alpha} \in \tau_0$ just in case $\pi(b) \models \bar{\alpha} \in \tau$. To see this, by induction on $n \in \omega$ argue that $b_n \models \bar{\alpha} \in \tau_0$. If this holds for $n = 2k$, then use the fact that $\tau_0$ is an $A_0$-name and so $b_{n+1} = \pi_0(b_n)$ must force $\bar{\alpha} \in \tau$; if this holds for $n = 2k + 1$, then use the fact that $\tau_1$ is an $A_1$-name forced to be equal to $\tau_0$ and so $b_{n+1} = \pi_1(b_n)$ must force $\bar{\alpha} \in \tau$. It follows that $b \models \bar{\alpha} \in \tau$ just on case $\pi(b) \models \bar{\alpha} \in \tau$, and so one can let $\langle \pi(b), \bar{\alpha} \rangle \in \tau_2$ if $b \models \bar{\alpha} \in \tau_0$. 

Theorem 3.5.6. Let $E$ be a Borel equivalence relation on a Polish space $X$. The statement “$E$ is virtually trim” is absolute among transitive models of set theory containing the code for $E$.

Proof. We will show that the statement “$E$ is not virtually trim” is in ZFC equivalent to the statement “there is a countable $\omega$-model $M$ of ZFC- containing the code for $E$ such that $M \models E$ is not virtually trim”. This is an analytic statement and therefore absolute among transitive models of set theory.

Now, the left-to-right implication is immediate: one just needs to take a countable elementary submodel of a large enough structure to get the requisite $M$. The right-to-left direction is more interesting. Suppose that $M$ is a countable $\omega$-model containing the code for $E$ which satisfies that $E$ is not virtually trim. Working in the model $M$, there must be complete algebra $B$, complete subalgebras $A_0, A_1 \subseteq B$, and respective $A_0, A_1$-names $\tau_0, \tau_1$ for elements of the underlying space $X$ such that $B \models V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$ and $\tau_0 \models E \tau_1$, and $B \times B \models \neg(\tau_0)_{\text{left}} E (\tau_0)_{\text{right}}$. The subalgebras $A_0, A_1 \subseteq B$ form a locally projective pair in $M$ since they are complete. By Proposition 3.5.5 applied in $M$ in the $(1) \rightarrow (2)$ direction, for every $b \in B$, the algebra $A_0 \upharpoonright b \cap A_1 \upharpoonright b$ has an atom.

Finally, we are in a position to give a succinct and principled proof of the main absoluteness result of this section.

Definition 3.5.4. Let $B$ be a Boolean algebra and $\{A_0, A_1\}$ be a pair of subalgebras. The pair is locally projective if for every nonzero element $b \in B$, in the algebra $B \upharpoonright b$ the subalgebras $A_0 \upharpoonright b = \{b\wedge a : a \in A_0\}$ and $A_1 \upharpoonright b = \{b\wedge a : a \in A_1\}$ form a projective pair.

The following is the central motivation of the notion of a projective pair of subalgebras:

Proposition 3.5.5. Let $B$ be a Boolean algebra and $A_0, A_1$ be a locally projective pair of subalgebras. The following are equivalent:

1. $B \models V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$;

2. for every $b \in B$, the intersection algebra $A_0 \upharpoonright b \cap A_1 \upharpoonright b$ has an atom.

Proof. Suppose that (2) fails for some $b \in B$. Write $C = A_0 \upharpoonright b \cap A_1 \upharpoonright b$ and use Proposition 3.5.3 to argue that $C$ is a projective subalgebra of $B \upharpoonright b$, and so $b \models G \cap C$ is a $C$-generic filter, and since $C$ is nonatomic, $G \cap C \notin V$. At the same time, $G \cap C$ belongs to $V[G \cap A_0] \cap V[G \cap A_1]$, and so (1) fails.

Suppose now that (2) holds. Suppose that $\tau_0, \tau_1$ are $A_0$- and $A_1$-names for sets of ordinals and some condition $b \in B$ forces $\tau_0 = \tau_1$. Write $C = A_0 \upharpoonright b \cap A_1 \upharpoonright b$ and let $c$ be an atom of $C$. To verify (1), it is enough to argue that $c$ decides the membership of any given ordinal in $\tau_0$. However, this follows immediately from Proposition 3.5.3 (2) applied below $b$.

The following notions have the slightly unpleasant feature that they do not necessarily survive localization well, so an additional local definition is needed.

Theorem 3.5.6. Let $E$ be a Borel equivalence relation on a Polish space $X$. The statement “$E$ is virtually trim” is absolute among transitive models of set theory containing the code for $E$.

Proof. We will show that the statement “$E$ is not virtually trim” is in ZFC equivalent to the statement “there is a countable $\omega$-model $M$ of ZFC- containing the code for $E$ such that $M \models E$ is not virtually trim”. This is an analytic statement and therefore absolute among transitive models of set theory.

Now, the left-to-right implication is immediate: one just needs to take a countable elementary submodel of a large enough structure to get the requisite $M$. The right-to-left direction is more interesting. Suppose that $M$ is a countable $\omega$-model containing the code for $E$ which satisfies that $E$ is not virtually trim. Working in the model $M$, there must be complete algebra $B$, complete subalgebras $A_0, A_1 \subseteq B$, and respective $A_0, A_1$-names $\tau_0, \tau_1$ for elements of the underlying space $X$ such that $B \models V[\dot{G} \cap A_0] \cap V[\dot{G} \cap A_1] = V$ and $\tau_0 \models E \tau_1$, and $B \times B \models \neg(\tau_0)_{\text{left}} E (\tau_0)_{\text{right}}$. The subalgebras $A_0, A_1 \subseteq B$ form a locally projective pair in $M$ since they are complete. By Proposition 3.5.5 applied in $M$ in the $(1) \rightarrow (2)$ direction, for every $b \in B$, the algebra $A_0 \upharpoonright b \cap A_1 \upharpoonright b$ has an atom.
Stepping out of the model $M$, we see that the pair $A_0, A_1 \subset B$ is a projective pair and for every $b \in B$, the algebra $A_0 \upharpoonright b \cap A_1 \upharpoonright b$ has an atom. By Proposition 3.5.5 applied in the $(2) \rightarrow (1)$ direction, $B \forces V[\mathcal{G} \cap A_0] \cap V[\mathcal{G} \cap A_1] = V$ holds. Moreover, $B \forces \tau_0 E \tau_1$, and $B \times B \forces \neg (\tau_0)_{\text{left}} E (\tau_0)_{\text{right}}$, since $E$ is absolute between generic extensions of $V$ and generic extensions of $M$ by Borel absoluteness. In conclusion, $E$ is not virtually trim in $V$.

**Corollary 3.5.7.** Let $E$ be a Borel equivalence relation on a Polish space $X$. The statement “$E$ is trim” is absolute among transitive models of set theory containing the code for $E$.

**Proof.** The trimness of $E$ is a conjunction of virtual trimness and the pinned property of $E$ by Proposition 3.3.4. The conjuncts are absolute by Theorem 3.5.6 and 2.6.1, and so is the conjunction. □

### 3.6 A variation for measure

The purpose of this section is to introduce a parallel for turbulence for measure, with attendant ergodicity results for measure. Curiously enough, the measure theoretic parallel for turbulence is intimately connected to the concentration of measure phenomenon. The following definition is close to the whirly actions of [11]:

**Definition 3.6.1.** Let $X$ be a Polish space with a Borel probability measure $\mu$ and a metric $d$. Let $\Gamma$ be a Polish group acting on $X$ in a measure preserving and distance preserving fashion. We say that the action has concentration of measure if for every open neighborhood $U \subset \Gamma$ containing the unit and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $d$-ball $B$ of radius $< \delta$ and every Borel set $C \subset B$ of relative $\mu$-mass $> \varepsilon$, the set $(U \cdot C) \cap B$ has relative $\mu$-mass $> 1/2$.

To formulate the main results of this section, let $P_\Gamma$ be the Cohen forcing on $\Gamma$ and $P_\mu$ be the random forcing with $\mu$, i.e. the forcing with $\mu$-positive Borel subset of $X$, ordered by inclusion. The poset $P_\Gamma$ adds a generic point $\dot{\gamma}_{\text{gen}} \in \Gamma$ while the poset $P_\mu$ adds a random point $\dot{x}_{\text{gen}} \in X$.

**Theorem 3.6.2.** Suppose that $\Gamma$ is a Polish group acting on a Polish space $X$ with a Borel probability measure $\mu$ and an ultrametric $d$ in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Then $P_\Gamma \times P_\mu \forces V[\dot{x}_{\text{gen}}] \cap V[\dot{\gamma}_{\text{gen}} \cdot \dot{x}_{\text{gen}}] = V$.

The proof of Theorem 3.6.2 hinges on a new walk concept and a proposition.

**Definition 3.6.3.** Let $U \subset \Gamma$ be an open neighborhood of the unit and $\delta > 0$. A $U, \delta$-walk is a sequence $\langle x_i : i \leq j \rangle$ of points in $X$ such that for every $i \in j$, either $d(x_i, x_{i+1}) < \delta$ or there is $\gamma \in U$ such that $\gamma \cdot x_i = x_{i+1}$.

**Definition 3.6.4.** Let $U \subset \Gamma$ be an open neighborhood of the unit and $D \subset X$ be a closed set. We say that the set $D$ is $U$-connected if for any two points $x_0, x_1 \in D$ and any $\delta > 0$ there is a $U, \delta$-walk from $x_0$ to $x_1$ using only points from $D$. 

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The main import of the concentration of measure assumption is that for any open neighborhood \( U \) of the unit, closed \( U \)-connected \( \mu \)-positive sets can be found under every stone.

**Proposition 3.6.5.** Suppose that the action of \( \Gamma \) has concentration of measure and \( d \) is an ultrametric. Then for every open neighborhood \( U \subset \Gamma \) of the unit and every \( \mu \)-positive Borel set \( C \subset X \) there is a \( \mu \)-positive \( U \)-connected compact set \( D \subset C \).

**Proof.** We will first fix useful terminology. For a symmetric neighborhood \( V \) containing the unit and \( d \) is an ultrametric, this is equivalent to the statement that there is \( \gamma \in V \) such that \( \gamma \cdot B_0 = B_1 \). Since \( d \) is an ultrametric, this is equivalent to the statement that there is \( \gamma \in V \) such that \( \gamma \cdot B_0 \cap B_1 \neq \emptyset \). A set \( B \) consisting of \( d \)-balls of the same radius will be called \( V \)-connected if for any two balls \( B_0, B_1 \in B \) one can find a sequence of balls in \( B \) which starts with \( B_0 \), ends in \( B_1 \), and successive balls in it are \( V \)-related.

Lastly, for Borel sets \( B, D \subset X \) write \( \mu_B(D) \) for the relative measure of \( D \) in \( B \), i.e. the ratio \( \frac{\mu(B \cap D)}{\mu(B)} \).

Let \( U \subset \Gamma \) be an open neighborhood of the unit, and let \( C \subset X \) be a Borel \( \mu \)-positive set. Find a symmetric open neighborhood \( V \subset \Gamma \) of the unit such that \( V^2 \subset U \), and thin down \( C \) if necessary to a compact \( \mu \)-positive set. Let \( \varepsilon_0 = 1/2 \) and for each \( n \in \omega \) let \( \varepsilon_{n+1} = \frac{\varepsilon_n}{2^n+2} \). Let \( \delta_n > 0 \) be the numbers witnessing the concentration of measure of the action for \( V \) and \( \delta_n \). Thinning down \( C \) if necessary, use the Lebesgue density theorem to find a \( d \)-ball \( B_{\text{ini}} \) of radius \( \delta_0 \) such that \( C \subset B_{\text{ini}} \) and \( \mu_B(C) > 0 \).

Let \( B_0 = \{ B_{\text{ini}} \} \) and by induction on \( n \in \omega \) find finite families \( B_n \) of pairwise disjoint \( d \)-balls of the same radius \( < \delta_n \) such that for every number \( n \in \omega \) \( B_{n+1} \) refines \( B_n \) and \( C \subset \bigcup B_n \). We will find families \( B'_n \subset B_n \) such that \( B'_{n+1} \) refines \( B'_n \), and writing \( Y(B) = \{ A \in B'_{n+1} : A \subset B \} \) for a ball \( B \in B'_{n+1} \), the following holds:

- for every \( A \in Y(B) \), \( \mu_A(C) > (2^n + 2)\varepsilon_{n+1} \);
- \( \mu_B(C \cap \bigcup Y(B)) > (1 - 2^{-n})\mu_B(C) \);
- the set \( Y(B) \) is \( V^2 \)-connected.

To begin, set \( B'_0 = \{ B_{\text{ini}} \} \). Now, suppose that \( B'_n \) has been constructed and \( B \in B'_n \) is a ball; we shall show how to construct \( Y(B) \) and therefore \( B'_{n+1} \).

First, let \( Y_0 = \{ A \in B_{n+1} : A \subset B \} \). Let \( Y_1 = \{ A \in Y_0 : \mu_A(C) > (2^n + 2)\varepsilon_{n+1} \} \).

Thus, \( \mu_B(C \cap \bigcup (Y_0 \setminus Y_1)) < 2^{n+1} \varepsilon_{n+1} \).

**Claim 3.6.6.** There is a \( V^2 \)-connected component \( Y_2 \subset Y_1 \) such that \( \mu_B(C \cap \bigcup Y_2) > (1 - 2^{-n})\mu_B(C) \).

**Proof.** We first show that there is a \( V^2 \)-connected component \( Y_2 \subset Y_1 \) such that \( \mu_B(C \cap \bigcup Y_2) > \varepsilon_n \). If this were not the case, it would be possible to divide \( Y_1 \) into sets \( Y' \) and \( Y'' \) which are both invariant under \( V^2 \)-connections, and
\( \mu_B(C \cap \bigcup Y') \) and \( \mu_B(C \cap \bigcup Y'') \) are both greater than \( \varepsilon_n \). By the concentration of measure assumption, the sets \( V \cdot (C \cap \bigcup Y') \) and \( V \cdot (C \cap \bigcup Y'') \) are both of \( \mu_B \)-mass greater than \( 1/2 \) and therefore intersect. It follows that \( V^2 \cdot (C \cap \bigcup Y') \cap (C \cap \bigcup Y'') \neq 0 \), contradicting the invariance of the sets \( Y' \) and \( Y'' \) under \( V^2 \)-connections.

Now, the \( V^2 \)-connected component \( Y_2 \) found in the first paragraph must in fact be such that \( \mu_B((C \cap \bigcup (Y_1 \setminus Y_2))) < \varepsilon_n \), by an argument identical to the one in the previous paragraph with \( Y' = Y_2 \) and \( Y'' = Y_1 \setminus Y_2 \). This completes the proof of the claim.

Letting \( Y(B) = Y_2 \) completes the construction. The \( V^2 \)-connectedness is clear from the choice of \( Y_2 \), the first item is clear from the choice of \( Y_1(B) \), and the second item follows from some arithmetic: \( \mu_B(C \cap \bigcup (Y \setminus Y_2)) < 2\varepsilon_n \) which is smaller than \( 2^{-(n+2)}\varepsilon_n \leq 2^{-n}\mu_B(C \cap B) \).

Now, let \( D = \bigcap_{1}^{n} \bigcup B_n' \). This is a closed subset of \( C \) of positive mass. Its mass distribution is governed by the following claim:

**Claim 3.6.7.** For every ball \( B \in B'_n \), \( \mu_B(D) > \varepsilon_n \).

**Proof.** To get the set \( D \cap B \), we subtracted from \( C \cap B \) a set of size at most \( \Sigma_{m=1}^{n} 2^{-m} \mu_B(C) \) by the second item above, and so \( \mu_B(D) > 1/2\mu_B(B) > \varepsilon_n \) by the first item above.

Towards the connectedness of the set \( D \), consider the following claim:

**Claim 3.6.8.** Let \( m \leq n \) be natural numbers, \( B \in B'_m \) a ball, and \( x_0, x_1 \in B \cap D \) be any points. Then there is a \( U, \delta_n \)-walk from \( x_0 \) to \( x_1 \) using only points in \( B \cap D \).

**Proof.** This is proved by induction on \( n - m \), which is the reason for the convoluted statement of the claim. The case \( n - m = 0 \) is trivial since then \( d(x_0, x_1) < \delta_n \). Now suppose that the statement is known for \( m + 1 \leq n \) and proceed to show that it is also true for \( m \leq n \). Let \( B_0, B_1 \in B'_{m+1} \) be balls such that \( x_0 \in B_0 \) and \( x_1 \in B_1 \). If \( B_0 = B_1 \) then the induction hypothesis immediately applies. Otherwise, by the third item above, the set \( Y(B) \) is \( V^2 \)-connected, and so there must be a sequence of balls in \( Y(B) \) starting at \( B_0 \) and ending at \( B_1 \) such that successive balls are \( V^2 \)-connected. Suppose for simplicity that this sequence has length \( 2 \) i.e. \( B_0 \) and \( B_1 \) are \( V^2 \)-connected. Fix an element \( \gamma \in V^2 \) such that \( \gamma \cdot B_0 = B_1 \).

Since \( \mu_{B_0}(D) \) and \( \mu_{B_1}(D) \) are both greater than \( \varepsilon_{m+1} \), the concentration assumption yields that the numbers \( \mu_{B_0}(V \cdot (D \cap B_0)) = \mu_{B_1}(V \cdot (D \cap B_1)) \) are both greater than \( 1/2 \). It follows that there is a point \( x \in (V \cdot (D \cap B_0)) \cap B_0 \) such that \( \gamma \cdot x \in (V \cdot (D \cap B_1)) \). Find points \( x_0' \in D \cap B_0 \) and \( x_1' \in D \cap B_1 \) and group elements \( \beta_0, \beta_1 \in V \) such that \( \beta_0 \cdot x_0' = x \) and \( \beta_1 \cdot f(x) = x_1' \). Use the induction hypothesis to find a \( U, \delta_n \)-walk from \( x_0 \) to \( x_0' \). Follow it by acting by \( \beta_0 \gamma \beta_1 \in U \) to get to the point \( x_1' \), and then use the induction hypothesis again to extend the walk from \( x_1' \) to \( x_1 \). The claim follows.
Since the numbers $\delta_n$ tend to 0 as $n$ tends to infinity and all points in $D$ belong to the unique ball in $E'_0$, the claim shows that the set $D$ is $U$-connected and completes the proof of the proposition. \qed

Proof of Theorem 3.6.2. Suppose towards contradiction that $\langle U, C \rangle$ is a condition in the product $P_\Gamma \times P_\mu$ which forces $V[\check{x}_{gen}] \cap V[\check{\gamma}_{gen} \cdot \check{x}_{gen}] \neq V$. The following claim follows from abstract forcing considerations.

Claim 3.6.9. $\langle U, C \rangle \Vdash 2^\omega \cap V[\check{x}_{gen}] \cap V[\check{\gamma}_{gen} \cdot \check{x}_{gen}] \neq 2^\omega \cap V$.

Proof. It will be enough to show that $P_\mu$ forces that for every ordinal $\alpha$ and every function $f: \alpha \to 2$ in the extension, if $f \notin V$ then there is a ground model countable set $b \subseteq \alpha$ such that $f \upharpoonright b \notin V$. It will follow immediately that $P_\Gamma \times P_\mu \Vdash$ if $f \in V[\check{x}_{gen}] \cap V[\check{\gamma}_{gen} \cdot \check{x}_{gen}] \setminus V$ then $f \upharpoonright b \in V[\check{x}_{gen}] \cap V[\check{\gamma}_{gen} \cdot \check{x}_{gen}] \setminus V$ for some ground model countable set $b$, and therefore $2^\omega \cap V[\check{x}_{gen}] \cap V[\check{\gamma}_{gen} \cdot \check{x}_{gen}] \neq 2^\omega \cap V$ as desired.

The forcing property of $P_\mu$ in question is well-known; we include a complete proof. Suppose towards contradiction that $B \in P_\mu$ and $B \Vdash \tau: \alpha \to 2$ is a function which is not in the ground model, and for every countable set $b \subseteq \alpha$, $\tau \upharpoonright b \notin V$. Let $\langle M_\beta : \beta \in \omega_1 \rangle$ be an $\mathcal{E}$-tower of countable elementary submodels of a large structure containing $B, \tau$ as elements. For each $\beta \in \omega_1$ use the contradictory assumption to find a function $\check{g}_\beta : M_\beta \cap \alpha \to 2$ in the model $M_{\beta+1}$ such that some condition below $B$ forces $\tau \upharpoonright M_\beta = \check{g}_\beta$. Let $B_\beta \subset B$ be the Borel set representing the nonzero Boolean value of the latter statement; $B_\beta \in M_{\beta+1}$ holds by elementarity of the model $M_{\beta+1}$, but also $B_\beta \notin M_\beta$ since $\tau$ is forced not to belong to the ground model. Use a counting argument and the Lebesgue density theorem to find a basic open set $O \subseteq X$ such that the set $C = \{ \beta \in \omega_1 : \mu(B_\beta \cap O) > \frac{1}{2} \mu(O) \}$ is uncountable. Since the conditions $\{B_\beta : \beta \in C\}$ are pairwise compatible, the functions $\{\check{g}_\beta : \beta \in C\}$ must form an increasing chain and so in fact the conditions $\{B_\beta : \beta \in C\}$ form a strictly decreasing chain in $P_\mu$. This contradicts the countable chain condition of $P_\mu$. \qed

Strengthening the condition $\langle U, C \rangle$ if necessary, we may find a continuous function $\check{f}: C \to 2^\omega$ and a name $\tau$ in the complete Boolean algebra generated by the name for $\check{\gamma}_{gen} \cdot \check{x}_{gen}$ such that the fibers of $\check{f}$ are $\mu$-null and $\langle U, C \rangle \Vdash \check{f}(\check{x}_{gen}) = \tau/\check{\gamma}_{gen} \cdot \check{x}_{gen}$. Let $M$ be a countable elementary submodel of a large structure containing $\check{f}$, $\tau$ in particular, and let $C' \subset M$ be a set of points $P_\mu$-generic over the model $M$. The set $C'$ is Borel and $\mu$-positive. Find open subsets $V, U'$ of $\Gamma$ such that $1 \in V$, $U' \subset U$, and $U' \cdot V \subset U$. Use Proposition 3.6.5 to find a $V$-connected compact $\mu$-positive subset $D \subset C'$.

Since the fibers of $f$ are $\mu$-null, there must be points $x_0, x_1 \in D$ such that $f(x_0) \neq f(x_1)$. Find a number $n \in \omega$ such that $f(x_0)(n) \neq f(x_1)(n)$. Let $O_0 = \{x \in D : f(x)(n) = f(x_0)(n)\}$ and $O_1 = \{x \in D : f(x)(n) = f(x_1)(n)\}$. These are complementary relatively open subsets of the compact set $D$ and so they are separated by some distance $\delta > 0$. Use the connectedness of the set $D$ to produce a $V, \delta$-walk from $x_0$ to $x_1$. There must be successive points $x'_0, x'_1$ on the walk such that $f(x'_0)(n) \neq f(x'_1)(n)$. The two points are at a distance
> δ by the choice of δ, and so there must be a group element β ∈ V such that β · x'_0 = x'_1.

By the Baire category theorem, there must be an element γ ∈ U' which belongs to no meager subset of Γ coded in the model V[x'_1] and also to no right β^{-1}-shift of any meager subset of Γ coded in the model V[x'_0]. As a result, the point γ is PΓ-generic over V[x'_1] and the point γ · β is PΓ-generic over the model V[x'_0]. Both of these points belong to the set U. By the product forcing theorem, the pairs ⟨γβ, x'_0⟩ and ⟨γ, x'_1⟩ are both PΓ × Pμ-generic over the model M, meeting the condition (U, C). However, f(x'_0) ≠ f(x'_1) while τ/γβx'_0 = τ/γx'_1, violating the forcing theorem in view of the initial contradictory assumption. Theorem 3.6.2 follows.

The main corollaries are encapsulated in the following ergodicity result.

**Corollary 3.6.10.** Suppose that Γ is a Polish group acting on a Polish space X with a Borel probability measure μ and an ultrametric d in a measure preserving and distance preserving fashion. Suppose that the action has concentration of measure. Suppose that E is the orbit equivalence relation, Y is a Polish space, F on Y is an analytic virtually trim equivalence relation, and h: X → Y is a Borel homomorphism from X to Y. Then there is an F-equivalence class with μ-positive h-preimage.

**Proof.** Let γ ∈ Γ and x ∈ X be mutually PΓ-generic and Pμ-generic points, and look at the models V[x] and V[γ · x]. Since h is a homomorphism of E to F and x ∈ E γ · x, h(x) F h(γ · x) must hold. Since F is virtually trim and V[x] ∩ V[γ · x] = V holds per Theorem 3.6.2, there must be a virtual F-class in the ground model such that h(x) and h(γ · x) realize it. Since the poset PΓ × Pμ is c.c.c., Theorem 2.8.2 shows that all virtual classes realized in its extension are in fact repreented in the ground model. Thus, there is a ground model element y ∈ Y such that h(x) F y holds. Since x is a Pμ-generic point, it belongs to no analytic ground model coded μ-small sets. Thus, μ(h^{-1}[y]_F) > 0 as desired.

Examples of actions with concentration of measure are not easy to identify. The following examples use Fσ P-ideals I on ω (which are Polish groups with the symmetric difference operation and a suitable topology by a result of Solecki [32]) and their standard action on 2^ω (a · x = y just in case {n ∈ ω: x(n) ≠ y(n)} = a), inducing the equivalence relation =_I. The action preserves the usual Borel probability measure μ on 2^ω and also the usual minimum difference metric d on 2^ω.

**Example 3.6.11.** Let a_n: n ∈ ω be positive real numbers such that Σ_n a_n is infinite while Σ_n a_n^2 is finite. Let I be the ideal of all sets b ⊆ ω such that Σ_n∈b a_n is finite. The standard action of I on 2^ω exhibits the concentration of measure.

To see this, let U be a neighborhood of the unit in Γ, and ε > 0 be a real number. Find a real number η > 0 such that the d, η-ball around the unit is a subset of U, and find a number m ∈ ω such that 2exp(−η^2/8Σ_0^∞ a_n) < ε. The concentration of measure formula in [30, Theorem 4.3.19] then shows that δ = 2^−m works as required in Definition 3.6.1.
Corollary 3.6.12. Let $I$ be the usual summable ideal on $\omega$. Let $h: 2^\omega \to X$ be a Borel homomorphism of $\equiv_I$ to a virtually trim analytic equivalence relation $F$ on a Polish space $X$. Then there is an $F$-class whose $h$-preimage has full $\mu$-mass.

Proof. By Corollary 3.6.10, there is an $F$-class whose $h$-preimage has positive $\mu$-mass. However, the $h$-preimage is an $\equiv_I$-invariant set, the equivalence relation $\equiv_I$ includes $\mathbb{E}_0$ as a subset, and by the usual $\mathbb{E}_0$-ergodicity considerations, the $h$-preimage must in fact have full $\mu$-mass.

The concentration of measure for actions fails in many cases. Typically, there is a homomorphism of the orbit equivalence relation which violates the conclusion of Corollary 3.6.10 and therefore witnesses the failure of the concentration of measure in a strong sense.

Example 3.6.13. There is a summable-type ideal $I$ on $\omega$ and a Borel homomorphism $h: B \to 2^\omega$ of $\equiv_I$ to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-equivalence classes are $\mu$-null.

Proof. The key tool is the following:

Claim 3.6.14. For every $i \in \omega$ and every $\varepsilon > 0$ there is a number $n \in \omega$ and sets $a, b \subset 2^n$ of the same relative size $> \frac{1 - \varepsilon}{2i}$ each, such that for every $x \in a$ and $y \in b$ the set $\{m \in n : x(m) \neq y(m)\}$ contains at least $i$ many elements.

Proof. Fix $i$ and $\varepsilon$. Elementary computation shows that there is $n \in \omega$ such that the size of the set $\{a \subset n : |a| - \frac{|a|}{2} < i + 1\}$ is less than $\varepsilon 2^n$. Let $a = \{x \in 2^n : \text{the set } \{m \in n : x(m) = 1\} \text{ contains at most } \frac{n}{2} - i \text{ many elements}\}$ and $b = \{x \in 2^n : \text{the set } \{m \in n : x(m) = 1\} \text{ contains at least } \frac{n}{2} + 1 \text{ many elements}\}$. This works.

Towards the proof of the example, find a partition $\omega = \bigcup_n I_n$ into finite sets such that for every $n \in \omega$, the set $2^{I_n}$ contains subsets $a_n, b_n$ of the same size such that their relative size in $2^{I_n}$ is greater than $1/2 - 2^{-n}$, and if $x \in a_n$ and $y \in b_n$ are arbitrary elements, then the set $\{i \in I_n : x(n) \neq y(n)\}$ has size at least $n$.

Now, define $w(m) = 1/n + 1$ if $m \in I_n$ and $I = \{a \subset \omega : \sum_{n \in a} w(n) < \infty\}$. Define $B = \{x \in 2^\omega : \forall n, x(n) = 0 \leftrightarrow x \upharpoonright I_n \in a_n\};$ this is a Borel set of full mass. For $x \in B$, define $h_0(x) \in 2^\omega$ by $h_0(x)(n) = 0 \leftrightarrow x \upharpoonright I_n \in a_n$. It is not difficult to check that $h_0: B \to 2^\omega$ is a continuous homomorphism from $B$ to $\mathbb{E}_0$ such that preimages of $\mathbb{E}_0$-classes are of zero mass. The rest of the proof only adjusts $h_0$ to a Borel homomorphism $h$ defined on the whole space.

To this end, let $C_n$ for $n \in \omega$ be inclusion increasing compact subsets of $B$ whose mass converges to 1. The set $\bigcup_n C_n$ is $K_\sigma$ and the equivalence relation $\equiv_I$ is $K_\sigma$, and so the saturation $D = [\bigcup_n C_n]_{\equiv_I}$ is $K_\sigma$ as well. Let $F \subset D \times 2^\omega$ be the Borel set of all pairs $(x, y)$ such that for some (equivalently, for all) $x' \in B$ such that $x' =_I x$, $h_0(x') \in \mathbb{E}_0$-related to $y$. $F$ is a Borel set with nonempty countable sections, and by the Lusin–Novikov theorem, it has a Borel uniformization $h$. Extend $h$ to all of $2^\omega$ by defining $h(x)$ for $x \notin D$ to be an arbitrary fixed element of $2^\omega$. It is not difficult to check that $h$ has the required properties. \qed
Example 3.6.15. There is a Tsirelson-type ideal $I$ on $\omega$ and a Borel homomorphism $h: B \to 2^\omega$ of $=_I$ to $E_0$ such that preimages of $E_0$-equivalence classes are $\mu$-null.

Proof. We will deal with a certain special kind of Tsirelson submeasures. Let $\alpha > 0$ be a real number and $f: \omega \to \mathbb{R}^+$ be a function. In a typical case, the function $f$ will converge to 0 and never increase. By induction on $n \in \omega$ define submeasures $\nu_n$ on $\omega$ by setting $\nu_0(a) = \sup_{i < \alpha} f(i)$, and $\nu_{n+1}(a) = \sup\{\nu_n(a), \alpha \sum_{b \in \vec{b}} \nu_n(b)\}$ where the variable $\vec{b}$ ranges over all finite sequences $\langle b_0, b_1, \ldots, b_j \rangle$ of finite subsets of $a$ such that $j < \min(b_0) \leq \max(b_0) < \min(b_1) \leq \max(b_1) < \ldots$. In the end, let the submeasure $\nu$ be the supremum of $\nu_n$ for $n \in \omega$. Some computations are necessary to verify that $\nu$ is really a lower semicontinuous submeasure on $\omega$. The Tsirelson ideal $I = \{a \subseteq \omega: \lim_m \nu(a \setminus m) = 0\}$ turns out to be an $F_\sigma$ $\mathcal{P}$-ideal [7].

By induction on $i \in \omega$ choose intervals $I_i \subseteq \omega$ such that $\max(I_i) > \min(I_{i+1})$ and such that $\min(I_i) > i/\alpha$ there are sets $a_i, b_i \subset 2^{I_i}$ of the same relative size $\geq 1 - 2^{-i}$ such that for any elements $x \in a_i, y \in b_i$, the set $\{m \in I_i: x(m) \neq y(m)\}$ has size at least $i/\alpha$. This is easily possible by Claim 3.6.14. Now, consider the function $f$ defined by $f(m) = 1/i$ for $m \in (\max(I_{i-1}), \max(I_i)]$ and let $\nu$ be the derived submeasure and $I$ the derived Tsirelson ideal. Observe that with this choice of the function $f$, for any $i \in \omega$ and elements $x \in a_i, y \in b_i$ the set $\{m \in I_i: x(m) \neq y(m)\}$ has $\nu$-mass at least 1, since it has $\nu_1$-mass at least 1. The rest of the proof follows the lines of Example 3.6.13.

Example 3.6.16. Let $J$ be the Rado graph ideal on $\omega$. There is a Borel homomorphism $h: 2^\omega \to 2^\omega$ from $=_J$ to $E_0$ such that preimages of $E_0$-classes are $\mu$-null.

Proof. Let $G$ be the Rado graph, interpreted so that $\omega$ is the set of its vertices; then $J$ is the ideal on $\omega$ generated by $J$-cliques and $J$-anticliques. To construct $h$, by induction on $n \in \omega$ find pairwise disjoint finite sets $I_n \subseteq \omega$ and sets $a_n, b_n \subset 2^{I_n}$ such that

- each $I_n$ is a $G$-anticlique;
- if $n \neq m$ then $I_n \times I_m \subset G$;
- for every $x \in a_n$ and every $y \in b_n$ the set $\{t \in I_n: x(t) \neq y(t)\}$ has size at least $n$;
- the sets $a_n, b_n \subset 2^{I_n}$ have the same relative size larger than $1/2 - 2^{-n}$.

This is easy to do using the universality properties of the Rado graph and Claim 3.6.14 repeatedly. Let $B = \{x \in 2^\omega: \forall n x \upharpoonright I_n \in a_n \cup b_n\}$ and let $h_0: B \to 2^\omega$ be the Borel map defined by $h_0(x) = 0 \leftrightarrow x \upharpoonright I_n \in a_n$. It is immediate that the set $B$ has full $\mu$-mass and the function $h$ is a homomorphism from $=_J$ to $E_0$. The rest of the proof follows the lines of Example 3.6.13.
There are numerous questions left open by the development of this section, of which we quote two.

**Question 3.6.17.** Can the assumption that $d$ be an ultrametric be eliminated from the assumptions of Theorem 3.6.2?

**Question 3.6.18.** Is there a Tsirelson ideal whose natural action on $2^\omega$ exhibits concentration of measure?
Chapter 4

Nested sequences of models

4.1 Prologue

The purpose of this chapter is to set up a calculus for infinite nested sequences of models of ZFC, which turn out to be critical for the treatment of the $E_1$ cardinal. As a motivation, we include a simple proof of the fact that $E_1$ is not Borel reducible to any orbit equivalence relation. It is quite different from the standard one [17, Theorem 11.8.1], and it has the advantage that it can be easily adapted to the context of inequalities between cardinalities of quotient spaces.

**Theorem 4.1.1.** $E_1$ is not Borel reducible to any orbit equivalence relation.

**Proof.** Let $Y$ be the Polish space $(2^\omega)^\omega$ and $\Gamma$ be a Polish group continuously acting on $Z$, inducing the orbit equivalence relation $F$. Suppose towards contradiction that there is a Borel reduction $h: Y \to Z$ of $E_1$ to $F$. Let $\langle x_n : n \in \omega \rangle$ be a sequence of mutually Cohen generic points in $2^\omega$. Let $y_n$ denote the element of $Y$ for which $y_n(m) = \text{the zero sequence in } 2^\omega$ if $m < n$, and $y_n(m) = x_m$ if $m \geq n$, and let $V_n = V[y_n]$.

Work in the model $V_0$. The reinterpretation of the Borel map $h$ in $V_0$ is still a reduction of $E_1$ to $F$. For each number $n > 0$ fix a group element $\gamma_n \in \Gamma$ such that $\gamma_n \cdot h(y_n) = h(y_0)$. Let $\gamma \in \Gamma$ be a point Cohen-generic over $V[y_0]$ and look into the model $V[\gamma \cdot h(y_0)]$. By a Mostowski absoluteness argument, there must be a point $y \in V[\gamma \cdot h(y_0)]$ such that $h(y) F \gamma \cdot h(y_0)$. Since the function $h$ is a reduction of $E_1$ to $F$ even in the model $V_0[\gamma]$, this point $y \in Y$ must be $E_1$-related to $y_0$, and so for all but finitely many numbers $n$ it must be the case that $y(n) = x_n$. Choose a number $n \in \omega$ such that $y(n) = x_n$, and look at the model $V_{n+1}$.

The point $\gamma \gamma_{n+1} \in \Gamma$ is Cohen generic over $V_0$ by the invariance of the meager ideal under right shift, and by the product forcing theorem it follows that the models $V_0$ and $V_{n+1}[\gamma \gamma_{n+1}]$ are mutually generic over the model $V_{n+1}$. Now, the points $y_{n+1}$ and $\gamma \gamma_{n+1} \cdot h(y_{n+1}) = \gamma \cdot h(y_0)$ belong to the model $V_{n+1}[\gamma \gamma_{n+1}]$ and so does $y$. Thus, even the point $y(n) = x_n \in 2^\omega$ belongs to this model; however, it is a point of $V_0$ Cohen generic over $V_{n+1}$. This contradicts the product forcing theorem.

\[ \square \]
Theorem 4.1.2. Let $P$ be the product of $\aleph_1$ many Cohen forcings with finite support. Let $E$ be an orbit equivalence relation of a continuous Polish group action. In the $P$-extension, there is no injection of the $E_1$-quotient space to the $E$-quotient space definable from ground model parameters.

4.2 Coherent sequences of models

It is clear from the argument that its generalizations will require codification of decreasing $\omega$-sequences of generic extensions. In addition to the approach from Theorem 4.1.1, we pay close attention to the intersection model. This is the contents of the following definitions and theorems.

Definition 4.2.1. Let $\langle V_n : n \in \omega \rangle$ be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is coherent if for every ordinal $\lambda$ and every natural number $n$, the intersection of the relation $R_\lambda = \{ \langle m, x \rangle : m \in \omega, \text{rank of } X \text{ is smaller than } \lambda, x \in V_m \}$ with $V_n$ belongs to $V_n$.

Example 4.2.2. Let $P$ be the countable support product of countably many copies of the Sacks forcing, adding a sequence $\langle x_m : m \in \omega \rangle$. Let $V_n = V[x_m : m \geq n]$. Then $\langle V_n : n \in \omega \rangle$ is a coherent sequence of models.

Example 4.2.3. Let $x$ be a set and let $M$ be a class containing all ordinals. By induction on $n \in \omega$, define models $V_n$ by $V_0 = V$ and $V_{n+1} =$the collection of all sets hereditarily definable from parameters in $M$ and the parameter $x \cap V_n$ in the model $V_n$. The sequence $\langle V_n : n \in \omega \rangle$ is coherent.

Example 4.2.4. Let $\kappa$ be a measurable cardinal, $U$ a measure on it, and $j : V \rightarrow M$ the $U$-ultrapower, with iterands denoted by $j_\alpha$ for every ordinal $\alpha$. For each $n \in \omega$ let $V_n = j_n(V)$. The sequence $\langle V_n : n \in \omega \rangle$ is coherent.

The coherent sequences of models are most often formed as generic extensions of the constant sequence $\langle V_n = V : n \in \omega \rangle$ using the following definition and theorem.

Definition 4.2.5. Let $P,Q$ be posets. A projection of $Q$ to $P$ is a pair of order-preserving functions $\pi : Q \rightarrow P$ and $\xi : P \rightarrow Q$ such that

1. $\pi \circ \xi$ is the identity on $P$;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

Definition 4.2.6. Let $\langle V_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC. A coherent sequence of posets is a sequence $\langle P_n, \pi_{nm}, \xi_n : P_n \rightarrow P_m, \xi_{mn} : P_m \rightarrow P_n : n \leq m \in \omega \rangle$ such that
1. the maps $\pi_{nm}, \xi_{mn}$ form a projection of $P_n$ to $P_m$;
2. the functions $\pi_{nm}$ form a commutative system, same for $\xi_{mn}$, and $\pi_{nn} = \xi_{nn}$ = the identity on $P_n$;
3. for every number $k \in \omega$, the sequence $\langle P_n, \pi_{nm} : P_n \to P_m, \xi_{mn} : P_m \to P_n : k \leq n \leq m \in \omega \rangle$ belongs to the model $V_k$.

In particular, every commutative sequence of projections is coherent over the constant coherent sequence $(V_n = V : n \in \omega)$.

**Theorem 4.2.7.** Let $\langle V_n : n \in \omega \rangle$ be a coherent sequence of models of ZFC and $\langle P_n, \pi_{nm} : P_n \to P_m, \xi_{mn} : P_m \to P_n : n \leq m \in \omega \rangle$ be a coherent sequence of posets. Let $G \subset P_0$ be a filter generic over $V_0$, and let $G_n = \xi_{n0}^{-1}G$. The sequence $\langle V_n[G_n] : n \in \omega \rangle$ is a coherent sequence of models of ZFC again.

**Proof.** Observe that for every number $k \in \omega$, in the model $V_k[G_k]$, one can form the sequence $(G_n : n \geq k)$ as $\langle \xi_{nk}^{-1}G_k : n \geq k \rangle$ by the commutativity of the $\xi$-maps. Thus, if $\lambda$ is a limit ordinal larger than the rank of all the posets on the coherent sequence, one can form the relation $\{ (n, x) : n \geq k, x \in V_n[G_n] \text{ has rank } < \lambda \}$ in the model $V_k[G_k]$ as the set $\{ (n, \tau/G_n) : n \geq k \text{ and } \tau \in V_n \text{ is a } P_n\text{-name of rank } < \lambda \}$ by the coherence of the original sequence $\langle V_n : n \in \omega \rangle$.

The critical object for understanding a coherent sequence of models is the intersection model $V_\omega = \cap_n V_n$. In Example 4.2.2, the intersection model is a model of ZFC, and it has been discussed in [18, Theorem 9.3.4]; we will come back to it below—Theorem 4.3.6. The evaluation of the intersection model in Example 4.2.3 is key to the surprising Theorem 13.4.7 below. In the context of general coherent sequences, the intersection model is a transitive model of ZF, and the Axiom of Choice may fail in it. This is the contents of the following theorem and example.

**Theorem 4.2.8.** If $\langle V_n : n \in \omega \rangle$ is a coherent decreasing sequence of generic extensions of $V$, then $V_\omega = \cap_n V_n$ is a class in all models $V_n$, and it is a model of ZF.

**Proof.** We will only show that $V_\omega$ is a class in $V_0$, the rest is routine. Let $\lambda$ be a large limit cardinal in $V$ so that $V_0$ is a generic extension of $V$ by a poset of rank $< \lambda$, and such that $V_\lambda$ satisfies a large fragment of ZFC. Note that then, all the models $V_n$ are also obtained from $V$ as generic extensions by posets of rank $< \lambda$. Move to the model $V_0$. Let $f$ be the function from $\omega$ to sets of rank $< \lambda$ such that $f(n) = \{ (P, G) : P \in V \text{ is a poset of rank } < \lambda, G \in V_n \text{ is a filter on } P \text{ generic over } V, \text{ and whenever } \langle Q, H \rangle \text{ is a pair consisting of a poset in } V \text{ of rank } < \lambda \text{ and a filter on } Q \text{ in } V_n \text{ generic over } V, \text{ there is a } P\text{-name } \sigma \text{ in } V \text{ such that } \sigma/G = H \}$. The coherence of the sequence $\langle V_n : n \in \omega \rangle$ shows that the function $f$ can be formed in $V_0$, and that the tail $f \upharpoonright (\omega \setminus n)$ belongs to $V_n$. One tricky point is that the function $f$ may not have a selector whose tails would belong to the respective models $V_n$. Now, $V_\omega$ is exactly the collection of all sets $x$ such that for every $n \in \omega$ and every pair $(P, G) \in f(n), x \in V[G]$. Since $V$ is a class in $V_0$, this shows that $V_\omega$ is a class in $V_0$ as well.
Example 4.2.9. Let \(\kappa\) be a measurable cardinal, with a normal measure \(U\) on \(\kappa\) and the associated ultrapower \(j : V \rightarrow M\). Let \(j_{nm} : M_n \rightarrow M_m\) be the iterands of \(j\) for \(n \leq m \leq \omega\). Then \(\bigcap_{n \in \omega} M_n = M_\omega[c]\) where \(c = (j_{0n}(\kappa)) : n \in \omega\).

It is well-known and follows from the geometric description of Prikry genericity by Mathias [29] that the set \(c\) is generic over the model \(M_\omega\) for the Prikry forcing associated with the measure \(j_0U\).

**Proof.** It is immediate that \(M_\omega \subset M_n\) and \(c \in M_n\) for all \(n \in \omega\), proving the right-to-left inclusion. For the left-to-right inclusion, suppose that \(a\) is a set of ordinals which belongs to all models \(M_n\), and work to show that \(a \in M_\omega[c]\).

For each \(n \in \omega\), the set \(b_n = j_{n0}^{-1}a\) belongs to the model \(M_n\), and as such it must be equal to \((j_{0n}f_n)(\kappa_i : i \in n)\) for some function \(f\). This follows from the fact that the embedding \(j_{0n}\) is an iterated \(U\)-ultrapower. The elementarity of the embedding \(j_{0n}\) shows that for every ordinal \(\alpha \in M_n\), \(\alpha \in b_n \Leftrightarrow j_{0n}(\alpha) \in j_{0n}(f_n)(\kappa_i : i \in n)\). Let \(F\) be the function assigning each number \(n \in \omega\) the function \(f_n\), and use the fact that \(M_\omega\) is a direct limit of the models \(M_n\) for \(n \in \omega\) to conclude that \(a = \limsup_n j_{0n}(F)(\kappa_i : i \in n)\) and therefore \(a \in M_\omega[c]\).

**Example 4.2.10.** Let \(c_0 : \omega_1 \times \omega \rightarrow 2\) be a Cohen-generic map, and let \(c_n = c_0 \upharpoonright \omega_1 \times (\omega \setminus n)\). Let \(V_n = V[c_n]\). In the model \(V_\omega = \bigcap_n V_n\), the chromatic number of \(G_0\) is greater than 2; thus, AC must fail in \(V_\omega\).

**Proof.** Work in \(V\). For each number \(n \in \omega\), let \(P_n\) be the poset of all finite functions from \(\omega_1 \times (\omega \setminus n)\) to 2 ordered by extension. Note that \(c_0\) is \(P_0\)-generic and the model \(V_\omega\) is equal to \(V[G \cap P_\omega]\). For each ordinal \(\alpha \in \omega_1\) and a number \(n \in \omega\), let \(d_{\alpha n}\) be a name for the function defined by \(d_{\alpha n}(m) = 0\) if \(m \in n\) and \(d_{\alpha n}(m) =\)the unique value of \(p(\alpha, m)\) for all conditions \(p\) in the generic filter with the pair \((\alpha, m)\) in their domain if \(m \geq n\). Note that \(d_{\alpha n}\) is really a \(P_n\)-name and it is forced to belong to the intersection model \(V_\omega\).

Now, let \(p \in P\) be a condition and \(\sigma\) be a name such that \(p \Vdash \sigma : 2^\omega \rightarrow 2\) is a function in \(V_\omega\); we will find an ordinal \(\alpha \in \omega_1\), a number \(n \in \omega\) and a condition strengthening \(p\) which forces \(d_{\alpha 0}\) and \(d_{\alpha n}\) to differ in an even number of entries if and only if \(\sigma(d_{\alpha 0}) \neq \sigma(d_{\alpha n})\). This cannot occur for a coloring of \(G_0\).

By a standard \(\Delta\)-system argument, strengthening \(p\) if necessary, we may find a stationary set \(S \subset \omega_1\), conditions \(p_\alpha \in P\) for \(\alpha \in S\) and a number \(n \in \omega\) such that the conditions \(p_\alpha\) for \(\alpha \in S\) form a \(\Delta\)-system with root \(p\), \(\text{dom}(p_\alpha) \subset \omega_1 \times n - 1\), and each \(p_\alpha\) decides the value of \(\sigma(d_{\alpha 0})\) to be some bit \(b_\alpha \in 2\). Find a condition \(q \leq p\) and a \(P_\alpha\)-name \(\tau\) such that \(q \Vdash \sigma = \tau\); this is possible as \(\sigma\) is forced to belong to \(V_\omega\). Find an ordinal \(\alpha \in S\) such that \(p_\alpha\) is compatible with \(q\). Find a condition \(r \in P_\alpha\) such that \(r \leq q \upharpoonright \omega_1 \times (\omega \setminus n)\) and \(r\) decides the value of \(\tau(d_{\alpha n})\) to be some specific bit \(b \in 2\). Note that \(p_\alpha\) and \(r\) are compatible in \(P\), and the pair \((\alpha, n - 1)\) does not belong to \(\text{dom}(p_\alpha \cup r)\).

Thus, it is possible to strengthen the condition \(p_\alpha \cup r\) to some \(s \in P\) such that \(\{\alpha\} \times n \subset \text{dom}(s)\), and cardinality of the set \(\{m \in n : s(\alpha, m) = 1\}\) is even if and only if \(b_\alpha \neq b\). This completes the proof.

\(\square\)
4.3 Choice-coherent sequences of models

In most of our examples, we will want to look at sequences of models which have a greater degree of coherence. Certain constructions arising from the axiom of choice will have to be performed in a coherent way. The following definition records the demands:

**Definition 4.3.1.** Let \( \langle V_n : n \in \omega \rangle \) be an inclusion decreasing sequence of transitive models of ZFC. We say that the sequence is **choice-coherent** if it is coherent and for every ordinal \( \lambda \) there is a well-ordering \( \leq \lambda \) of sets of rank \( < \lambda \) (in \( V_0 \)) such that its intersection with each model \( V_n \) belongs to \( V_n \).

**Theorem 4.3.2.** If \( \langle V_n : n \in \omega \rangle \) is a choice-coherent decreasing sequence of generic extensions of \( V \), then \( V_\omega = \bigcap_n V_n \) is a model of ZFC.

**Proof.** Let \( \lambda \) be any ordinal. In view of Theorem 4.2.8, we only need to produce a well-ordering \( \leq \) of the collection of sets of rank \( < \lambda \) in the model \( V_\omega \) such that \( \leq \in V_\omega \). Fix a well-ordering \( \leq \lambda \) witnessing the fact that \( \langle V_n : n \in \omega \rangle \) is a choice-admissible decreasing sequence, and note that \( \leq = \leq \lambda \cap V_\omega \) works as desired. \( \square \)

The theorem makes the following definition possible:

**Definition 4.3.3.** The sequence \( \langle V_n : n \in \omega \rangle \) is **generic over** \( V \) if \( V \) is a model of ZFC contained in all \( V_n \) for \( n \in \omega \) and \( V_0 \) is a generic extension of \( V \).

The usual abstract forcing arguments (Fact 1.3.7 show that if the sequence of models is generic over \( V \) then all models \( V_n \) and also \( V_\omega \) are generic extensions of \( V \) and if \( n < m \leq \omega \) then \( V_n \) is a generic extension of \( V_m \). Most examples of choice-coherent sequences are generic and obtained from the trivial one \( \langle V_n = V : n \in \omega \rangle \) by a coherent forcing which satisfies a certain degree of completeness.

**Definition 4.3.4.** Let \( \langle P_n, \pi_{nm}, \xi_{mn} : P_n \to P_m, \xi_{mn} : P_m \to P_n : n \leq m \in \omega \rangle \) be a commutative system of projections from posets \( P_n \) to \( P_m \) for \( n \leq m \).

1. The **diagonal game** is the following infinite game between Players I and II, at round \( n \) Player I plays \( p_n \in P_n \) and Player II responds by \( q_n \leq p_n \). Additionally, \( p_{n+1} \leq \pi_{n+1}(q_n) \). In the end, Player II wins if there is a condition \( r \in P_0 \) such that \( \pi_{0n}(r) \leq q_n \).

2. The sequence is **diagonally distributive** if Player I has no winning strategy in the diagonal game.

**Example 4.3.5.** Suppose that \( \langle Q_m : m \in \omega \rangle \) are arbitrary posets, and let \( P_n = \prod_{m \geq n} Q_m \) be the countable support product with the natural projection maps from \( P_n \) to \( P_m \) for \( n \leq m \). Player II has a simple winning strategy in the diagonal game in this setup: set \( q_n = p_n \).
Theorem 4.3.6. Let \( \langle V_n : n \in \omega \rangle \) be a choice-coherent sequence of models of ZFC. Let \( \langle P_n, \pi_{nn} : P_n \to P_m, \xi_{nn} : P_m \to P_n : n \leq m \in \omega \rangle \) be a coherent sequence of posets which is diagonally distributive in \( V_0 \). Let \( G \subseteq P_0 \) be a filter generic over \( V_0 \), and let \( G_n = \xi_n^{-1} G \). Then

1. the sequence \( \langle V_n[G_n] : n \in \omega \rangle \) is choice-coherent;

2. the models \( \bigcap_n V_n \) and \( \bigcap_n V_n[G] \) contain the same \( \omega \)-sequences of ordinals.

Proof. Let \( V_\omega = \bigcup_n V_n \). We start with (1). The main task is to find a poset \( P_\omega \in V_\omega \) and a filter \( G_\omega \subset P_\omega \) generic over \( V_\omega \) such that \( \bigcap_n V_n[G_n] = V_\omega[G_\omega] \). This is an uneventful case of diagram chasing.

Using the choice coherence of the original sequence \( \langle V_n : n \in \omega \rangle \), we may assume that there is a sequence \( \langle \alpha_n : n \in \omega \rangle \) such that the underlying set of the poset \( P_n \) is exactly \( \alpha_n \). For each condition \( p \in P_0 \), the ordinal \( \omega \)-sequence \( \langle \pi_{0n}(p) : n \in \omega \rangle \) belongs to \( V_\omega \), since for each number \( k \in \omega \), the tail \( \langle \pi_{kn}(p) : n \in \omega \rangle \) is reconstructed as \( \langle \xi_{kn}(\pi_{0n}(p)) : n \in \omega \rangle \) in the model \( V_k \). Similarly, the set \( P_\omega = \{ q \in \prod_n P_n : \exists k \in \omega \forall n \geq k \ q(n) = \pi_{kn}(p(k)) \} \) belongs to the model \( V_\omega \). For elements \( q_0, q_1 \in P_\omega \), we have \( q_1(n) \leq q_0(n) \) in the poset \( P_n \), and conclude that the poset \( \langle P_\omega, \leq \rangle \) belongs to the model \( V_\omega \).

Define a function \( \pi_{0\omega} : P_0 \to P_\omega \) by \( \pi_{0\omega}(p) = q \) where \( q(n) = \pi_{0n}(p) \), and a function \( \xi_{0\omega} : P_\omega \to P_0 \) by \( \xi_{0\omega}(q) = \xi_{0\omega}(\pi_{kn}(p(k))) \) where \( k \in \omega \) is such that for all \( n \geq k \), \( \pi_{kn}(p(k)) = q(n) \). One can also similarly define maps \( \pi_{n\omega} : P_n \to P_\omega \) and \( \xi_{n\omega} : P_\omega \to P_n \). It is a matter of trivial diagram chasing to show that the maps form a commuting system of projections from \( P_n \) to \( P_\omega \) and moreover \( \pi_{n\omega}, \xi_{n\omega} \in V_n \). Thus, letting \( G_\omega = \xi_{0\omega}^{-1} G_0 \), one can conclude that \( G_\omega \subset P_\omega \) is a filter generic over \( V_0 \) and therefore over \( V_\omega \). Also \( G_\omega \in \bigcap_n V_n[G_n] \) since \( G_\omega \) can be reconstructed in \( V_n[G_n] \) as \( G_\omega = \xi_{0\omega}^{-1} G_\omega \). In conclusion, \( G_\omega \in \bigcap_n V_n[G_n] \).

Finally, we have to prove that every element of the intersection \( \bigcap_n V_n[G_n] \) belongs to \( V_\omega[G_\omega] \). This is where the diagonal distributivity of the original poset sequence is used. Suppose that \( \tau \in V_0 \) is a \( P_0 \)-name for a set of ordinals and \( p \in P_0 \) is a condition forcing \( \tau \in \bigcap_n V_n[G_n] \); we must produce a condition \( p' \leq p \) and a \( P_\omega \)-name \( \tau_\omega \in V_\omega \), such that \( p' \forces \tau = \tau_\omega[G_\omega] \). Consider a strategy by Player I in the diagonalization game in which he plays \( p_n \) so that \( p_0 \leq p_n \), \( \tau_0 = \tau \), and there is a \( P_{n+1} \)-name \( \tau_{n+1} \in V_{n+1} \) such that \( p_n \forces \tau_{n+1} = \tau_{n+1}[G_{n+1}] \). This is possible by the assumption on the name \( \tau \). By the diagonalization assumption, Player II has a counterplay with conditions \( q_n \leq p_n \) such that there is a condition \( p' \leq p \) for which \( \pi_{0n}(p) \leq q_n \) for all \( n \in \omega \). Let \( \tau_\omega \) be the \( P_\omega \)-name defined by \( q \forces \bar{\alpha} \in \tau_\omega \) just in case \( \xi_{0\omega}(q) \forces \tau_0 \bar{\alpha} \in \tau \). The name \( \tau_\omega \) can be reconstructed in every model \( V_n \) by the definition \( q \forces \bar{\alpha} \in \tau_\omega \) just in case \( \xi_{0\omega}(q) \forces \tau_0 \bar{\alpha} \in \tau \) by the choice of the strategy for Player I in the diagonalization game. As a result, \( \tau_\omega \in V_\omega \). It is immediate from the definition of \( \tau_\omega \) that \( p' \forces \tau = \tau_\omega[G_\omega] \) as desired.

Now we are ready to construct the requisite well-orderings verifying the choice-coherence of the models \( \langle V_n[G_n] : n \in \omega \rangle \). Let \( \lambda \) be an ordinal larger than the ranks of all the posets \( P_n \) for \( n \in \omega \). Let \( \leq \) be a coherent well-ordering
of sets of rank $< \lambda$ in $V_0$. We will now describe a coherent well-ordering $\leq'$ of sets of rank $< \lambda$ in the model $V_0[G_0]$. In this well-ordering, the sets in $V_\omega[G_\omega]$ come first, ordered by some well-ordering in the model $V_\omega[G_\omega]$. The sets in $V_0[G_0] \setminus V_1[G_1]$ come next, well-ordered by their $\leq$-first $P_0$-name in the model $V_0$ representing them. The sets in $V_1[G_1] \setminus V_2[G_2]$ come next with a similar well-order, and so on. The coherence of the resulting well-ordering $\leq'$ is due to the fact that for each $k \in \omega$, the sequence $\langle G_n : n \geq k \rangle$ belongs to the model $V_k[G_k]$.

(2) is much easier. Suppose that $\tau \in V_0$ is a $P_0$-name for an $\omega$-sequence of ordinals in the model $V_\omega[G_\omega]$ and $p \in P_0$ is a condition; we must find a condition $r \leq p$ and an $\omega$-sequence $z \in V_\omega$ such that $r \Vdash \tau = \check{z}$. Consider a strategy for Player I in the diagonal game in which he plays conditions $p_n \in P_n$ and on the side produces $P_n$-names $\tau_n \in V_n$ so that $p_0 \leq p$, $\tau_0 = \tau$ and $p_n \Vdash r \tau_n = \tau_{n+1}$ evaluated by the $\chi_{n+1}$-image of the generic filter on $P_n$, and also $p_n$ decides the value $\tau_n(n)$ to be some ordinal $z(n)$. The assumptions on the name $\tau$ shows that this is a valid strategy. The initial assumptions on the coherent sequence of posets show that this is not a winning strategy and so there must be a play against it such that in the end there is a condition $r \leq p$ with $\chi_{0n}(r) \leq p_n$ for all $n \in \omega$. Let $\tau_n \in V_n$ be the names produced during that counterplay, and let $z$ be the $\omega$-sequence of ordinals obtained. The definitions show that for all $n \in \omega$, $\chi_{0n}(r) \Vdash p_n \tau_n = \check{z}$. It follows that $z \in V_n$ for all $n \in \omega$, and therefore $r, z$ are as required in (2).

The main feature of choice-coherent sequences of models we use later is the following theorem connecting them with orbit equivalence relations:

**Theorem 4.3.7.** Let $\langle V_n : n \in \omega \rangle$ be a choice-coherent, generic sequence of models. Let $E$ be an orbit equivalence relation on a Polish space $X$ with code in $V_\omega = \bigcap_n V_n$. If a virtual $E$-class is represented in $V_n$ for every $n \in \omega$, then it is represented in $V_\omega$.

Note that a virtual $E$-class is an equivalence class of $E$-pins. Thus, the theorem says that if there are pairwise equivalent $E$-pins $\langle P_n, \tau_n \rangle \in V_n$ for all $n \in \omega$, then there is an $E$-pin equivalent to them in the intersection model.

**Proof.** Let $\Gamma$ be a Polish group continuously acting on the space $X$, inducing the equivalence relation $E$. Let $d$ be a compatible right-invariant metric on $\Gamma$. Let $\langle P_0, \tau_0 \rangle \in V_0$ be an $E$-pin which has an equivalent in the model $V_n$ for every $n \in \omega$. Let $\lambda$ be a cardinal so large that for each $n \in \omega$, $V_0$ is a generic extension of $V_n$ by a poset of size $< \lambda$, and $V_n$ contains an $E$-pin on a poset of size $< \lambda$ equivalent to the pin $\langle P_0, \tau_0 \rangle$.

Let $P_\Gamma$ be the Cohen forcing on the Polish group $\Gamma$, with its name $\dot{\gamma}_{\text{gen}}$ for a generic point. Let $\gamma \in \Gamma$ be a Cohen-generic point, $H \subseteq P_0$ be a generic filter and $K \subseteq \text{Coll}(\omega, \lambda)$ be a generic filter, mutually generic over $V_0$; let $x_0 = \sigma_0/H$. In the model $V_0[\gamma, H, K]$, form the model $M$ as the class of all sets hereditarily definable from $\gamma \cdot x_0$ and parameters in $V_\omega$. The model $M$ is an intermediate model of ZFC between $V_\omega$ and $V_0[\gamma, H]$, and so by Fact 1.3.7, the model $M$ is
a forcing extension of $V_\omega$. We will argue that $M, V_0[H]$ are mutually generic extensions of $V_\omega$.

First note that this will prove the theorem. Let $Q, \tau \in V_\omega$ be a poset and a name and $L \subset Q$ be a filter generic over the model $V_0[H]$ such that $V_\omega[L] = M$ and $\tau/L = \gamma \cdot x_0$. By the forcing theorem in the model $V_0$, there have to be conditions $p \in H$ and $q \in L$ such that $\langle p, q \rangle \Vdash \tau_0 E \tau$. It is immediate that $\tau$ as a name on $Q \upharpoonright q$ is $E$-pinned, and the $E$-pin $\langle Q \upharpoonright q, \tau \rangle$ is equivalent to $\langle P_0, \tau_0 \rangle$. This confirms the conclusion of the theorem.

To argue that $M, V_0[H]$ are mutually generic extensions of $V_\omega$, we use the criterion of Theorem 13.3.2. In other words, if $a \in V_0[H]$ and $b \in M$ are disjoint subsets of some ordinal $\kappa$, we must find a set $c \in V_\omega$ of ordinals such that $a \subset c$ and $b \cap c = 0$. Towards this end, move back to the model $V_0$. Suppose that $O \subset \Gamma$ is a nonempty open set, $p \in P_0$ is a condition, $\dot{a}$ is a $P_0$-name for a set of ordinals, and $\phi$ is a formula with parameters in $V_\omega$ such that in the poset $P_\Gamma \times P_0$, $\langle O, p \rangle \Vdash \text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \dot{a} \phi(\beta, \dot{\gamma}_{\text{gen}} \cdot \tau_0)$ holds. Due to the definition of the model $M$, it will be enough to find a set $c \in V_\omega$ and a condition $\langle O', p' \rangle \leq \langle O, p \rangle$ which forces $\dot{a} \subset c$ and $\text{Coll}(\omega, \lambda) \Vdash \forall \beta \in \phi(\beta, \dot{\gamma}_{\text{gen}} \cdot \tau_0)$ holds.

Finally, we are in a position to use some coherence arguments. Let $\prec$ be a coherent well-ordering of the collection of sets of rank $< \lambda$; i.e. the restriction of $\prec$ to each $V_n$ belongs to $V_n$. We will use the ordering to perform some coherent constructions. A typical construction of a coherent sequence of elements of $V_n$ proceeds by induction on $n \in \omega$. If $\langle v_n : n \in \omega \rangle$ is coherent, $w_0 \in V_0$, and $\phi$ is some formula with parameters in $V_\omega$, one can select the $\prec$-least $w_{n+1} \in V_{n+1}$ such that $V_\omega \models \phi(v_n, w_n, w_{n+1})$ if it exists; then, the sequence $\langle w_n : n \in \omega \rangle$ is coherent. The routine details of these constructions will be suppressed below.

Find a coherent sequence $\langle P_n, \tau_n : n \in \omega \rangle$ of pairwise equivalent $E$-pins starting with $P_0, \tau_0$; i.e. for every number $n \in \omega$ it is the case that $P_n \times P_{n+1} \Vdash \sigma_n E \sigma_{n+1}$. Find a coherent sequence $\langle \dot{\gamma}_n : n \in \omega \rangle$ such that for each $n \in \omega$, $\dot{\gamma}_n$ is a $P_n \times P_{n+1}$-name for an element of the group $\Gamma$ such that $\tau_n = \dot{\gamma}_n \cdot \tau_{n+1}$. Let $D \subset \Gamma$ be a fixed countable dense set in the model $V_\omega$, and let $\delta_0 \in D$ and $\varepsilon > 0$ be such that the open $d$-ball $B(\delta_0, \varepsilon) \subset \Gamma$ is a subset of the open set $O$. Find a coherent sequence $\langle p_n, \delta_n : n \in \omega \rangle$ such that $p_0 \leq p, p_n \in P_n, \delta_n \in D$, and in the poset $P_n \times P_{n+1}, \langle p_n, p_{n+1} \rangle \Vdash d(\delta_n : \dot{\gamma}_n, \delta_{n+1}) < \varepsilon < 2^{-n-3}$. Let $O_n = B(\delta_n, \varepsilon/2)$. The point in these definitions is the following claim:

**Claim 4.3.8.** Let $n > 0$. The condition $\langle p_i : i \leq n \rangle$ forces in the product $\prod_{i \leq n} P_i$ the following:

1. $O_n \subset B(\delta_0, \varepsilon) \cdot \dot{\gamma}_0 \dot{\gamma}_1 \ldots \dot{\gamma}_{n-1}$;
2. $B(\delta_0, \varepsilon/4) \cdot \dot{\gamma}_0 \dot{\gamma}_1 \ldots \dot{\gamma}_{n-1} \subset O_n$.

**Proof.** Use the right invariance of the metric $d$ to argue by induction on $i \in n$ that $d(\delta_{i+1}, \delta_0 \cdot \dot{\gamma}_0 \dot{\gamma}_1 \ldots \dot{\gamma}_i)$ is forced to be smaller than $\varepsilon \cdot 2^{-j-3}$. In conclusion, $d(\delta_n, \delta_0 \cdot \dot{\gamma}_0 \dot{\gamma}_1 \ldots \dot{\gamma}_{n-1})$ is forced to be smaller than $\varepsilon/4$. The two items then follow immediately by the right invariance of the metric $d$ again. \(\square\)
Now, for every number $n \in \omega$, in the model $V_n$ form the set $c_n = \{ \beta \in \kappa : \text{in the poset } P_i \times P_n, \langle \beta, n \rangle \models \text{Coll}(\omega, \lambda) \models \phi(\bar{\beta}, \check{\gamma}_n \cdot \tau_n) \}$. Finally, let $c = \limsup_n c_n = \{ \beta \in \kappa : \exists n \beta \in c_n \}$. It is immediate that the sequence $\langle c_n : n \in \omega \rangle$ is coherent and therefore the set $c$ belongs to the model $V_\omega$. Let $O' = B(\delta_0, \varepsilon/4)$ and $p' = p_0$. The following two claims stated in the model $V_0$ complete the proof of the theorem.

Claim 4.3.9. In the poset $P_0$, $p' \Vdash \dot{\alpha} \subseteq \dot{c}$.

Proof. Let $p'' \leq p'$ be a condition and $\beta \in \kappa$ an ordinal such that $p'' \Vdash \dot{\beta} \in \dot{\alpha}$. It will be enough to show that for all $n > 0$, $\beta \in c_n$. To this end, fix a number $n > 0$ and let $H_i : i \leq n$ be filters on the respective posets $P_i$ mutually generic over the model $V_0$ such that $p_i \in H_i$ and moreover $p'' \in H_0$. Write $x_i = \tau_i / H_i$ and $\gamma_i = \gamma_i / H_i, H_i+1$; so $x_0 = \gamma_0 \gamma_1 \ldots \gamma_{n-1} x_n$.

Let $\gamma \in O_n$ be a point $P_0$-generic over the model $V_0[H_i : i \leq n]$. Let $\gamma' = \gamma_{\gamma_1-1} \gamma_{\gamma_2-1} \ldots \gamma_{\gamma_0-1}$. By the invariance of the meager ideal on $\Gamma$ under right translations, $\gamma' \in \Gamma$ is a point Cohen generic over the model $V_0[H_i : i \leq n]$. By Claim 4.3.8(1), $\gamma' \in B(\delta_0, \varepsilon) \subseteq O$; moreover, $\gamma \cdot x_0 = \gamma' \cdot x_0$.

Let $K \subseteq \text{Coll}(\omega, \lambda)$ be a filter generic over the model $V_0[H_i : i \leq n][\gamma]$. The model $N = V_0[H_i : i \leq n][\gamma][K]$ is a $\text{Coll}(\omega, \lambda)$-extension of both $V_0[\gamma', H_0]$ and $V_0[\gamma, H_n]$ by the choice of $\lambda$ and Fact 1.3.9. By the forcing theorem in the model $V_0$ and the initial assumptions on the name $\dot{\alpha}$ and the formula $\phi$, $N \models \phi(\bar{\beta}, \check{\gamma} \cdot \tau_0)$. This means that $\beta \in c_n$ as required.

Claim 4.3.10. In the poset $P_1 \times P_0$, for every ordinal $\beta \in c$, $\langle O', p' \rangle \Vdash$ Coll$(\omega, \lambda) \models \phi(\bar{\beta}, \check{\gamma} \cdot \tau_0)$.

Proof. Find a number $n > 0$ such that $\beta \in c_n$. Let $H_i \subseteq P_i$ for $i \leq n$ be filters mutually generic over $V_0$ containing the conditions $p_i$ respectively. Write $x_i = \tau_i / H_i$ and $\gamma_i = \gamma_i / H_i, H_i+1$; so $x_0 = \gamma_0 \gamma_1 \ldots \gamma_{n-1} x_n$.

Let $\gamma' \in O'$ be a point $P_1$-generic over the model $V_0[H_i : i \leq n]$. Let $\gamma = \gamma' \gamma_0 \gamma_1 \ldots \gamma_{n-1}$; by the invariance of the meager ideal on $\Gamma$ under right translations, $\gamma \in \Gamma$ is a point Cohen generic over the model $V_0[H_i : i \leq n]$. By Claim 4.3.8(2), $\gamma \in O_n$; moreover, $\gamma \cdot x_0 = \gamma' \cdot x_0$.

Let $K \subseteq \text{Coll}(\omega, \lambda)$ be a filter generic over the model $V_0[H_i : i \leq n][\gamma]$. The model $N = V_0[H_i : i \leq n][\gamma][K]$ is a $\text{Coll}(\omega, \lambda)$-extension of both $V_0[\gamma', H_0]$ and $V_0[\gamma, H_n]$ by the basic properties of collapse posets–Fact 1.3.9. By the forcing theorem in the model $V_0$ and the initial assumptions on the name $\dot{\alpha}$ and the formula $\phi$, $N \models \phi(\bar{\beta}, \check{\gamma} \cdot x_0)$. By the forcing theorem in the model $V_0$, the filter on $P_1 \times P_0$ given by $\gamma', H_0$ must contain a condition forcing $\text{Coll}(\omega, \lambda) \models \phi(\bar{\beta}, \check{\gamma} \cdot \tau_0)$. However, $\gamma', H_0$ were arbitrary meeting the condition $(O', p')$, so it must be the case that this condition forces $\text{Coll}(\omega, \lambda) \models \phi(\bar{\beta}, \check{\gamma} \cdot \tau_0)$ as required.

As a final remark, in general it is necessary to consider virtual $E$-classes as opposed to just $E$-classes in the wording of Theorem 4.3.7. To see this, start with the trivial coherent sequence $V_n = V$ for all $n \in \omega$ and let $P_n$ be the countable support product of copies of the poset $\text{Coll}(\omega, 2^\omega)$ indexed by natural numbers $\geq n$, with the natural projections from $P_n$ to $P_m$ added. This is a diagonally complete sequence of posets, and by Theorem 4.3.7 it induces a choice-coherent sequence of models $\langle V_n[G_n] : n \in \omega \rangle$ such that the model $\bigcap_n V_n[G_n]$ contains only ground model $\omega$-sequences of ordinals. Now, every model $V_n[G_n]$ contains an enumeration of the set $(2^\omega \cap V)$ by natural numbers, and all of these enumerations are $F_2$-related. Clearly, there is no $F_2$-equivalent of them in the intersection model. Yet, there is a virtual $F_2$-class related to these enumerations in the intersection model, and even in the ground model $V$. 
Chapter 5

Balanced Suslin forcing

5.1 Virtual conditions

We look at the class of Suslin posets from an angle quite distinct from the standard treatment in [2]; in particular, the center of attention is on \(\sigma\)-closed Suslin forcings as opposed to c.c.c. or proper forcings adding reals. Recall:

**Definition 5.1.1.** A poset \(\langle P, \leq \rangle\) is *Suslin* if there is a Polish space \(X\) such that

1. \(P\) is an analytic subset of \(X\);
2. the ordering \(\leq\) is an analytic subset of \(X^2\);
3. the incompatibility relation is an analytic subset of \(X^2\).

We will need to consider various *virtual* versions of Suslin posets. They are related to the separative quotients. Recall that if \(\langle P, \leq \rangle\) is a partially ordered set, conditions \(p_0, p_1\) are called *inseparable* if for every condition \(q \leq p_0, q, p_1\) have a lower bound, and vice versa, for every condition \(q \leq p_1, q, p_0\) have a lower bound. The inseparability relation \(E_P\) is an equivalence; the partial order \(P\) is separative if this equivalence relation is the identity. The separative quotient of \(P\) is the poset of inseparability classes ordered by the relation \(\leq_s\), where \(p_1 \leq_s p_0\) if for every condition \(q \leq p_1, q, p_0\) have a lower bound. It is not difficult to see that \(\leq_s\) induces a separative ordering on the separative quotient.

**Definition 5.1.2.** Let \(\langle P, \leq \rangle\) be a Suslin forcing on a Polish space \(X\). Suppose that the set \(P\) is Borel and the relation \(\leq_s\) is analytic. The pair \(\langle P^{**}, \leq^{**} \rangle\) denotes the virtual version of the analytic quotient structure \(\langle P, E_P, \leq_s \rangle\). The elements of \(P^{**}\) are called *combinatorial virtual conditions* of \(P\).

It should be noted that under the assumption of the definition, the relation \(\leq_s\) is automatically Borel by the Suslin theorem: the relation \(q \leq_s p\) is analytic by the assumption, and its complement \(q \not\leq_s p\) is restated as there is a condition

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below \( q \) incompatible with \( p \), and this is an analytic again. Thus, Theorem 2.5.6 shows that \( P^{**} \) is a set with fewer than \( \aleph_1 \) elements. We will be interested in the classification of elements of \( P^{**} \).

A simple yet informative example is in order. Consider the poset of countable partial functions from \( 2^\omega \) to \( 2 \) ordered by reverse extension. As stated, it is not a Suslin forcing, since it is not an analytic subset of a Polish space. One has to make an innocuous adjustment: \( P \) is in fact the set of all functions from \( \omega \) to \( 2^\omega \times 2 \) whose range is a function, and order \( P \) by setting \( q \leq p \) if \( \text{rng}(p) \subseteq \text{rng}(q) \).

The price paid for this adjustment is that the equivalence relation \( E_P \) becomes unpinned. Theorem 2.3.4 now shows that the combinatorial virtual conditions of \( P \) are exactly classified by (possibly uncountable) functions from \( 2^\omega \) to \( 2 \) ordered by reverse extension. Note that total functions from \( 2^\omega \) to \( 2 \) are atoms in the ordering \( P^{**} \).

One disadvantage of the poset \( P^{**} \) is that it depends too much on the precise combinatorial description of the poset \( P \) in question as opposed to its forcing equivalence class. The following example elucidates the difficulties:

**Example 5.1.3.** Let \( P \) be the subset of \( \mathcal{P}(\omega)^\omega \) consisting of those elements \( p \) such that intersections of finite subcollections of \( \text{rng}(p) \) are infinite. The ordering on \( P \) is defined by \( p_1 \leq p_0 \) if \( \text{rng}(p_0) \subseteq \text{rng}(p_1) \). Let \( Q \) be \( [\omega]^\aleph_0 \) ordered by coordinatewise inclusion. The separative quotient of \( Q \) is naturally densely embedded to the separative quotient of \( P \). However, \( E_Q \) is the modulo finite equality of sets, therefore pinned and so \( Q = Q^{**} \). On the other hand, \( E_P \) is very complicated and \( P^{**} \) has an interesting, informative analysis performed in Section 7.1.

To remove the dependence of the virtual version on the presentation of the Suslin forcing \( P \) and harvest some other highly useful features, we have to consider names for analytic subsets of \( P \) as opposed to names for conditions.

**Definition 5.1.4.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( Q \) be a poset and \( \tau \) a \( Q \)-name for an analytic subset of \( P \). We say that the name \( \tau \) is \( P \)-pinned if \( Q \times Q \vDash \Sigma_\tau \text{left} = \Sigma_\tau \text{right} \) in the completion of the separative quotient of the poset \( P \). If \( \tau \) is \( P \)-pinned, then the pair \( \langle P, \tau \rangle \) is called a \( P \)-pin.

The class of \( P \)-pins is equipped with a natural and absolute equivalence and ordering:

**Definition 5.1.5.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) be \( P \)-pins. Define \( \langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle \) if \( Q_0 \times Q_1 \vDash \Sigma_\tau_0 \leq \Sigma_\tau_1 \) in the separative quotient of the poset \( P \), and \( \langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle \) if \( \langle Q_0, \tau_0 \rangle \leq \langle Q_1, \tau_1 \rangle \) and \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \).

For the proofs of the following propositions note that for analytic sets \( A_0, A_1 \subseteq P \), the statement \( \Sigma A_1 \leq \Sigma A_0 \) is equivalent to \( \forall p \in P \forall p_1 \in A_1 p \leq p_1 \rightarrow \exists q \leq p \exists p_0 \in A_0 q \leq p_0 \). Since the ordering \( \leq \) on \( P \) is analytic, this is a \( \Pi^1_2 \) statement.

**Proposition 5.1.6.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \langle Q_0, \tau_0 \rangle \rangle \), \( \langle Q_1, \tau_1 \rangle \rangle \) be posets and names for analytic subsets of \( P \). The following statements are absolute among all forcing extensions:
1. \( \langle Q_0, \tau_0 \rangle \) is a \( P \)-pin;
2. \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \);
3. \( \langle Q_0, \tau_0 \rangle \equiv \langle Q_1, \tau_1 \rangle \).

Proof. We will prove (2), the other items are parallel. Suppose first that \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \) holds, and \( V[H] \) is a generic extension of \( V \); we must verify that the \( \leq \)-relation transfers to \( V[H] \). To see this, suppose that \( G_0 \subset Q_0, G_1 \subset Q_1 \) are filters mutually generic over \( V[H] \), and let \( A_0 = \tau_0/G_0 \) and \( A_1 = \tau_1/G_1 \). Thus, \( A_0, A_1 \subset P \) are analytic sets with codes in \( V[G_0] \) and \( V[G_1] \), respectively. By the assumption, \( V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0 \) in the completion of the separative quotient of the poset \( P \). This \( \Pi^1_2 \) statement transfers to \( V[H][G_0, G_1] \) by the Shoenfield absoluteness. In conclusion, \( V[H][G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0 \) and therefore \( V[H] \models \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \).

For the other direction, if \( \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \) fails in \( V \) and \( V[H] \) is a generic extension of \( V \), we must show that the failure of the \( \leq \) relation transfers to \( V[H] \). Fix conditions \( q_0 \in Q_0, q_1 \in Q_1 \) such that in \( V \), \( q_0, q_1 \Vdash \tau_0, \tau_1 \notin \Sigma \tau_0 \); we will show that this property of \( q_0, q_1 \) transfers to the model \( V[H] \). This follows from the same Shoenfield absoluteness argument as the one used in the previous paragraph. \qed

**Proposition 5.1.7.** \( \equiv \) is an equivalence relation on \( P \)-pins and \( \leq \) is an ordering on \( \equiv \)-equivalence classes.

Proof. We will show that \( \leq \) is transitive, the other statements are similar. Suppose that \( \langle Q_2, \tau_2 \rangle \leq \langle Q_1, \tau_1 \rangle \leq \langle Q_0, \tau_0 \rangle \); we need to show that \( \langle Q_2, \tau_2 \rangle \leq \langle Q_0, \tau_0 \rangle \) holds. Let \( G_0 \subset Q_0, G_2 \subset Q_2 \) are mutually generic filters and \( A_2 = \tau_2/G_2 \) and \( \tau_0/G_0 \); we must show that \( V[G_0, G_2] \models \Sigma A_2 \leq \Sigma A_0 \). Let \( G_1 \subset Q_1 \) be a filter generic over \( V[G_0, G_2] \) and let \( A_1 = \tau_1/G_1 \). By the assumption, \( V[G_0, G_1] \models \Sigma A_1 \leq \Sigma A_0 \) and \( V[G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1 \). By the Shoenfield absoluteness, \( V[G_0, G_1, G_2] \models \Sigma A_2 \leq \Sigma A_1 \leq \Sigma A_0 \), in particular \( \Sigma A_2 \leq \Sigma A_0 \). By another application of the Shoenfield absoluteness, the inequality \( \Sigma A_2 \leq \Sigma A_0 \) transfers from \( V[G_0, G_1, G_2] \) to \( V[G_0, G_2] \). \qed

**Definition 5.1.8.** Let \( \langle P, \leq \rangle \) be a Suslin poset. The **virtual conditions of** \( P \) are the equivalence classes of \( \equiv \).

Clearly, the poset of combinatorial virtual conditions is naturally viewed as a subset of the poset of all virtual conditions. The disadvantage of the general virtual conditions over the combinatorial ones is that they are not naturally viewed as a virtual space of an analytic equivalence structure, and that they form a proper class with many elements of no earthly interest:

**Proposition 5.1.9.** Suppose that \( \langle P, \leq \rangle \) is a \( \sigma \)-closed Suslin poset such that below any element \( p \in P \) there are two incompatible ones. Then the equivalence \( \equiv \) has proper class many equivalence classes.
Proof. Let $h: 2^{<\omega} \to P$ be a function such that for every $t \in \omega^{<\omega}$, the conditions $h(t^\frown 0)$ and $h(t^\frown 1)$ are incompatible and stronger than $h(t)$. Let $g: \omega \to \omega^2$ be a bijection. For every ordinal $\alpha$, consider the Coll$(\omega, \alpha)$ name $\tau_\alpha$ for the set of all conditions $p \in P$ such that for every $n \in \omega$ there is (exactly one) string $t(p, n) \in 2^n$ such that $p \leq h(t)$, and the binary relation $g''\{n \in \omega: t(p, n + 1)(n) = 1\}$ is isomorphic to $\alpha$. It is not difficult to see that the pair $(\text{Coll}(\omega, \alpha), \tau_\alpha)$ is a $P$-pin; in fact the evaluation of $\tau_\alpha$ yields the same analytic set no matter what the generic filter on Coll$(\omega, \alpha)$ is. The $P$-pins obtained in this way are $\equiv$-inequivalent; in fact, for distinct ordinals $\alpha \neq \beta$, Coll$(\omega, \alpha) \times$ Coll$(\omega, \beta) \models \Sigma \tau_\alpha$ and $\Sigma \tau_\beta$ are incompatible elements of the completion of the separative quotient of $P$.  

The advantage of the general virtual conditions is that they are independent of the Suslin presentation of the poset $P$, and that the analytic complexity is provably optimal for harvesting the critical features isolated in the following Section 5.2.

5.2 Balanced conditions

This section isolates the notion of a balanced condition in a given Suslin poset and a natural equivalence of balanced conditions. It turns out that every balanced class is represented by a virtual condition.

Definition 5.2.1. Let $\langle P, \leq \rangle$ be a Suslin forcing. Let $Q$ be a poset and $\tau$ a $Q$-name for an analytic subset of $P$. We say that $\langle Q, \tau \rangle$ is a balanced pair in $P$ if for all posets $R_0, R_1$ and all $R_0 \times Q$- and $R_1 \times Q$- names $\sigma_0, \sigma_1$ for elements of $P$ such that $R_0 \times Q \models \sigma_0 \leq \Sigma \tau$ and $R_1 \times Q \models \sigma_1 \leq \Sigma \tau$, it is the case that $(R_0 \times Q) \times (R_1 \times Q) \models \sigma_0, \sigma_1$ are compatible conditions in the poset $P$.

Note that the definition of a balanced pair depends on the Suslin poset $P$. The Suslin poset is not mentioned as it will be always understood from the context and no opportunity for confusion arises. The expressions of the type $\sigma \leq \Sigma \tau$ will be shortened to $\sigma \leq \tau$ below as there can be no confusion. Balanced pairs and virtual conditions are quite different things; the main topic of interest in this book are pairs $\langle Q, \tau \rangle$ which are simultaneously balanced and virtual conditions. The following proposition restates the notion of a balanced pair in terms of the generic extension as opposed to the forcing relation.

Proposition 5.2.2. Let $\langle P, \leq \rangle$ be a Suslin forcing. Let $Q$ be a poset and $\tau$ a $Q$-name for an analytic subset of $P$. The following are equivalent:

1. $\langle Q, \tau \rangle$ is a balanced pair;

2. whenever $V[H_0]$ and $V[H_1]$ are mutually generic extensions, $G_0, G_1 \subset Q$ are filters generic over the ground model in the respective extensions and $p_0 \leq \tau/G_0, p_1 \leq \tau/G_1$ are conditions in $P$ in the respective extensions $V[H_0], V[H_1]$, then $p_0, p_1$ are compatible in $P$. 


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The class of balanced conditions is greatly simplified by introducing the following equivalence relation on it:

**Definition 5.2.3.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. If \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are balanced pairs, we say that \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \) if for all posets \( R_0, R_1 \) and all \( R_0 \times Q_0 \)-names \( \sigma_0, \sigma_1 \) for elements of \( P \) such that \( R_0 \times Q_0 \models \sigma_0 \leq \tau \) and \( R_1 \times Q_1 \models \sigma_1 \leq \tau \), it is the case that \( (R_0 \times Q_0) \times (R_1 \times Q_1) \models \sigma_0, \sigma_1 \) are compatible conditions in the poset \( P \).

**Proposition 5.2.4.** The relation \( \equiv \) is an equivalence on balanced pairs. Moreover, if \( \langle Q, \tau \rangle \) is a balanced pair, then

1. if \( Q \models \sigma \leq \tau \) then \( \langle Q, \sigma \rangle \) is a balanced pair \( \equiv_b \)-related to \( \langle Q, \tau \rangle \);
2. if \( Q \) is a regular subposet of \( R \) then \( \langle R, \tau \rangle \) is a balanced pair \( \equiv_b \)-related to \( \langle Q, \tau \rangle \).

**Proof.** We argue for the first sentence; items (1) and (2) are immediate consequences of the definition. The relation \( \equiv_b \) is clearly symmetric and contains the identity; we will show that it is transitive. Let \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \equiv_b \langle Q_2, \tau_2 \rangle \); it must be shown that \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_2, \tau_2 \rangle \) follows. To this end, let \( R_0, R_2 \) be posets and \( \sigma_0, \sigma_2 \) be \( R_0 \times Q_0 \) and \( R_2 \times Q_2 \)-names for conditions in \( P \) stronger than \( \tau_0, \tau_1 \) respectively. By the assumption about \( Q_0, Q_1 \), there is a \( (R_0 \times Q_0) \times (R_1 \times Q_1) \)-name \( \sigma_1 \) for an element of \( P \) which is a lower bound of \( \sigma_0 \) and \( \tau_1 \). By the assumption about \( Q_1, Q_2 \), there is a \( (R_0 \times Q_0) \times (R_1 \times Q_1) \times (R_2 \times Q_2) \)-name \( \eta \) which is a lower bound of \( \sigma_1 \) and \( \sigma_2 \). Thus, in the long product extension \( (R_0 \times Q_0) \times (R_1 \times Q_1) \times (R_2 \times Q_2) \), the conditions \( \sigma_0, \sigma_2 \) are compatible, and by Mostowski absoluteness this statement transfers to the \( (R_0 \times Q_0) \times (R_2 \times Q_2) \)-extension. This completes the proof of the transitivity of \( \equiv \).

**Proposition 5.2.5.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle \) be balanced pairs. The following are equivalent:

1. \( \langle Q_0, \tau_0 \rangle \equiv_b \langle Q_1, \tau_1 \rangle \);
2. whenever \( V[H_0] \) and \( V[H_1] \) are mutually generic extensions, \( G_0 \subset Q_0, G_1 \subset Q_1 \) are filters generic over the ground model in the respective extensions and \( p_0 \leq \tau_0/G_0, p_1 \leq \tau_1/G_1 \) are conditions in \( P \) in the respective extensions \( V[H_0], V[H_1] \), then \( p_0, p_1 \) are compatible in \( P \).

It is now time for a simple and informative example. Let \( P \) be the poset of countable functions from \( 2^\omega \) to \( 2 \), ordered by reverse inclusion. On one hand, if \( f: 2^\omega \to 2 \) is a total function, then \( \langle \text{Coll}(\omega, 2^\omega), f \rangle \) is a balanced pair. On the other hand, if \( \langle Q, \tau \rangle \) is a balanced pair, then for every point \( x \in 2^\omega \) in the ground model, it must be the case that \( Q \) forces \( \langle x \rangle \in \text{dom}(\tau) \) and in fact \( Q \) has to decide the value of \( \tau(x) \) as well—otherwise it would be easy to violate the balance of the pair. Let \( f: 2^\omega \to 2 \) be the total function given by \( \forall x \in 2^\omega Q \models \tau(x) = f(x) \) and note that \( \langle Q, \tau \rangle \equiv \langle \text{Coll}(\omega, 2^\omega), f \rangle \). Thus, the balanced classes for \( P \) are exactly classified by total functions from \( 2^\omega \) to \( 2 \).
One of the main concerns of this book is the classification of balanced classes for various Suslin posets \( (P, \leq) \). The following theorem is the basic contribution in this direction. To enable its statement, note that similarly to Section 5.1 we can extend the definition of balanced pairs to names for analytic subsets of \( P \). By Proposition 5.2.4, such an extension does not add new balanced classes.

**Theorem 5.2.6.** Let \( (P, \leq) \) be a Suslin forcing. Every \( \equiv_b \)-class contains a virtual condition. The condition is unique up to the virtual condition equivalence.

**Proof.** Let \( \langle Q, \tau \rangle \) be a balanced pair; by Proposition 5.2.4 we may assume that \( \tau \) is a name for a single element of \( P \). Consider the poset \( Q' = \text{Coll}(\omega, \mathcal{P}(Q)) \) and the name \( \tau' \) for the analytic set \( \{ p \in P : \text{for some filter } G \subset Q \text{ meeting all open dense sets enumerated by the } Q' \text{-generic, } p = \tau/G \} \). We will first show that the pair \( \langle Q', \tau' \rangle \) is balanced via Proposition 5.2.2. To this end, suppose that \( V[H_0], V[H_1] \) are mutually generic extensions, \( G'_0, G'_1 \subset Q' \) are filters in the respective extensions generic over \( V \), and \( r_0 \leq \tau'/G'_0 \) and \( r_1 \leq \tau'/G'_1 \) are conditions in the poset \( P \); we must show that \( r_0, r_1 \in \Sigma \) are compatible conditions. By the definition of the name \( \tau' \), in \( V[H_0] \) it is possible to find a filter \( G_0 \subset Q \) generic over \( V \) such that \( p_0 = \tau/G_0 \) is compatible with \( r_0 \), and similarly in the model \( V[G_1] \). Let \( s_0, s_1 \in P \) be lower bounds of \( r_0 \) and \( p_0 \) in \( V[G_0] \) and \( r_1 \) and \( p_1 \) in \( V[G_1] \) respectively. Since \( \langle Q, \tau \rangle \) was a balanced condition in \( P \), the conditions \( s_0, s_1 \) must be compatible in \( P \), and their lower bound is a lower bound of \( r_0 \) and \( r_1 \) as well.

To show that \( \langle Q, \tau \rangle \equiv_b \langle Q', \tau' \rangle \), use Proposition 5.2.4. To show that the pair \( \langle Q', \tau' \rangle \) is a virtual condition, suppose that \( G'_0, G'_1 \subset Q' \) are mutually generic filters. Looking at the model \( V[G_0, G_1] \), it in fact turns out that \( \tau'/G'_0 \) and \( \tau'/G'_1 \) are in fact equal as analytic subsets of \( P \).

Finally, suppose that \( \langle Q'', \tau'' \rangle \) is another virtual condition which is an element of the \( \equiv_b \)-class of the balanced class of \( \langle Q, \tau \rangle \); we must show that \( Q' \times Q'' \models \Sigma \tau' = \Sigma \tau'' \). For the \( \leq \) direction, suppose that \( G' \subset Q' \) and \( G'' \subset Q'' \) are mutually generic filters and \( \tau \in P \) is a condition in \( V[G', G''] \) which is below \( \Sigma \tau'/G' \); we must show that it is compatible with some element of \( \tau''/G'' \). Let \( H \subset Q'' \) be a filter generic over the model \( V[G', G''] \). Since the pairs \( \langle Q', \tau' \rangle \) and \( \langle Q'', \tau'' \rangle \) come from the same balanced class, it must be the case that the condition \( \tau \) must be compatible with every element of \( \tau''/H \); let \( s \) be such a lower bound. Since the pair \( \langle Q'', \tau'' \rangle \) is a virtual condition, the condition \( s \leq \Sigma \tau''/H \) must be compatible with some condition of \( \tau''/G'' \). By the Mostowski absoluteness between the models \( V[G', G''] \) and \( V[G', G'', H] \), \( \tau \) is compatible with some element of \( \tau''/G'' \) in the model \( V[G', G''] \) as required. \( \square \)

**Question 5.2.7.** Let \( (P, \leq) \) be a Suslin forcing. Is it necessarily the case that the equivalence relation \( \equiv_b \) has only set many classes? Is it necessarily the case that every balance class has a representative on a poset of size \( \aleph_1 \)?

It is now time to state the central definition of this book.

**Definition 5.2.8.** Let \( P \) be a Suslin poset. \( P \) is balanced if for every condition \( p \in P \) there is a balanced virtual condition below \( p \).
A definition of this sort immediately begs a question: which Suslin posets are balanced? We should immediately douse the flames of entirely misguided hopes:

**Proposition 5.2.9.** The following Suslin posets do not have any balanced virtual conditions and therefore are not balanced:

1. nonatomic c.c.c. posets;
2. nonatomic tree posets;
3. the posets $P(\omega)/I$ where $I$ is a countably separated Borel ideal on $\omega$

Here, a tree poset on a Polish space $X$ is an analytic family of closed subsets of $X$ closed under nonempty intersections with closures of basic open sets, ordered by inclusion. An ideal $I$ on $\omega$ is countably separated if there is a countable collection $A$ of subsets of $\omega$ such that for every $b \in I$ and $c \notin I$ there is $a \in A$ such that $b \cap a = 0$ and $c \cap a \notin I$. Countably separated ideals include for example the ideal of nowhere dense subsets of the rationals; a characterization theorem is proved in [24]. Note that the instrumental property of the set $A$ is coanalytic, and by Mostowski absoluteness it will hold in all forcing extensions.

**Proof.** For (1), suppose towards contradiction that $\langle P, \leq \rangle$ is a Suslin c.c.c. poset and $\langle Q, \tau \rangle$ is a balanced pair. For every maximal antichain $A \subset P$, its maximality is a coanalytic statement and therefore absolute to the $Q$-extension. Thus, $Q \models \tau$ is compatible with some element of $A$. The balance of $\tau$ immediately shows that there can be only one element of $A$ with which $\tau$ is compatible, and the largest condition in $Q$ identifies this element. It follows that the set $\{p \in P: Q \models \tau \leq \check{p} \in$ the separative quotient of $P\}$ is a filter on $P$ which meets all maximal antichains, an impossibility in nonatomic posets.

For (2), suppose towards contradiction that $\langle P, \leq \rangle$ is a tree poset on a Polish space $X$ and $\langle Q, \tau \rangle$ is a balanced pair. For every basic open set $O \subset X$, the statement $\tau \cap \bar{O} \neq 0$ must be decided by the largest condition in $Q$ by the balance of $\tau$. This decision cannot be positive for two disjoint basic open subsets of $X$ by the balance of $\tau$ again. Thus, $\tau$ would have to be forced by $Q$ to be a singleton (even a specific singleton in the ground model), an impossibility in nonatomic tree posets.

For (3), suppose towards contradiction that $I$ is a countably separated ideal on $\omega$ as witnessed by a countable separating set $A \subset P(\omega)$ and $\langle Q, \tau \rangle$ is a balanced pair in the poset $P(\omega)/I$; so $\tau$ is a name for an $I$-positive subset of $\omega$. By the balance of $\tau$, for every set $a \in A$ it must be decided by the largest condition in $Q$ whether $\tau \cap \check{a} \in I$ or not. Let $B = \{a \in A: Q \models \tau \cap \check{a} \notin I\}$. Use the balance of the name $\tau$ to argue that any intersection of finitely many elements of $B$ is an infinite set. Let $b \subset \omega$ be an infinite set such that for every $a \in B$, $b \setminus a$ is finite. Use the density of the ideal $I$ to argue that thinning out the set $b$ if necessary, we may assume that $b \in I$. It is now immediate that no set $a \in A$ can separate the $I$-small set $b$ from the $I$-positive set $\tau$ in the
Q-extension: if $a \in A \setminus B$ then $\tau \cap a \in I$ is forced, and if $a \in B$ then $b \cap a \neq 0$. This is a contradiction.

Most balanced Suslin posets used in this book are $\sigma$-closed. There are $\sigma$-closed posets which are not balanced, such as the one which adds a maximal almost disjoint family in $\mathcal{P}(\omega)$ by countable approximations, cf. Theorem 11.1.1. There are some posets which are balanced and in ZFC even collapse $\aleph_1$, cf. Example 6.5.4. Even such posets are valuable; remember that they prove their worth in the choiceless symmetric Solovay extension. The balanced status of certain posets is nonabsolute, see Example 6.6.10. Thus, even though typically the balanced conditions correspond to traditional objects of combinatorial set theory, the balanced status is a complicated matter. There is only one general preservation theorem, which is nevertheless extremely useful for obtaining consistency results:

**Theorem 5.2.10.** Let $P = \prod_n P_n$ be a countable support product of Suslin forcing notions. Balanced conditions in $P$ are exactly classified by sequences $\langle p_n : n \in \omega \rangle$ where for every $n \in \omega$, $p_n$ is a balanced condition in $P_n$.

**Proof.** On one hand, if $\langle Q_n, \tau_n \rangle$ are balanced pairs for each $n \in \omega$, the name for the sequence $\langle \tau_n : n \in \omega \rangle$ in the product $Q = \prod_n Q_n$ is balanced for the poset $P$ essentially by the definitions. The choice of the support in the product $Q$ is immaterial. On the other hand, if $Q$ is a poset and $\tau = \langle \tau_n : n \in \omega \rangle$ is a balanced $Q$-name for the poset $P$, it must be the case that each of the names $\tau_n$ for $n \in \omega$ is balanced for the poset $P_n$. It equally easy to see that equivalent balanced names for $P$ give equivalent balanced names on each coordinate, and sequences of balanced names on the posets $P_n$ which are coordinatewise equivalent yield equivalent balanced names for the product forcing.

**Corollary 5.2.11.** The countable support product of balanced Suslin forcings is balanced.

One issue that is constantly present in this book is the lack of absoluteness of the notions surrounding balance. As long as Question 5.2.7 remains open, it will also be necessary to relativize the definition of a balanced poset to an inaccessible cardinal.

**Definition 5.2.12.** Let $\langle P, \leq \rangle$ be a Suslin poset and $\kappa$ an inaccessible cardinal.

1. $P$ is balanced below $\kappa$ if $V_\kappa$ satisfies that $P$ is balanced in every forcing extension;

2. $P$ is balanced cofinally below $\kappa$ if $V_\kappa$ satisfies that any ordinal can be collapsed by a poset which makes $P$ balanced.

**Definition 5.2.13.** Let $\phi$ be a sentence of the language of set theory, with possible real parameters.
5.2. BALANCED CONDITIONS

1. The phrase “In every balanced extension of a symmetric Solovay model, \( \phi \) holds” means that “For every Suslin forcing \( P \) and every inaccessible cardinal \( \kappa \), if \( V_\kappa \models P \) is balanced in every forcing extension, then in the symmetric Solovay model derived from \( \kappa \), \( P \models \phi \) holds”.

2. The phrase “In every cofinally balanced extension of a symmetric Solovay model, \( \phi \) holds” means that “For every Suslin forcing \( P \) and every inaccessible cardinal \( \kappa \), if \( V_\kappa \models \) the cardinality of every ordinal can be collapsed to \( \aleph_0 \) in a forcing extension in which \( P \) balanced, then in the symmetric Solovay model derived from \( \kappa \), \( P \models \phi \) holds”.

One may think that with suitable large cardinal hypothesis on \( \kappa \), one could use reflection to show that relativization is unnecessary. The best result we have in this direction is

**Proposition 5.2.14.** Let \( \langle P, \leq \rangle \) be a Suslin forcing. Let \( \kappa \) be a strong cardinal. The following are equivalent:

1. \( P \) is balanced;
2. \( V_\kappa \models P \) is balanced.

**Proof.** The argument uses two simple absoluteness claims of independent interest:

**Claim 5.2.15.** If \( M \) is a transitive model of ZFC, \( \langle Q, \tau \rangle \) is a pair in \( M \) and the pair is balanced in \( V \), then it is balanced in \( M \).

**Proof.** Immediate by Mostowski absoluteness between generic extensions of \( M \) and \( V \).

**Claim 5.2.16.** If \( M \) is a transitive model of ZFC, \( \langle Q, \tau \rangle \) is a pair in \( M \), \( M \models \langle Q, \tau \rangle \) is balanced, and \( P(\mathcal{Q}) \subset M \), then the pair \( \langle Q, \tau \rangle \) is balanced in \( V \).

**Proof.** Work in \( V \). Suppose that the conclusion fails, as witnessed by posets \( R_0, R_1 \) and names \( \sigma_0, \sigma_1 \) on \( R_0 \times Q \) and \( R_1 \times Q \) respectively. Take an elementary submodel \( N \) of a large enough structure such that \( Q \subset N \) and \( |N| = |Q| \). The posets \( R_0 \cap N, R_1 \cap N \) and names \( \sigma_0 \cap N \) and \( \sigma_1 \cap N \) still witness the failure of \( P \) to be balanced of the pair \( \langle Q, \tau \rangle \): if \( G_0 \subset (R_0 \cap N) \times Q \) and \( G_1 \subset (R_1 \cap N) \times Q \) are mutually generic filters, \( p_0 = \sigma_0/G_0 \) and \( p_1 = \sigma_1/G_1 \in P \), then \( N[G_0, G_1] \models p_0, p_1 \in P \) are incompatible conditions by the forcing theorem applied in the model \( N \), and by the Mostowski absoluteness between \( N[G_0, G_1] \) and \( V[G_0, G_1] \), this is still true in \( V[G_0, G_1] \). Now, the assumption \( P(\mathcal{Q}) \subset M \) shows that the posets \( R_0 \cap N, R_1 \cap N \) and names \( \sigma_0 \cap N \) and \( \sigma_1 \cap N \) have isomorphic copies in the model \( M \), obtaining the failure of balance of the pair \( \langle Q, \tau \rangle \) in \( M \).

Now, Claim 5.2.16 applied to \( M = V_\kappa \) immediately yields the implication \( (2) \rightarrow (1) \). For the converse, suppose that \( P \) is balanced, \( p \in P \), and find a balanced pair \( \langle Q, \tau \rangle \) below \( p \). We have to deal with the unseemly possibility that
\[|Q| > \kappa \text{ holds. Use the large cardinal hypothesis to find an elementary embedding } j: V \rightarrow M \text{ with critical point } \kappa \text{ such that } \langle Q, \tau \rangle \in M \cap V_{j(\kappa)}. \text{ Claim 5.2.15 shows that } M \models \langle Q, \tau \rangle \text{ is a balanced pair below } p. \text{ By elementarity of the embedding } j, \text{ there must be a balanced pair } \langle Q', \tau' \rangle \in V_\kappa \text{ below } p. \text{ Applying Claim 5.2.15 again with } M = V_\kappa, \text{ we see that } V_\kappa \models \langle Q', \tau' \rangle \text{ is a balanced pair below } p. \text{ Since the condition } p \in P \text{ was arbitrary, (2) follows.} \]

5.3 Variations and generalizations

There is an interesting generalization of balanced Suslin forcing which can realize additional effects in extensions of the symmetric Solovay model. The basic definitions can be stated as a minor variation of the work done in the previous sections.

**Definition 5.3.1.** Let \( P \) be a Suslin forcing. A pair \( \langle Q, \tau \rangle \) is **weakly balanced** if \( Q \vdash \tau \subset P \) is an analytic set and whenever \( R_0, R_1 \) are posets, \( \sigma_0, \sigma_1 \) are \( R_0 \times Q \) and \( R_1 \times Q \)-names for elements of \( P \) below \( \Sigma \tau \) and \( \langle r_0, q_0 \rangle \in R_0 \times Q \) and \( \langle r_1, q_1 \rangle \in R_1 \times Q \) are conditions then in some forcing extension there are filters \( H_0 \subset R_0 \times Q \) and \( H_1 \subset R_1 \times Q \) which are separately generic over \( V, \langle r_0, q_0 \rangle \in H_0, \langle r_1, q_1 \rangle \in H_1, \text{ and } \sigma_0/H_0, \sigma_1/H_1 \text{ are compatible elements of } P. \)

**Definition 5.3.2.** Let \( P \) be a Suslin forcing and \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are weakly balanced pairs. Say that the pairs are **equivalent** and write \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \) if whenever \( R_0, R_1 \) are posets, \( \sigma_0, \sigma_1 \) are \( R_0 \times Q_0 \) and \( R_1 \times Q_1 \)-names for elements of \( P \) below \( \tau_0 \) and \( \tau_1 \) respectively and \( \langle r_0, q_0 \rangle \in R_0 \times Q_0 \) and \( \langle r_1, q_1 \rangle \in R_1 \times Q_1 \) are conditions then in some forcing extension there are filters \( H_0 \subset R_0 \times Q_0 \) and \( H_1 \subset R_1 \times Q_1 \) which are separately generic over \( V, \langle r_0, q_0 \rangle \in H_0, \langle r_1, q_1 \rangle \in H_1, \text{ and } \sigma_0/H_0, \sigma_1/H_1 \text{ are compatible elements of } P. \)

**Proposition 5.3.3.** The relation \( \equiv_{wb} \) is an equivalence relation on weakly balanced pairs.

**Proof.** The relation is clearly symmetric and contains the identity as a subset. For the transitivity, let \( \langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle \text{ and } \langle Q_2, \tau_2 \rangle \) be weakly balanced pairs such that \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \text{ and } \langle Q_1, \tau_1 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \). To show that \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \) holds, let \( R_0, R_2 \) be posets and \( \sigma_0, \sigma_2 \text{ be } R_0 \times Q_0 \text{- and } R_2 \times Q_2 \)-names for elements of \( P \) which are forced to be smaller than some element of \( \tau_0 \) and \( \tau_2 \) respectively, and let \( \langle r_0, q_0 \rangle \text{ and } \langle r_2, q_2 \rangle \) be conditions in the two product posets. Let \( \kappa \) be a cardinal bigger than \( |P(R_0 \times Q_0)| \). Since \( \langle Q_0, \tau_0 \rangle \equiv_{wb} \langle Q_1, \tau_1 \rangle \) holds, an absoluteness argument shows that in the \( \text{Coll}(\omega, \kappa) \times Q_0 \)-extension, there is a filter \( H_0 \subset R_0 \times Q_0 \) generic over the ground model containing \( \langle r_0, q_0 \rangle \) such that \( \sigma_0/H_0 \) and \( \tau_1 \) as conditions in \( P \) have a lower bound; call the \( \text{Coll}(\omega, \kappa) \times Q_0 \)-name for the bound \( \sigma_1 \). Since \( \langle Q_1, \tau_1 \rangle \equiv_{wb} \langle Q_2, \tau_2 \rangle \) holds, in some forcing extension there are filters \( H_1 \subset \text{Coll}(\omega, \kappa) \times Q_1 \text{ and } H_2 \subset R_2 \times Q_2 \) separately generic over the ground model, such that \( \langle r_2, q_2 \rangle \in H_2 \text{ and } \sigma_1/H_1, \sigma_2/H_2 \text{ are compatible elements of } P. \)
Let $H_0 = \hat{H}_0/H_1 \subset R_0 \times Q_0$; this is a filter generic over the ground model, containing the condition $(r_0, q_0)$ such that $\sigma_0/H_0 \leq \sigma_1/H_1$.

In total, the conditions $\sigma_0/H_0, \sigma_2/H_2 \in P$ are compatible, and the relation $(Q_0, \tau_0) \equiv_{wb} (Q_2, \tau_2)$ has been verified. \hfill \Box

The notion of a weakly balanced pair is a faithful extension of the notion of a balanced pair. This is the contents of the following proposition.

**Proposition 5.3.4.** Let $P$ be a Suslin forcing.

1. Every balanced pair is weakly balanced;
2. the class of balanced pairs is invariant under the $\equiv_{wb}$ equivalence;
3. the relations $\equiv_b$ and $\equiv_{wb}$ coincide on balanced pairs.

**Proof.** (1) is immediate from the definitions. For (2), suppose towards contradiction that $(Q_0, \tau_0)$ and $(Q_1, \tau_1)$ are $\equiv_{wb}$-equivalent pairs and $(Q_0, \tau_0)$ is balanced, while $(Q_1, \tau_1)$ is not. The latter statement is witnessed by some posets $R_0, R_1$, names $\sigma_0, \sigma_1$, and conditions $(r_0, q_0) \in R_0 \times Q_1$ and $(r_1, q_1) \in R_1 \times Q_1$ which force in the product $(R_0 \times Q_0) \times (R_1 \times Q_1)$ that $\sigma_0, \sigma_1$ are incompatible elements of $P$.

Use the $\equiv_{wb}$-equivalence assumption to find posets $S_0, S_1$ and $S_0, S_1$-names $\hat{G}_0, \hat{H}_0, K_0$ and $G_1, H_1, K_1$ respectively so that $S_0 \Vdash \hat{G}_0 \subset Q_0$ and $H_0 \times K_0 \subset R_0 \times Q_1$ are filters generic over $V$ such that $\hat{r}_0 \in H_0$, $\hat{q}_0 \in K_0$, and $\tau_0/\hat{G}_0, \sigma_0/\hat{H}_0 \times K_0$ are conditions compatible in $P$, with a lower bound $\chi_0 \in P$. Similar objects exist on the $S_1$-side.

Now, let $L_0 \subset S_0, L_1 \subset S_1$ be filters mutually generic over $V$. The balance of the pair $(Q_0, \tau_0)$ shows that $\chi_0/L_0, \chi_1/L_1$ are compatible conditions in the poset $P$. Consider the filters $H_0 = \hat{H}_0/L_0 \subset R_0$, $K_0 = K_0/L_0 \subset Q_1$, and $H_1 = \hat{H}_1/L_1 \subset R_0$, $K_1 = K_1/L_1 \subset Q_1$. The filters $H_0 \times K_0 \subset R_0 \times Q_1$ in $V[L_0]$ and $H_1 \times K_1 \subset R_1 \times Q_1$ in $V[L_1]$ are generic over $V$. Since the filters $L_0, L_1$ are mutually generic, so are $H_0 \times K_0$ and $H_1 \times K_1$ by Corollary 13.3.3. At the same time, the conditions $\sigma_0/H_0 \times K_0$ and $\sigma_1/H_1 \times K_1$ in $P$ must be compatible, because they are weaker than the compatible conditions $\chi_0/L_0$ and $\chi_1/L_1$ respectively. This is a contradiction with the initial choice of $\sigma_0, \sigma_1$.

(3) is proved in a similar way; the argument is left to the patient reader. \hfill \Box

**Proposition 5.3.5.** Let $P$ be a Suslin forcing. Every $\equiv_{wb}$-class contains a virtual condition, which is unique up to $\equiv$-equivalence.

**Proof.** Let $(Q, \tau)$ be a weakly balanced condition; strengthening $\tau$ if necessary, we may assume that $\tau$ is in fact a name for an element of $P$. Let $\kappa$ be an ordinal such that $P(Q) \subset V_\kappa$ and let $\sigma$ be a Coll$(\omega, V_\kappa)$-name for the set $\{ p \in P : \exists G \subset Q G \text{ is a filter meeting all the dense subsets of } Q \in V_\kappa^V \text{ such that } p = \tau/G \}$. It is clear that the pair $(\text{Coll}(\omega, V_\kappa), \sigma)$ is a $P$-pin. We will show that the pair $(\text{Coll}(\omega, V_\kappa), \sigma)$ is $\equiv_{wb}$-related to $(Q, \tau)$. 

Suppose that \( R_0, R_1 \) are posets and \( \sigma_0, \sigma_1 \) are \( R_0 \times \text{Coll}(\omega, V_\kappa) \)- and \( R_1 \times Q \)-names for elements of \( P \) stronger than \( \sigma \) and \( \tau \) respectively, and \( \langle r_0, q_0 \rangle, \langle r_1, q_1 \rangle \) are conditions in the respective products. 

For the uniqueness part, suppose that \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are \( P \)-pins which are both \( \equiv_{wb} \)-related to \( \langle Q, \tau \rangle \); we must show that they are \( \equiv \)-related. Suppose towards contradiction that this fails. Then, in the \( Q_1 \times Q_0 \)-extension, there must be an element \( p \in P \) which is below some element of \( \tau_0 \) and incompatible with any element of \( \tau_1 \) (or vice versa). By a Mostowski absoluteness argument, this element \( p \) will maintain its incompatibility property in every further forcing extension. Let \( \sigma_0 \) be a \( Q_1 \times Q_0 \)-name for this element, let \( \sigma_1 \) be a \( Q_1 \)-name for any element of \( \tau_1 \), and observe that \( \langle Q_1 \times Q_0, \sigma_0 \rangle \) and \( \langle Q_1, \sigma_1 \rangle \) witness that the the pairs \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) are \( \equiv_{wb} \)-unrelated, contradicting the initial assumptions.

Unlike the balanced conditions, the weakly balanced virtual conditions can be actually recognized in the ordering of virtual conditions by a natural first order property.

**Proposition 5.3.6.** Let \( P \) be a Suslin forcing. Let \( \langle Q, \tau \rangle \) be a \( P \)-pin. The following are equivalent:

1. \( \langle Q, \tau \rangle \) is weakly balanced;
2. \( \langle Q, \tau \rangle \) is an atom in the ordering of virtual conditions.

Recall that an element \( \bar{p} \) of a partial order is an atom if every element compatible with \( \bar{p} \) is in fact above \( \bar{p} \).

**Proof.** For \( (1) \to (2) \) direction, let \( \langle R, \sigma \rangle \) be a \( P \)-pin which is not above \( \langle Q, \tau \rangle \); we must show that it is incompatible with \( \langle Q, \tau \rangle \). Suppose towards contradiction that it is compatible, with a lower bound \( \langle S, \chi \rangle \). In the \( R \times Q \) extension, the inequality \( \Sigma_\tau \leq \Sigma_\sigma \) must fail, and so there is a condition \( \bar{p}_0 \) for an element of \( P \) which is below \( \Sigma_\tau \) but incompatible with \( \Sigma_\sigma \). In the \( S \times R \times Q \)-extension, there is a condition \( \bar{p}_1 \) of \( P \) which is below \( \Sigma_\chi \), and therefore also below \( \Sigma_\sigma \) and \( \Sigma_\tau \). Use the weak balance of the pair \( \langle Q, \tau \rangle \) to find, in some generic extension, filters \( H_0 \times G_0 \subset R \times Q \) and \( K_1 \times H_1 \times G_1 \subset S \times R \times Q \) separately generic over \( V \) such that the conditions \( \bar{p}_0 / H_0 \times G_0 \) and \( \bar{p}_1 / K_1 \times H_1 \times G_1 \) are compatible in \( P \), with a lower bound \( p \). Note that the sums \( \Sigma_\sigma / H_0 \) and \( \Sigma_\sigma / H_1 \) in the completion of the poset \( P \) must coincide, as \( \langle R, \sigma \rangle \) is a \( P \)-pin. However, the condition \( p \) should be incompatible with the former and below the latter by the forcing theorem. This is a contradiction.

For the \( (2) \to (1) \) direction, suppose that \( \langle Q, \tau \rangle \) is a \( P \)-pin which is an atom in the ordering of virtual conditions. To prove the weak balance, suppose that \( R_0, R_1 \) are posets, \( \sigma_0, \sigma_1 \) are \( R_0 \times Q \) and \( R_1 \times Q \)-names for elements of \( P \) stronger than \( \tau \), and let \( \langle r_0, q_0 \rangle \) and \( \langle r_1, q_1 \rangle \) be conditions in the products. To find the instrumental generic filters, let \( \kappa \) be an ordinal such that \( \mathcal{P}(R_0 \times Q) \) and \( \mathcal{P}(R_1 \times Q) \) are both subsets of \( V_\kappa \), and consider the Coll(\( \omega, V_\kappa \))-names \( \chi_0, \chi_1 \) for analytic subsets of \( P \) defined by \( \chi_0 = \{ p \in P : \exists G \subset R_0 \times Q \text{ such that } G \) is...
a filter meeting all open dense subsets of $R_0 \times Q$ in $V_\kappa^V$ such that $\langle r_0, q_0 \rangle \in G$ and $p = \sigma/G$. The name $\chi_1$ is defined in the same way.

It is not difficult to see that $\langle \text{Coll}(\omega, V_\kappa), \chi_0 \rangle$ and $\langle \text{Coll}(\omega, V_\kappa), \chi_1 \rangle$ are both $P$-pins. They are also both $\leq \langle Q, \tau \rangle$ by their definition. By the assumption on the $P$-pin $\langle Q, \tau \rangle$, they must both be $\equiv$-equivalent to $\langle Q, \tau \rangle$. This means that in the $\text{Coll}(\omega, V_\kappa)$-extension, there must be conditions $p_0 \in \chi_0$ and $p_1 \in \chi_1$ which are compatible in $P$. Reviewing the definition of the names $\chi_0$ and $\chi_1$, we get the filters $G_0 \subset H_0 \times K_0$ and $G_1 \subset H_1 \times K_1$ separately generic over $V$ such that $\langle r_0, q_0 \rangle \in G_0$, $\langle r_1, q_1 \rangle \in G_1$, and $\sigma_0/G_0$ and $\sigma_1/G_1$ are compatible conditions in $P$ as desired.
Chapter 6

Simplicial complex forcings

Many examples of $\sigma$-closed Suslin partially ordered sets in this book are presented in the same way:

Definition 6.0.1. Let $X$ be a set.

1. A set $\mathcal{K}$ of finite subsets of $X$ is a simplicial complex if it is closed under subset;

2. a set $A \subset X$ is a $\mathcal{K}$-set if $[A]^{<\aleph_0} \subset \mathcal{K}$. It is maximal if it is not a proper subset of another $\mathcal{K}$-set;

3. the poset $P_\mathcal{K} \subset X^\omega$ consists of countable $\mathcal{K}$-sets ordered by reverse inclusion;

4. $\dot{A}_{\text{gen}}$ is the $P_\mathcal{K}$-name for the union of all sets in the generic filter.

When the simplicial context $\mathcal{K}$ is understood from the context, we put $P = P_\mathcal{K}$ in this section. The poset $P$ is obviously $\sigma$-closed. By an elementary density argument, the set $\dot{A}_{\text{gen}}$ is forced to be a maximal $\mathcal{K}$-set. The poset $P$ can be naturally presented as a Suslin forcing by replacing the countable $\mathcal{K}$-sets with their enumerations by natural numbers, and replacing the reverse inclusion ordering by reverse inclusion of ranges of the enumerations. We will neglect this innocuous step in this section as it merely complicates the notation.

Nearly every poset considered in this book can be presented as a poset of the form $P_\mathcal{K}$ for a Borel simplicial complex $\mathcal{K}$ on a Polish space $X$. Namely, for a poset $Q$ let $\mathcal{K}$ be the simplicial complex of the finite subsets of $Q$ which have a common lower bound. Under suitable assumptions on definability and existence of lower bounds (which are invariably satisfied), the posets $Q$ and $P_\mathcal{K}$ are naturally forcing equivalent. However, this point of view rarely brings any new insight. In this chapter, we deal with simplicial complexes that are in some way algebraically natural, and their algebraic structure leads to the classification of the balanced conditions. We discovered a number of possibilities.
6.1 Flag complexes

In this section, we present the case of flag or clique complexes, consisting of cliques in some underlying graph.

Definition 6.1.1. A Borel simplicial complex $K$ on a Polish space $X$ is a countable Borel flag complex if there is a Borel function $f: X \to [X]^{\omega_0}$ such that $x \notin f(x)$ for every $x \in X$ and a finite set $a \subseteq X$ is in $K$ just in case for every pair $x_0, x_1 \in a$ of distinct points, $x_0 \notin f(x_1)$ and $x_1 \notin f(x_0)$.

The balanced virtual conditions for the poset $P$ are classified by $K$-sets of a certain type.

Definition 6.1.2. Let $K$ be a countable Borel flag complex on a Polish space $X$. A set $A \subseteq X$ is balanced if $\text{Coll}(\omega, X) \Vdash \exists P \in P \text{ such that } A \cap V = \bar{A}$. It is of course useful to find a description of balanced subsets of $X$ which does not involve the forcing relation. In the case of flag complexes, this is immediate:

Proposition 6.1.3. Let $K$ be a countable Borel flag complex on a Polish space $X$. Let $A \subseteq X$ be a set. The following are equivalent:

1. $A$ is balanced;

2. $A$ is a $K$-set, and for every $x \in X \setminus A$, either $A \cup \{x\}$ is not a $K$-set, or for every countable set $B \subseteq X$ there is $y \notin B$ such that $(A \cap B) \cup \{y\}$ is a $K$-set while $\{x, y\} \notin K$.

In particular, every maximal $K$-set is balanced. In the common situation where $y \in f(x)$ is equivalent to $x \in f(y)$, every balanced set must be a maximal $K$-set.

Proof. Fix a Borel function $f: X \to [X]^{\omega_0}$ witnessing that $K$ is a countable flag complex. For (1)$\implies$(2) implication, let $A \subseteq X$ be balanced. Consider a generic extension $V[G]$ making $X \cap V$ countable and in it, a condition $p \in P$ such that $p \Vdash A \cap V = \bar{A}$. Using the $\sigma$-closedness of the poset $P$ in the extension $V[G]$, one can strengthen $p$ if necessary so that $A \subseteq p$ and for every $x \in X \cap V \setminus A$ there is $y \in p$ such that either $x \in f(y)$ or $y \in f(x)$. It follows that $A$ is a $K$-set. Now, if $x \in X \cap V \setminus A$ then pick a point $y \in p$ such that either $x \in f(y)$ or $y \in f(x)$. If $y \in V$, then $y \in A$ and the first disjunct in (2) holds for $x$. If $y \notin V$, then $y \in f(x)$ is impossible and so $x \in f(y)$ holds. But then, the point $y$ witnesses the second disjunct in (2) for any countable set $B \subseteq V$ in the extension $V[G]$, and by the Mostowski absoluteness between $V$ and $V[G]$, the second disjunct is satisfied for $x$ already in $V$.

For (2)$\implies$(1) implication, let $A \subseteq X$ be a set satisfying (2). Let $C \subseteq X$ be the set of all points $x \in X \setminus A$ satisfying the second disjunct of (2).

Claim 6.1.4. For every $x \in C$, in some forcing extension, there is a point $y \in X \setminus V$ such that $x \in f(y)$ and $A \cup \{y\}$ is a $K$-set.
6.1. FLAG COMPLEXES

Proof. Let $Q$ be the poset of all stationary subsets of $[X]^{\aleph_0}$, let $G \subseteq Q$ be a generic filter, and let $j: V \rightarrow M$ be its associated generic ultrapower. The model $M$ may be ill-founded, but it is well-founded up to $\omega_1^V$ and the set $X \cap V$ belongs to $M$ and is countable there, being represented by the identity function. By elementarity, in the model $M$ there is a point $y$ such that $y \notin X \cap V$, $A \cup \{y\}$ is a $\mathcal{K}$-set and $\{x, y\} \notin \mathcal{K}$. The latter condition means that either $x \in f(y)$ or $y \in f(x)$. The latter disjunct is impossible as $y \notin V$, and so $x \in f(y)$ holds as required. 

Now, for each point $x \in C$ a point as in the claim must exist in every forcing extension in which the set $X \cap V$ is countable by the Mostowski absoluteness. To verify (1), note that the poset Coll($\omega, X$) is isomorphic to the finite support product of countably many copies of itself. Thus, if $G \subseteq \text{Coll}(\omega, X)$ is a generic filter, in $V[G]$ it is possible to find a collection $G_n \subseteq \text{Coll}(\omega, X)$ for $n \in \omega$ pairwise mutually generic over the ground model. Still working in $V[G]$, find a list $C = \{x_n: n \in \omega\}$ and for each $n \in \omega$ find a point $y_n \in X$ such that $y_n \notin V$, $A \cup \{y_n\}$ is a $\mathcal{K}$-set, and $\{x_n, y_n\} \notin \mathcal{K}$. Let $p = A \cup \{y_n: n \in \omega\}$; it will be enough to show that $p$ is a $\mathcal{K}$-set.

To verify this, it is only necessary to check that for $n \neq m$, $y_n \notin f(y_m)$. However, this follows from the fact that $f(y_m) \subseteq V[G_m]$ and $y_n \notin V[G_m]$ by the product forcing theorem. 

We want to show that the balanced conditions in $P$ are classified by balanced subsets of $X$. This is arranged through the following definition:

Definition 6.1.5. Let $\mathcal{K}$ be a countable Borel flag complex on a Polish space $X$. Let $A \subseteq X$ be a balanced set. Define $\tau_A$ to be the Coll($\omega, X$)-name for the set of all conditions $p \in P_X$ such that for every $x \in V$, if $x \in A$ then $x \in p$, and if $x \notin A$ then $p \cup \{x\}$ is not a $\mathcal{K}$-set.

Note that in the Coll($\omega, X$)-extension, $\Sigma \tau_A$ is simply the Boolean sum of all conditions in the poset $P$ which force $A_{\text{gen}} \cap V = A$. We can now make the promised classification conclusion:

Theorem 6.1.6. Let $\mathcal{K}$ be a countable Borel flag complex on a Polish space $X$.

1. If $A \subseteq X$ is a balanced set then $\langle \text{Coll}(\omega, X), \tau_A \rangle$ is a balanced virtual condition;

2. if $\langle Q, \sigma \rangle$ is a balanced pair then there is a balanced set $A \subseteq X$ such that $\langle \text{Coll}(\omega, X), \tau_A \rangle$ is equivalent to $\langle Q, \sigma \rangle$;

3. if $A \neq B$ are balanced sets then $\langle \text{Coll}(\omega, X), \tau_A \rangle$ and $\langle \text{Coll}(\omega, X), \tau_B \rangle$ are inequivalent balanced virtual conditions.

Proof. Fix a Borel function $f: X \rightarrow [X]^{\aleph_0}$ witnessing the fact that $\mathcal{K}$ is a countable Borel flag complex. The proof uses the following simple claim:
Claim 6.1.7. Let \( A \subseteq X \) be a set. Let \( \langle Q_0, \tau_0 \rangle \) and \( \langle Q_1, \tau_1 \rangle \) be posets and their names for elements of \( P \) such that \( Q_0 \models \tau_0 \models_P \tilde{A}_{\text{gen}} \cap V = \tilde{A} \) and \( Q_1 \models \tau_1 \models_P \tilde{A}_{\text{gen}} \cap V = \tilde{A} \). Then \( Q_0 \times Q_1 \models \tau_0, \tau_1 \) are compatible in \( P \).

Proof. Let \( G_0 \subseteq Q_0, G_1 \subseteq Q_1 \) be mutually generic filters. Write \( r_0 = \sigma_0 \setminus G_0 \) and \( r_1 = \sigma_1 / G_1 \); we must show that \( r_0 \cup r_1 \) is a \( \mathcal{K} \)-set. To this end, let \( x_0 \in r_0 \) and \( x_1 \in r_1 \) be points and argue that \( x_0 \notin f(x_1) \) and \( x_1 \notin f(x_0) \).

This is clear if one of the points \( x_0, x_1 \), say \( x_0 \), is in \( V \). In such a case, necessarily \( x_0 \in A \). In the model \( V[G_1], r_1 \models \tilde{x}_0 \in \tilde{A}_{\text{gen}} \) and so \( x_0 \notin f(x_1) \) and \( x_1 \notin f(x_0) \) must hold.

If, on the other hand, neither of the points \( x_0, x_1 \) belongs to \( V \), the product forcing theorem implies that \( x_0 \in V[G_0] \setminus V[G_1] \) and \( x_1 \in V[G_1] \setminus V[G_0] \). Since the values of the function \( f \) are countable sets, we have \( f(x_0) \subseteq V[G_0] \neq x_1 \) and \( f(x_1) \subseteq V[G_1] \neq x_0 \) as required again.

To see (1), it is immediate that \( \langle \text{Coll}(\omega, X), \tau_A \rangle \) is a \( P \)-pin. The balance of this \( P \)-pin is an immediate corollary of Claim 6.1.7. To see (2), suppose that \( \langle Q, \sigma \rangle \) is a balanced pair. The balance shows that for every point \( x \in X \), either \( Q \models \sigma \models_P \tilde{x} \in \tilde{A}_{\text{gen}} \) or \( Q \models \sigma \models_P \tilde{x} \notin \tilde{A}_{\text{gen}} \) holds. Let \( A \subseteq X \) be the set of those points \( x \in X \) for which the former alternative holds. By the Mostowski absoluteness between the \( Q \)-extension and the \( \text{Coll}(\omega, X) \)-extension, \( A \) is a balanced set. To conclude the proof of (2), observe that the balanced pair \( \langle Q, \sigma \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X), \tau_A \rangle \) by Claim 6.1.7 again.

Finally, (3) is immediate.

Example 6.1.8. Let \( P \) be the poset of all countable functions from \( 2^{\omega} \) to \( 2 \), ordered by inclusion. The poset \( P \) can be presented using a flag complex on the set \( X = 2^{\omega} \times 2 \), with the function \( f \) defined by \( f(\langle x, 0 \rangle) = \{ \langle x, 1 \rangle \} \) and vice versa. The balanced virtual conditions are classified by total functions from \( 2^{\omega} \) to \( 2 \).

Example 6.1.9. Let \( g : 2^{\omega} \to (2^\omega)^\omega \) be any Borel function in which every point in \( (2^\omega)^\omega \) has uncountable preimage. Let \( f : 2^{\omega} \to [2^{\omega}]^{\aleph_0} \) be defined by \( f(x) = \text{rng}(g(x)) \setminus \{ x \} \) and consider the associated flag complex and a poset \( P \). The poset \( P \) adds a maximal \( f \)-free set. Any \( f \)-independent set \( A \subseteq X \) is balanced, since in any extension making \( 2^{\omega} \cap V \) countable, there is a point \( y \notin V \) such that \( f(y) = 2^{\omega} \cap V \setminus A \).

Example 6.1.10. Let \( E \) be a countable Borel equivalence relation on a Polish space \( X \). Let \( f(x) = [x]_E \setminus \{ x \} \). Consider the associated flag complex and a poset \( P \). The poset \( P \) adds an \( E \)-transversal. The balanced virtual conditions are classified by \( E \)-transversals.

### 6.2 Modular complexes

In this section, we investigate simplicial complexes with a structure familiar from geometric model theory:
6.2. MODULAR COMPLEXES

Definition 6.2.1. A Borel simplicial complex $K$ on a Polish space $X$ is modular if there is a Borel function $f : K \to [Y]^\aleph_0$ for some Polish space $Y$ such that

1. (monotonicity) $f(0) = 0$ and $a \subset b$ implies $f(a) \subset f(b)$;
2. (modularity) for all $a, b \in K$, $a \cup b \in K$ if and only if $f(a) \cap f(b) = f(a \cap b)$.

For a $K$ set $A \subset X$ we set $f(A) = \bigcup \{ f(a) : a \in [A]^{<\aleph_0} \}$.

Note that the properties of the function $f$ are coanalytic, and therefore transfer to any transitive model of set theory by the Mostowski absoluteness, in particular to all generic extensions. It is not easy to see which Borel simplicial complexes are modular and which are not. We will introduce two important classes of modular complexes and then a couple of other examples.

Definition 6.2.2. Let $K$ be a Borel simplicial complex on a Polish space $X$. The complex is

1. locally countable if there is a Borel function $g : K \to [X]^{\aleph_0}$ such that for all sets $a, b \in K$, $a \cup b \notin K$ if and only if $a \cup (b \cap g(a)) \notin K$.
2. locally finite if the function $g$ as in (1) takes only finite values.

Note that the definitory properties of the function $f$ are coanalytic and therefore absolute among all transitive models containing the codes for $f$ and $K$ by the Mostowski absoluteness.

Example 6.2.3. Let $\Gamma$ be a locally finite graph on $X$ and let $K$ be the simplicial complex of $\Gamma$-independent sets. Then $K$ is locally finite, as witnessed by the function $g(a) = \{ x \in X : \exists y \in a : (x, y) \in \Gamma \}$.

Example 6.2.4. If there is a countable Borel equivalence relation $E$ on $X$ such that for every finite set $a \subset X$, $a \in K$ if and only if $a \cap c \in K$ for all $E$-classes $c \subset X$, then $K$ is locally countable—just let $g(a)$ be the saturation $[a]_E$.

Example 6.2.5. Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $K$ be the simplicial complex on $X$ consisting of sets of pairwise $E$-unrelated elements. As per the previous example, the complex $K$ is modular.

Example 6.2.6. The simplicial complex $K$ of finite subsets of $X = \mathbb{R}$ linearly independent over $\mathbb{Q}$ is not locally countable. To see this, suppose that $g : K \to [X]^\aleph_0$ is a Borel function. Pick a nonzero $x \in X$. Use a counting argument to find a point $y \in X$ which is not a rational multiple of $x$, such that $\{ y, x - y \} \cap g(\{ x \}) = 0$. The set $\{ x \} \cup \{ y, x - y \}$ does not belong to $K$, proving that the $g$ does not witness the local countability of $K$.

The examples above suggest a certain characterization of locally countable complexes, which can be supported by a theorem:

Theorem 6.2.7. Suppose that $K$ is a Borel simplicial complex on a Polish space $X$. Exactly one of the following occurs:
1. \( \mathcal{K} \) is locally countable;

2. there is a set \( a \in \mathcal{K} \) and a perfect collection \( B \subset \mathcal{K} \) consisting of pairwise disjoint sets such that \( b \in B \rightarrow a \cup b \notin \mathcal{K} \).

**Proof.** It is clear that (2) implies the negation of (1), since the incompatibility of the sets in \( B \) with \( a \) cannot be inherited by their intersections with any fixed countable set. Now, suppose that (2) fails and work to confirm (1).

The failure of (2) together with the work on puncture sets [4, Theorem 21] shows that there is a Borel function \( h: \mathcal{K} \rightarrow [X]^{\omega_0} \) such that for all \( a, b \in \mathcal{K} \) with \( a \cup b \notin \mathcal{K} \), \( b \cap h(a) \neq 0 \) holds. Now, define functions \( g_n: \mathcal{K} \rightarrow [X]^{\omega_0} \) by recursion as follows: \( g_0(a) = a \cup h(a) \) and \( g_{n+1}(a) = g_n(a) \cup \bigcup \{ h(c): c \in \mathcal{K}, c \subset g_n(a) \} \). Let \( g(a) = \bigcup_n g_n(a) \) and argue that the function \( g \) works as desired.

Indeed, suppose that \( a, b \in \mathcal{K} \) are such that \( a \cup b \notin \mathcal{K} \), and towards contradiction suppose that \( c = a \cup (b \cap g(a)) \in \mathcal{K} \). There is a number \( n \in \omega \) such that \( c \subset g_n(a) \) holds. Since \( c \cup (b \setminus c) \notin \mathcal{K} \) holds, it must be the case that \( (b \setminus c) \cap h(c) \neq 0 \), which means that \( (b \setminus c) \cap g_{n+1}(a) \neq 0 \). This contradicts the choice of \( n \).

\[ \square \]

Now it is time for the theorem connecting local countability and modularity.

**Theorem 6.2.8.** Every locally countable Borel complex is modular.

**Proof.** Let \( \mathcal{K} \) be a locally countable Borel complex on a Polish space \( X \), as witnessed by a Borel function \( g: \mathcal{K} \rightarrow [X]^{\omega_0} \). Let \( f \) be the Borel function on \( \mathcal{K} \) defined by \( f(a) = \{ \{ c, d \in [\mathcal{K}]^2: c \cup d \notin \mathcal{K}, c \subset a \text{ or } d \subset a \text{ and } c \subset g(d) \text{ and } d \subset g(c) \} \). We will show that \( f \) is a function witnessing the modularity of the complex \( \mathcal{K} \).

First of all, it is clear that the function \( f \) is monotone. To confirm the modularity, let \( a, b \in \mathcal{K} \) be arbitrary sets. Suppose first that \( a \cup b \notin \mathcal{K} \). By recursion on \( n \in \omega \) define sets \( a_n, b_n \) as \( a = a_0, b = b_0 \) and \( a_{n+1} = a_n \cap g(b_{n+1}) \) and \( b_{n+1} = b_n \cap g(a_{n+1}) \). By induction argue that \( a_{n+1} \subseteq a_n, b_{n+1} \subseteq b_n, \) and \( a_n \cup b_n \notin \mathcal{K} \). It follows that the two sequences have to stabilize at some \( n \in \omega \). Write \( c = a_n \) and \( d = b_n \). It is clear that the pair \( \{ c, d \} \) belongs to both \( f(a) \) and \( f(b) \). It also must be the case that \( \{ c, d \} \notin f(a \cap b) \): if this were not the case, then either \( c \subset a \cap b \) or \( d \subset a \cap b \), and accordingly either \( b \notin \mathcal{K} \) or \( a \notin \mathcal{K} \), a contradiction.

Suppose now that \( a \cup b \in \mathcal{K} \) and \( \{ c, d \} \in f(a) \cap f(b) \). It cannot be the case that \( c \subset a \) and \( d \subset b \) (or vice versa), because this would violate the assumption that \( a \cup b \in \mathcal{K} \). Thus, one of \( c, d \) must be a subset of both \( a \) and \( b \), confirming that \( \{ c, d \} \in f(a \cap b) \) as required.

\[ \square \]

An important class of examples of modular simplicial complexes come from modular pre-geometries. In order to formulate the general pattern, we include the definition of a pre-geometry.

**Definition 6.2.9.** Let \( X \) be a set and \( f: [X]^{<\omega_0} \rightarrow [X]^{\omega_0} \) be a function. The pair \( \langle X, f \rangle \) is a pre-geometry if
1. \( a \subset f(a) \);

2. (idempotence) \( b \subset f(a) \) implies \( f(b) \subset f(a) \) for all finite sets \( a \subset b \subset X \);

3. (exchange principle) if \( x \in f(a \cup \{y\}) \setminus f(a) \) then \( y \in f(a \cup \{x\}) \).

A finite set \( a \subset X \) is free if for every \( x \in a \), \( x \notin f(a \setminus \{x\}) \). The pre-geometry \( \langle X, f \rangle \) is modular if for all finite sets \( a, b \subset X \) and every point \( x \in f(a \cup b) \) there are points \( x_a \in f(a) \) and \( x_b \in f(b) \) such that \( x \in f(\{x_a, x_b\}) \).

Standard examples of pre-geometries include the following. If \( E \) is an equivalence relation on \( X \) with all classes countable and \( f(a) = [a]_E \), then \( \langle X, f \rangle \) is a pre-geometry. It is modular and trivial: if \( x \in f(a) \) then there is a singleton \( b \subset a \) such that \( x \in f(b) \). If \( X \) is a vector space over a countable field \( F \) and \( f(a) = \)the linear span of \( a \), then \( \langle X, f \rangle \) is a modular pre-geometry. If \( X \) is a field over a countable field \( F \) and \( f(a) = \)the algebraic closure of \( a \) then \( \langle X, f \rangle \) is a pre-geometry which is typically not modular.

**Theorem 6.2.10.** Let \( \langle X, f \rangle \) be a Polish space with a Borel pre-geometry. Let \( K \) be the Borel simplicial complex on \( X \) consisting of finite free sets. Then \( K \) is modular as witnessed by the function \( f \upharpoonright K \) assigning to every finite set its algebraic closure.

**Proof.** To verify the modularity of the function \( f \), suppose that \( a, b \in K \) are finite sets. Suppose first that \( f(a) \cap f(b) \neq f(a \cap b) \) and work to show that \( a \cup b \) is not free. Let \( x \in X \) be a point in \( f(a) \cap f(b) \) which is not in \( f(a \cap b) \). Let \( a' \subset a \) be inclusion minimal superset of \( a \cap b \) such that \( x \in f(a') \). Note that \( a' \not\subset a \cap b \), choose a point \( y \in a' \setminus (a \cap b) \), and let \( a'' = a' \setminus \{y\} \). By the exchange property of the pre-geometry, it follows that \( y \in f(a'' \cup \{x\}) \). By the monotonicity and idempotence of the pre-geometry, \( y \in f(a'' \cup b) \) and since \( y \in a \cup b \) and \( y \notin a'' \cup b \), this means that \( a \cup b \) is not free. This direction does not use the modularity assumption on the pre-geometry.

Now suppose that \( a \cup b \) is not free and work to show that \( f(a) \cap f(b) \neq f(a \cap b) \). Let \( c \) be an inclusion minimal subset of \( a \cup b \) such that \( c \cap b \subset c \), \( c \) is free, and for some \( x \in a \cup b \), \( c \cup \{x\} \) is not free: pick the witness \( x \). First argue that \( x \in f(c) \).

To this end, let \( d \subset c \cup \{x\} \) be minimal such that there is \( y \in (c \cup \{x\}) \setminus d \) which is in \( f(d) \). If \( x \notin d \) then \( y \) must be equal to \( x \) by the freeness of the set \( c \); and if \( x \in d \), then by the exchange property \( x \in f(d \cup \{y\} \setminus \{x\}) \) and so \( x \in f(c) \) again.

Now, assume for definiteness that \( x \in b \). Let \( a' = c \setminus b \) and \( b' = c \cap b \). Since \( x \in f(a' \cup b') \), the modularity assumption on the pre-geometry yields points \( x_0 \in f(a') \) and \( x_1 \in f(b') \) such that \( x \in f(x_0, x_1) \). Note that the set \( \{x_0, x_1\} \) is free: if for example \( x_1 \in f(\{x_0\}) \), then by idempotence \( x_1 \in f(a') \), by the exchange property there would have to be some element \( z \in a' \) such that \( z \in f(\{x_1\} \cup (a' \setminus \{z\})) \), and by idempotence again \( z \in f(c \setminus \{z\}) \), contradicting the freeness of the set \( c \). By the freeness of the set \( b \), \( x \notin f(\{x_1\}) \). By the exchange property applied again, \( x_0 \in f(\{x_1, x\}) \setminus f(b) \). It follows that \( x_0 \notin f(a) \cap f(b) \). At the same time, \( x_0 \notin f(a \cap b) \), since then (as \( a \cap b \subset c \) holds) \( x_0 \in f(b') \) and so \( x \in f(b') \), violating the freeness assumption on \( b \). \( \square \)
**Example 6.2.11.** Let $X$ be a Polish vector space over a countable field $F$, and let $\mathcal{K}$ be the simplicial complex of linearly independent sets. Then $\mathcal{K}$ is modular. The poset $P_\mathcal{K}$ is designed to add a basis for the space $X$.

One important point of the present section is that there are interesting modular complexes which do not arise from modular geometries, as the following graph theoretic examples shows:

**Example 6.2.12.** Let $G$ be a Borel graph on a Polish space $X$. Let $\mathcal{K}$ be the simplicial complex on $G$ consisting of finite acyclic subsets of $G$. Then $\mathcal{K}$ is modular. The poset $P_\mathcal{K}$ is designed a maximal acyclic subgraph to $G$ by countable approximations.

**Proof.** First, the description of the modular function $f$. If $a \subset G$ is a finite acyclic set of edges and $E_a$ is its path connectivity equivalence relation, then the functional value $f(a)$ is the union $f_0(a) \cup f_1(a)$. Here, $f_0(a)$ consists of all finite sequences $\langle x_i : i \in 2n+1 \rangle$ for $n > 0$ of vertices in $X$ mentioned in the edges in $a$, which are pairwise distinct except for $x_{2n} = x_0$ and $\forall j \in n \ x_{2j} E_a x_{2j+1}$. The set $f_1(a)$ consists of all finite sequences $\langle x_i : i \in 2n+1 \rangle$ for $n > 0$ of vertices in $X$ mentioned in the edges in $a$, which are pairwise distinct except for $x_{2n} = x_0$ and $\forall j \in n \ x_{2j+1} E_a x_{2j+2}$. We will show that $f$ is modular. The function is clearly monotone with respect to inclusion. To verify the modularity demand, suppose that $a, b$ are acyclic sets.

**Case 1.** $a \cup b$ is acyclic; we must check that $f(a) \cap f(b) = f(a \cap b)$. Well, in the first place $E_a \cap E_b = E_{a \cap b}$: if vertices $x_0, x_1$ are connected by injective paths $p_a, p_b$ in $a$ and in $b$ but by no path in $a \cap b$, then $p_a \cup p_b$ forms a cycle containing at least two edges which are not in $a \cap b$, and so the set $\{ e \in G : e$ belongs to exactly one of $p_a, p_b \}$ forms a nonempty injective cycle in $a \cup b$, contradicting the case assumption. Now, suppose that $\vec{x} = \langle x_i : i \in 2n+1 \rangle$ is a sequence in both $f(a)$ and $f(b)$. If $\vec{x} \in f_0(a) \cap f_0(b)$, then $\vec{x} \in f_0(a \cap b)$ since $E_a \cap E_b = E_{a \cap b}$. If $\vec{x} \in f_1(a) \cap f_1(b)$ then $\vec{x} \in f_1(a \cap b)$ for the same reason. The remaining case, when $\vec{x} \in f_0(a)$ and $\vec{x} \in f_1(b)$ (or vice versa) leads to a cycle in $a \cup b$ which is impossible by the case assumption.

**Case 2.** $a \cup b$ contains a cycle; we must verify that $f(a) \cap f(b) \neq f(a \cap b)$. Let $c$ be an injective cycle in $a \cap b$. There must be a sequence $\vec{x} = \langle x_i : i \in 2n+1 \rangle$ of vertices on that cycle such that $n > 0$, $x_{2n} = x_0$, and the edges in $c$ between $x_{2i}$ and $x_{2i+1}$ are all in $a$ and the edges on $c$ between $x_{2i+1}$ and $x_{2i+2}$ are all in $b$. The sequence $\vec{x}$ belongs to $f(a) \cap f(b)$ since $\vec{x} \in f_0(a)$ and $\vec{x} \in f_1(b)$. The proof in this case will be complete once we prove that $\vec{x} \notin f(a \cap b)$. An indeed, if $\vec{x}$ belonged to $f(a \cap b)$, then it would yield a cycle in $a$ or $b$ depending on whether $\vec{x} \in f_0(a \cap b)$ or $\vec{x} \in f_1(a \cap b)$.

**Example 6.2.13.** Let $G$ be a Borel graph on a Polish space $X$. Let $\mathcal{K}$ be the simplicial complex on $G$ consisting of finite sets of edges in which no two distinct edges share a vertex. Then $\mathcal{K}$ is modular. The poset $P_\mathcal{K}$ is designed to add a maximal (perhaps perfect) matching to $G$. To exhibit the modular function, let $f(a) = \{ x \in X :$ there is an edge $e \in a$ such that $x$ is one of the two vertices of $e \}$.
Finally, we need to present a couple of examples of simplicial complexes which are not modular. To rule out the existence of a function with the modular property, we use the following simple criterion reminiscent of Theorem 6.2.7:

(*) there are pairwise disjoint sets \( \{a, b_\alpha : \alpha \in \omega_1\} \) in \( \mathcal{K} \) such that for any \( \alpha, \beta \in \omega_1 \) \( b_\alpha \cup b_\beta \in \mathcal{K} \) holds and for every \( \alpha \in \omega_1 \), \( a \cup b_\alpha \notin \mathcal{K} \) holds.

It is clear that if (*) holds, then no function \( f \) can have the modularity property: the values of \( f(b_\alpha) \) for \( \alpha \in \omega_1 \) would have to be pairwise disjoint in view of the first demand in (*), and then there would have to be an ordinal \( \alpha \in \omega_1 \) such that \( f(a) \cap f(b_\alpha) = 0 \), which violates the modularity property in view of the second demand in (*).

**Example 6.2.14.** Let \( X \) be an uncountable Polish field, and let \( \mathcal{K} \) be the simplicial complex on \( X \) consisting of sets algebraically free over some countable subfield \( F \subset X \). The simplicial complex \( \mathcal{K} \) satisfies (*) and therefore is not modular. To see this, let \( B \subset X \) be an uncountable algebraically free set, let \( a \subset B \) be an arbitrary set of size 2, \( a = \{s, t\} \), and for each \( x \in B \setminus a \) let \( b_x = \{x, s, t\} \). We will show that \( \{a, b_x : x \in X \setminus a\} \) exemplify (*). Clearly, for each \( x \in B \setminus a \), \( a \cup b_x \) is not free. Also, for distinct elements \( x, y \in B \setminus a \), the set \( b_x \cup b_y \) is free. To see that, it is enough to show that the field \( Q(b_x, b_y) \) contains \( s, t, x, y \) and therefore has transcendence degree 4. To recover \( s, t, x, y \) from \( b_x \cup b_y \), write \( w = x - y \) and \( z = xy^{-1} \); both clearly belong to \( Q(b_x, b_y) \). Then the recovery follows from the string of equalities \( y = w(z - 1)^{-1}, x = zy, t = (xt)x^{-1} \) and \( s = (x + s) - x \). (This elegant argument was pointed out to us by Peter Sin.)

**Example 6.2.15.** Let \( X \) be an uncountable Polish space and let \( \mathcal{K} \) be the simplicial complex on \( [X]^2 \) consisting of those finite sets \( a \subset [X]^2 \) such that there is no set \( b \in [X]^3 \) such that \( |b|^2 \subset a \). The associated poset \( P_\mathcal{K} \) adds a maximal triangle-free graph on \( X \). The simplicial complex \( \mathcal{K} \) satisfies (*) and so is not modular. To see this, let \( x_0, x_1 \in X \) be any distinct points and let \( a = \{\{x_0, x_1\}\} \). For each \( z \in X \) distinct from \( x_0, x_1 \) let \( b_z = \{\{x_0, z\}, \{x_1, z\}\} \). It is easy to check that \( \{a, b_z : z \in X\} \) exemplify (*).

**Example 6.2.16.** Let \( X = \mathcal{P}(\omega) \) and let \( \mathcal{K} \) be the simplicial complex of all finite sets \( a \subset X \) such that \( \bigcap a \) is infinite. The associated poset \( P_\mathcal{K} \) adds a Ramsey ultrafilter. The simplicial complex \( \mathcal{K} \) satisfies (*) and so is not modular. To see this, let \( F \) be any filter on \( \omega \) which is not generated by countably many sets. Let \( a \subset \omega \) be an infinite set such that the complement of \( a \) is in \( F \), and let \( b_\alpha : \alpha \in \omega_1 \) be distinct elements of \( F \) disjoint from \( a \). It is not difficult to verify that the sets \( \{a\}, \{b_\alpha : \alpha \in \omega_1\} \) exemplify (*).

Since the criterion (*) rules out an arbitrary modular function as opposed to a Borel modular function, and in general the modular functions seem to be very difficult to construct by transfinite recursion procedures, the following question suggests itself:
Question 6.2.17. Is there a Borel simplicial complex which possesses a modular function, but no Borel modular function?

The balanced pairs in the poset $P = P_K$ for modular complexes $K$ are again classified by certain $K$-function, but no Borel modular function?

Definition 6.2.18. Let $K$ be a modular simplicial complex on a Polish space $X$. A $K$-set $A \subseteq X$ is modular-balanced if for every Borel collection $B \subseteq K$ at least one of the following occurs:

1. there are $a_0 \neq a_1$ in $B$ such that $a_0 \cup a_1 \in K$;
2. there is $b \in B$ such that $b \subseteq A$;
3. there is a countable set $p \subseteq A$ such that for no $b \in B$, $p \cup b$ is a $K$-set.

Theorem 6.2.19. Suppose that $K$ is a modular Borel simplicial complex on a Polish space $X$.

1. Whenever $A \subseteq X$ is a modular-balanced set then the pair $\langle \text{Coll}(\omega, X), A \rangle$ is balanced;
2. every balanced pair is equivalent to $\langle \text{Coll}(\omega, X), A \rangle$ for some modular-balanced set $A \subseteq X$;
3. distinct modular-balanced sets give rise to inequivalent balanced pairs.

Proof. Let $f: K \to [Y]^\aleph_0$ be a Borel modular function. For (1), first argue that a modular-balanced set $A \subseteq X$ must be a maximal $K$-set. For this, given $x \in X$ consider the set $B = \{\{x\}\}$ and consult the three options of Definition 6.2.19. The first option is impossible, the second option yields $x \in A$, and the third option shows gives that $A \cup \{x\}$ is not a $K$-set. The maximality of the set $A$ follows.

Now, suppose that $V[H_0]$ and $V[H_1]$ are mutually generic extensions of the ground model, and $a_0 \in V[H_0]$ and $a_1 \in V[H_1]$ are elements of $K$ such that $A \cup a_0, A \cup a_1$ are $K$-sets; we must show that $a_0 \cup a_1 \in K$. Suppose towards contradiction that this fails. Using the maximality of the set $A$, enlarge the sets $a_0, a_1$ so that $a_0 \cap a_1 = a_0 \cap V = a_1 \cap V$. The modularity of the function $f$ implies that the set $f(a_0) \cap f(a_1) \cap f(a_0 \cap a_1)$ is nonempty, containing some element $y \in Y$. Since $y \in V[H_0] \cap V[H_1]$, the product forcing theorem implies that $y \in V$.

Back in $V$, consider the set $B \subseteq K$, $B = \{b \subseteq K: y \in f(b) \text{ and for no proper subset } c \subseteq b, y \not\in f(c)\}$. Apply Definition 6.2.18 to $A, B$. The first option is impossible by the modularity of the function $f$. The second option gives a set $b \subseteq K$ such that $b \subseteq A$ and $y \in f(b)$. Increasing the set $b$ if necessary, we may arrange that $a_0 \cap V \subseteq b$. But then, $y \in f(a_0) \cap f(b)$ while $y \not\in f(a_0 \cap b) = f(a_0 \cap V) = f(a_0 \cap a_1)$. The modularity of the function $f$ shows that $b \cup a_0 \not\in K$ and so $A \cup a_0$ is not a $K$-set, contradicting the initial assumptions. In the third option of the trichotomy, there is (in $V$) a countable set $p \subseteq A$ such
that no element of $B$ can be added to it. Let $b \subset a_0$ be an inclusion-minimal subset of $a_0$ such that $y \in f(b)$. Then $b \in B$ and $p \cup b$ is a $\mathcal{K}$-set, contradicting the choice of $p$. (1) has been proved.

For (2), assume that $\langle Q, \tau \rangle$ is a balanced pair; strengthening $\tau$ if necessary, we may assume that it is in fact a name for an element of $P_\mathcal{K}$. By the balance, for each $x \in X$ it has to be the case that $Q \models \tau \models \dot{x} \in \dot{A}_{\text{gen}}$ or $Q \models \tau \models \dot{y} \notin \dot{A}_{\text{gen}}$; let $A \subset X$ be the set of all points $x \in X$ for which the former alternative occurs. We will show that $\langle Q, \tau \rangle$ is equivalent to $\langle \text{Coll}(\omega, X), A \rangle$. Since in the separative quotient of $P$, $\tau \leq \mathcal{K}$ is forced, in view of (1) it is enough to argue that the set $A \subset X$ is modular-balanced. We start with a small claim. Suppose towards contradiction that $A \subset X$ is not modular-balanced, as witnessed by some Borel set $B \subset \mathcal{K}$.

Claim 6.2.20. There is a poset $R$ and an $R$-name $\dot{b}$ such that $R \models \dot{b} \in \mathcal{K}$ and $A \cup \dot{b}$ is a $\mathcal{K}$-set.

Proof. Let $R$ be the poset of all stationary subsets of $[X]^{\aleph_0}$. Now, if $G \subset R$ is a generic filter and $j: V \rightarrow M$ is the associated generic ultrapower, the set $X \cap V$ belongs to $M$ and it is countable in $M$, as it is represented by the identity function. Thus, $A = j(A) \cap (X \cap M)$ is a countable subset of $j(A)$ in the model $M$, and by the failure of the third option in Definition 6.2.18 in $V$ and elementarity of $j$, $M \models \exists b \in B$ and $A \cup b$ is a $\mathcal{K}$-set. Since $M$ is an $\omega$-model of set theory, these two statements about $b$ transfer without change to the generic extension $V[G]$.

The treatment now divides into two cases.

Case 1. For some condition $q \leq Q$ and a $Q$-name $\dot{b}$ for an element of $\dot{B}$, $q \models \tau \cup \dot{b}$ is not a $\mathcal{K}$-set. In this case, note that since the second option in Definition 6.2.18 fails for $A, B$, it must be the case that $\dot{b}$ is forced not to be a subset of the ground model. Let $H_0, H_1 \subset Q$ be mutually generic filters meeting the condition $q$. The sets $\dot{b}/H_0, \dot{b}/H_1 \in B$ must be distinct by the product forcing theorem. Since the first option of Definition 6.2.18 fails, it must be that $\dot{b}/H_0 \cup \dot{b}/H_1 \notin \mathcal{K}$. Thus, $\tau/H_0 \cup \dot{b}/H_0$ and $\tau/H_1 \cup \dot{b}/H_1$ are incompatible conditions in the poset $P$ in the respective models $V[H_0]$ and $V[H_1]$ contradicting the balance of the pair $\langle Q, \tau \rangle$.

Case 2. $Q \models \forall b \in B \tau \cup b$ is not a $\mathcal{K}$-set. Let $R$ and $\dot{b}$ be a poset and a name as in the claim. Note that $Q \times R \models \tau \cup \dot{b}$ is not a $\mathcal{K}$-set by the Mostowski absoluteness between the $Q$- and $Q \times R$-extension. By passing to a condition in $Q$ and $R$ if necessary, find a $Q$-name $\dot{c}$ and a finite set $d \subset A$ such that $Q \models \dot{c} \subset \tau, \dot{d} = \dot{c} \cap V$, and $Q \times R \models \dot{c} \cup \dot{b} \notin \mathcal{K}$. Note that $R \models \dot{d} \cup \dot{b} \in \mathcal{K}$. By the modularity of the function $f$, $Q \times R \models f(\dot{d} \cup \dot{b}) \cap f(\dot{c} \setminus V) \neq 0$ holds, and by the product forcing theorem, all elements in the intersection must be in the ground model. In particular, there must be a point $y \in Y$ and a condition $q \in Q$ forcing $y \in f(\dot{c} \setminus V)$. Let $H_0, H_1 \subset Q$ be filters mutually generic over the ground model, containing the condition $q \in Q$. Then $y \in f(\dot{c}/H_0 \setminus V) \cap f(\dot{c}/H_1 \setminus V)$ while $y \notin f(\dot{c}/H_0 \cap H_1 \setminus V) = f(0) = 0$. The modularity of the function $f$ shows that
$\mathcal{P}\mathcal{H}_0 \cup \mathcal{C}\mathcal{H}_0 \setminus \mathcal{V} \notin \mathcal{K}$, in particular $\tau/\mathcal{H}_0, \tau/\mathcal{H}_1$ are incompatible conditions in the poset $P$. This contradicts the balance of the pair $\langle Q, \tau \rangle$.

(2) has just been proved. (3) is obvious.

While the definition of a balanced set may seem to be difficult to check in specific cases, in fact we nearly always achieve a complete classification of balanced sets.

**Theorem 6.2.21.** Let $\mathcal{K}$ be a modular Borel complex.

1. If $\mathcal{K}$ is the collection of finite free sets in a modular Borel pre-geometry, then every maximal $\mathcal{K}$-set is modular-balanced;

2. if $\mathcal{K}$ is locally countable then every maximal $\mathcal{K}$-set is modular-balanced;

3. if $\mathcal{K}$ is an acyclic complex of a Borel graph, then every maximal $\mathcal{K}$-set is modular balanced;

4. if $\mathcal{K}$ is the matching complex of a Borel graph in which every vertex has uncountable degree, a set is modular-balanced if and only if it is a perfect matching;

5. the $P = P_\mathcal{K}$-generic maximal $\mathcal{K}$-set is modular-balanced.

**Proof.** For (1), let $X$ be the Polish domain of the complex $\mathcal{K}$, and let $f: [X]^{<\aleph_0} \to [X]^\aleph_0$ be defined as $f(a)$ =the closure of the set $a$ in the pregeometry; as in Theorem 6.2.10, $f \upharpoonright \mathcal{K}$ is a modular function. Let $A \subset X$ be a maximal $\mathcal{K}$-set. Suppose that $B \subset \mathcal{K}$ is a set. Let $M$ be a countable elementary submodel of a large structure containing $B$ and $A$. If the third option of Definition 6.2.18 fails for $B$, there must be a set $b \in B$ such that $(M \cap A) \cup b$ is a $\mathcal{K}$-set. If $b \in M$ then $b \subset A$ by the maximality of $A$ and the elementarity of the model $M$ and the second option of Definition 6.2.18 holds. Suppose then that $b \notin M$. By the elementarity of the model $M$, there must be a set $a \in B \cap M$ such that $b \cap M \subset a$. We claim that $a \cup b \in \mathcal{K}$, confirming the first option of Definition 6.2.18. Suppose towards contradiction that this fails; by the exchange property, there have to be sets $a' \subset a$, $M \cap b \subset b' \subset b$ and a point $x \in b \setminus b'$ such that $x \in f(a' \cup b')$. By the modularity, there have to be points $y, z$ such that $y \in f(a')$, $z \in f(b')$ and $x \in f(y, z)$. By the exchange property, $y \in f(z, x) \subset f(b)$. At the same time, $y \in M$ and so by the maximality of the set $A$ and the elementarity of the model $M$, there has to be $d \subset A$ in the model $M$ such that $y \in f(d)$. In total, $y \in f(b) \cap f(d)$, by the initial assumption on the set $b \cap d \in \mathcal{K}$ holds, and by the modularity of the function $f$, $y \in f(b \cap d) \subset f(b')$. Then $x \in f(y, z) \subset f(b')$, contradicting the freeness of the set $b$.

For (2), let $X$ be the Polish domain of $\mathcal{K}$, let $f: \mathcal{K} \to [X]^\aleph_0$ be a Borel function witnessing the local countability and let $A \subset X$ be a maximal $\mathcal{K}$-set. Suppose that $B \subset \mathcal{K}$ is a set. Let $M$ be a countable elementary submodel of a large structure containing $B$, $A$ and $f$. If the third option of Definition 6.2.18 fails for $B$, there must be a set $b \in B$ such that $(M \cap A) \cup b$ is a $\mathcal{K}$-set. If $b \in M$ then $b \subset A$ by the maximality of $A$ and the elementarity of the model
By the elementarity of the model $M$ and the second option of Definition 6.2.18 holds. Suppose then that $b \notin M$. By the elementarity of $M$, there must be a set $a \subseteq B \cap M$ such that $b \cap M \subseteq a$. By the elementarity of $M$ again, $f(a) \subseteq M$ and so $b \cap f(a) \subseteq a$. By the choice of the function $f$, $a \cup b \in \mathcal{K}$ and the first option of Definition 6.2.18 holds.

For (3), let $G$ be a Borel graph on a Polish space $X$ such that $\mathcal{K}$ is the simplicial complex of acyclic subsets of $G$. Suppose that $B \subseteq \mathcal{K}$ is a set. Let $M$ be a countable elementary submodel of a large structure containing $B$ and $A$. If the third option of Definition 6.2.18 fails for $B$, there must be a set $b \in B$ such that $(M \cap A) \cup b$ is a $\mathcal{K}$-set. If $b \in M$ then $b \subseteq A$ by the maximality of $A$ and the elementarity of the model $M$ and the second option of Definition 6.2.18 holds. Suppose then that $b \notin M$. By the elementarity of the model $M$, there must be a set $a \subseteq B \cap M$ such that $b \cap M \subseteq a$. We claim that $a \cup b \in \mathcal{K}$, confirming the first option of Definition 6.2.18. Suppose towards contradiction that this fails, and let $c \subseteq a \cup b$ be a cycle. The cycle has to contain some edges in $b \setminus M$; let $c' \subseteq c$ be a maximal contiguous part of $c$ containing only edges in $b \setminus M$. The beginning and ending vertex of $c'$ (denoted by $v_0,v_1$ respectively) must belong to $M$. Thus, $v_0,v_1$ are connected by a $G$-path, they are also connected by $A$-path by the maximality of $A$, and such a path $a \subseteq A$ must be found in the model $M$ by the elementarity of $M$. Then $d \cup c'$ is a cycle in $(M \cap A) \cup b$, contradicting the choice of the set $b$.

For (4), let $G$ be a Borel graph on a Polish space $X$ in which every vertex has uncountable degree. Every modular-balanced set $A \subseteq G$ is a $\mathcal{K}$-set, and therefore a matching. To see that $A$ has to be a perfect matching, for every vertex $x \in X$ consider the set $B_x = \{e : x \text{ is one of the vertices in } e\}$. Consider the alternatives of Definition 6.2.18 for $B_x$. (1) fails and (3) is impossible as the vertex $x$ has uncountable degree in $G$; therefore, (2) has to hold, meaning that the set $A$ contains an edge with the vertex $x$ on it and so $A$ is a perfect matching. For the other implication, suppose that $A$ is a perfect matching and $B \subseteq \mathcal{K}$ is a Borel set. To verify Definition 6.2.18 for $A,B$, let $M$ be a countable elementary submodel of a large structure containing $G,A,B$. If the third option of Definition 6.2.18 fails, there must be $b \in B$ such that $(M \cap A) \cup b$ is a matching. Let $b' = b \cap M$ and use the elementarity of the model $M$ to find $a \subseteq B \cap M$ such that $b' \subseteq a$. There are several possibilities now. If $a = b = b'$ then $a \cup A$ is a matching by the elementarity of $M$, and by the maximality of $A$, $a \subseteq A$ and the option (2) has been verified. If $a \neq b$ and $a \cup b$ is a matching, then option (1) follows. Otherwise, there has to be an edge $e_0 \subseteq b$ connecting some vertex $x$ mentioned in $a$ with a vertex outside of $M$. Since $A$ is a perfect matching and $M$ is elementary, there is an edge $e_1 \in A \cap M$ containing $x$; since the other vertex of $e_1$ is in $M$, it must be the case that $e_0 \neq e_1$ and so $(M \cap A) \cup b$ is not a matching, contradicting the choice of the set $b$.

For the last item, let $B \subseteq \mathcal{K}$ be a Borel set in the $P$-extension. The poset $P$ is $\sigma$-closed, so the Borel set $B$ is in fact in the ground model. The disjunction of the second and third option of Definition 6.2.18 for the set $B$ is an immediate consequence of the genericity. 

$\square$
6.3 Cofinal variations

There is a family of simplicial complexes $K$ which are modular or locally finite or locally countable except the function witnessing that property is not defined on all of $K$ but only on a certain judiciously chosen cofinal subset of it.

**Definition 6.3.1.** A simplicial complex $K$ on a space $X$ is **tight** if

1. for every set $a \in K$, its closure, the set $\bar{a} = \{ x \in X : \forall b \in K \ a \cup b \in K \leftrightarrow a \cup b \cup \{ x \} \in K \}$, is finite;
2. the set $\{ a \in K : \bar{a} = a \}$ of all closed sets is Borel.

The following proposition records the basic properties of tight simplicial complexes.

**Proposition 6.3.2.** Let $K$ be a tight simplicial complex.

1. The closure of any element of $K$ is still in $K$;
2. whenever $a, b \in K$ then $a \cup b \in K \leftrightarrow \bar{a} \cup b \in K$.
3. the closure is an idempotent monotone operator;
4. the intersection of any family of closed sets is closed.

*Proof.* For (1), list the elements of $\bar{a} \setminus a$ as $\{ x_i : i \in n \}$ and by induction on $m \leq n$ show that $a \cup \{ x_i : i \in m \}$ is in $K$. To perform the induction step, note that $a \cup (a \cup \{ x_i : i \in m \}) \in K$ by the induction hypothesis and so $a \cup (a \cup \{ x_i : i \in m \}) \cup \{ x_m \} \in K$ by the definition of the membership in $\bar{a}$ applied to $x_m$.

(2) is proved by an identical induction argument. For the idempotent part of (3), let $b = \bar{a}$ and suppose that $x \in \bar{b}$; we must prove that $x \in \bar{a}$. Let $c \in K$ be a finite set such that $a \cup c \in K$. By (2), $b \cup c \in K$ must hold. Since $x \in \bar{b}$, $b \cup c \cup \{ x \}$ holds, and by the closure of $K$ under subset, $a \cup c \cup \{ x \} \in K$ holds as well. In conclusion, $x \in \bar{a}$ as required.

For the monotonicity part of (3), suppose that $a_0 \subseteq a_1 \in K$ and $x \in \bar{a}_0$. Towards the conclusion $x \in \bar{a}_1$, suppose that $b \in K$ is a set such that $a_1 \cup b \in K$. Then $a_0 \cup (a_1 \cup b) \in K$, by the assumption on $x$, $a_0 \cup (a_1 \cup b) \cup \{ x \} \in K$, and so $a_1 \cup b \cup \{ x \} \in K$. The property $x \in \bar{a}_1$ has just been verified.

(4) now follows immediately from (3).

In contradistinction to the last item above, the union of two closed sets may not be closed, and this is the main reason why the tight complexes need to be introduced in the first place. Note that the properties and the calculation of the closure operator are all $\Pi_1^3$ and therefore absolute among all forcing extensions.

Most simplicial complexes are defined to be tight and the closure operation is simply the identity. However, there are some important exceptions to this rule:
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Example 6.3.3. Let $B \subset X^2$ be a Borel set and let $\mathcal{K}$ be the complex on $X^2$ of all finite sets $a \subset X^2$ such that $a^0 \times a^1 \subset B$, where $a^0$ is the projection of $a$ into the first coordinate and $a^1$ is the projection of $a$ into the second coordinate. The poset $P_\mathcal{K}$ would add a maximal rectangle which is a subset of $B$. Clearly, the closure of the set $a$ must contain the set $a^0 \times a^1$ at the very least.

For several classes of complexes introduced above, it is possible to define a cofinal variation in a natural way as follows.

Definition 6.3.4. Let $\mathcal{K}$ be a tight Borel complex on a Polish space $X$. Let $\mathcal{L} \subset \mathcal{K}$ be the collection of closed elements of $\mathcal{K}$. $\mathcal{K}$ is cofinally modular if there is a function $f: \mathcal{L} \to [\mathcal{Y}]^{\aleph_0}$ for some Polish space $\mathcal{Y}$ such that

1. (monotonicity) $f(0) = 0$ and $a \subset b$ implies $f(a) \subset f(b)$;
2. (modularity) for all $a, b \in \mathcal{L}$, $a \cup b \in \mathcal{K}$ if and only if $f(a) \cap f(b) = f(a \cap b)$.

Definition 6.3.5. Let $\mathcal{K}$ be a tight Borel simplicial complex on a Polish space $X$. Let $\mathcal{L}$ denote the collection of closed elements of $\mathcal{K}$. The complex is

1. cofinally locally countable if there is a Borel function $f: \mathcal{L} \to [X]^{\aleph_0}$ such that for all sets $a, b \in \mathcal{L}$, $a \cup b \notin \mathcal{K}$ if and only if $a \cup (b \cap f(a)) \notin \mathcal{K}$.
2. cofinally locally finite if the function $f$ as in (1) takes only finite values.

Example 6.3.6. Let $X$ be an uncountable Polish space. Consider the simplicial complex $\mathcal{K}$ on $X^2$ consisting of finite acyclic subsets of $X^2$. $\mathcal{K}$ is cofinally locally finite and cofinally modular but not modular. The poset $P_\mathcal{K}$ adds a linear ordering of the space $X$.

Proof. For $a \in \mathcal{K}$, the closure operation yields the set $\bar{a}$ = the transitive closure of $a$. Thus, the complex $\mathcal{K}$ is tight. Write $\mathcal{L} \subset \mathcal{K}$ for the collection of transitive elements of $\mathcal{K}$. To identify the modular function $f$, for every set $a \in \mathcal{L}$ let $f_0(a) = \{(x_i : i \leq 2n) :$ the points $x_i \in X$ are pairwise distinct except for $x_0 = x_{2i}$, and $(x_{2i}, x_{2i+1}) \in a$ for all $i \in n\}$, and $f_1(a) = \{(x_i : i \leq 2n) :$ the points $x_i \in X$ are pairwise distinct except for $x_0 = x_{2i}$, and $(x_{2i+1}, x_{2i+2}) \in a$ for all $i \in n\}$. Finally, let $f(a) = f_0(a) \cup f_1(a)$.

To check the modularity of the function $f$, suppose that $a, b \in \mathcal{L}$ are finite transitive acyclic sets. Observe that $a \cup b$ contains a cycle if and only if there is a sequence $\bar{x} = (x_i : i \in 2n)$ of points in $X$ which are pairwise distinct except for $x_{2n} = x_0$, and $\bar{x} \in f_0(a)$ and $\bar{x} \in f_1(b)$. Note that the transitivity of $a, b$ is used to establish this equivalence. Note also that such a sequence $\bar{x}$ cannot belong to $f(a \cap b)$ because it would induce a cycle in $a$ or $b$ depending on whether $\bar{x} \in f_0(a \cap b)$ or $\bar{x} \in f_1(a \cap b)$.

To check the cofinal locally finite property of $\mathcal{K}$, for a transitive element $a \in \mathcal{K}$ let $g(a) = \{(x, y) : x, y \in X$ are points mentioned in some edge in $a\}$. The function $g: \mathcal{L} \to [X^2]^{\aleph_0}$ clearly has the requested properties. Finally, to disprove the modularity of $\mathcal{K}$, let $a = \{(x_0, x_1)\}$ for some distinct points $x_0, x_1 \in X$, and for some pairwise distinct points $y_\alpha \in X$ for $\alpha \in \omega_1$ let $b_\alpha = \{(x_1, y_\alpha), (y_\alpha, x_0)\} \in \mathcal{K}$. The elements $a, b_\alpha : \alpha \in \omega_1$ witness the condition (*) above and so the complex $\mathcal{K}$ is not modular. \qed
6.4 Quotient variations

Many simplicial complex forcings are naturally connected with a Borel equivalence relation.

**Definition 6.4.1.** Let $E$ be a Borel equivalence relation on a Polish space $X$. A Borel simplicial complex $K$ on $X$ is an $E$-quotient complex if the membership of any finite set $a \subset X$ in $K$ depends only on $[a]_E$. The letter $E$ is left out if $E$ is understood from the context or will be specified later.

It is possible to define quotient modular or quotient flag complexes in parallel to Sections 6.2 or 6.1; however, the resulting theorems characterizing the balanced conditions become more cumbersome. We investigate several more specific classes of quotient complexes where the balanced conditions are easy to classify and use.

**Definition 6.4.2.** Let $E,F$ be Borel equivalence relations on Polish spaces $X,Y$ with uncountably many classes. Let $K_{E,F} = K$ be the simplicial complex on the set $(X \times Y) \cup \bar{Y}$ consisting of finite sets $a$ such that for all $\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in a$, if $x_1 E x_0$ then $y_1 \not\in \bar{Y}$ holds, and for all $\langle x_0, y_0 \rangle, y_2 \in a$, $y_0 E y_2$ holds. Clearly, $K$ is an $(E \times F) \cup \bar{F}$-quotient complex. The associated poset $P_K$ is referred to as the collapse poset of $|E|$ to $|F|$.

If $A \subset (X \times Y) \cup \bar{Y}$ is a generic set, the part $A \cap (X \times Y)$ is an injection from $X/E$ to $Y/F$, while the part $A \cap \bar{Y}$ is the complement of the range of $A \cap (X \times Y)$. The balanced conditions are neatly classified by injections from $X^{**}$ to $Y^{**}$. If $g : X^{**} \to Y^{**}$ is an injection, let $\tau_g$ be the $\text{Coll}(\omega, \mathcal{P}(\omega))$-name for the set of those elements $p \in P_K$ such that for each pair $\langle c,d \rangle \in g$, there is a pair $\langle x,y \rangle \in p$ such that $x \in X$ is a realization of the virtual $E$-class $c$ and $y \in Y$ is a realization of the virtual $F$-class $d$. Moreover, if $d$ is a virtual $F$-class not in the range of $g$, then $p$ is required to contain an element $y \in Y$ which is a realization of the virtual $F$-class $d$.

**Theorem 6.4.3.** Let $E,F$ be Borel equivalence relations on respective Polish spaces $X,Y$. Let $P$ be the collapse poset of $|E|$ to $|F|$.

1. For every total injection $g : X^{**} \to Y^{**}$, the pair $(\text{Coll}(\omega, \mathcal{P}(\omega)), \tau_g)$ is a balanced $P$-pin;

2. for every balanced pair $\langle Q, \tau \rangle$ there is a total injection $g : X^{**} \to Y^{**}$ such the pair $(\text{Coll}(\omega, \mathcal{P}(\omega)), \tau_g)$ is equivalent to $\langle Q, \tau \rangle$;

3. distinct injections as in (1) yield inequivalent balanced conditions.

**Proof.** Write $R = \text{Coll}(\omega, \mathcal{P}(\omega))$. For (1), the pair $(\text{Coll}(\omega, \mathcal{P}(\omega)), \tau_g)$ is clearly a $P$-pin. For the balance, let $V[H_0], V[H_1]$ be mutually generic extensions of the ground model and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be conditions in the analytic set $\tau_g/H_0 = \tau_g/H_1$; we must show that $p_0, p_1 \in P$ are compatible.
To see this, suppose for example that \( \langle x_0, y_0 \rangle \in p_0 \), \( \langle x_1, y_1 \rangle \in p_1 \), and \( x_0 E x_1 \); we must show that \( y_0 E y_1 \). By a mutual genericity argument, the points \( x_0, x_1 \) must be realizations of the same virtual \( E \)-class \( c \) in the ground model. Since both \( p_0, p_1 \) are \( \mathcal{K} \)-sets, the definition of the name \( \tau_0 \) shows that both \( y_0, y_1 \) must be realizations of the virtual \( F \)-class \( g(c) \) and therefore \( y_0 F y_1 \) as desired.

Similarly, if \( \langle x_0, y_0 \rangle \in p_0 \) and \( y_2 \in p_1 \), we need to show that \( y_0 F y_2 \) holds. Suppose towards contradiction that \( y_0 F y_2 \) fails. By a mutual genericity argument, the points \( y_0, y_2 \) are realizations of the same virtual \( F \)-class \( d \) in the ground model. Now, if \( d \in \text{rng}(g) \) then \( p_1 \) cannot be a \( \mathcal{K} \)-set, and if \( d \notin \text{rng}(g) \) then \( p_0 \) cannot be a \( \mathcal{K} \)-set. There are no other options, and this contradiction completes the proof of (1).

For (2), let \( g : X^\ast \rightarrow Y^\ast \) be the collection of all pairs \( \langle c, d \rangle \in X^\ast \times Y^\ast \) so that \( R \times Q \models \tau \models P \langle x, y \rangle \in \hat{A}_{\text{gen}} \) for some (all) realizations \( x, y \) of the virtual classes \( c, d \). It will be enough to show that \( g \) is a total injection from \( X^\ast \) to \( Y^\ast \) and the pair \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, \bigtriangleup_n), \tau_0 \rangle \).

To see that \( \text{dom}(g) = X^\ast \) and \( g \) is an injection, let \( c \in X^\ast \) be an arbitrary virtual \( E \)-class. Use the balance of the pair \( \langle Q, \tau \rangle \) to show that there must be a unique virtual \( F \)-class \( d \in Y^\ast \) such that \( R \times Q \models \tau \models P \langle x, y \rangle \in \hat{A}_{\text{gen}} \) for some (all) realizations \( x, y \) of the virtual classes \( c, d \). Use the balance of the pair \( \langle Q, \tau \rangle \) again to show that for every virtual \( F \)-class \( d \in Y^\ast \), there either must be a virtual \( E \)-class \( c \in X^\ast \) such that \( R \times Q \models \tau \models P \langle x, y \rangle \in \hat{A}_{\text{gen}} \) for some (all) realizations \( x, y \) of the virtual classes \( c, d \), or it must be the case that \( R \times Q \models \tau \models P \langle x, y \rangle \notin \text{rng}(\hat{A}_{\text{gen}}) \) for some (all) realizations \( y \) of the virtual class \( d \).

To see that \( \langle R, \tau_0 \rangle \) is equivalent to \( \langle Q, \tau \rangle \), strengthen \( \tau \) if necessary so it is a name for an actual element of \( P \) and notice that in the \( R \times Q \) extension, \( \tau \in \tau_0 \) holds. The equivalence then follows from Proposition 5.2.4.

Finally, (3) is obvious.

\[\square\]

**Corollary 6.4.4.** Let \( E, F \) be Borel equivalence relations on respective Polish spaces \( X, Y \) with uncountably many classes. The poset collapsing \( |E| \) to \( |F| \) is balanced if and only if \( \lambda(E) \leq \lambda(F) \).

The collapse posets exemplify an important phenomenon: a \( \sigma \)-closed Suslin forcing whose balanced status cannot be decided in ZFC. There are Borel equivalence relations \( E, F \) for which the status of the inequality \( \lambda(E) \leq \lambda(F) \) cannot be decided in ZFC (see ???) and for them the balanced status of the corresponding collapse forcing is undecidable as well.

**Definition 6.4.5.** Let \( E, F \) be Borel equivalence relations on Polish spaces \( X, Y \) and let \( B \subset X \times Y \) be an \( E \times F \)-invariant Borel set with all vertical sections nonempty. Let \( \mathcal{K}_{E,F,B} = \mathcal{K} \) be the simplicial complex on \( X \times Y \) consisting of all finite sets \( a \subset X \times Y \) such that for all \( \langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in a \), if \( x_0 E x_1 \) then \( y_0 F y_1 \). The poset \( P_\mathcal{K} \) is referred to as the *uniformization poset* for \( E, F, B \).

It is obvious that the generic set for the uniformization poset is a uniformization of the set \( B/E \times F \subset X/E \times Y/F \). To classify the balanced conditions, consider the virtual \( E \)-quotient space \( X^\ast \), the virtual \( F \)-quotient space \( Y^\ast \) and the
virtual version $B^{**} \subset X^{**} \times Y^{**}$ of the set $B$. For a total function $f : X^{**} \to Y^{**}$ which is a subset of $B^{**}$ let $\tau_f$ be the Coll($\omega, \mathbb{Z}_\omega$)-name for the set of all conditions $p$ in the uniformization poset such that for each virtual $E$-class $c \in X^{**}$, $p$ contains some pair $\langle x, y \rangle$ where $x$ is a realization of $c$ and $y$ is a realization of $f(c)$. It is not difficult to see that the pair $\langle \text{Coll}(\omega, \mathbb{Z}_\omega), \tau_f \rangle$ is a $P$-pin.

**Theorem 6.4.6.** Let $E, F$ be Borel equivalence relations on Polish spaces $X, Y$ and let $B \subset X \times Y$ be an $E \times F$-invariant Borel set with all vertical sections nonempty. Let $P$ be the uniformization poset. Then

1. for every total function $f : X^{**} \to Y^{**}$ such that $f \subset B^{**}$, the pair $\langle \text{Coll}(\omega, \mathbb{Z}_\omega), \tau_f \rangle$ is balanced;

2. for every balanced pair $\langle Q, \tau \rangle$ there is a total function $f$ as in (1) such that $\langle \text{Coll}(\omega, \mathbb{Z}_\omega), \tau_f \rangle$ is equivalent to $\langle Q, \tau \rangle$;

3. distinct functions yield nonequivalent balanced pairs.

**Proof.** Write $R = \text{Coll}(\omega, \mathbb{Z}_\omega)$. To see (1), let $V[H_0]$ and $V[H_1]$ be mutually generic extensions of the ground model and $p_0 \in V[H_0]$ and $p_1 \in V[H_1]$ be conditions in the set $\tau_f/H_0 = \tau_f/H_1$; we have to show that $p_0, p_1 \in P$ are compatible. To this end, suppose that $\langle x_0, y_0 \rangle \in p_0$ and $\langle x_1, y_1 \rangle \in p_1$ are ordered pairs. We must show that if $x_0 E x_1$ then $y F y_1$ holds. A mutual genericity argument shows that $x_0, x_1$ are realizations of the same virtual $E$-class $c$. By the definition of the name $\tau_f$, it must be the case that $y_0, y_1$ are both realizations of the virtual $F$-class $f(c)$ and therefore $F$-related.

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. To find the total function $f$, let $c$ be a virtual $E$-class. There must be a virtual $F$-class $d_c$ such that $Q \vdash$ for some pair $\langle x, y \rangle \in \tau$ $x$ is a realization of $c$ and $y$ is a realization of $d_c$; otherwise, one could find conditions $q_0, q_1 \in Q$ and names for strengthenings $\tau_0, \tau_1$ of $\tau$ and names $\hat{x}_0, \hat{y}_0, \hat{x}_1, \hat{y}_1$ such that $q_0 \vdash \hat{x}_0$ is a realization of $c$ and $\langle \hat{x}_0, \hat{y}_0 \rangle \in \tau_0$, $q_1 \vdash \hat{x}_1$ is a realization of $c$ and $\langle \hat{x}_1, \hat{y}_1 \rangle \in \tau_1$ and $\langle q_0, q_1 \rangle \vdash Q \times Q \neg \hat{y}_0 F \hat{y}_1$. This would violate the balance of the pair $\langle Q, \tau \rangle$ as $\langle q_0, q_1 \rangle \vdash \tau_0, \tau_1 \in P$ are incompatible conditions. Let $f : c \mapsto d_c$ be the resulting function.

To see that $\langle R, \tau_f \rangle$ is a balanced pair equivalent to $\langle Q, \tau \rangle$, strengthen $\tau$ if necessary to ensure that $\tau$ is a name for an actual element of $P$, and observe that $Q \times R \vdash \tau \in \tau_f$. The equivalence of the two balanced pairs then follows from Proposition 5.2.4.

Finally, (3) is obvious. □

**Corollary 6.4.7.** Let $E, F$ be Borel equivalence relations on Polish spaces $X, Y$ and let $B \subset X \times Y$ be an $E \times F$-invariant Borel set with all vertical sections nonempty. Suppose moreover that the equivalence relation $E$ is pinned. Let $P$ be the uniformization poset. Then $P$ is balanced.

**Proof.** The additional assumption that $E$ is pinned makes sure that $X^{**} = X/E$. Thus, in this case, all vertical sections of $B^{**}$ are nonempty and total functions as in (1) of Theorem 6.4.6 actually exist. □
Example 6.4.8. A poset introducing a transversal to a pinned Borel equivalence relation $E$ on a Polish space $X$. Consider the Borel relation $B = E$ on the space $X \times X$, and the uniformization poset uniformizing $B$ as a relation on the quotient space $X/E \times X$. The generic set is an $E$-transversal. By Theorem 6.4.6, the balanced conditions are classified by $E$-transversals.

Example 6.4.9. A poset introducing a complete countable section to a pinned Borel equivalence relation $E$ on a Polish space $X$. Let $Y = \omega$, equipped by the equivalence relation $F$ connecting $y_0, y_1$ if $\text{rng}(y_0) = \text{rng}(y_1)$. Consider the Borel relation $B \subset X \times Y$ defined by $\langle x, y \rangle \in B$ if $\text{rng}(y) \subset [x]_E$. Consider the uniformization poset uniformizing $B$ as a relation on the quotient space $X/E \times Y/F$. The generic set is a countable complete section of the relation $E$. Note that since every subset of $X$ corresponds to a virtual $\mathbb{P}_2$-class, Theorem 6.4.6 says that the balanced conditions are classified by arbitrary sets $A \subset X$ which have nonempty intersection with every $E$-class.

An important class of examples appears in which one starts with a countable Borel equivalence relation $E$ and adds a function selecting a certain structure on each $E$-class where no such Borel function may exist.

Example 6.4.10. Let $E$ be a countable Borel equivalence relation on a Polish space $X$ with all equivalence classes infinite. Let $B \subset (X^2)^\omega$ be the Borel set of all pairs $\langle x, y \rangle$ such that $\text{rng}(y)$ is a linear ordering of $[x]_E$ isomorphic to $\mathbb{Z}$. Clearly, the set $B$ is invariant under the equivalence relation $E \times =^+$. The poset uniformizing $B$ as a subset of $X/E \times X/\mathbb{Z}$ adds a $\mathbb{Z}$-type ordering to each $E$-class and therefore adds a (discontinuous) action of $\mathbb{Z}$ on $X$ which induces $E$ as its orbit equivalence relation. Theorem 6.4.6 says that the balanced conditions are classified by all functions $f$ assigning each $E$-class a linear ordering of its elements isomorphic to $\mathbb{Z}$.

Example 6.4.11. Let $X = (2^\omega)^\omega$ and consider the relation $B \subset X \times X$ defined by $\langle x_0, x_1 \rangle \in X$ if $\text{rng}(x_0) \cap \text{rng}(x_1) = 0$. Let $E = \mathbb{P}_2$, note that the set $B$ is $E \times E$-invariant and consider the uniformization poset uniformizing $B$ as a subset of $X/E \times X/E$. The equivalence relation $E$ is not pinned and so Theorem 6.4.6 does not apply to show that the resulting poset is balanced. Indeed, in ZF one can prove that existence of a uniformization of the quotient set $B/E \times E$ implies the existence of an $\omega_1$-sequence of pairwise distinct $E$-classes. To see this, view the quotient space as the set of nonempty countable subsets of $2^\omega$, suppose that $g$ is the uniformization, let $c_0 \subset 2^\omega$ be an arbitrary nonempty set, and by transfinite recursion on $\alpha \in \omega_1$ define $c_{\alpha+1} = c_\alpha \cup g(c_\alpha)$ and $c_\alpha = \bigcup_{\beta \leq \alpha} c_\beta$.

We will need to consider situations where unpinned equivalence relations occur, where Corollary 6.4.7 does not apply. To do this, we reach for quotient local finiteness and countability:

Definition 6.4.12. Let $E$ be a Borel equivalence relation on a Polish space $X$. A Borel simplicial complex $K$ on a Polish space $X$ is $E$-quotient locally countable if there is a relation $f$ so that
Theorem 6.4.13. Let $E$ be a Borel equivalence relation on a Polish spaces $X$ and let $K$ be a $E$-quotient locally countable Borel complex. Let $P = P^*$. Then

1. for every maximal $K^*$-set $A \subset X^*$, the pair $(\text{Coll}(\omega, \mathbb{D}_{\omega_1}), \tau_A)$ is balanced;

2. for every balanced pair $(R, \tau)$ there is a maximal $K^*$-set $A$ as in (1) such that $(\text{Coll}(\omega, \mathbb{D}_{\omega_1}), \tau_A)$ is equivalent to $(R, \tau)$;

3. distinct maximal $K^*$-sets yield nonequivalent balanced pairs.

Proof. Write $Q = \text{Coll}(\omega, \mathbb{D}_{\omega_1})$ and let $f$ be the relation witnessing the local countability of the complex $K$. For (1), let $R_0, R_1$ be posets and $\sigma_0, \sigma_1$ be respective $R_0 \times Q$- and $R \times Q_1$-names for elements of $\tau_A$; we must show that $(R_0 \times Q) \times (R_1 \times Q) \models \sigma_0, \sigma_1$ are compatible elements of $P$. To this end, let $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ be filters mutually generic over the ground model, and let $a_0 \subset \sigma_0/H_0$ and $a_1 \subset \sigma_1/H_1$ be finite sets; we must show that $a_0 \cup a_1 \in K$.

To this end, consider the vertical section $f_{a_0} \subset X$; it consists of countably many $E$-classes represented in $V[H_0]$. Its intersection with $a_1$ consists of finitely many elements whose $E$-classes are represented in both models $V[H_0]$ and $V[H_1]$ and therefore must be realizations of some virtual $E$-classes in the ground model. Since $A$ is a maximal $K^*$-set and $a_1$ is compatible with it, $a_1 \cap f_{a_0}$ must consist of realizations of some virtual $E$-classes in the set $A$. Since $a_1$ is compatible with $A$, it must be that $(a_1 \cap f_{a_0}) \cup a_0 \in K$. The definitory properties of the relation $f$ then imply that $a_0 \cup a_1 \in K$ as desired.

For (2), let $(R, \tau)$ be a balanced pair. The balance of $\tau$ shows that for each virtual $E$-class $c$, either $R \times Q \models \tau \models_P$ some realization of $c$ belongs to the $P$-generic set or $R \times Q \models \tau \models_P$ no realization of $c$ belongs to the $P$-generic set. Let $A$ be the set of all virtual $E$-classes $c$ for which the former alternative prevails. Clearly, $A$ is a $K^*$-set; it will be enough to show that it is a maximal $K^*$-set.
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To this end, let \( c \) be a virtual \( E \)-class which is not in \( A \). Thus, \( R \times Q \models \tau_P \) there is a finite set \( a \) in the \( P \)-generic set such that for no realization \( x \) of \( c \), \( \{x\} \cup a \notin \mathcal{K} \). By the definitory properties of the relation \( f \), \( \{x\} \cup (a \cap f_{\{x\}}) \notin \mathcal{K} \). The set \( f_{\{x\}} \) consists of countably many \( E \)-classes all of which must be realizations of virtual \( E \)-classes since \( \{x\} \) is. Thus, there is a finite set \( b \) of virtual \( E \)-classes in \( V \) such that the sets \( a \cap f_{\{x\}} \) consists exactly of realizations of the virtual \( E \)-classes in \( b \). It follows that \( b \subset A \) and \( b \cup \{c\} \notin \mathcal{K}^\ast\ast \), confirming the maximality of the set \( A \).

Finally, (3) is obvious.

Example 6.4.14. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). We wish to add a tournament to the quotient space \( X/E \). To this end, let \( f \) be the relation on \( X^2 \) defined by \( \langle x_0, y_0 \rangle f \langle x_1, y_1 \rangle \) if \( x_0 E y_1 \) and \( y_0 E x_1 \). Let \( \mathcal{K} \) be the complex of finite \( f \)-free sets. Then \( \mathcal{K} \) is a quotient locally finite complex. The poset \( P_\mathcal{K} \) performs the requested task. The balanced conditions are classified by tournaments on the virtual \( E \)-space.

Example 6.4.15. Let \( G \) be a Borel graph on a Polish space \( X \) which contains no odd length cycles such that the \( G \)-path connectedness equivalence relation \( E \) on \( X \) is Borel. We wish to add a 2-coloring of the graph \( G \). To this end, let \( F \) be the Borel equivalence relation on \( X \) relating \( x \) to \( y \) if \( x E y \) holds and some (equivalently, every) \( G \)-path from \( x \) to \( y \) is of even length. Let \( f = E \setminus F \) and let \( \mathcal{K} \) be the flag complex consisting of \( f \)-free sets. Then \( \mathcal{K} \) is a quotient locally finite complex. The poset \( P_\mathcal{K} \) selects exactly one \( F \)-class from each \( E \)-class and therefore adds a bipartization of the graph \( G \). The balanced conditions are classified by functions which select exactly one virtual \( F \)-class from each virtual \( E \)-class.

It is possible to define quotient cofinally modular complexes in parallel to the work of Section 6.3. We omit the rather obvious definitions for brevity and include only one useful example.

Example 6.4.16. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). Let \( \mathcal{K} \) be the Borel simplicial complex of acyclic relations on \( X \) whose \( E \)-saturation contains no cycle. This is a quotient cofinally modular complex by a proof identical to Example 6.3.6. Consider the poset \( P = P_\mathcal{K} \). The poset is designed to add a maximal acyclic relation on \( X \) respecting the equivalence relation \( E \), i.e. a linear order of \( E \)-classes. The balanced conditions are classified by linear orderings of the space of virtual \( E \)-classes.

The classification of the balanced conditions transfers from Theorem 6.4.13 without change, with only a somewhat more verbose proof.

Theorem 6.4.17. Let \( \mathcal{K} \) be a quotient cofinally locally countable Borel simplicial complex on a Polish space \( X \) as witnessed by a Borel equivalence relation \( E \) on \( X \).

1. For every maximal \( \mathcal{K}^\ast\ast \)-set \( A \subset X^\ast\ast \), the pair \( \langle \text{Coll}(\omega, \beth_1), \tau_A \rangle \) is a balanced virtual condition in \( P \);
2. for every balanced pair there is a maximal $K^{**}$-set $A \subseteq X^{**}$ such that 
$(\text{Coll}(\omega, \mathcal{D}_\omega), \tau_A)$ is equivalent to it;

3. distinct $K^{**}$-sets yield inequivalent balanced conditions.

### 6.5 Variations violating DC

There is a class of balanced partial orders which violate the Axiom of Dependent Choices.

#### Definition 6.5.1

Let $X$ be a Polish space. An ideal $I$ of countable subsets of $X$ is **Borel** if the set \( \{ y \in X^\omega : \text{rng}(y) \in I \} \) is Borel.

A Borel ideal of countable subsets of $X$ can be reinterpreted in any forcing extensions. Note that the statement that $I$ is an ideal is coanalytic and therefore transfers to any forcing extension by the Mostowski absoluteness. One can immediately define the virtual version $I^{**}$ of $I$ as the set of all subsets $A \subseteq X$ such that $\text{Coll}(\omega, X) \vDash \bar{A} \in I$. It follows immediately that $I^{**}$ is an ideal on $X$.

#### Definition 6.5.2

Let $X$ be a Polish space, $\mathcal{K}$ a Borel simplicial complex on $X$, and $I$ a Borel ideal of countable subsets of $X$ containing all singletons. The poset $P_{\mathcal{K}, I}$ consists of all countable $\mathcal{K}$-sets $p \subseteq X$ such that $p \in I$. The order is that of reverse inclusion.

A rather mechanical repetition of the arguments in Section 6.2 yields the following:

#### Theorem 6.5.3

Let $\mathcal{K}$ be a locally countable Borel simplicial complex on a Polish space $X$. Let $I$ be a Borel ideal of countable subsets of $X$. The balanced conditions in the forcing $P_{\mathcal{K}, I}$ are classified by maximal $\mathcal{K}$-sets which are in the ideal $I^{**}$.

#### Example 6.5.4

Let $Y$ be a Polish space and write $X = Y \times 2$. Let $\mathcal{K}$ be the complex on $X$ consisting of all finite partial functions from $Y$ to 2. Let $I$ be the ideal of all countable subsets $A \subseteq X$ such that $A \cap (Y \times \{0\})$ is finite. The partial order $P = P_{\mathcal{K}, I}$ consists essentially of all pairs $p = \langle a_p, b_p \rangle$ where $a_p, b_p \subseteq Y$ are a finite and countable disjoint subsets of $Y$ respectively. The ordering is that of coordinatewise reverse inclusion.

It is immediate that the complex $\mathcal{K}$ is locally countable. Theorem 6.5.3 then classifies all balanced conditions of the poset $P$: these are all pairs $\langle A, B \rangle$ where $A \cup B = X$, $A \cap B = 0$, and $A$ is finite. In particular, the poset $P$ is balanced. In the symmetric Solovay model, the poset $P$ adds a partition of $X$ into two parts: the union of the left coordinates of conditions in the generic filter (which contains no countably infinite subset) and the union of the right coordinates of conditions in the generic filter (which contains no perfect subset). Note that as a balanced forcing, the poset $P$ adds no new $\omega$-sequences of ordinals and therefore no new elements of $X$. The generic $P$-partition then violates DC.
6.6 Operations

The families of simplicial complexes introduced in this chapter are closed under several natural operations. All the operations have natural counterparts in the quotient spaces which we are not going to state for the sake of brevity.

Definition 6.6.1. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$, and let $B \subset X$ be a Borel set. The restriction $\mathcal{K} \upharpoonright B$ is just the set $\mathcal{K} \cap [B]^{<\aleph_0}$.

While the operation may look trivial from the complex point of view, the relationship between the posets $P_\mathcal{K}$ and $P_{\mathcal{K} \upharpoonright B}$ may be quite complicated. It is immediate that the families of countable flag complexes, modular complexes, and locally countable or finite complexes are closed under restriction.

Definition 6.6.2. Let $X$ be a Polish space and for each $n \in \omega$, let $\mathcal{K}_n$ be a Borel complex on a Borel set $B_n \subset X$. $\mathcal{K} = \bigwedge_n \mathcal{K}_n$ is the simplicial complex of all finite sets $a \subset X$ such that $\forall n \ a \cap B_n \in \mathcal{K}_n$.

One way to use this operation is when $B_n = X$ for all $n \in \omega$, when $\mathcal{K} = \bigcap_n \mathcal{K}_n$. In the language of model theory, this corresponds to countable intersection of formulas defining the complexes $\mathcal{K}_n$. Another way to use this operation appears when the Borel sets $B_n$ for $n \in \omega$ are pairwise disjoint. In this case, the poset $P_\mathcal{K}$ corresponds to the countable support product of the posets $P_{\mathcal{K}_n}$ for $n \in \omega$.

Proposition 6.6.3. If the simplicial complexes $\mathcal{K}_n$ on sets $B_n \subset X$ for $n \in \omega$ are all locally countable or modular or countable flag complexes, then $\mathcal{K} = \bigwedge_n \mathcal{K}_n$ belongs to the same class of complexes.

Proof. Suppose first that the complexes are all locally countable, as witnessed by functions $f_n : \mathcal{K}_n \to [X]^{\aleph_0}$ for $n \in \omega$. Then the function $f : \mathcal{K} \to [X]^{\aleph_0}$ defined by $f(a) = \bigcup_n f_n(a_n \cap B_n)$ shows that the complex $\mathcal{K}$ is locally countable.

Suppose now that the complexes are all countable flag complexes, as witnessed by a Borel function $f_n : B_n \to [B_n]^{\aleph_0}$ for each $n \in \omega$. Then the function $f : X \to [X]^{\aleph_0}$ defined by $f(x) = \bigcup_n f_n(x)$ witnesses the fact that $\mathcal{K}$ is a countable flag complex as well.

Finally, suppose that all the complexes are modular, and let $Y_n$ for $n \in \omega$ be pairwise disjoint Polish spaces and $f_n : \mathcal{K}_n \to [Y_n]^{\aleph_0}$ be Borel functions witnessing the modularity of the respective complexes $\mathcal{K}_n$. Let $Y$ be the union $\bigcup_n Y_n$ and define $f : \mathcal{K} \to [Y]^{\aleph_0}$ by $f(a) = \bigcup_n f_n(a \cap X_n)$. It is not difficult to see that the function $f$ witnessed the modularity of the complex $\mathcal{K}$.

A more sophisticated operation on simplicial complexes leads to a partial order which shoots a large $\mathcal{K}$-set and a set complementary to it. This is useful in a number of applications.

Definition 6.6.4. Let $\mathcal{K}$ be a simplicial complex on a Polish space $X$. The complemented variation $\mathcal{K}^c$ is the complex on $X \times 2$ consisting of all finite sets $a \subset X \times 2$ such that the set $a^0 = \{ x \in X : (x,0) \in a \}$ belongs to $\mathcal{K}$ and is disjoint from the set $a^1 = \{ x \in X : (x,1) \in a \}$.
Proposition 6.6.5. If a simplicial complex $\mathcal{K}$ on a Polish space $X$ is locally finite or locally countable or modular or countable flag complex, then the complemented variation $\mathcal{K}^c$ belongs to the same class of complexes.

Proof. Suppose that $\mathcal{K}$ is a countable flag simplex, as witnessed by a Borel function $f : X \to [X]^\aleph_0$. Then let $f^c : X \times 2 \to [X \times 2]^\aleph_0$ be defined by $f^c(x, 0) = \{(x, 1)\} \cup (f(x) \times \{0\})$ and $f^c(x, 1) = \{(x, 0)\}$. It is immediate that a finite set $a \subset X \times 2$ belongs to $\mathcal{K}^c$ just in case it is $f^c$-free.

Suppose that $\mathcal{K}$ is a modular simplex, as witnessed by a Borel function $f : \mathcal{K} \to [Y]^\aleph_0$ for some Polish space $Y$ disjoint from $X$. Then let $f^c : \mathcal{K}^c \to [X \cup Y]^\aleph_0$ be defined by $f^c(a) = f(a^0) \cup \{x \in X : \langle x, 0 \rangle \in a \lor \langle x, 1 \rangle \in a\}$. It is immediate that for sets $a, b \in \mathcal{K}^c$, the set $a \cup b$ belongs to $\mathcal{K}^c$ if and only if $f^c(a) \cap f^c(b) = f^c(a \cap b)$.

Suppose that $\mathcal{K}$ is a locally countable complex, as witnessed by a Borel function $f : \mathcal{K} \to [X]^\aleph_0$. Let $f^c : \mathcal{K}^c \to [X \times 2]^\aleph_0$ be defined by $f^c(a) = (f(a^0) \cup a^0 \cup a^1) \times 2$. It is immediate that the function $f^c$ witnesses the local countability of $\mathcal{K}^c$. Moreover, if the values of $f$ were finite, so are the values of $f^c$.  

The purpose of the complemented variations is to simplify the description of balanced conditions and introduce new ones in some situations. We illustrate the situation in the case of countable flag complexes.

Definition 6.6.6. Let $\mathcal{K}$ be a Borel simplicial complex on a Polish space $X$. Whenever $A \subset X$ is a $\mathcal{K}$-set, let $\tau_A$ be the Coll($\omega, X$)-name for the set of all conditions $p \in P_{\mathcal{K}^c}$ for which $\forall x \in A \langle x, 0 \rangle \in p$ and $\forall x \in (X \cap V) \setminus A \langle x, 1 \rangle \in p$.

Theorem 6.6.7. Let $\mathcal{K}$ be a countable Borel flag complex on a Polish space $X$. Let $P = P_{\mathcal{K}^c}$.

1. For every $\mathcal{K}$-set $A \subset X$, the pair $\langle \text{Coll}(\omega, X), \tau_A\rangle$ is a balanced virtual condition in $P$;

2. for every balanced pair $\langle Q, \tau \rangle$ there is a $\mathcal{K}$-set $A \subset X$ such that $\langle Q, \tau \rangle$ is equivalent to $\langle \text{Coll}(\omega, X), \tau_A\rangle$;

3. distinct $\mathcal{K}$-sets yield inequivalent balanced conditions.

Proof. To prove (1), fix a function $f : X \to [X]^\aleph_0$ which witnesses the fact that $\mathcal{K}$ is a countable flag complex. Fix a $\mathcal{K}$ set $A \subset X$. Suppose that $V[H_0], V[H_1]$ are mutually generic extensions and $p_0 \in V[H_0], p_1 \in V[H_1]$ are conditions in $P$ such that $\forall x \in A \langle x, 0 \rangle \in p_0$ and $\forall x \in (X \cap V) \setminus A \langle x, 1 \rangle \in p_0$ and similarly for $p_1$. We must show that $p_0, p_1$ are compatible.

To show that $p_0 \cup p_1$ is a $\mathcal{K}^c$-set, by symmetricity there are two items to check:

- if $\langle x_0, 0 \rangle \in p_0$ and $\langle x_1, 0 \rangle \in p_1$ then $x_0 \notin f(x_1)$;

- if $\langle x_0, 0 \rangle \in p_0$ and $\langle x_1, 1 \rangle \in p_1$ then $x_0 \neq x_1$. 

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To see the first item, note that if \( x_0 \in V \) then \( x_0 \in A \). Therefore, \( \langle x_0, 0 \rangle \in p_1 \) and the conclusion follows from the fact that \( p_1 \) is a \( K' \)-set. If, on the other hand, \( x_0 \notin V \), then \( x_0 \notin V[H_1] \) by the product forcing theorem, and so \( x_0 \notin f(x_1) \) as desired. The second item is proved by a split into the same two cases. In conclusion, \( p_0, p_1 \) are compatible and so the pair \( \langle \text{Coll}(\omega, X), \tau_A \rangle \) is balanced.

To prove (2), first use a density argument to prove that \( P \) forces that for each \( x \in X \), exactly one of the pairs \( \langle x, 0 \rangle \) and \( \langle x, 1 \rangle \) is in the generic set \( \hat{A}_{\text{gen}} \subset X \times 2 \). Use the balance of the pair \( \langle Q, \tau \rangle \) to show that for every point \( x \in X \), either \( Q \models \tau \models_P (\hat{x}, 0) \in \hat{A}_{\text{gen}} \) or \( Q \models \tau \models_P (\hat{x}, 0) \in \hat{A}_{\text{gen}} \). Let \( A \subset X \) be the set of all points for which the former alternative occurs, and note that \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, X), \tau_A \rangle \).

Finally, (3) is immediate. \( \square \)

In the case of modular complexes, there is a more sophisticated way of defining the complemented variation:

**Definition 6.6.8.** Let \( K \) be a modular Borel simplicial complex on a Polish space \( X \), witnessed by a Borel function \( f: K \to [Y]^{|\omega|} \) for some Polish space \( Y \) disjoint from \( X \). The simplicial complex \( K^{mc} \) on \( X \times Y \) consists of finite sets \( a \subset X \cup Y \) such that the set \( a^0 = a \cap X \) belongs to \( K \) and the set \( a^1 = a \cap Y \) is disjoint from \( f(a^0) \).

The definition clearly depends on the function \( f \); the function will be identified separately or understood from the context.

**Proposition 6.6.9.** Let \( K \) be a modular Borel simplicial complex on a Polish space \( X \). The complex \( K^{mc} \) is modular.

**Proof.** Let \( f: K \to [Y]^{|\omega|} \) be a Borel function witnessing the modularity of \( K \), used in the construction of \( K^{mc} \). Let \( f^{mc}: K^{mc} \to [Y \times 2]^{|\omega|} \) be a function defined by \( f^{mc}(a) = (f(a^0) \times \{0\}) \cup ((f(a^0) \cup a^1) \times \{1\}) \). It is not difficult to check that the function \( f^{mc} \) witnesses the modularity of the complex \( K^{mc} \). \( \square \)

The complemented variations can be defined in the same way in the quotient context. This is the situation in the following example.

**Example 6.6.10.** Let \( E, F \) be Borel equivalence relations on respective Polish spaces \( X, Y \), each with uncountably many classes. Let \( K \) be a simplicial complex on the set \( X \times Y/E \times F \) consisting of all finite sets \( a \) which are finite injections from \( X/E \) to \( Y/F \). This is a quotient modular complex as witnessed by the function \( f: K \to (X/E) \cup (Y/F) \) defined by \( f(a) = \text{dom}(a) \cup \text{rng}(a) \). The balanced conditions of the poset \( P_K \) are classified by bijections between the spaces of virtual \( E \)-classes and virtual \( F \)-classes, and so the poset is not balanced unless \( \lambda(E) = \lambda(F) \).

The complemented variation allows more freedom. Consider the complement complex \( K^{mc} \) on \( (X \times Y/E \times F) \cup (X/E) \cup (Y/F) \). The resulting poset adds a partial injection from \( X/E \) to \( Y/F \); the balanced conditions of the resulting poset are classified by partial injections from \( X^{**} \) to \( Y^{**} \) (plus the complement
of the domain and the range). In order to force a total injection from $X/E$ to $Y/F$, consider the restriction $K^{mc} | (X \times Y/E \times F) \cup (Y/F)$ and the resulting poset $P$. This is exactly the collapse poset isolated in Definition 6.4.2.
Chapter 7

Ultrafilter forcings

Many applications of the axiom of choice rely on nonprincipal ultrafilters on \(\omega\) and their combinatorial properties. In this section, we will show that in several cases, natural attempts to add an ultrafilter result in balanced forcings and moreover the resulting generic ultrafilter can be characterized in a simple way.

7.1 A Ramsey ultrafilter

The most elementary attempt at adding a nonprincipal ultrafilter is the poset \(P\) of infinite subsets of \(\omega\) ordered by inclusion. Here, the classification of balanced conditions is particularly appealing and simple. It starts with a central definition and proposition.

Definition 7.1.1. Let \(U\) be a nonprincipal ultrafilter on \(\omega\). An infinite set \(a \subset \omega\) in some forcing extension diagonalizes \(U\) if for every \(b \in U\), \(a \setminus b\) is finite.

Proposition 7.1.2. Let \(U\) be a nonprincipal ultrafilter on \(\omega\). Suppose that \(Q_0, Q_1\) are posets and \(\tau_0, \tau_1\) are respective names for \(I\)-positive subsets of \(\omega\) which diagonalize \(U\). Then \(Q_0 \times Q_1 \Vdash \tau_0, \tau_1\) are compatible in the poset \(P\).

Proof. The most appealing argument uses the general Theorem 13.3.2 on product forcing extensions. Let \(H_0, H_1 \subseteq Q_0, Q_1\) be mutually generic filters and let \(a_0 = \tau_0/H_0\) and \(a_1 = \tau_1/H_1\). Suppose toward contradiction that \(a_0, a_1\) are incompatible as conditions in \(P\); i.e. the intersection \(a_0 \cap a_1\) is finite. By Theorem 13.3.2, there are sets \(b_0, b_1 \subset \omega\) in the ground model such that \(a_0 \subset b_0\), \(a_1 \subset b_1\), and \(b_0 \cap b_1\) is finite. At most one of the sets \(b_0, b_1\) belongs to the ultrafilter \(U\). If, say, \(b_0 \notin U\), then \(\omega \setminus b_0 \in U\), violating the assumption that \(a_0\) diagonalizes the ultrafilter \(U\). \(\square\)

Definition 7.1.3. Let \(U\) be a nonprincipal ultrafilter on \(\omega\). The symbol \(\tau_U\) denotes the Coll\((\omega, U)\)-name for the analytic collection of all infinite sets \(a \subset \omega\) which diagonalize the ultrafilter \(U\).

Finally, we can state and prove the promised classification theorem.
Theorem 7.1.4. 1. The pair \( \langle \text{Coll}(\omega, U), \tau_U \rangle \) is a balanced virtual condition for \( P \) for every nonprincipal ultrafilter \( U \) on \( \omega \); 

2. If \( \langle Q, \tau \rangle \) is a balanced pair for \( P \), then there is a nonprincipal ultrafilter \( U \) such that \( \langle Q, \tau \rangle \) is equivalent to \( \langle \text{Coll}(\omega, U), \tau_U \rangle \); 

3. distinct nonprincipal ultrafilters yield inequivalent balanced virtual conditions.

Proof. For (1), note that the pair in fact is a virtual condition: in \( \text{Coll}(\omega, U) \), the set \( U \) is countable and so \( \tau_U \) is an analytic subset of \( P \), and the evaluation of \( \tau_U \) does not depend on the particular generic filter on the collapse poset. The balance of the pair follows immediately from Proposition 7.1.2 applied to \( U \).

For (2), first use the balance of the name \( \tau \) to note that for every set \( a \subset \omega \), it must be the case that either \( Q \models \tau \subset \check{a} \) up to a finite set, or \( Q \models \tau \cap \check{a} \) is finite. Let \( U \) be the set of all \( a \subset \omega \) for which the first option occurs. It is immediate that \( U \) is a nonprincipal ultrafilter on \( \omega \). The equivalence of \( \langle Q, \tau \rangle \) with \( \langle \text{Coll}(\omega, U), \tau_U \rangle \) is immediately clear from Proposition 7.1.2.

For (3), note that if \( U_0, U_1 \) are distinct ultrafilters on \( \omega \) then there is a set \( a \subset \omega \) such that \( a \in U_0 \) and \( \omega \setminus a \in U_1 \). Thus the balanced names \( \tau_{U_0} \) and \( \tau_{U_1} \) represent incompatible virtual conditions as the former is below \( a \) and the other is below \( \omega \setminus a \).

Corollary 7.1.5. The poset \( P \) is balanced.

Proof. For any infinite set \( a \subset \omega \) there is a nonprincipal ultrafilter \( U \) containing \( a \) as an element. A reference to Theorem 7.1.4(1) then completes the argument.

7.2 An ultrafilter disjoint from an \( F_\sigma \)-ideal

Let \( I \) be an \( F_\sigma \)-ideal on \( \omega \). Consider the poset \( P \) of all \( I \)-positive subsets of \( \omega \), ordered by inclusion. The poset \( P \) adds an ultrafilter disjoint from \( I \). This section provides a classification of the balanced conditions in \( P \) and description of the generic ultrafilter. For the duration of this section, write \( \nu \) for the space \( \nu(\text{Coll}(\omega, U), \tau_U) \) which are closed under taking a subset. \( \nu \) is ordered by inclusion. The classification of the balanced conditions is facilitated by the following definition and proposition.

Definition 7.2.1. Let \( F \subset \nu \) be a filter. An \( I \)-positive set \( a \subset \omega \) in some forcing extension diagonalizes \( F \) if for every ground model set \( C \in F \) there is \( b \in I \ a \setminus b \subset C \), and for all ground model sets \( C \in \nu \setminus F \), \( \nu(a) \cap C \subset I \).

Proposition 7.2.2. Let \( F \subset \nu \) be a filter. Suppose that \( Q_0, Q_1 \) are posets and \( \tau_0, \tau_1 \) are respective names for \( I \)-positive subsets of \( \omega \) which diagonalize \( F \). Then \( Q_0 \times Q_1 \models \tau_0, \tau_1 \) are compatible in the poset \( P \).
Proof. Note that the compatibility of the sets \( \tau_0, \tau_1 \) is equivalent to the intersection \( \tau_0 \cap \tau_1 \) being an \( I \)-positive set. Let \( I = \bigcup_n I_n \) be a presentation of \( I \) as an increasing union of closed sets, each closed under subset. Let \( q_0 \in Q_0, q_1 \in Q_1 \) and \( n \in \omega \). We have to produce a finite set \( b \subset \omega \) and a condition \( q_0' \leq q_0 \) and \( q_1' \leq q_1 \) such that \( q_0' \models \bar{b} \subset \tau_0, q_1' \models \bar{b} \subset \tau_1 \), and there is no element of \( I_n \) which contains \( b \) as a subset. The proposition will then follow by an elementary density argument.

To find the set \( b \), let \( a_0 = \{ b \in [\omega]^{<\aleph_0} : \exists q_0' \leq q_0 \ q_0' \models \bar{b} \subset \tau_0 \} \) and \( a_1 = \{ b \in [\omega]^{<\aleph_0} : \exists q_1' \leq q_1 \ q_1' \models \bar{b} \subset \tau_1 \} \). Also, let \( C_0, C_1 \subset \mathcal{P}(\omega) \) be the closures of \( a_0, a_1 \). Note that \( C_0, C_1 \) are closed under subset since \( a_0, a_1 \) are, and so \( C_0, C_1 \in \nu \).

Since \( q_0 \models \tau_0 \in C_0 \) and \( q_1 \models \tau_1 \in C_1 \), by the definition of the diagonalization it must be the case that \( C_0 \cap C_1 \in F \). Let \( C = C_0 \cap C_1 \in F \). There must be an \( I \)-positive set \( c \subset C \) since \( Q_0 \) forces the existence such a set, and a Mostowski absoluteness yields the existence of such a set in the ground model. Let \( b \subset c \) be a finite set such that no superset of \( b \) belongs to \( I_n \). By the definitions of the sets \( C_0, C_1 \), it must be the case that \( b \in a_0 \cap a_1 \). Choose conditions \( q_0' \leq q_0 \) and \( q_1' \leq q_1 \) which force \( \bar{b} \) to be a subset of \( \tau_0 \) and \( \tau_1 \) respectively. This completes the proof.

Definition 7.2.3. A filter \( F \subset \nu \) is a balanced if in the \( \text{Coll}(\omega, F) \)-extension there is a set diagonalizing \( F \) such that \( a \notin I \). If \( F \) is a balanced filter then let \( \tau_F \) be the \( \text{Coll}(\omega, F) \)-name ???

Theorem 7.2.4.  
1. For every balanced filter \( F \subset \nu \), the pair \( \langle \text{Coll}(\omega, F), \tau_F \rangle \) is a virtual balanced condition in \( P \);

2. whenever \( \langle Q, \tau \rangle \) is a balanced pair then there is a balanced filter \( F \subset \nu \) such that \( \langle Q, \tau \rangle \) is equivalent to \( \langle Q, \tau \rangle \);

3. distinct balanced filters yield distinct balanced conditions.

Definition 7.2.5. Let \( F \) be a weak \( I \)-point and \( R \) a poset. A \( R \)-name \( \sigma \) is \( F \)-good if \( P \) forces \( \tau \) to be an \( I \)-positive subset of \( \omega \) such that for every point \( C \in Y \), if \( C \in F \) then there is \( b_C \in C \) such that \( \sigma \setminus b_C \in I \), and if \( C \notin F \) then \( \{ b \cap a : b \in C \} \in I \).

Proposition 7.2.6. If \( R_0, R_1 \) are posets and \( \sigma_0, \sigma_1 \) are respective \( F \)-good names on them, then \( R_0 \times R_1 \models \sigma_0 \cap \sigma_1 \notin I \).

Proof. Express \( I = \bigcup_n I_n \) as a countable union of closed sets. Suppose that \( r_0 \in R_0, r_1 \in R_1 \) are conditions and \( n \in \omega \) is a number; we must find conditions \( r_0' \leq r_0, r_1' \leq r_1 \), and also a finite set \( b \subset \omega \) such that no element of \( I_n \) contains \( b \) as a subset, and \( r_0' \models R_0 b \subset \sigma_0 \) and \( r_1' \models R_1 \bar{b} \subset \sigma_1 \). To do this, let \( a_0 = \{ b \in [\omega]^{<\aleph_0} : \exists r_0 \ r \models b \subset \sigma_0 \} \) and \( a_1 = \{ b \in [\omega]^{<\aleph_0} : \exists r_1 \ r \models \bar{b} \subset \sigma_1 \} \); let also \( C_0, C_1 \subset \mathcal{P}(\omega) \) be the closures of \( a_0, a_1 \).

Example 7.2.7. Let \( I \) be the ideal on \( \omega \times \omega \) generated by graphs of functions. There is an ultrafilter disjoint from \( I \) which does not represent a balanced virtual condition in \( P(I) \). In particular, the simplicial complex \( \mathcal{K}(I) \) is not separated.
Proof. Let $G$ be the random graph on $\omega$. It is well-known that $J = \{a \subseteq \omega: G \upharpoonright a$ does not contain an isomorphic copy of $G\}$ is a coanalytic ideal on $\omega$. Let $K$ be the coanalytic ideal on $\omega \times \omega$ consisting of all sets $a \subseteq \omega \times \omega$ such that all but finitely many vertical sections of $a$ belong to $J$. It is immediately clear that $I \subseteq K$. Use the axiom of choice to find an ultrafilter disjoint from $K$. We claim that the ultrafilter $U$ does not represent a balanced virtual condition in $P(I)$.

To prove this, it will be enough to find posets $R_0, R_1$ with corresponding names $\sigma_0$ and $\sigma_1$ for subsets of $\omega \times \omega$ such that for every set $a \subseteq U$, $R_0 \Vdash \sigma_0 \cap \dot{a} \notin I$ holds, similarly for $R_1$ and $\sigma_1$, and $R_0 \times R_1 \Vdash \sigma_0 \cap \sigma_1 \in I$.

The poset $R_0$ consists of pairs $r = \langle w_r, a_r \rangle$ where $w_r \subseteq \omega \times \omega$ is finite and every vertical section of $w_r$ is a $G$-anticlique, and the set $a_r \subseteq \omega \times \omega$ is in the ultrafilter $U$. The ordering is defined by $s \leq r$ if $w_r \subseteq w_r$, $a_r \subseteq a_s$, and $w_s \setminus w_r \subseteq a_r$. The name $\sigma_0$ is just the $R_0$-name for the union of the first coordinates of conditions in the generic filter. To see that for every set $a \subseteq U$, $R_0 \Vdash \sigma_0 \cap \dot{a} \notin I$, suppose that $r \in R_0$ is a condition and $n \in \omega$ is a number; it will be enough to find a condition $s \leq r$ such that for every set $a \subseteq U$, $R_0 \Vdash \sigma_0 \cap \dot{a} \notin I$. To find the condition $s$, consider the set $b = a \cap a_r \in U$. Since $b \in U$, it follows that $b \notin K$ and so there must be $m \in \omega$ such that the vertical section $(w_r)_m$ is empty and the vertical section $b_m$ contains an isomorphic copy of $G$; in particular, it contains a $G$-anticlique $c \subseteq \omega$ of size $n$. Let $w_s = w_r \cup (c \times \{m\})$ and let $s = \langle w_s, a_r \rangle$. It is not difficult to see that the condition $s \leq r$ works as desired.

The poset $R_1$ and the name $\sigma_1$ are similar to $R_0, \sigma_0$ except that the vertical sections of $w_r$ are demanded to be $G$-cliques. It is also clear that $R_0 \times R_1 \Vdash \sigma_0 \cap \sigma_1 \in I$, since the vertical sections of $\sigma_0 \cap \sigma_1$ must be simultaneously $G$-cliques and $G$-anticlques; in other words, they have to be either empty or singletons. \hfill \Box

The ideal $I$ in Example 7.2.7 is $F_\sigma$. For $F_\sigma$-ideals, we do not have any classification of virtual conditions or balanced virtual conditions, but at least we can find many balanced virtual conditions under the Continuum Hypothesis:

Definition 7.2.8. Let $I$ be an ideal on $\omega$. An ultrafilter $U$ on $\omega$ is $I$-quasi-generic if $I \cap U = 0$ and for every analytic ideal $J$ on $\omega$, if $J \cap U = 0$ then there is a set $a \subseteq U$ such that $J \cap \mathcal{P}(a) \subseteq I \cap \mathcal{P}(a)$.

Proposition 7.2.9. Let $I$ be an $F_\sigma$-ideal on $\omega$. Let $U$ be an $I$-quasi-generic ultrafilter on $\omega$. Then $U$ represents a balanced virtual condition in $P(I)$.

Proof. To begin, fix a representation $I = \bigcup_m I_m$ of the $F_\sigma$-ideal as a union of countably many closed sets closed under subset. In any generic extension, call a set $a \subseteq \omega$ positive if its intersection with any element of the ultrafilter $U$ does not belong to $I$. The following simple claim is critical for the argument:

Claim 7.2.10. Suppose that $Q$ is a poset and $\sigma$ is a $Q$-name for a positive subset of $\omega$. Then there is a filter $g \subseteq Q$ such that the set $b_g = \{n \in \omega: \exists q \in g \forall \dot{n} \in \sigma\}$ belongs to the ultrafilter $U$. 

7.2. AN ULTRAFILTER DISJOINT FROM AN $F_\sigma$-IDEAL

Proof. Suppose towards contradiction that the conclusion fails. Let $M$ be a countable elementary submodel of a large structure containing $I, U, Q, \sigma, R$. Let $J$ be the $\sigma$-ideal generated by sets $b_q$ where $q \in Q \cap M$ is a filter. This is an analytic ideal and by the contradictory assumption, it is disjoint from $U$. Use the quasi-genericity of the ultrafilter $U$ to find a set $a \in U$ such that $J \cap P(a) \subset I \cap P(a)$. Note that the set $\sigma \cap \hat{a}$ is forced to not belong to the ideal $I$, and so for every condition $q \in Q$ and every $m \in \omega$ there is a condition $r \leq q$ and a finite set $c \subset a$ such that $r \Vdash \hat{c} \subset \sigma$ and no set containing $c$ belongs to $I_m$. Note that by elementarity of the model $M$, if $q \in M$ then one can find such a condition $r \leq q$ in $M$ even though the set $a$ may not belong to $M$. Thus, by induction it is possible to produce a descending sequence $\langle q_m : m \in \omega \rangle$ of conditions in $M \cap Q$ such that for each $m \in \omega$, $c_m \subset a$, no set containing $c_m$ belongs to $I_m$, and $q_m \Vdash \hat{c}_m \subset \sigma_m$. Let $g \in M \cap Q$ be the filter generated by the conditions $q_m$ for $m \in \omega$. We have that $b_j \in J$ and $b_j \cap a \notin I$; this contradicts the choice of the set $a$.

Now, consider the $\text{Coll}(\omega, U)$-name $\tau_U$ for a generic enumeration of the ultrafilter $U$. Clearly, this is a virtual condition in the poset $P(I)$; we must show that it is balanced. Since the ideal $I$ is $F_\sigma$, it is enough to show that if $R_0, R_1$ are posets and $\sigma_0, \sigma_1$ are respective names on them for positive subsets of $\omega$, then $R_0 \times R_1 \Vdash \sigma_0 \cap \sigma_1$ is a positive subset of $\omega$. To see this, suppose that $r_0 \in R_0, r_1 \in R_1$ are conditions, $b \in U$ is a set and $m \in \omega$ is a number; we must find conditions $r_0', r_1' \leq r_0, r_1$, a finite set $c \subset b$ such that no set containing $c$ belongs to $I_m$, and $r_0' \Vdash \hat{c} \subset \sigma_0$ and $r_1' \Vdash \hat{c} \subset \sigma_1$. To do this, apply the claim to find filters $g_0 \in R_0$ and $g_1 \in R_1$ containing $r_0, r_1$ respectively such that the sets $d_0 = \{ n \in \omega : \exists r \in g_0 \ r \Vdash \hat{n} \in \sigma_0 \}$ and $d_1 = \{ n \in \omega : \exists r \in g_1 \ r \Vdash \hat{n} \in \sigma_1 \}$ both belong to the ultrafilter $U$. Then $d_0 \cap d_1 \cap b \in U$ and so there is a finite set $c \subset d_0 \cap d_1 \cap b$ such that no set containing $c$ belongs to $I_m$. Since $g_0, g_1$ are filters, there must be a condition $r_0' \leq r_0$ in $g_0$ and $r_1' \leq r_1$ such that $r_0' \Vdash \hat{c} \subset \sigma_0$ and $r_1' \Vdash \hat{c} \subset \sigma_1$. The proof is complete.

Corollary 7.2.11. (ZFC+CH) Let $I$ be an $F_\sigma$-ideal. The poset $P(I)$ is balanced.

Proof. We produce a single balanced virtual condition; a minor adjustment of the argument produces a balanced virtual condition below any given condition. Let $\langle J_\alpha : \alpha \in \omega_1 \rangle$ be an enumeration of all analytic ideals on $\omega$. By recursion on $\alpha \in \omega_1$ build sets $a_\alpha \subset \omega$ which are modulo finite decreasing, $I$-positive, and for each ordinal $\alpha$, either $J_\alpha \cap P(a_\alpha) \subset I \cap P(\alpha)$ or $a_{\alpha+1} \in J_\alpha$. This is easy to do; at any limit stage $\alpha$ the assumption that $I$ is $F_\sigma$ makes it possible to find the $I$-positive diagonal intersection of the sets $\{a_\beta : \beta \in \alpha \}$. In the end, the sets $a_\alpha$ for $\alpha \in \omega_1$ together with the filter dual to $I$ generate an $I$-quasi-generic ultrafilter and therefore a balanced virtual condition in $P(I)$.

Note that Proposition 7.2.9 does not yield a complete classification of balanced virtual conditions: for $I = \text{the ideal of finite sets}$, all ultrafilters, not just the quasi-generic ones, yield balanced virtual conditions. Sadly, as soon as one
steps out of the realm of $F_\sigma$-ideals, even this limited understanding of balanced virtual conditions disappears.

### 7.3 Semigroup ultrafilters

In this section, we show that in a natural forcing connected with a countable semigroup, the balanced conditions exist and are classified by idempotent ultrafilters. This connects the theory of balanced forcing extensions with Ramsey theory and dynamics.

**Definition 7.3.1.** Let $\Gamma$ be a countable semigroup. The poset $P(\Gamma)$ is defined as follows. The conditions of $P(\Gamma)$ are elements of $\Gamma^\omega$. The ordering is defined by $q \leq p$ if there are nonempty finite sets $a_n \subset \omega$ for all $n \in \omega$ such that $\max(a_n) < \min(a_{n+1})$ and $\prod_{m \in a_n} p(m) = q(n)$, where the products are always taken in the increasing order.

It is not difficult to see that given a condition in $P(\Gamma)$, shifting its entries to the left and/or changing finitely many entries does not change the separative quotient equivalence class of the condition. It follows that the separative quotient of $P(\Gamma)$ is a $\sigma$-closed poset. The purpose of the poset $P(\Gamma)$ is clear from the following definition and proposition.

**Definition 7.3.2.** Let $A$ be a collection of subsets of $\Gamma$. A sequence $p \in P(\Gamma)$ diagonalizes $A$ if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\prod_{m \in a} p(m) \in b$ holds. The sequence $p \in P(\Gamma)$ sorts out $A$ if for every set $b \in A$ there is $n \in \omega$ such that for every nonempty finite set $a \subset \omega \setminus n$, $\prod_{m \in a} p(m) \not\in b$ holds. If $b \in A$ and the former alternative occurs, we say that $p$ accepts $b$, if the latter alternative occurs then $p$ declines $b$.

**Proposition 7.3.3.** For every condition $p \in P(\Gamma)$ and every countable set $A \subset P(\Gamma)$, there is a condition $q \leq p$ which sorts out $A$.

**Proof.** Let $A = \{b_n : n \in \omega\}$. By induction on $n \in \omega$ build a descending sequence $p_n$ of conditions in $P(\Gamma)$ such that $p = p_0$ and for all $n \in \omega$, for all nonempty finite sets $a \subset \omega$ it is the case that $\prod_{m \in a} p_{n+1}(m) \in b_n$ holds, or for all nonempty finite sets $a \subset \omega$ it is the case that $\prod_{m \in a} p_{n+1}(m) \not\in b_n$ holds. To perform the induction step, use Hindman’s theorem on the partition $\pi_n$ of nonempty finite subsets of $\omega$ defined by $\pi_n(a) = 1$ if $\prod_{m \in a} p_n(m) \in b_n$ holds.

In the end, let $q \in P(\Gamma)$ be defined by $q(n) = p_n(n)$ and observe that the conclusion of the proposition is satisfied.

Suppose that $G \subset P(\Gamma)$ is a generic filter. By Proposition 7.3.3 and a density argument, the set $U \subset P(\Gamma)$ of all sets $b \subset \Gamma$ such that some condition $p \in G$ accepts $b$, is a ultrafilter. Also, $G$ can be recovered from $U$ as the set of all conditions $p \in P(\Gamma)$ such that for all $n \in \omega$, the set $\{\prod_{m \in a} p(m) : a \subset \omega$ is a nonempty set and $\min(a) > n\}$ belongs to the ultrafilter $U$. 

7.3. SEMIGROUP ULTRAFILTERS

It turns out that the balanced conditions in the poset $P(\Gamma)$ correspond to a well-known concept from topological dynamics and Ramsey theory.

**Definition 7.3.4.** [13, Section 4.1] Let $(\Gamma, \cdot)$ be a countable semigroup.

1. for ultrafilters $U_0, U_1$ on $\Gamma$, let $U_0 \cdot U_1 = \{ A \subseteq \Gamma : \{ \gamma \in \Gamma : \{ \delta \in \Gamma : \gamma \cdot \delta \in A \} \in U_1 \} \in U_0 \}.$

2. an ultrafilter $U$ on $\Gamma$ is an idempotent if $U \cdot U = U$.

It turns out that $\cdot$ is a semi-continuous operation on the space $\beta\Gamma$ of all ultrafilters on $\Gamma$. The fundamental Ellis–Numakura theorem [13, Section 2.4] says that every closed subsemigroup of $\beta\Gamma$ contains an idempotent ultrafilter. The idempotent ultrafilters have an alternative diagonalization description.

**Proposition 7.3.5.** Let $U$ be an ultrafilter on $\Gamma$. The following are equivalent:

1. in $\text{Coll}(\omega, P(\Gamma))$ extension there is a condition in $P(\Gamma)$ diagonalizing $U$;

2. for every countable subset of $U$, there is a condition in $P(\Gamma)$ diagonalizing it;

3. $U$ is an idempotent.

**Proof.** For the implication (1) $\rightarrow$ (2), if $U' \subset U$ is a countable set, then in some forcing extension there is a condition $p \in P$ which diagonalizes $U$ and therefore $U'$. By a Mostowski absoluteness argument between the ground model and the forcing extension, there must be a condition $p \in P$ diagonalizing $U'$.

For the implication (2) $\rightarrow$ (1), consider the partial order $Q = [P(U)]^{\aleph_0}$ modulo the nonstationary ideal. Let $G \subset Q$ be a filter generic over $V$ and let $j : V \rightarrow M$ be the generic ultrapower. Then $M$ is an $\omega$-model which is possibly illfounded. Moreover, $M$ contains the set $j''U$ as the equivalence class of the identity function on $[P(U)]^{\aleph_0}$, and $M \models j''U \subset U$ is a countable set by the Loś theorem. Thus, if the ground model satisfies (2), then by elementarity $M$ (and so $V[G]$) contains a condition $p \in P$ which diagonalizes $j''U$ and therefore $U$. The existence of such a condition then transfers to any $\text{Coll}(\omega, P(\Gamma))$ extension of the ground model. This confirms (1).

(3) implies (2): this is the contents of Galvin–Glazer theorem or its folkloric variation for countable sets, [13, Chapter 5]. Finally, negation of (3) implies the negation of (2). To see this, if $U$ is not an idempotent, there must be a set $A \in U$ such that $A \notin U \cdot U$. This means that the set $B = \{ \gamma \in \Gamma : \{ \delta \in \Gamma : \gamma \cdot \delta \notin A \} \in U \}$ belongs to $U$. Consider the countable set including $A, B$, as well as all the sets $C_\gamma = \{ \delta \in \Gamma : \gamma \cdot \delta \notin A \}$ for $\gamma \in B$; we will show that it cannot be diagonalized. Suppose towards contradiction that some condition $p \in P(\Gamma)$ does diagonalize it. This means that there is $m_0 \in \gamma$ such that for all $k > m \geq m_0$, $p(m) \in B$ and $p(m) \cdot p(k) \in A$. There must be also some $k_0 \in \omega$ such that for all $k \geq k_0$, $p(k) \in C_{p(m_0)}$. Then $p(m_0) \cdot p(k_0)$ should belong simultaneously to $A$ and to the complement of $A$, which is a contradiction. \qed
Finally, the statement of the classification theorem for the balanced virtual conditions in the poset $P(\Gamma)$ is at hand.

**Definition 7.3.6.** Whenever $\langle \Gamma, \cdot \rangle$ is a countable semigroup and $U$ is an idempotent ultrafilter, let $\tau_U$ denote the $\text{Coll}(\omega, P(\Gamma))$-name for the (nonempty analytic) set of all conditions diagonalizing $U$.

**Theorem 7.3.7.** Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup.

1. Whenever $U$ is an idempotent ultrafilter then $\langle \text{Coll}(\omega, P(\Gamma)), \tau_U \rangle$ is a balanced virtual condition;

2. every balanced pair is equivalent to $\langle \text{Coll}(\omega, P(\Gamma)), \tau_U \rangle$ for some idempotent ultrafilter $U$;

3. distinct idempotent ultrafilters give rise to inequivalent balanced virtual conditions.

**Proof.** (1) follows immediately from the following claim:

**Claim 7.3.8.** Let $U$ be an ultrafilter on $\Gamma$ and $\langle Q_0, \tau_0 \rangle, \langle Q_1, \tau_1 \rangle$ be posets and their respective names for elements of $P(\Gamma)$ diagonalizing $U$. Then $Q_0 \times Q_1 \Vdash \tau_0, \tau_1$ are compatible in $P(\Gamma)$.

**Proof.** Let $q_0 \in Q_0$ and $q_1 \in Q_1$ be conditions and $n \in \omega$ be a natural number. We must find an element $\gamma \in \Gamma$ and conditions $q'_0 \leq q_0$ and $q'_1 \leq q_1$ and finite nonempty sets $a_0, a_1 \subseteq \omega$ with minimum larger than $n$ such that $q'_0 \Vdash \gamma = \Pi_{m \in a_0} \tau_0(m)$ and $q'_1 \Vdash \gamma = \Pi_{m \in a_1} \tau_1(m)$. The compatibility of $\tau_0, \tau_1$ is then granted by a genericity argument.

Let $A_0 \subseteq \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subseteq \omega$ with minimum greater than $n$ and a condition $q' \leq q_0$ in $Q_0$ forcing $\gamma = \Pi_{m \in a} \tau_0(m)$. Since $\tau_0$ is forced to diagonalize $U$, it must be the case that $A_0 \in U$. Similarly, let $A_1 \subseteq \Gamma$ be the set of all $\gamma \in \Gamma$ such that there exists a finite nonempty set $a \subseteq \omega$ with minimum greater than $n$ and a condition $q' \leq q_1$ in $Q_1$ forcing $\gamma = \Pi_{m \in a} \tau_1(m)$. Since $\tau_1$ is forced to diagonalize $U$, it must be the case that $A_1 \in U$. Choose $\gamma \in A_0 \cap A_1$ and choose $q'_0 \leq q_0, q'_1 \leq q_1$ and sets $a_0, a_1$ witnessing the membership of $\gamma$ in $A_0, A_1$. This completes the proof.

For (2), suppose that $\langle Q, \tau \rangle$ is a balanced pair. Without loss of generality we may assume that $Q$ collapses the size of $P(\Gamma) \cap V$ to $\aleph_0$. Strengthening $\tau$ in the $Q$-extension repeatedly by an application of Proposition 7.3.3, we may assume that $\tau$ sorts out $P(\Gamma) \cap V$. Let $\sigma$ be the $Q$-name for the collection of all sets in $P(\Gamma) \cap V$ which $\tau$ accepts.

**Claim 7.3.9.** The membership of every set $A \subseteq \Gamma$ in $\dot{U}$ is decided by the largest condition in $Q$.

**Proof.** If not, then there is a set $A \subseteq \Gamma$ and conditions $q_0, q_1 \in Q$ such that $q_0 \Vdash \dot{A} \in \dot{U}$ and $q_1 \Vdash \dot{\Gamma} \setminus \dot{A} \notin \dot{U}$. Plainly, the condition $\langle q_0, q_1 \rangle$ forces in the product $Q \times Q$ that $\tau_{\text{left}}, \tau_{\text{right}}$ are conditions incompatible in $P(\Gamma)$, contradicting the balance assumption on the name $\tau$. 

$\square$
Let $U = \{A \subseteq \Gamma: Q \vDash \check{A} \in \sigma\}$. This is an ultrafilter on $\Gamma$. By Proposition 7.3.5, it is an idempotent ultrafilter. By Claim 7.3.8, the pair $\langle Q, \tau \rangle$ is equivalent to $\langle \text{Coll}(\omega, P(\Gamma)), \tau_U \rangle$. (2) follows.

Finally (3) is immediate.

\[ \square \]

**Corollary 7.3.10.** Let $\langle \Gamma, \cdot \rangle$ be a countable semigroup. The poset $P(\Gamma)$ is balanced.

**Proof.** The Ellis–Numakura theorem yields an idempotent ultrafilter below any condition $p \in P(\Gamma)$.

\[ \square \]
Chapter 8

Cardinalities in $W[G]$

8.1 Preserving the well-ordered divide

The main feature of the balanced Suslin forcing is that it does not add any well-ordered sequences of elements of the symmetric Solovay extension. This has a number of cardinality corollaries for the resulting extensions.

Theorem 8.1.1. In a cofinally balanced extension of a symmetric Solovay model $W$, every well-ordered sequence of elements of $W$ belongs to $W$.

Proof. Let $P$ be a Suslin forcing and $\kappa$ be an inaccessible cardinal such that $P$ is balanced cofinally below $\kappa$. Let $W$ be a symmetric Solovay model derived from $\kappa$ and work in $W$. Suppose towards a contradiction that there is a condition $p \in P$, an ordinal $\alpha$ and a $P$-name $\tau$ such that $p \Vdash \tau$ is an $\alpha$-sequence of elements of $W$ which does not belong to $W$. The name $\tau$ is definable from some real parameter $z \in 2^\omega$ and some elements of the ground model. Find an intermediate model $V[K]$ which is an extension of the ground model by a poset of size $< \kappa$ such that $p, z \in V[K]$ and $P$ is balanced in $V[K]$.

Work in $V[K]$. Let $\tilde{p}$ be a balanced virtual condition in $P$ below $p$. Since in $W$, $\tau$ is forced not to belong to $W$, there must be in $V[K]$ an ordinal $\beta \in \alpha$ and a posets $R$ of size $< \kappa$, $R$-names $\sigma_0, \sigma_1$ for conditions in $P$ stronger than $\tilde{p}$ $R$-names $\eta_0, \eta_1$ for elements of $2^\omega$ and formulas $\phi_0, \phi_1$ such that

- $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash_P \tau(\tilde{\beta})$ is the unique $w$ in the symmetric extension satisfying $\phi_0(w, \eta_0)$;
- an identical formula for $\sigma_1, \phi_1, \eta_1$;
- $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash_P \tau(\tilde{\beta})$ is not the unique $w$ in the symmetric extension satisfying $\phi_0(w, \eta_0)$ is not equal to the unique $w$ in the symmetric extension satisfying $\phi_1(w, \eta_1)$.

Working in the model $W$, let $H_0, H_1 \subset R$ be mutually generic filters, and for bits $b, c \in 2$ let $p_{bc} = \sigma_b/H_c \in P$ and $y_{bc} = \eta_b/H_c \in 2^\omega$. By the third item above, in
the model $V[K][H_0, H_1]$ there must be a bit $c \in 2$ such that $\Coll(\omega, < \kappa) \vdash$ the unique $w$ in the symmetric extension satisfying $\phi_0(w, y_{00})$ is not the same as the unique $w$ in the symmetric extension satisfying $\phi_c(w, y_{1c})$. The conditions $p_{00}, p_{1c} \in P$ are compatible in $W$ by the balance of the condition $\bar{p}$, with a lower bound $q \in P$. In the model $W$, let $w_0$ be the unique element satisfying $\phi_0(w, y_{00})$ and $w_1$ be the unique element satisfying $\phi_c(w, y_{1c})$. Since $W$ is the symmetric Solovay extension of the model $V[K][H_0]$, the forcing theorem applied with that model yields that $w_0$ is well-defined and $q \Vdash \tau(\bar{\beta}) = \bar{w}_0$. Since $W$ is the symmetric Solovay extension of $V[K][H_1]$, the forcing theorem applied with $V[K][H_1]$ yields that $w_1$ is well-defined and $q \Vdash \tau(\bar{\beta}) = \bar{w}_1$. Finally, since $W$ is the symmetric Solovay extension of $V[K][H_0][H_1]$, $w_0 \neq w_1$ holds. Thus, the condition $q$ forces two distinct values to the name $\tau(\bar{\beta})$, an impossibility.

**Corollary 8.1.2.** In a cofinally balanced extension of a symmetric Solovay model, there is no transfinite uncountable sequences of pairwise distinct Borel sets of bounded rank.

**Proof.** Since cofinally balanced extensions add no countable sequences of elements of the Solovay model $W$ by Theorem 8.1.1, all Borel sets in $W[G]$ belong to $W$ and have the same Borel rank there as in $W[G]$. Thus, an uncountable sequence of distinct Borel sets of bounded Borel rank in $W[G]$ would have to belong to $W$ by Theorem 8.1.1 again. However, there are no such sequences in the Solovay model $W$ by a seminal result of Stern [34].

**Corollary 8.1.3.** Let $E$ be a Borel equivalence relation on a Polish space $X$. In a cofinally balanced extension of a symmetric Solovay model, $\aleph_1 \not\leq |E|$ holds.

**Proof.** An $\omega_1$-sequence of distinct $E$-classes would constitute an $\omega_1$-sequence of distinct Borel sets of Borel rank bounded by the rank of $E$. Such sequences are ruled out by the previous corollary.

**Corollary 8.1.4.** Let $E$ be a Borel equivalence relation on a Polish space $X$. In a cofinally balanced extension of a symmetric Solovay model, $|E^+| \not\leq |E|$ holds.

**Proof.** Suppose towards contradiction that $|E^+| \leq |E|$ holds in the extension. The ZF theorem 13.4.1 yields $|\mathbb{H}| \leq |E|$. Clearly, $\aleph_1 \leq |\mathbb{H}|$ holds in ZF, and the concatenation of the cardinal inequalities contradicts the conclusion of Corollary 8.1.3.

A very strong version of the well-ordered divide in the Solovay model is a general version of the Silver dichotomy for Borel equivalence relations:

**Definition 8.1.5.** The full Silver dichotomy is the following statement: If $E$ is a Borel equivalence relation on a Polish space $X$ and $A \subset X$ is an $E$-invariant set, then either $A$ contains only countably many $E$-classes or $A$ contains a perfect set consisting of pairwise $E$-unrelated elements.
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As a consequence, the full Silver dichotomy implies that among all sets whose cardinality is smaller than a Borel quotient space, $2^\omega$ is the set with the smallest uncountable cardinality. The terminology refers to the classical Silver dichotomy [17, Theorem 5.7.1], a theorem of ZF+DC which says in particular that the dichotomy holds for analytic sets $A \subset X$. The Solovay model satisfies the full Silver dichotomy. Certain balanced Suslin forcings preserve this desirable feature as well and in fact, there is a simple and useful criterion in this direction:

**Definition 8.1.6.** A Suslin forcing $P$ is perfect if for every Borel function $f : 2^\omega \to P$, either there is a finite set $a \subset 2^\omega$ such that the set $f'' a \subset P$ has no lower bound, or else there is a perfect set $B \subset 2^\omega$ such that $f'' B$ has a lower bound in the separative quotient.

If the nonexistence of lower bounds for finite subsets of $P$ is an analytic relation (which is universally true for all classes of partial orders discussed in this paper) then perfectness is a $\Pi_1^\mathbb{I}$ condition and therefore absolute to all forcing extensions.

**Theorem 8.1.7.** Let $P$ be a Suslin forcing and $\kappa$ be an inaccessible cardinal. Suppose that $P$ is perfect and balanced cofinally below $\kappa$. Let $W$ be the symmetric Solovay model derived from $\kappa$ and let $G \subset P$ be a filter generic over $W$. In the model $W[G]$, the full Silver dichotomy holds.

**Proof.** Work in the model $W$. Let $p \in P$ be a condition and $\tau$ be a $P$-name such that $p \Vdash \tau \subset X$ is an $E$-invariant set containing uncountably many $E$-classes. The condition $p$ as well as the name $\tau$ have to be definable from parameters in $V$ as well as some parameter $z \in 2^\omega$. Use the assumptions to find an intermediate model $V[K]$ obtained as a generic extension of $V$ by a poset of size $< \kappa$ such that $z \in V[K]$ and $P$ is strongly balanced in $V[K]$.

Work in the model $V[K]$. Find a virtual balanced condition $\bar{p} \leq p$. Note that the equivalence relation $E$ is Borel and therefore it has fewer than $\beth_1^V[K]$ many virtual classes. The cardinality $2_\omega^V[K]$ is countable in the model $W$, while the set $\tau$ is forced to contain uncountably many $E$-classes. It follows that in the model $V[K]$ there has to be a poset $R$ of size $< \kappa$, an $R$-name $\sigma$ for a condition in $P$ stronger than $\bar{p}$ and an $R$-name $\eta$ for an element of $X$ which is forced not to be a realization of any virtual $E$-class in the model $V[K]$, and $R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash R \eta \in \tau$.

Back in the model $W$, use Thoerem 13.3.4 to find a function continuous function $y \mapsto H_y$ which to each element $y \in 2^\omega$ assigns a filter $H_y \subset R$ generic over $V[K]$ so that for every finite set $a \subset 2^\omega$, the filters $\{ H_y : y \in a \}$ are mutually generic over $V[K]$. Let $p_y = \sigma/H_y \in P$ and $x_y = \eta/H_y$ for all $y \in 2^\omega$. The map $y \mapsto p_y$ is Borel, and the map $y \mapsto x_y$ is a continuous injection from $2^\omega$ to $X$. By the assumption on the name $\eta$, the points $x_y$ for $y \in 2^\omega$ are pairwise $E$-unrelated.

**Claim 8.1.8.** For every finite set $a \subset 2^\omega$, the set $\{ p_y : y \in a \} \subset P$ has a common lower bound.
Proof. Let \( \langle y_j : j \in i \rangle \) enumerate the set \( a \) without repetitions. By induction on \( j \in i \), construct a descending sequence \( \langle q_j : j \in i \rangle \) of conditions in \( P \) such that for each \( j \in i \) \( \bar{p}_j \in V[K][H_k : k \leq j] \) and \( q_j \leq p_{y_j} \). This is easy to do using the balance of the condition \( \bar{p} \) at every stage of the induction, noting that \( q_j, p_{q_{j+1}} \) are both strengthenings of \( \bar{p} \) in mutually generic extensions of the model \( V[K] \).

In the end, the condition \( q_{i-1} \) is a lower bound of the set \( \{ p_y : y \in a \} \subset P \). \( \square \)

Since the poset \( P \) is perfect, there is a perfect set \( C \subset 2^\omega \) such that the set \( \{ p_y : y \in C \} \) has a lower bound, say \( q \in P \) in the separative quotient of \( P \). By the forcing theorem applied in every model \( V[K][H_y] \) for \( y \in B \) it is the case that in the model \( W \), the condition \( q \) forces the perfect set \( \{ x_y : y \in C \} \) consisting of pairwise \( E \)-unrelated elements to be a subset of \( \tau \). The proof is complete. \( \square \)

Example 8.1.9. Let \( I \) be an \( F_\sigma \)-ideal on \( \omega \). The poset \( P(I) \) of Subsection 7.2 designed to add an ultrafilter on \( \omega \) which is disjoint from \( I \), is perfect and therefore the full Silver dichotomy holds in the \( P(I) \)-extension of the symmetric Solovay model.

As a special case, this includes the poset of infinite subsets of \( \omega \) ordered by inclusion, used to add a Ramsey ultrafilter over the symmetric Solovay model.

Proof. Recall that \( P(I) \) is the poset of all \( I \)-positive subsets of \( \omega \), ordered by inclusion. Write \( I = \bigcup_n I_n \) as a countable union of closed sets, each of which is closed under taking subset. Let \( f : 2^\omega \to \mathcal{P}(\omega) \) be a Borel function such that for any finite set \( a \subset 2^\omega \), \( \bigcap f''a \notin I \) holds; we must find a perfect set \( B \subset 2^\omega \) such that the set \( f''B \subset P \) has a lower bound. Thinning the domain of \( f \) if necessary we may assume that the function \( f \) is in fact continuous. By induction on \( n \in \omega \) build nodes \( u_t \in 2^{<\omega} \) for all \( t \in 2^n \) and finite sets \( b_n \subset \omega \) such that

- \( b_n \notin I_n \);
- \( s \subset t \) implies that \( u_s \subset u_t \) and \( s \) is incompatible with \( t \) implies \( u_s \) is incompatible with \( u_t \);
- for each \( t \in 2^{n+1} \) and every \( y \in [u_t] \) it is the case that \( b_n \subset f(y) \).

Once the induction is performed, let \( b = \bigcup_n b_n \), let \( B \subset 2^\omega \) be the perfect set of all points \( y \in 2^\omega \) such that \( \forall n \exists t \in 2^n u_t \subset y \) and use the continuity of the function \( f \) to prove that \( b \) is the lower bound of the set \( f''B \).

To start the induction, let \( u_0 = 0 \). Now suppose that the nodes \( u_t \in 2^{<\omega} \) for \( t \in 2^n \) as well as sets \( b_m \) for \( m \in n \) have been constructed. For each \( t \in 2^n \) choose distinct points \( y_{t0}, y_{t1} \in [u_t] \) and use the initial assumption on the function \( f \) to observe that \( c = \bigcap_{t \in 2^n} f(y_{t0}) \cap \bigcap_{t \in 2^n} f(y_{t1}) \) is an \( I \)-positive set. Since the set \( I_n \subset \mathcal{P}(\omega) \) is closed, there must be a finite subset \( b_n \subset c \) which is not in \( I_n \). Use the continuity of the function \( f \) to find initial segments \( u_{t-0} \subset y_{t0} \) and \( u_{t-0} \subset y_{t1} \) satisfying the second item of the induction hypothesis. This concludes the inductive step. \( \square \)
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Example 8.1.10. Let \( \Gamma \) be a countable semigroup. The poset \( P = P(\Gamma) \) of Subsection 7.3 designed to add a union ultrafilter is perfect and therefore the full Silver dichotomy holds in the \( P \)-extension of the symmetric Solovay model.

Proof. Recall that elements of \( P \) are just sequences in \( \Gamma^\omega \) and the ordering is defined by \( q \leq p \) if there are pairwise disjoint nonempty finite sets \( a_n \subset \omega \) such that \( q(n) = \prod_{m \in a_n} p(m) \). The proof of the perfect property of the poset \( P \) is another fusion argument. Let \( f : 2^\omega \rightarrow P \) be a Borel function such that for every finite set \( b \subset 2^\omega \), the conditions in the set \( f''b \) have a common lower bound in \( P \); we must find a perfect set \( B \subset 2^\omega \) such that the set \( f''B \subset P \) has a lower bound.

Thinning the domain of \( f \) if necessary we may assume that the function \( f \) is continuous. By induction on \( n \in \omega \) build nodes \( u_t \in 2^{<\omega} \) for all \( t \in 2^n \), finite sets \( a_t \subset \omega \) and elements \( \gamma_n \) so that

- \( \min(a_{n+1}) > \max(a_n) \);
- \( s \subset t \) implies that \( u_s \subset u_t \) and \( s \) is incompatible with \( t \) implies \( u_s \) is incompatible with \( u_t \);
- \( a_0 = 0 \) and if \( s \subset t \) then \( a_s \cap a_t = 0 \);
- for each \( t \in 2^{n+1} \) and each \( y \in [u_t] \), \( \gamma_n = \prod_{m \in a_t} f(y)(m) \).

Once the induction is performed, let \( p = \langle \gamma_n : n \in \omega \rangle \), let \( B \subset 2^\omega \) be the perfect set of all points \( y \in 2^\omega \) such that \( \forall n \exists t \in 2^n : u_t \subset y \) and observe that \( p \) is the lower bound of the set \( f''B \).

To start the induction, let \( u_0 = 0 \) and \( a_0 = 0 \). Now suppose that the nodes \( u_t \in 2^{<\omega} \) for \( t \in 2^n \) as well as sets \( a_s \) for \( s \in 2^{\leq n} \) and elements \( \gamma_m \) for \( m \in n \) have been constructed. For each \( t \in 2^n \) choose distinct points \( y_{t0}, y_{t1} \in [u_t] \) and use the initial assumption on the function \( f \) to observe that the set \( e = \{ f(y_{t0}), f(y_{t1}) : t \in 2^n \} \) has a lower bound in the poset \( P \). This means that there are nonempty finite sets \( a_{t-0}, a_{t-1} \subset \omega \) disjoint from \( \bigcup_{s \in 2^{\leq n}} a_s \) and a semigroup element \( \gamma_n \in \Gamma \) such that \( \prod_{m \in a_{t-0}} f(y_{t0})(m) = \gamma_n \) and \( \prod_{m \in a_{t-1}} f(y_{t1})(m) = \gamma_n \) holds for all \( t \in 2^n \). Use the continuity of the function \( f \) to find initial segments \( u_{t-0} \subset y_{t0} \) and \( u_{t-1} \subset y_{t1} \) such that \( f(y_{t0})(m) = f(y)(m) \) holds for all \( m \in a_{t0} \) and all \( y \in [u_{t-0}] \) and \( f(y_{t1})(m) = f(y)(m) \) holds for all \( m \in a_{t1} \) and all \( y \in [u_{t-1}] \). This concludes the inductive step. \( \square \)

Not many Suslin forcings are perfect, and in a typical balanced extension of the Solovay model the cardinalities below \( 2^\omega \) exhibit chaotic structure. We include one example illustrating the difficulties.

Example 8.1.11. Let \( P \) be the poset of all countable functions from \( 2^\omega \) to \( 2 \), ordered by reverse extension. Let \( \kappa \) be an inaccessible cardinal, let \( W \) be the symmetric Solovay model derived from \( \kappa \), and let \( G \subset P \) be a filter generic over \( W \). In \( W[G] \), let \( a_0 = \{ y \in 2^\omega : \exists p \in G : p(y) = 0 \} \) and \( a_1 = \{ y \in 2^\omega : \exists p \in G : p(y) = 1 \} \). In \( W[G] \), \( |a_0| \neq |a_1| \).
Thus, $2^{<\omega}$ can be decomposed into two uncountable sets of distinct cardinalities, contradicting the full Silver dichotomy. In fact, one can prove all kinds of pathologies regarding the cardinalities of $a_0$ and $a_1$ respectively; for example $|a_0^2| \not\leq |a_0|$ holds in $W[G]$.

Proof. Work in $W$. Suppose towards contradiction that $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \Vdash: \dot{a}_0 \rightarrow \dot{a}_1$ is a bijection. The condition $p$ as well as the name $\tau$ have to be definable from some ground model parameters and some parameter $z \in 2^{<\omega}$. Find an intermediate model $V[K]$ obtained as a generic extension of the ground model by a poset of size $\kappa$ such that $z \in V[K]$.

Work in the model $V[K]$. Let $\bar{p}$ be the $\text{Coll}(\omega, \kappa)$-name for the set $\{q \in P: p \subset q \land \forall x \in V[K] \cap 2^{<\omega} \setminus \text{dom}(p) \ q(x) = 0\}$. This is a balanced virtual condition in $P$ for the model $V[K]$. For every point $x \in 2^{<\omega} \setminus \text{dom}(p)$ let $y_x \in 2^{<\omega}$ be a point such that $\text{Coll}(\omega, \kappa) \Vdash \bar{p} \Vdash_P \tau(x) = y_x$, if it exists. As $\tau$ is forced to be an injection from $\dot{a}_0$ to $\dot{a}_1$, for each $x$ there can be at most one $y_x$ of this kind, and the function $g: x \mapsto y_x$ must be an injection from $2^{<\omega} \setminus \text{dom}(p)$ to $\text{dom}(p)$. Since the former set is uncountable and the latter is countable, there must be a point $x \in 2^{<\omega} \setminus \text{dom}(p)$ which is not in the domain of $g$.

Still in the model $V[K]$, the statement that $x \notin \text{dom}(g)$ means that $\text{Coll}(\omega, < \kappa)$ forces that either there is a condition below $\bar{p}$ in $P$ which forces $\tau(\check{x})$ out of $V[K]$, or there are two distinct conditions in $P$ stronger than $\bar{p}$ which force the value $\tau(\check{x})$ to be two distinct points in $V[K]$. Suppose for definiteness that the former is the case. Then there must be a poset $R$ of size $\kappa$, an $R$-name $\sigma$ for a condition in $P$ stronger than $\bar{p}$ and an $R$-name $\eta$ for an element of $2^{<\omega} \setminus V[K]$ such that $R \Vdash \text{Coll}(\omega, < \kappa) \Vdash: \sigma \Vdash_P \tau(\check{x}) = \eta$.

In the model $W$, pick filters $H_0, H_1 \subset R$ mutually generic over the model $V[K]$. Let $p_0 = \sigma/H_0 \in P$ and $p_1 = \sigma/H_1 \in P$; by the balance of the condition $\bar{p}$, $p_0$ and $p_1$ are conditions compatible in $P$, with some lower bound $q \in P$. Let $y_0 = \eta/H_0 \in 2^{<\omega}$ and $y_1 = \eta/H_1 \in 2^{<\omega}$; by the product forcing theorem and the choice of the name $\eta$, $y_0 \neq y_1$ holds. But then, $q \Vdash \tau(\check{x}) = y_0$ and $\tau(\check{x}) = y_1$, an impossibility. \hfill \Box

8.2 Preserving the pinned divide

No unpinned equivalence relation can be reduced by a Borel function to a pinned one, see Fact 2.3.2. This feature persists to the cardinality computations in balanced extensions of the choiceless Solovay model, with a small proviso.

Theorem 8.2.1. Let $E, F$ be Borel equivalence relations on respective Polish spaces $X, Y$, $E$ pinned and $F$ unpinned. In a balanced extension of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, $|F| \not\leq |E|$ holds.

We do not know if the increase in the large cardinal strength of the large cardinal hypothesis is necessary. We do know though that one has to consider Suslin posets which are balanced everywhere below $\kappa$ as opposed to just cofinally balanced. This is clear from Theorem 8.2.2 below.
8.2. PRESERVING THE PINNED DIVIDE

Proof. Let $\kappa$ be an inaccessible cardinal which is a limit of inaccessibles. Let $P$ be a Suslin poset which is balanced below $\kappa$. Let $W$ be the symmetric Solovay model derived from $\kappa$ and work in the model $W$. Towards a contradiction, suppose that $p \in P$ is a condition and $\tau$ a $P$-name for an injection from $F$-classes to $E$-classes. Both $p, \tau$ are definable from some parameter $z \in 2^\omega$ and parameters in the ground model. The assumptions imply that there is an inaccessible cardinal $\lambda < \kappa$ of $V$ and a filter $K \subset \text{Coll}(\omega, < \lambda)$ generic over $V$ such that $z \in V[K]$.

Work in the model $V[K]$. The balanced assumption on the poset $P$ implies that there is a balanced virtual condition $\bar{p} \leq p$ in $P$. Theorem 2.6.1 shows that $E$ is pinned in $V[K]$ and therefore has at most $\mathfrak{c} = \aleph_1$ many virtual classes, while Theorem 2.5.11 shows that $F$ has at least $2^{2^{\aleph_1}}$ many virtual classes. The argument splits into two cases:

Case 1. For every poset $R$ of size $< \kappa$, every $R$-name for a condition $\sigma \leq \bar{p}$ in the poset $P$, every $F$-pinned $R$-name $\eta$ for an element of $X$ and every $R$-name $\chi$ for an element of $X$ such that $R \models \text{Coll}(\omega, < \kappa) \models \sigma \Vdash_P \tau(\langle \eta \rangle_F) = [\chi]_E$, the name $\chi$ is $E$-pinned.

In this case, by a counting argument with the virtual $E$- and $F$-classes, it must be the case that there are posets $R_0, R_1$ and respective names $\sigma_0, \eta_0, \chi_0$ and $\sigma_1, \eta_1, \chi_1$ on them such that $\eta_0, \eta_1$ are $F$-pinned, $\chi_0, \chi_1$ are $E$-pinned, $R_0 \models \text{Coll}(\omega, < \kappa) \models \sigma_0 \Vdash_P \tau_0(\langle \eta_0 \rangle_F) = [\chi_0]_E$ and $R_1 \models \text{Coll}(\omega, < \kappa) \models \sigma_1 \Vdash_P \tau_1(\langle \eta_1 \rangle_F) = [\chi_1]_E$ and $\langle R_0, \eta_0 \rangle \in E(R_1, \chi_1)$ holds while $\langle R_0, \eta_0 \rangle \in F(R_1, \eta_1)$ fails. Let $H_0 \subset R_0, H_1 \subset R_1$ be filters generic over $V[K]$, and write $r_0 = \sigma_0/H_0 \in P$, $r_1 = \sigma_1/H_1 \in P$, $x_0 = \chi_0/H_0 \in X$, $x_1 = \chi_1/H_1 \in X$, and $y_0 = \eta_0/H_0 \in Y$ and $y_1 = \eta_1/H_1 \in Y$. The balance of the virtual condition $\bar{p}$ in $P$ implies that $r_0, r_1$ are compatible in $P$ with some lower bound $r \in P$, and the pinned assumptions imply that $x_0 \equiv x_1$ holds and $y_0 \equiv y_1$ fails. Since $W$ is the symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem in these two models implies that in $W$, $r \Vdash_P \tau_1(\langle \eta_1 \rangle_F) = [\bar{x}_1]_E$ and $\tau_1(\langle \eta_1 \rangle_F) = [\bar{x}_1]_E$; this contradicts the assumption that $\tau$ is forced to be an injection.

Case 2. Case 1 fails. Then, there has to be poset $R$ of size $< \kappa$, every $R$-name for a condition $\sigma \leq \bar{p}$ in the poset $P$, every $F$-pinned $R$-name $\eta$ for an element of $X$ and every $R$-name $\chi$ for an element of $X$ such that $R \models \text{Coll}(\omega, < \kappa) \models \sigma \Vdash_P \tau(\langle \eta \rangle_F) = [\chi]_E$, such that the name $\chi$ is not $E$-pinned. In such a case, there must be filters $H_0, H_1 \subset R$ mutually generic over $V[K]$ such that the points $x_0 = \chi/H_0, x_1 = \chi/H_1 \in X$ are $E$-unrelated. Let $r_0 = \sigma/H_0, r_1 = \sigma/H_1$, and let $y_0 = \eta/H_0, y_1 = \eta/H_1 \in Y$. The balance of the virtual condition $\bar{p}$ implies that the conditions $r_0, r_1 \in P$ are compatible with a lower bound $r$, and the pinned assumptions imply that $y_0 \equiv y_1$ holds. Since $W$ is the symmetric Solovay extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem in these two models implies that in $W$, $r \Vdash_P \tau_1(\langle \eta_1 \rangle_F) = [\bar{x}_1]_E$ and $\tau_1(\langle \eta_1 \rangle_F) = [\bar{x}_1]_E$; this contradicts the assumption that $\tau$ is forced to be a function.

The following two characterization theorems illustrate the possibilities of violating the pinned divide if the Suslin poset in question is only cofinally balanced as opposed to balanced in every forcing extension.
Theorem 8.2.2. Let \( E \) be a Borel equivalence relation on a Polish space \( X \), Borel reducible to \( \mathbb{F}_2 \). The following are equivalent:

1. \( \mathbb{F}_2 \) is Borel reducible to \( E \);
2. in any cofinally balanced extension of a symmetric Solovay model, \(|E| \leq |2^\omega|\) holds.

We do not know if it is possible to eliminate the assumption that \( E \) is Borel reducible to \( \mathbb{F}_2 \).

Proof. If (1) occurs then (2) holds by Corollary 8.1.4. Thus, assume that (1) fails. Let \( P \) be the collapse poset of Definition 6.4.2, collapsing \(|E|\) to \(|2^\omega|\). To confirm the failure of (2), we only have to show that \( P \) is cofinally balanced below \( \kappa \) where \( \kappa \) is any inaccessible cardinal. To do this, given an ordinal \( \lambda < \kappa \), consider the generic extension \( V[K] \) obtained by \( \text{Coll}(\omega, \lambda) \) followed by adding \( \sum_{\omega_1} \) many Cohen reals. In the model \( V[K] \), \( E \) has only \( \aleph_1 \) many virtual classes by Theorem 2.7.6. The poset \( P \) is then balanced in \( V[K] \) by Corollary 6.4.4.

Theorem 8.2.3. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). The following are equivalent:

1. \( E \) is pinned;
2. in any cofinally balanced extension of a symmetric Solovay model derived from an inaccessible limit of inaccessibles, \( |\mathbb{F}_2| \nleq |E| \) holds.

Proof. Suppose first that (1) fails. To confirm the failure of (2), consider the poset \( P \) from Definition 6.4.2 collapsing the cardinality of \( \mathbb{F}_2 \) to \(|E|\). Let \( \kappa \) be any inaccessible limit of inaccessible cardinals. To show that \( P \) witnesses the failure of (2), let \( \lambda < \kappa \) be an inaccessible cardinal and consider the generic extension \( V[K] \) by a generic filter \( K \subset \text{Coll}(\omega, \lambda) \). Since \( E \) is unpinned, Theorem 2.5.11 shows that it has at least \( \aleph_1 \) many virtual classes in \( V[K] \). The equivalence relation \( \mathbb{F}_2 \) has likewise \( \aleph_1 = \aleph_1 \) many virtual classes in \( V[K] \). Corollary 6.4.4 shows that the poset \( P \) is balanced in the model \( V[K] \) as desired.

Suppose now (1) holds. To confirm (2), let \( \kappa \) be any inaccessible cardinal and suppose that \( P \) is a Suslin forcing balanced cofinally below \( \kappa \); we must show that in the symmetric Solovay model \( W \), \( P \) forces \( |\mathbb{F}_2| \nleq |E| \). To this aim, suppose towards contradiction that some condition in \( P \) forces the opposite. Consider the poset \( Q \) collapsing the cardinality of \(|E|\) to \(|2^\omega|\) as isolated in Definition 6.4.2. In every forcing extension, the virtual quotient space of \( E \) is identical to its quotient space, so it has size \( \aleph_1 \) and by Corollary 6.4.4 the poset \( Q \) is balanced. Thus \( P \times Q \) is a Suslin forcing balanced cofinally below \( \kappa \) by Theorem 5.2.10 and over the symmetric Solovay model it forces \( |\mathbb{F}_2| \leq |E| \leq |2^\omega| \). This contradicts Corollary 8.1.4.
8.3 Preserving the turbulent divide

We wish to transfer the ergodicity theorem 3.3.5 to cardinal inequalities in generic extensions of the Solovay model. The central transfer tool is the following natural variation of balance.

Definition 8.3.1. Let $P$ be a Suslin forcing. A pair $\langle Q, \tau \rangle$ is trim balanced if $Q \Vdash \tau \in P$ and if $R_0, R_1$ are posets and $\sigma_0, \sigma_1$ are $R_0 \times Q$ and $R_1 \times Q$-names respectively such that $R_0 \times Q \Vdash \sigma_0 \leq p$ and $R_1 \times Q \Vdash \sigma_1 \leq p$, and in some ambient forcing extension, $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ are filters separately generic over $V$ such that $V[H_0] \cap V[H_1] = V$, then $\sigma_0/H_0$ and $\sigma_1/H_1$ are compatible conditions in $P$.

Note that if $H_0 \subset R_0 \times Q$ and $H_1 \subset R_1 \times Q$ are mutually generic over the ground model, then $V[H_0] \cap V[H_1] = V$ holds by the product forcing theorem. Therefore, trim balance is a strengthening of balance. It is also not difficult to show that trim balance is invariant under the equivalence of balanced pairs.

Definition 8.3.2. Let $P$ be a Suslin forcing and let $\kappa$ be an inaccessible cardinal.

1. The poset $P$ is trim balanced if for every condition $p \in P$ there is a virtual condition $q \leq p$ in $P$ which is trim-balanced;

2. the poset is trim balanced cofinally below $\kappa$ if $V_{\kappa}$ satisfies that every ordinal can be collapsed by a poset making $P$ trim balanced.

Theorem 8.3.3. Let $E$ be an orbit equivalence relation of a turbulent Polish group action, with all orbits meager. Let $F$ be a virtually trim analytic equivalence relation on a Polish space $Y$. In cofinally trim balanced extensions of a symmetric Solovay model, $|E| \not\leq |F|$ holds.

Proof. Let $\Gamma$ be a Polish group, turbulently acting on a Polish space $X$, resulting in the equivalence relation $E$. Let $P$ be a Suslin forcing and let $\kappa$ be an inaccessible cardinal such that $P$ is cofinally trim balanced below $\kappa$. Let $W$ be a symmetric Solovay model derived from $\kappa$ and work in the model $W$. Suppose towards contradiction that there is a condition $p \in P$ and a $P$-name for a function which is an injection from the $E$-quotient space to the $F$-quotient space. The condition $p$ as well as the name $\tau$ must be definable from some ground model parameters together with a parameter $z \in 2^\omega$. Use the assumptions to find an intermediate model $V[K]$, which is obtained from $V$ by a poset of size $\leq \kappa$, contains $z$ and in which the poset $P$ is trim balanced.

Work in the model $V[K]$. Let $\bar{p} \leq p$ be a virtual condition in $P$ which is trim balanced. Let $Q$ be the poset for adding a Cohen point of the space $X$. That is, $Q$ is the poset of all nonempty open subsets of $X$ ordered by inclusion, adding a point $\dot{x} \in X$. There are two cases.

Case 1. There is a condition $q \in Q$ and a poset $R$ of size $< \kappa$ and a $Q \times R$-name $\sigma$ for an element of $P$ stronger than $\bar{p}$ and a virtual $F$-class $y \in V[K]$.
such that \( \langle q, 1 \rangle \forces_{Q \times R} \text{Coll}(\omega, < \kappa) \vdash \sigma \forces_P \tau([\vec{x}]_E) = [\vec{y}]_F \). In this case, let \( x_0, x_1 \in X \) and \( H_0, H_1 \subset R \) be points in \( X \) and filters on \( R \) mutually generic over the model \( V[K] \). Let \( r_0 = \sigma/x_0 \times H_0 \in P \cap V[K][x_0][H_0] \) and \( r_1 = \sigma/x_1 \times H_1 \in P \cap V[K][x_1][H_1] \). By the balance of the condition \( \vec{p} \), these conditions are compatible in the poset \( P \), with some lower bound \( r \). The points \( x_0, x_1 \in X \) are mutually Cohen generic, and since their respective \( E \)-classes are meager, it must be the case that \( x_0 E x_1 \) fails.

Now, move back to the model \( W \). The model \( W \) is a symmetric Solovay extension of all models under consideration, and so by the forcing theorem applied in the models \( V[K][x_0][H_0] \) and \( V[K][x_1][H_1] \), it must be the case that \( r \vdash \tau([x_0]_E) = [\vec{y}]_F \) and \( \tau([x_1]_E) = [\vec{y}]_F \). This contradicts the assumption that \( \tau \) is an injection.

**Case 2.** Case 1 fails. There must be a poset \( R \) of size \( < \kappa \) and \( Q \times R \)-names \( \sigma \) for an element of the poset \( P \) stronger than \( \vec{p} \) and \( \eta \) for an element of \( Y \) such that \( Q \times R \models \text{Coll}(\omega, < \kappa) \vdash \sigma \forces_P \tau([\vec{x}]_E) = [\vec{y}]_F \). The case assumption shows that \( \eta \) is not an \( F \)-pinned name. Now, it is time to use the turbulence assumption. Let \( x_0, \gamma \) be mutually Cohen-generic elements of \( X \) and \( \Gamma \), and let \( x_1 = \gamma \cdot x_0 \). By Theorem 3.2.2, \( x_1 \) is a point of \( X \) Cohen generic over \( V[K] \), and \( V[K][x_0] \cap V[K][x_1] = V[K] \). Let \( H_0, H_1 \subset R \) be filters mutually generic over the model \( V[K][x_0, \gamma] \). Use the mutual genericity of these filters to argue that \( V[K][x_0][H_0] \cap V[K][x_1][H_1] = V[K] \).

Let \( r_0 = \sigma/x_0 \times H_0 \in P \cap V[K][x_0][H_0] \), \( r_1 = \sigma/x_1 \times H_1 \in P \cap V[K][x_1][H_1] \) and use the trim balance of the condition \( \vec{p} \) to argue that the conditions \( r_0, r_1 \) are compatible, with a lower bound \( r \). Let \( y_0 = \eta/x_0 \times H_0 \) and \( y_1 = \eta/x_1 \times H_1 \); either of these two points is not a realization of any virtual \( F \)-class in \( V[K] \) by the case assumption, and so by the virtual trimness assumption on the equivalence relation \( F \), \( y_0 F y_1 \) fails. The model \( W \) is a symmetric Solovay extension of all models under consideration, and so by the forcing theorem applied in the models \( V[K][x_0][H_0] \) and \( V[K][x_1][H_1] \), it must be the case that \( r \models \tau([x_0]_E) = [\vec{y}_0]_F \) and \( \tau([x_1]_E) = [\vec{y}_1]_F \). However, \( x_0 E x_1 \) holds by the choice of \( x_0, x_1 \), and \( y_0 F y_1 \) fails. This is a contradiction completing the proof of the theorem. \( \square \)

It turns out that large classes of balanced Suslin forcings are in fact trim balanced.

**Proposition 8.3.4.** Suppose that \( K \) is a countable Borel flag complex on a Polish space \( X \). Then the poset \( P_K \) is trim balanced, and every balanced condition is trim balanced.

**Proof.** Revisiting the work of Subsection 6.1, it is enough to observe that the proof of Theorem 6.1.6 uses the product forcing theorem only to ascertain that if \( V[H_0], V[H_1] \) are mutually generic extensions of the ground model, then \( V[H_0] \cap V[H_1] = V \). \( \square \)

**Proposition 8.3.5.** Suppose that \( K \) is a modular Borel complex on a Polish space \( X \). Then the poset \( P_K \) is cofinally trim balanced, and every balanced condition is trim balanced.
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Proof. It is enough to observe that the proof of Theorem 6.2.19 uses the product forcing theorem only to ascertain that if $V[H_0], V[H_1]$ are mutually generic extensions of the ground model, then $V[H_0] \cap V[H_1] = V$. □

The forcings associated with quotient complexes, as studied in Section 6.4, are trim balanced as soon as the equivalence relations involved are virtually trim. The proofs of the following propositions follow the lines of Theorem 6.4.3, 6.4.6, and 6.4.13.

**Proposition 8.3.6.** Suppose that $E, F$ are virtually trim Borel equivalence relations on the respective Polish spaces $X, Y$. Suppose that $\lambda(E) \leq \lambda(F)$. Then the collapse poset of $|E|$ to $|F|$ is trim balanced.

**Proposition 8.3.7.** Suppose that $E, F$ are a trim and a virtually trim Borel equivalence relation on Polish spaces $X, Y$ respectively. Let $B \subset X \times Y$ be an $E \times F$-invariant Borel set with all vertical sections nonempty. The poset uniformizing the set $B/E \times F \subset X/E \times Y/E$ is trim balanced.

**Proposition 8.3.8.** Suppose that $K$ is a quotient cofinally locally countable Borel complex on a Polish space $X$. Suppose that the associated Borel equivalence relation is virtually trim. Then the poset $P_K$ is trim balanced, and every balanced condition is trim balanced.

**Corollary 8.3.9.** The following Suslin posets are trim balanced:

1. adding a transversal to a trim equivalence relation;
2. adding a maximal acyclic subgraph to a Borel graph;
3. adding a Hamel basis to a Polish vector space;
4. linearizing the quotient space of a virtually trim equivalence relation;
5. bipartizing a Borel graph which is odd cycle free, and whose path connectedness equivalence relation relation is Borel and virtually trim.

**Example 8.3.10.** The ultrafilter poset of Subsection 7.1 is balanced but not trim balanced. This observation should be compared with Corollary 10.3.7 below.

**Question 8.3.11.** Let $P$ be the poset of Subsection 7.1 for adding an ultrafilter. Let $W$ be a symmetric Solovay extension and $G \subset P$ be a filter generic over $W$. Does $P$ preserve the turbulent divide?

8.4 Preserving the orbit divide

It is well-known that $E_1$ is not Borel reducible to any orbit equivalence relation. It is then tempting to think that in many models which we study, $|E_1|$ cannot be smaller than $|E|$ where $E$ is an orbit equivalence relation. This question is with some success addressed in the present section. The following definition connects coherent sequences of generic extensions with Suslin forcings.
**Definition 8.4.1.** Let $P$ be a Suslin forcing. An orbit charm below a condition $p \in P$ is a choice-coherent sequence $\langle V_n : n \in \omega \rangle$ of generic extensions of $V$ and a sequence $\langle \bar{p}_n : n \leq \omega \rangle$ so that

1. $2^n \cap V_n \neq 2^n \cap V_{n+1}$ for all $n \in \omega$;
2. $\bar{p}_0 \leq \bar{p}_1 \leq \cdots \leq \bar{p}_\omega \leq p$ where $\bar{p}_n$ for each $n \in \omega$ is a balanced virtual condition in $V_n$ and $\bar{p}_\omega$ is a balanced virtual condition in the intersection model $\bigcap_n V_n$.

**Definition 8.4.2.** Let $\kappa$ be a Suslin forcing and $\kappa$ an inaccessible cardinal. The poset $P$ is orbit charming below $\kappa$ if $V_\kappa$ satisfies that every ordinal can be collapsed by a poset which forces $P$ to have an orbit charm below every condition.

**Theorem 8.4.3.** Let $\kappa$ be an inaccessible cardinal and $P$ a Suslin forcing. If $P$ is orbit charming below $\kappa$ then in the $P$-extension of the Solovay model derived from $\kappa$, $|\mathcal{E}_1| \nleq |E|$ for every orbit equivalence relation $E$ holds.

**Proof.** Towards contradiction, suppose that $Y$ is a Polish space, $\Gamma$ is a Polish group and $\Gamma$ continuously acts on $Y$, inducing an orbit equivalence relation $E$. Suppose also that $W$ is a symmetric Solovay model derived from $\kappa$ and in $W$, $p \in P$ is a condition and $\tau$ is a $P$-name such that $p \Vdash \tau$ is an injection from $\mathcal{E}_1$-classes to $E$-classes.

The objects $\Gamma$, $Y$, $p$ and $\tau$ are all definable in the model $W$ from parameters in the ground model plus a parameter $z \in 2^\omega$. Use the assumptions to find an intermediate forcing extension $V[K]$ by a poset of size $< \kappa$ such that $z \in V[K]$ and $V[K]$ has an orbit charm below $p$. Still working in $W$, find the charm. This is a sequence $\langle V_n : n \in \omega \rangle$ of generic extensions of $V[K]$ together with a sequence $\langle \bar{p}_n : n \leq \omega \rangle$ of balanced virtual conditions below $p$ satisfying the demands of Definition 8.4.1.

First, use assumption (1) on the charm to find a coherent sequence $\langle z_n : n \in \omega \rangle$ such that $z_n \in 2^n \cap V_n \setminus V_{n+1}$ for every $n \in \omega$. Let $x_n \in X = (2^\omega)^\omega$ be the point defined by $x_n(m) =$ the zero binary sequence if $m < n$ and $x_n(m) = z_m$ if $m \geq n$; thus $x_n \in V_n$.

By the forcing theorem, for each number $n \in \omega$, in the model $V_n$ there must be a poset $R_n$ of size $< \kappa$, an $R_n$-name $\sigma_n$ for a condition in $P$ stronger than $\bar{p}_n$ and an $R_n$-name $\eta_n$ for an element of $Y$ such that $R_n \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_n \Vdash p \tau([x_n]_{E_1}) = [\eta_n]_E$.

**Claim 8.4.4.** For $n \in \omega$, the pair $\langle R_n, \eta_n \rangle$ is an $E$-pin.

**Proof.** If this failed for some number $n \in \omega$, then in the model $W$ there would be filters $H_0, H_1 \subset R_n$ mutually generic over the model $V_n$ such that the points $y_0 = \sigma_n/H_0, y_1 = \sigma_n/H_1 \in Y$ are $E$-unrelated. Let $p_0 = \sigma_n/H_0$ and $p_1 = \sigma_n/H_1$ be conditions in the poset $P$; they are compatible by the balance of $\bar{p}_n$.

By the forcing theorem applied in the models $V_n[H_0]$ and $V_n[H_1]$, their lower bound forces in $P$ that $\tau([x_n]_{E_1})$ is equal simultaneously to $[\bar{y}_0]_E$ and $[\bar{y}_1]_E$. This is impossible. \qed
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Claim 8.4.5. For \( n < m < \omega \), the \( E \)-pins \( \langle R_n, \eta_n \rangle \) and \( \langle R_m, \eta_m \rangle \) are equivalent.

Proof. If this failed for some numbers \( n < m < \omega \), then in the model \( W \) there would be filters \( H_0 \subseteq R_n, H_1 \subseteq R_m \) mutually generic over the model \( V_n \) such that the points \( y_0 = \sigma_n / H_0, y_1 = \sigma_m / H_1 \in Y \) are \( E \)-unrelated. Let \( p_0 = \sigma_n / H_0 \) and \( p_1 = \sigma_m / H_1 \) be conditions in the poset \( P \); they are compatible by the balance of \( p_m \) since \( p_0 \leq p_n \leq p_m \) by demand (2) of Definition 8.4.1. By the forcing theorem applied in the models \( V_n[H_0] \) and \( V_m[H_1] \), their lower bound forces in \( P \) that \( \tau([x_n]_{E_1}) = [y_0]_E \) and \( \tau([x_m]_{E_1}) = [y_1]_E \). This is impossible since \( x_n E_1 x_m \).

By Theorem 4.3.7, there is an \( E \)-pin \( \langle R, \eta \rangle \in V_\omega \) which is equivalent to all the pins \( \langle R_n, \eta_n \rangle \) for \( n < \omega \). Still in \( V_\omega \), find a poset \( Q \) generating the model \( V_0 \), a \( Q \)-name \( \dot{x} \) for the sequence \( x_0 \), and a \( Q \)-name \( \dot{R}_0 \) for the poset \( R_0 \). By the forcing theorem, there must be a condition \( q \in Q \) which forces the following: \( \dot{x} \) has no \( E_1 \)-equivalent in \( V_\omega \), \( \dot{R}_0 \vdash \sigma_0 \leq p_\omega \) and \( \dot{R} \times \dot{R}_0 \vdash \eta E \eta_0 \). Note that the last statement means that the name \( \eta_0 \) is \( E \)-pinned on the iteration \( Q \upharpoonright q * \dot{R}_0 \). In the model \( W \), let \( H_0, H_1 \subseteq Q \upharpoonright q * \dot{R}_0 \) be filters mutually generic over the model \( V_\omega \), and write \( p_0 = \sigma_0 / H_0, p_1 = \sigma_0 / H_1, x_0 = \dot{x} / H_0, x_1 = \dot{x} / H_1, y_0 = \eta_0 / H_0 \) and \( y_1 = \eta_0 / H_1 \). The balance of the condition \( p_\omega \) implies that \( p_0, p_1 \in P \) are compatible conditions. Since \( E_1 \) is a pinned equivalence relation, the points \( x_0, x_1 \in X \) are \( E_1 \)-unrelated. Since the name \( \eta_0 \) was pinned on the iteration \( Q \upharpoonright q * \dot{R}_0 \), the points \( y_0, y_1 \) are \( E \)-related. In total, in the model \( W \), the lower bound of the conditions \( p_0, p_1 \) forces that \( \tau([x_0]_{E_1}) = \tau([x_1]_{E_1}) = [y_0]_E \), which is impossible since \( \tau \) was forced to be an injection.

A number of examples of orbit charms are in this or that way connected with ultrafilters.

Example 8.4.6. The poset \( P \) of infinite subsets of \( \omega \) ordered by inclusion has orbit charms below every condition.

Proof. Recall (Theorem 7.1.4) that the balanced virtual conditions in \( P \) are classified by nonprincipal ultrafilters on \( \omega \). Let \( p \in P \) be a condition. Let \( p \in U \subseteq P \) and \( \{x_n : n \in \omega \} \) be mutually generic ultrafilter and Sacks reals over \( V \); the product of the Sacks reals is taken with countable support. It is well known [31] that the Sacks real product adds no independent reals, and therefore, by a genericity argument, \( U \) generates an ultrafilter on \( \omega \) in the model \( V[U] \). Let \( V_n = V[U]|x_m : m \geq n | \) for all \( n \in \omega \). Then \( \langle V_n : n \in \omega \rangle \) is a choice-coherent sequence of models by Theorem 4.3.6 or Example 4.3.5 applied in the model \( V[U] \). Now, for each \( n \in \omega \) let \( \bar{p}_n \) be the balanced virtual condition in the model \( V_n \) obtained from the ultrafilter generated by \( U \). Similarly, in the intersection model \( V = \bigcap_n V_n \), let \( \bar{p}_\omega \) be the balanced virtual condition obtained from the ultrafilter generated by \( U \). Clearly, \( \langle V_n : n \in \omega, \bar{p}_n : n \leq \omega \rangle \) is an orbit charm below \( p \). In this case, in fact all the balanced virtual conditions considered are equivalent.
Corollary 8.4.7. 1. In the extension of the symmetric Solovay model by a Ramsey ultrafilter $U$ on $\omega$, $|E_1| \not\leq |E|$ holds for every orbit equivalence relation $E$.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a Ramsey ultrafilter on $\omega$, and $|E_1| \not\leq |E|$ holds for every orbit equivalence relation $E$.

Proof. This is a conjunction of Example 8.4.6 and Theorem 8.4.3.

Question 8.4.8. Is there a Borel (or even $F_\sigma$) ideal $I$ on $\omega$ such that the existence of an ultrafilter disjoint from $I$ implies in ZF+DC that $|E_1| \leq |E|$ for some orbit equivalence relation $E$?

Example 8.4.9. Let $K$ be a quotient cofinally locally finite Borel simplicial complex on a Polish space $X$. The poset $P = P_K$ has orbit charms below every condition.

Proof. Let $F$ be a Borel equivalence relation on the Polish space $X$ associated with the complex $K$. Recall (Theorem 6.4.17) that the balanced virtual conditions in $P$ are classified by maximal $K^{**}$-subsets of the virtual $F$-quotient space $X^{**}$.

Now, let $p \in P$ be an arbitrary condition. As in Example 8.4.6, let $U \subset [\omega]^\aleph_0$ and $\{x_n : n \in \omega\}$ be mutually generic ultrafilter and Sacks reals over $V$; the product of the Sacks reals is taken with countable support. Thus, $U$ generates an ultrafilter on $\omega$ in the model $V[U][x_n : n \in \omega]$. Let $V_n = V[U][x_m : m \geq n]$ for all $n \in \omega$. Then $\langle V_n : n \in \omega \rangle$ is a choice-coherent sequence of models; write $V_\omega = \bigcap_n V_n$.

To build the balanced virtual conditions for the models $V_n$, let $\prec$ be a coherent well-ordering of the collection of all sets of rank $< \beth_\omega$; so $\prec | V_n$ belongs to $V_n$ for every $n \in \omega$. Let $A_\omega \subset V_\omega$ be a maximal $K^{**}$-set extending $p$. For every number $m \in \omega$, by downward induction on $n \leq m$ build sets $A_n \in V_n$ as follows: $A_n^m$ is the $\prec$-first maximal $K^{**}$-set in the model $V_m$ extending $A_{n-1}$, and $A_n^m$ is the $\prec$-first maximal $K^{**}$-set extending $A_n^m$. Note that $(X^{**})^V_\omega$ is naturally embedded in $(X^{**})^V_{n-1}$ to make sense of this definition.

For each $n \in \omega$, observe that the sequence $\langle A_n^m : m \geq n \rangle$ belongs to the model $V_n$, and in the model $V_n$ let $A_n = \bigcap_m A_n^m \ d_n(x) = \{x \in X^{**} : \langle m \in \omega : x \in A_n^m \rangle \in U \}. The set $A_n \subset X^{**}$ is a maximal $K^{**}$-set as proved in Claim ?? below. Since for each $m \geq n$, $A_n^m \subset A_n^{m-1}$ holds, it must be the case that $A_n \subset A_{n-1}$ holds as well. In addition, $A_n \subset A_n$ occurs for each $n \in \omega$. As a result, the sets $A_n$ for $n \leq \omega$ generate a chain of balanced virtual conditions, completing the construction of the orbit charm for the poset $P$.

Corollary 8.4.10. Let $F$ be a Borel equivalence relation on a Polish space $Y$.

1. In the extension of the Solovay model by the linearization poset of Example 6.4.16 for the quotient $Y/F$, $|E_1| \leq |E|$ holds for every orbit equivalence relation $E$;
2. It is consistent relative to an inaccessible cardinal that $\text{ZF+DC}$ holds, the quotient $Y/F$ is linearly ordered, and $|\mathbb{E}_1| \nleq |E|$ holds for every orbit equivalence relation $E$.

**Proof.** This is a conjunction of Example 6.4.16, Example 8.4.9 and Theorem 8.4.3. 

Another group of orbit charming posets is obtained from orbit equivalence relations:

**Example 8.4.11.** Let $F$ be a Borel pinned orbit equivalence relation on a Polish space $X$. Let $P$ be the poset of countable subsets of $X$ consisting of pairwise $F$-unrelated elements, ordered by reverse inclusion. Then $P$ has orbit charms below every condition.

**Proof.** Recall (Example 6.4.8) that the balanced virtual conditions are classified by $F$-transversals. Let $p \in P$ be a condition. To find an orbit charm below $p$, consider any choice-coherent sequence $\langle V_n : n \in \omega \rangle$ consisting of generic extensions of the ground model such that for every $n \in \omega$, $2^{\omega} \cap V_n \setminus V_{n+1} \neq 0$—the choice-coherent sequence of models obtained from the infinite product of Sacks reals will do. To find a coherent sequence of balanced virtual conditions, work in the model $V_0$. Let $\prec$ be a coherent well-ordering of $X$. For each $F$-class $C \subset X$, either $C$ has a representative in $V_\omega$, or there is a largest number $n \in \omega$ such that $C$ has a representative in $V_n$—this follows from Theorem 4.3.7 together with the assumption that $E$ is pinned and the fact that the pinned property of Borel equivalence relations is absolute (Corollary 2.6.3). Let $x_C \in C$ be the element defined as follows:

- if $p \cap C \neq 0$ then $x_C$ is the unique element of $p \cap C$;
- if the first item fails and $C$ has a representative in $V_\omega$ then $x_C$ is the $\prec$-first element of $C \cap V_\omega$;
- if the first two items fail, let $n \in \omega$ be the largest number such that $C$ has a representative in $V_n$ and let $x_C$ be the $\prec$-first element of $C \cap V_n$.

Let $A = \{ x_C : C \text{ is an } F\text{-class} \}$. It is not difficult to see that $A$ is an $F$-transversal, and moreover for each $n \leq \omega$, the set $A \cap V_n$ belongs to $V_0$ and is an $F$-transversal there. Thus, for each $n \leq \omega$ let $\bar{p}_n$ be the balanced virtual condition derived from the set $A \cap V_n$. The system $\langle V_n : n \in \omega, \bar{p}_n : n \leq \omega \rangle$ is the sought orbit charm below $p$. 

**Corollary 8.4.12.** Let $F$ be a Borel pinned orbit equivalence relation on a Polish space $X$.

1. In the $P$-extension of the symmetric Solovay model, $|\mathbb{E}_1| \nleq |E|$ holds for every orbit equivalence relation $E$. 

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, $F$ has a transversal, and $|\mathcal{E}_1| \not\leq |E|$ holds for every orbit equivalence relation $E$.

Proof. This is a conjunction of Example 8.4.11 and Theorem 8.4.3.

Similar conclusions can be obtained for many posets connected with orbit equivalence relations. For example, for a Borel pinned orbit equivalence relation $F$, one can consider the poset adding countable complete section to $F$ by countable approximations—Example 6.4.9. A countable complete section of $F$ induces a cardinal equivalence $|F| \leq |\mathbb{F}_2|$. The poset has orbit charms obtained in a way similar to Example 8.4.11 and so in the resulting extension of the symmetric Solovay model, $|\mathcal{E}_1| \not\leq |E|$ holds for every orbit equivalence relation $E$.

The determination of the orbit charm status of the posets adding a maximal $\mathcal{K}$-set for a Borel simplicial complex $\mathcal{K}$ is much more challenging. The ultimate limitation is the surprising Theorem 13.4.7 below, showing in ZF+DC that if $R$ has a Hamel basis, then $\mathcal{E}_1$ has a complete countable section, and therefore $|\mathcal{E}_1| \leq |\mathbb{F}_2|$ holds. Thus, we have to resort to partial results.

Example 8.4.13. Let $\mathcal{K}$ be a countable Borel flag complex on a Polish space $X$. Let $P = P_\mathcal{K}$ be the poset of countable $\mathcal{K}$-sets ordered by reverse inclusion. Then $P$ has orbit charm below every condition $p \in P$.

Proof. To cut the notational clutter, we ignore the condition $p \in P$. Let $f : X \to [X]^\omega$ be a Borel function such that $\mathcal{K}$ consists of all $f$-free sets. The balanced virtual conditions in the poset $P$ were described in Theorem 6.1.6; for the purposes of this proof, a more specific type of balanced conditions will be used. To begin, consider the set $B \subset X$ of all points $x \in X$ for which there is a countable set $a \subset X$ such that

- either the set $\{ y \in X : x \in f(y) \subset a \}$ is uncountable;
- or the collection $\{ f(y) \setminus a : y \in X, x \in f(y) \} \subset [X]^\omega$ cannot be punctured by a countable set $a \subset X$.

A careful computation using ??? shows that the set $B$ is $\Sigma^1_2$. Let $A$ be a maximal $\mathcal{K}$-subset of $X \setminus B$. Let $\tau_A$ be the $\text{Coll}(\omega, X)$-name for the set $\{ p \in P : p \forces \check{A}_{\text{gen}} \cap V = \check{A} \}$. The following claim follows from Proposition 6.1.3:

Claim 8.4.14. The pair $(\text{Coll}(\omega, X), \tau_A)$ is a balanced $P$-pin.

Now, let $\langle V_n : n \in \omega \rangle$ be a choice-coherent sequence of generic extensions such that $V_0$ is an extension of the ground model by proper forcing, and for all $n \in \omega$ the set $2^{<\omega} \cap V_n \setminus V_{n+1}$ is nonempty. The following claim uses the properness assumption:

Claim 8.4.15. Whenever $n \in \omega$ is a number and $y \in X \cap V_n \setminus V_{n+1}$ is a point, then $f(y) \setminus B \subset V_n \setminus V_{n+1}$.
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Proof. Suppose that \( x \in V_{n+1} \) and \( x \in f(y) \); we must show that \( x \in B \). The set \( f(y) \cap V_{n+1} \) is a subset of \( V_{n+1} \) which is countable in \( V_n \); by the properness assumption, it is covered by a set \( a \in V_{n+1} \) which is countable in \( V_{n+1} \). We claim that \( a \) is a witness to \( x \in B \). Indeed, if \( f(y) \subset a \) then \( x \in B \) by the first item in the definition of \( B \) (the point \( y \) is not in \( V_{n+1} \) and so it does not belong to any countable set in \( V_{n+1} \)), and if \( f(y) \setminus a \neq 0 \) then \( x \in B \) by the second item in the definition of \( B \) \( ((f(y) \setminus a) \cap V_{n+1} = 0 \) and so \( f(y) \setminus a \) is disjoint from any countable set in \( V_{n+1} \)). \( \square \)

Now, work in \( V_0 \). Let \( \prec \) be a coherent well-ordering of \( P(X) \), and for every number \( n \in \omega \) let \( C_n \subset V_n \) be the \( \prec \)-least maximal \( K \)-subset of \( (X \setminus B) \cap V_n \setminus V_{n+1} \). Let also \( C_\omega \subset V_\omega \) be the \( \prec \)-maximal \( K \)-subset of \( X \cap V_\omega \setminus V_\omega \). For \( n \leq \omega \), let \( A_n = \bigcup_{m \geq n} C_m \). It is immediate that \( A_n \subset V_n \) and if \( n \in m \) then \( A_m = A_n \setminus V_m \). The claim implies that \( A_n \) is a \( K \)-set and so a maximal \( K \)-subset of \( (X \setminus B) \cap V_n \). Let \( \bar{p}_n \) be the balanced virtual condition in the model \( V_n \) derived from \( A_n \) as in Claim 8.4.14. It is immediate that \( \langle V_n : n \in \omega, \bar{p}_n : n \leq \omega \rangle \) is the desired orbit charm. \( \square \)

Corollary 8.4.16. Let \( K \) be a countable Borel flag complex on a Polish space \( X \).

1. In the \( P_K \) extension of the symmetric Solovay model, \( |E_1| \not\leq |E| \) holds for every orbit equivalence relation \( E \);

2. it is consistent relative to an inaccessible cardinal that \( \text{ZF+DC} \) holds, there is a maximal \( K \)-set, and \( |E_1| \not\leq |E| \) holds for every orbit equivalence relation \( E \).

Example 8.4.17. Let \( X \) be a Polish space and let \( G \) be a Borel graph on \( X \). Let \( K \) be the Borel simplicial complex on \( G \) of finite acyclic subsets of \( G \), and let \( P = P_K \). Then \( P \) is orbit-charming.

Proof. Let \( Q_m \) for each \( m \in \omega \) be the countable support product of \( R_1 \) many Sacks posets, and for each \( n \in \omega \) let \( P_n \) be the countable support product \( \Pi_{m \geq n} Q_m \), with the natural projection maps. Let \( H_0 \subset P_0 \) be a generic filter, and for each \( n \in \omega \) let \( H_n = H \cap P_n \). By Example 4.3.5 and Theorem 4.3.6, the sequence \( \langle V[H_n] : n \in \omega \rangle \) is a choice-coherent sequence of models of \( \text{ZFC} \); write \( V_n = V[H_n] \) and \( \omega_n = \bigcap V_n \).

Write \( X_\omega = X \cap V_\omega \), and for each \( m \leq n \) write \( X_{mn} = X_\omega \cup (X \cap V_m \setminus V_n) \). The following elusive claim is the main reason for the choice of the posets \( Q_m \).

Claim 8.4.18. Let \( l < m < n \) be natural numbers and \( x_0, x_1 \in X_{mn} \) be vertices. If \( x_0, x_1 \) are connected by a path in the graph \( G \upharpoonright X_{lm} \), then they are connected by a path in the graph \( G \upharpoonright X_{mn} \).

Proof. It will be enough to show that if \( x_0, x_1 \) are connected by a path in the graph \( G \upharpoonright X_{lm} \) whose vertices except for \( x_0, x_1 \) all belong to \( X_{lm} \), then they are connected by a path in the graph \( G \upharpoonright X_{mn} \). To this end, work in \( V_m \) and let...
$A = \{ a \in [X]<^{\aleph_0} : \text{there is a } G\text{-path connecting } x_0, x_1 \text{ using only the vertices in } a \}$. 

The set $A \subset [X]<^{\aleph_0}$ is Borel. It cannot be punctured by a countable set by a Mostowski absoluteness argument: in the model $V_1$ there is a $G$-path between $x_0, x_1$ using no vertices in $V_m$ and therefore no vertices in any given countable set in $V_m$. By the work on puncture sets [4, Theorem 21] there is a perfect set $B \subset A$ consisting of pairwise disjoint sets. By a standard fusion argument with the product of Sacks forcing, there is a countable set $c$ of Sacks reals added by the filter $H_m$ such that $B$ has a code in the model $V[c]$. Since the poset $Q_m$ is a product of uncountably many copies of Sacks forcing, in $V_m$ there has to be an element $a \in B$ which does not belong to the model $V_n[c] \subset V_m$. Since the sets in $B$ are pairwise disjoint, every element of $a$ would reconstruct $a$ over the model $V_n[c]$. Therefore, no element of $a$ belongs to the model $V_n[c]$; in particular, $a \cap V_n = 0$ and $a$ yields the desired $G$-path between $x_0$ and $x_1$ using only vertices in $X_{mn}$. 

By Theorem 6.2.21, it will be enough to find sets $A_m \subset G$ for $m \leq \omega$ forming an inclusion-descending sequence such that $V_m \models A_m \subset G$ is a maximal acyclic subset of $G$. To this end, let $\prec$ be a coherent well-ordering of all subsets of $G$ in $V_0$. Let $A_\omega \subset G$ be a maximal acyclic subset of $G$ in the model $V_\omega$. For each $m \in \omega$, let $A_{mm+1} \subset V_m$ be the $\prec$-first set in the model $V_m$ which is a maximal acyclic subset of $G \upharpoonright X_{mm+1}$ and extends $A_\omega$.

Now, by induction on $n - m$, for $m < n$ define $A_{mn} \subset V_m$ to be the $\prec$-first set in the model $V_m$ which is a maximal acyclic subset of $G \upharpoonright X_{mn}$ and extends both $A_{m+1,n}$ and $A_{m,n-1}$. To see that this is possible, simultaneously argue by induction on $n - m$ that if $m \leq m' < n' \leq n$ then $A_{m',n'} \subset A_{mn}$ and moreover, the set $A_{m+1,n} \cup A_{m,n-1}$ does not contain a cycle. The former statement is clear. The latter statement requires the claim. A putative cycle $c \subset A_{m+1,n} \cup A_{m,n-1}$ has to contain some edges from both sets by the acyclicity induction hypothesis. Choose an inclusion-maximal contiguous part $d \subset c$ of the cycle consisting of edges in $A_{m,n-1} \setminus A_{m+1,n}$. The end-nodes of $d$, denote them by $x_0, x_1$, must be distinct because the cycle must use some edges from the set $A_{m+1,n}$ as well. It must also be the case that $x_0, x_1 \in X_{mm+1} \cap X_{m+1,n} = X_{m+1,n-1}$. The path $d$ connects the nodes $x_0, x_1$ in the graph $G \upharpoonright X_{mn-1}$; by the claim then, they have to be connected by a path in the graph $G \upharpoonright X_{m+1,n-1}$ as well, and therefore by a path $e$ in the set $A_{m+1,n-1}$. Then $e \cup d$ forms a cycle in the set $A_{m,n-1}$, violating the induction hypothesis.

In the end, let $A_m = \bigcup_{n > m} A_{mn}$. It is clear from the coherence of the well-ordering $\prec$ that $A_m \subset V_m$. The construction also guarantees that $V_m \models A_m \subset G$ is a maximal acyclic subset, and $m < n$ implies $A_n \subset A_m$. The proof is complete.

**Corollary 8.4.19.** Let $X$ be a Polish space and let $G$ be a Borel graph on $X$. Let $K$ be the Borel simplicial complex on $G$ of finite acyclic subsets of $G$.

1. In the $P_K$ extension of the symmetric Solovay model, $|E| \not\leq |E|$ holds for every orbit equivalence relation $E$;
it is consistent relative to an inaccessible cardinal that ZF+DC holds, G has a maximal acyclic subset, and $|E_1| \not\leq |E|$ holds for every orbit equivalence relation $E$.

8.5 Cardinal charms

In this section, we provide a methodology for some more limited quotient cardinal preservation results.

**Definition 8.5.1.** Let $P$ be a Suslin forcing. Let $E, F$ be Borel equivalence relations on respective Polish spaces $X, Y$. An $E, F$-charm is a pair $(\bar{V}[G], \bar{p})$ where $\bar{p}$ is a balanced virtual condition in $V[G]$, and in $V[G]$ there is no injection from the set of virtual $E$-classes to the set of virtual $F$-classes.

**Definition 8.5.2.** Let $P$ be a Suslin forcing, $E, F$ be Borel equivalence relations, and $\kappa$ be an inaccessible cardinal.

1. $P$ is $E, F$-charming if it has an $E, F$-charm below every condition in $P$;

2. $P$ is $E, F$-charming cofinally below $\kappa$ if $V_\kappa$ satisfies that every ordinal can be collapsed by a forcing which makes $P$ $E, F$-charming.

**Theorem 8.5.3.** Let $\kappa$ be an inaccessible cardinal. Let $E, F$ be Borel equivalence relations on respective Polish spaces $X, Y$, Let $P$ be a Suslin forcing which is $E, F$-charming cofinally below $\kappa$. Let $W$ be the Solovay model derived from $\kappa$, and let $G \subset P$ be a filter generic over $W$. In the model $W[G]$, $|E| \not\leq |F|$ holds.

**Proof.** Suppose towards contradiction that the conclusion fails. By the forcing theorem, in the model $W$ there has to be a $P$-name $\tau$ and a condition $p \in P$ such that $p \Vdash \tau$ is a total injection from $E$-classes to $F$-classes. The condition $p \in P$ as well as the name $\tau$ must be definable from ground model parameters and some extra parameter $z \in 2^{\omega}$. Use the assumption on the poset $P$ to find an intermediate model $V[K]$ which is a generic extension of the ground model by a poset of size $< \kappa$, such that $z \in V[K]$ and $V[K] \models P$ is $E, F$-charming.

In the model $W$, there is some generic extension $V[K][L]$ by a poset of size $< \kappa$ such that in the model $V[K][L]$, there is a balanced virtual condition $\bar{p} \leq p$ and no injection from virtual $E$-classes to virtual $F$-classes definable from $\bar{p}$ and parameters in $V[K]$. Work in the model $V[K][L]$. Let $X^{**}, Y^{**}$ be the virtual $E$ and $F$ spaces respectively, and let $h \in X^{**} \times Y^{**}$ be the relation defined by the following: $(\bar{x}, \bar{y}) \in h$ if there is a poset $R$ of size $< \kappa$ and $R$-names $\dot{x}, \dot{y}$, and $\sigma$ for elements of $X, Y$ and $P$ respectively so that $R$ forces that $\sigma \leq p$, $\dot{x}$ is a realization of the virtual $E$-class $\bar{x}$, $\dot{y}$ is a realization of the virtual $F$-class $\bar{y}$, and $\text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash P(\bar{x}E) = [\bar{y}].$ We will show that $h$ is in fact an injection from $X^{**}$ to $Y^{**}$, which contradicts the assumption on the model $V[K][L]$ since $h$ is definable from $\bar{p}$ and parameters from the ground model.

The checking of the properties of the relation $h$ is easy and somewhat repetitive. The most important point is that $\text{dom}(h) = X^{**}$ holds. To see this, fix
a virtual $E$-class $\bar{x}$. In the model $V[K][L]$, there must be a poset of size $< \kappa$ and $R$-names $\bar{x}, \bar{y}$ and $\sigma$ for elements of $X, Y$, and $P$ respectively such that $R$ forces $\bar{x}$ to be a realization of $\bar{x}, \sigma \leq \bar{p}$, and $\text{Coll}(\omega, < \kappa) \models \sigma_{\bar{p}} \tau([\bar{x}]_E) = [\bar{y}]_F$.

We claim that in fact the name $\bar{y}$ is $F$-pinned on the poset $R$. Then, writing $\bar{y} \in Y^{**}$ for the virtual $F$-class of the pair $\langle R, \bar{y} \rangle$, it is immediate that $\langle \bar{x}, \bar{y} \rangle \in h$.

To see that $\bar{y}$ is an $F$-pinned name, suppose that $r_0, r_1 \in R$ are conditions. In the model $W$, find filters $H_0, H_1 \subseteq R$ mutually generic over the model $V[K][L]$ such that $r_0 \in H_0$ and $r_1 \in H_1$. Write $x_0 = \bar{x}/H_0$, $y_0 = \bar{y}/H_0$ and $p_0 = \sigma/H_0$, and similarly for subscript $1$; we must show that $y_0 \ V y_1$. To see this, use the balance of the condition $\bar{p}$ to argue that the conditions $p_0, p_1 \ V P$ have a lower bound $q$. Since the model $W$ is the symmetric Solovay extension of both models $V[K][L][H_0]$ and $V[K][L][H_1]$, the forcing theorem applied in these models shows that in $W$, $q \ V \tau([x_0]_E) = \tau([x_1]_E) = [y_0]_F = [y_1]_F$, in other words $y_0, y_1$ are $F$-related as desired.

It is now necessary to show that the function $h$ is a function and an injection; we prove the latter. Suppose towards contradiction that there are distinct virtual $E$-classes $\bar{x}_0, \bar{x}_1 \in X^{**}$ and a virtual $F$-class $\bar{y} \in Y^{**}$ such that $\langle \bar{x}_0, \bar{y} \rangle$ and $\langle \bar{x}_1, \bar{y} \rangle$ are both elements of $h$. There must be the corresponding posets $R_0, R_1 \subseteq V[K][L]$ of size $< \kappa$, and $R_0$-names $\bar{x}_0, y_0, \sigma_0$ and $R_1$-names $\bar{x}_1, y_1, \sigma_1$. In the model $W$, find filters $H_0 \subseteq R_0$ and $H_1 \subseteq R_1$ which are mutually generic over $V[K][L]$. Let $x_0 = \bar{x}_0/H_0$, $y_0 = \bar{y}/H_0$, $p_0 = \sigma_0/H_0$ and similarly for the subscript $1$. Since $\bar{x}_0, \bar{x}_1$ were distinct virtual $E$-classes, it is the case that $x_0, x_1$ are $E$–unrelated elements while $y_0, y_1$ are $F$-related elements. The balance of the condition $\bar{p}$ shows that $p_0, p_1 \ V P$ are compatible conditions, with lower bound some $q \in P$. Since $W$ is the symmetric Solovay extension of both models $V[K][L][H_0]$ and $V[K][L][H_1]$, the forcing theorem applied in these models shows that in $W$, $q \ V \tau([x_0]_E) = \tau([x_1]_E) = [y_0]_F = [y_1]_F$. This contradicts the assumption that $\tau$ was forced to be an injection.

We will illustrate the power of the theorem on several independence results regarding pure cardinal inequalities among the quotient spaces.

**Definition 8.5.4.** Let $E$ be a Borel equivalence relation on a Polish space $X$. The poset $P_E$ consists of countable sets $p \subseteq X$, with the ordering defined by $p_1 \ V p_0$ if $p_0 \subset p_1$ and $p_1 \cap [p_0]_E = p_0$.

It is not difficult to see that the poset $P_E$ is $\sigma$-closed. The union of the conditions in the generic filter is a set which meets every $E$-class in a nonempty countable set—a complete countable $E$-section. This means in particular that in the $P_E$ extension, $\|E\| \leq |F_2|$ holds—to each $E$-class one can assign the $F_2$ class of all enumerations of the intersection of the generic set with the $E$-class. It is not difficult to see that if $E$ is pinned then the poset $P_E$ is balanced, and every set $A \subseteq X$ intersecting every $E$-class is a balanced condition. If $E$ is not pinned then there are no balanced conditions in this forcing.

If $P_E$ is a pinned equivalence relation then $P_E$ introduces the inequality $\|E\| \leq |F_2|$ without introducing most any other cardinal inequality between quotient spaces. This is the content of the following theorem.
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**Theorem 8.5.5.** Let \( \kappa \) be an inaccessible cardinal. Let \( E \) be a pinned equivalence relation on a Polish space \( X \). In the \( P_E \)-extension of the Solovay model, the following holds:

1. \( E \) has a complete countable section, in particular \( |E| \leq |\mathbb{F}_2| \);
2. whenever \( D \) is the orbit equivalence relation of a turbulent group action and \( F \) is a Borel trim equivalence relation, then \( |D| \leq |F| \);
3. whenever \( F \) is a pinned orbit equivalence relation then \( \mathbb{E}_1 \leq |F| \);

**Definition 8.5.6.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \).

1. A \( E, E_0 \)-map is a partial function \( h : X \to 2^\omega \) such that \( E \)-related inputs yield \( E_0 \)-related outputs and \( E \)-unrelated inputs yield \( E_0 \)-unrelated outputs;
2. The collapse poset of \( E \) to \( \mathbb{E}_0 \) is the poset \( P_E \) consisting of all pairs \( p = (a_p, b_p) \) where \( a_p \) is a countable \( E, E_0 \)-function, and \( b_p \subset 2^\omega \) is a countable set such that \( b_p \cap \text{rng}(a_p) = \emptyset \). The ordering is defined by \( p_1 \leq p_0 \) if \( a_{p_0} \subset a_{p_1}, b_{p_0} \subset b_{p_1}, \) and \( \text{dom}(a_{p_1}) \cap |\text{dom}(a_{p_0})|_E = \text{dom}(a_{p_0}) \).

Clearly, the poset \( P_E \) is designed to add an injection from the \( X/E \) quotient to the \( 2^\omega/\mathbb{E}_0 \) quotient as the union of the first coordinates of conditions in the generic filter. At the same time, the poset \( P_E \) keeps the cardinality of the \( \mathbb{E}_0 \)-quotient larger than \( 2^\omega \). This is the contents of the following theorem.

**Theorem 8.5.7.** Let \( \kappa \) be an inaccessible cardinal. Let \( E \) be a pinned Borel equivalence relation on a Polish space \( X \). In the \( P_E \)-extension of the symmetric Solovay model, the following holds:

1. \( |E| \leq |\mathbb{E}_0| \);
2. \( |\mathbb{E}_0| > |2^\omega| \).

**Proof.** It is not difficult to see that the poset \( P \) is balanced. The pinned assumption on the equivalence relation \( E \) shows that the balanced virtual conditions are classified by partial \( E, E_0 \)-functions \( h : X \to 2^\omega \). To verify the inequality \( |\mathbb{E}_0| > |2^\omega| \), one has to find an \( \mathbb{E}_0 \)-id-charm for the poset \( P \).

Towards this aim, let \( Q \) be the forcing adding an \( E, E_0 \)-function from \( X \cap V \) to \( 2^\omega \) of the extension with finite conditions. More precisely, let \( q \in Q \) if there is a number \( m \in \omega \) such that \( q : X \to 2^m \) is a finite function. The ordering is defined by \( q_1 \leq q_0 \) if \( \text{dom}(q_0) \subset \text{dom}(q_1) \), for all \( x \in \text{dom}(q_0) \) \( q_0(x) \subseteq q_1(x) \) holds, and for all \( E \)-related points \( x_0, x_1 \in \text{dom}(q_0), q_1(x_0) \setminus q_0(x_0) = q_1(x_1) \setminus q_0(x_0) \) holds. If \( H \subset Q \) is a generic filter then the function associated with \( H \) is the function \( h : X \cap V \to 2^\omega \) defined by \( h(x)(n) = b \) if there is a condition \( q \in H \) such that \( q(x)(n) = b \). The definition of the poset \( Q \) guarantees that \( h \) is and \( E, E_0 \)-function with \( \text{dom}(h) = X \cap V \).

**Claim 8.5.8.** The poset \( Q \) is c.c.c. very Suslin, and it is Suslin \( \sigma \)-centered.
Proof. It is clear that the poset $Q$ is Suslin. To verify the very Suslin property, if $a \subset Q$ is a countable set, if there is a condition in $Q$ incompatible with all elements of $a$, then there must be one whose domain is a subset of $\bigcup_{q \in a} \text{dom}(q)$. Thus, to verify the predensity of the set $a \subset Q$, one needs to check compatibility with only countably many conditions in $Q$, which is a Borel task.

To see that the poset $Q$ is Suslin $\sigma$-centered, for every finite collection $b$ of pairwise disjoint basic open subsets of $X$ and every function $f : b \to 2^\omega$ for $m \in \omega$, let $B_{bf} = \{ q \in Q : \text{dom}(q) \subset \bigcup b \land \forall O \in b \forall x \in \text{dom}(q) \cap O q(x) = f(O) \}$. It is not difficult to check that the sets $B_{bf} \subset Q$ are Borel and centered, and each condition in $Q$ appears in at least one of them. 

Let $z_0, z_1 \in 2^\omega$ be points mutually Cohen generic over $V$. In $V[z_0, z_1]$ let $R$ be the finite support iteration of the poset $Q$ of length $\omega_1$ and let $H \subset R$ be a filter generic over $V[z_0, z_1]$. For every countable ordinal $\alpha \in \omega_1$ let $H_\alpha$ be the filter on $Q$ added by the $\alpha$-th stage of the iteration, and let $h_\alpha : X \cap V_\alpha \to 2^\omega \cap V_{\alpha+1}$ be the function associated with $H_\alpha$, where $V_\alpha = V[z_0, z_1][H_\beta : \beta \in \alpha]$. In the model $V[z_0, z_1][H]$, let $h$ be the partial function from $X$ to $2^\omega$ defined by $x \in \text{dom}(h)$ if $x \in V_\alpha$ for the smallest ordinal $\alpha$ such that $V_\alpha$ contains any representatives of the class $[x]_E$, and $h(x) = h_\alpha(x)$. It is clear that $h$ is an $E, E_0$-function with $\| \text{dom}(h) \|_E = X$ and therefore represents a balanced virtual condition in the poset $P_E$. Note that $h$ depends only on $H$ and the model $V[z_0, z_1]$, not on the points $z_0, z_1$ per se.

Claim 8.5.9. In $V[K][H]$, there is no injection from $2^\omega/\mathbb{E}_0$ to $2^\omega$ which is definable from $h$ and elements of the ground model.

Proof. Return to $V$ and assume that there is a condition $(t_0, t_1, \dot{r})$ in the iteration Cohen $\times$ Cohen $\times R$ and a name $\dot{k}$ for an injection from $2^\omega/\mathbb{E}_0$ to $2^\omega$ which is definable by some specific formula from $h$ and parameters in the ground model. Since the two Cohen generic points $\dot{z}_0, \dot{z}_1$ added in the beginning are forced to be $\mathbb{E}_0$-unrelated, strengthening the condition if necessary we can find a number $n \in \omega$ so that $\dot{k}(\dot{z}_0)(n) = 0$ and $\dot{k}(\dot{z}_1)(n) = 1$ is forced, or vice versa. There also has to be a countable ordinal $\alpha \in \omega_1$ such that the condition $\dot{r}$ is forced to be an element of $R_\alpha$, the finite support iteration of $Q$ of length $\alpha$.

By Theorem 13.2.16, the poset $R_\alpha$ is Suslin $\sigma$-centered, and so by strengthening $t_0, t_1$ if necessary there has to be an analytic set $A \subset R_\alpha$ which is centered so that $(t_0, t_1) \Vdash \dot{r} \in A$. Let $z_0, z_1 \in 2^\omega$ be Cohen points mutually generic over $V$ extending the binary strings $t_0, t_1$ respectively. Let $z_0'$ be $z_1$ with an initial segment rewritten by $t_0$, and let $z_1'$ be $z_0$ with an initial segment rewritten by $t_1$. Thus, $z_0', z_1' \in 2^\omega$ are points mutually generic over $V$ extending $t_0, t_1$ respectively, and moreover $V[z_0, z_1] = V[z_0', z_1']$. Let $r = r/z_0, z_1$ and $r' = r/z_0', z_1'$. These are two conditions in the poset $R_\alpha$ which both belong to $A$, and therefore are compatible. Let $H \subset R$ be a filter generic over the model $V[z_0, z_1]$ containing their lower bound. The evaluation of the functions $h, k$ using their respective definitions yields the same objects whether the filters obtained from $z_0, z_1, H$ or the filters obtained from $z_0', z_1', H$ are used. By the forcing theorem, it should be the case that $k(z_0')(n) = k(z_0)(n) = 0$ and $k(z_1')(n) = k(z_1)(n) = 1$. 

and also, since $z_0, z'_1$ are $E_0$ related and so are $z_1, z'_0$, it should be the case that $k(z_0) = k(z'_1)$. This is impossible.

A reference to Theorem 8.5.3 now concludes the argument.
Chapter 9

Uniformization in $W[G]$

The question whether various forms of uniformization hold in the models within purview of this book is one of the more slippery issues we set out to resolve.

9.1 Adequate Suslin forcing

In order to prove all forms of uniformization in a clean sweep, several definitions must be stated.

**Definition 9.1.1.** Let $P$ be a Suslin forcing and $\lambda$ be a cardinal. Let $Q$ be a poset and $\tau$ be a $Q$-name for an element of $P$. Say that the pair $\langle Q, \tau \rangle$ is $\lambda$-adequate if for every virtual condition $\sigma$ in $P$ represented on a poset of size $< \lambda$, either $Q \Vdash \tau \leq \sigma$ or $Q \Vdash \tau, \sigma$ are incompatible in $P$.

**Definition 9.1.2.** A Suslin forcing $P$ is $\lambda$-adequate if every $\lambda$-adequate pair is balanced. $P$ is adequate if it is $\lambda$-adequate for some $\lambda$. If $\kappa$ is an inaccessible cardinal then $P$ is adequate below $\kappa$ if $V_\kappa \models P$ is adequate in every forcing extension.

Note that every balanced pair is $\lambda$-adequate for every cardinal $\lambda$. Thus, a Suslin poset is adequate if every reasonable attempt at a balanced condition actually works. In the way of an initial example, consider the poset $P$ of all infinite subsets of $\omega$ ordered by inclusion and $\lambda = \aleph_0$. If $\langle Q, \tau \rangle$ is a $\lambda$-adequate condition, then for every infinite set $a \subset \omega$ it is the case that either $Q \Vdash \tau \setminus \check{a}$ is finite or $Q \Vdash \tau \cap \check{a}$ is finite. Let $U = \{ a \subset \omega : Q \Vdash \tau \setminus \check{a} \text{ is finite} \}$ and observe that $U$ is a nonprincipal ultrafilter on $\omega$. Then, Theorem 7.1.4 shows that $\langle Q, \tau \rangle$ is balanced as required.

It is important to understand that Suslin forcing can be adequate for more complex reasons than the ultrafilter example in the previous paragraph. The $\lambda$-adequacy statement becomes weaker as $\lambda$ grows, and some Suslin forcings become adequate only at higher values of $\lambda$. The following upper bound question remains open.
Question 9.1.3. Let $P$ be an adequate Suslin forcing. Is $P$ $\lambda$-adequate for some $\lambda < \beth_1$?

One immediate corollary of adequacy is that it places an upper bound on the number of balanced classes, which we do not know how to obtain in general.

Proposition 9.1.4. Let $P$ be a Suslin forcing. If $P$ is $\lambda$-adequate then there are at most $2^{2^\lambda}$ many balanced classes.

Proof. Let $A$ be the set of all virtual conditions of $P$ represented on posets of size $< \lambda$. It is a matter of elementary cardinal arithmetic to conclude that $|A| \leq 2^\lambda$. Whenever $(Q, \tau)$ is a balanced pair, write $B_{\tau} = \{ \sigma \in A : Q \Vdash \tau \leq \sigma \}$.

To prove the proposition, it will be enough to show that the set $B_{\tau}$ is a complete invariant of the equivalence of balanced pairs.

It is clear that if balanced pairs $(Q_0, \tau_0)$ and $(Q_1, \tau_1)$ are equivalent, then the sets $B_{\tau_0}, B_{\tau_1}$ are equal. For the opposite implication, if the sets $B_{\tau_0}, B_{\tau_1}$ are equal, then consider the poset $Q_2$ which is the disjoint union of $Q_0, Q_1$, and the $Q_2$-name $\tau_2$ which is the disjoint union of $\tau_0, \tau_1$. Clearly, the pair $(Q_2, \tau_2)$ is $\lambda$-adequate and therefore balanced by the initial assumption on the poset $P$. It follows that $Q_0 \times Q_1 \Vdash \tau_0, \tau_1$ are compatible in $P$ and therefore the balanced pairs $(Q_0, \tau_0)$ and $(Q_1, \tau_1)$ are equivalent as desired. \qed

9.2 Pinned uniformization

In this section, we show that sets whose vertical sections are $E$-classes for a suitably regular equivalence relation $E$ can be uniformized.

Definition 9.2.1. Let $E$ be an equivalence relation on a Polish space $X$. The $E$-uniformization is the statement: if $B \subset 2^\omega \times X$ is a set whose vertical sections are $E$-classes, then there is a function $f \subset B$ such that for every $y \in 2^\omega$, if $B_y \neq \emptyset$ then $f(y)$ is defined (and is an element of $B_y$).

Theorem 9.2.2. Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin poset which is balanced cofinally below $\kappa$ and adequate below $\kappa$. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$. Let $W$ the symmetric Solovay model derived from $\kappa$, let $G \subset P$ be a $W$-generic filter. In the model $W[G]$, the $E$-uniformization holds.

Proof. The argument uses two claims. The first contains the critical use of the adequacy assumption and will be referenced later.

Claim 9.2.3. Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin poset which is balanced cofinally below $\kappa$ and adequate below $\kappa$. Let $W$ the symmetric Solovay model derived from $\kappa$, let $G \subset P$ be a $W$-generic filter, and let $z \in 2^\omega \cap W$. Then $G$ contains a realization of a balanced virtual condition in $\text{HOD}_{V,z,G}$.

Proof. Let $M = \text{HOD}_{V,z,G}$; $M$ is a model of ZFC. Use the balance of the poset $P$ to argue that $M$ is a generic extension of $V$ by a poset of size $< \kappa$ and
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Therefore belongs to \( W \) by Theorem 8.1.1. Use the adequacy assumption to find a cardinal \( \lambda \in \kappa \) such that \( M \models P \) is \( \lambda \)-adequate. Working in \( M \), let \( D \) be the set of all virtual conditions in \( P \) represented on posets of size \( < \lambda \). In \( W[G] \), let \( C \subset D \) be the set of all virtual conditions in the set \( D \) whose realization belongs to the generic filter \( G \). The set \( C \) has just been defined from the ordinal \( \lambda \) and the ultrafilter \( G \), so \( C \in M \). By the genericity of the filter \( G \), there must be a condition \( p \in G \) which is below all conditions in \( C \) and incompatible with all the conditions in \( D \setminus C \).

By the forcing theorem, in \( M \) there has to be a poset \( Q \) of size \( < \kappa \) and a \( Q \)-name \( \tau \) such that \( Q \models \tau \) is below all conditions in \( C \) and incompatible with all conditions in \( D \setminus C \), and such that there is a filter \( H \subset Q \) generic over the model \( M \) with \( p = \tau / H \). By the adequacy assumption, the pair \( \langle Q, \tau \rangle \) is balanced in \( M \). Still working in the model \( M \), let \( R \) be the poset \( \text{Coll}(\omega, \mathcal{P}(Q)) \) and let \( \sigma \) be the \( R \)-name for the analytic set \( \{ p' \in P : \exists H' \subset Q \ K \text{ meets all the open dense sets on the } R \text{-generic enumeration and } p' = \tau / H' \} \). Then \( \langle R, \sigma \rangle \) is a balanced virtual condition in \( M \) and the filter \( G \) contains a realization of it.

**Claim 9.2.4.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing and \( \bar{p} \) a balanced virtual condition in \( P \). Let \( E \) be a pinned equivalence relation on a Polish space \( X \). Let \( W \) be a symmetric Solovay extension derived from \( \kappa \) and \( \sigma \) in \( W \), let \( \tau \) be a \( \bar{P} \)-name definable from ground model elements for an \( E \)-class. Let \( \bar{p} \) be a balanced condition in \( P \). Then \( W \models \bar{p} \vDash \tau \cap V \neq 0 \).

*Proof.* Suppose not. Then in \( V \), there has to be a poset \( Q \) of size \( < \kappa \) and \( R \)-names \( \sigma \) for a condition in \( P \) and \( \eta \) for an element of \( X \) such that \( R \vDash \sigma \leq \bar{p}, R \vDash \eta \) has no ground model \( E \)-equivalent and \( R \vDash \text{Coll}(\omega, \mathcal{P}(Q)) \vDash \sigma \vDash P \eta \in \tau \). In \( W \), let \( H_0, H_1 \subset R \) be mutually generic filters. Write \( p_0 = \sigma / H_0, p_1 = \sigma / H_1, x_0 = \eta / H_0 \) and \( x_1 = \eta / H_1 \). The balance of the condition \( \bar{p} \) implies that \( p_0, p_1 \in P \) are compatible conditions with a lower bound \( q \). The pinned assumption on the equivalence relation implies that \( x_0, x_1 \in X \) are not \( E \)-related.

Since \( W \) is also a symmetric Solovay extension of both \( V[H_0] \) and \( V[H_1] \), the forcing theorem applied in these two models implies that \( q \vDash \bar{x}_0, \bar{x}_1 \in \tau \). This is impossible as \( \tau \) is forced to be an \( E \)-class and \( x_0, x_1 \) are not \( E \)-related.

To complete the argument for the theorem, suppose that \( E \) is a pinned Borel equivalence relation on a Polish space \( X \). Let \( W \) be a symmetric Solovay extension derived from the cardinal \( \kappa \). In the model \( W \), let \( p \in P \) be a condition and \( \tau \) be a \( \bar{P} \)-name such that \( p \vDash \tau \subset 2^\omega \times X \) is a set whose vertical sections are \( E \)-classes. Let \( z \in 2^\omega \) be a parameter such that \( p, \tau \) are both definable from \( z \) and some additional ground model parameters. Let \( G \subset P \) be a generic filter meeting the condition \( p \) and let \( B = \tau / G \). We claim that every nonempty vertical section \( B_y \) contains an element of the model \( M_y = \text{HOD}_{V[y,z,G]} \). Once this is proved, it can be concluded that the function assigning to each \( y \in 2^\omega \) the element of \( M_y \cap B_y \) with the lexicographically first definition is the required uniformization of the set \( B \).

To find an element of \( B_y \) in the model \( M_y \), first use Claim 9.2.3 to see that the filter \( G \) contains a realization of a balanced virtual condition in \( M_y \). Then
use Claim 9.2.4 in $M_y$ to conclude that $B_y$ must contain an element of $M_y$ as desired. Note that the Borel equivalence relation $E$ is still pinned in $M_y$ by Theorem 2.6.1.

Theorem 9.2.2 is the strongest possible result of its kind, as the following observation shows.

**Theorem 9.2.5.** Let $E$ be an unpinned Borel equivalence relation on a Polish space $X$. Then $E$-uniformization fails in balanced extensions of the Solovay model.

**Proof.** Let $W[G]$ be a balanced extension of the Solovay model. For each parameter $z \in 2^\omega$ the model $M_z = \text{HOD}_{V,z,G}$ is well-ordered and therefore by Theorem 8.1.1, it is a well-ordered subclass of $W$ and so an extension of $V$ by a poset $< \kappa$. The Borel equivalence relation $E$ is unpinned in $M_z$ by Theorem 2.6.1. By Theorem 2.8.3, $M_z \models$ there is a nontrivial $E$-pinned name on the poset $\text{Coll}(\omega, \omega_1)$. Note that $(\mathcal{P}(\omega_1))^{M_z}$ is a countable set in $W$. Thus, one can successfully define the set $B \subset 2^\omega \times X$ by setting $(z, x) \in B$ if there is a filter $g \subset \text{Coll}(\omega, \omega_1)^{M_z}$ generic over $M_z$ such that $\tau_z/g E x$, where $\tau_z$ is the first $E$-unpinned name in the canonical well-ordering of the model $M_z$. Every vertical section of the set $B$ then consists of precisely one $E$-equivalence class.

To see that the set $B$ cannot be uniformized, note that every $P$-name in the model $W$ is definable from a real parameter and some additional parameters in the ground model, and therefore every element of $W[G]$ is definable from a real parameter, the generic filter $G$, and some additional parameters in the ground model. Thus, if $f : 2^\omega \to X$ is a putative uniformization of the set $B$, one can find a real parameter $z \in 2^\omega$ such that $f$ is definable from $z, G$ and some elements of the ground model. Then $f(z)$ should be an element of the model $M_z$ by the definition of the model $M_z$. At the same time, the vertical sections $B_z$ contains no elements of $M_z$ by the definition of the set $B$. \hfill \Box

### 9.3 Well-orderable uniformization

In this section, we will prove that a strong version of the countable-to-one uniformization statement holds in the extensions of the Solovay model by adequate forcings.

**Definition 9.3.1.** Let $E$ be an equivalence relation on a Polish space $X$. The $E$-well-orderable uniformization is the statement: if $E$ is a Borel equivalence relation on a Polish space $X$ and $B \subset 2^\omega \times X$ is a set, then the following are equivalent:

1. for every $y \in 2^\omega$, $B_y$ is a union of well-orderable collection of $E$-classes;
2. $B = \bigcup_\alpha B_\alpha$ where each $B_\alpha$ is a set whose vertical sections are either empty or $E$-classes.
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Clearly, only the implication (1)→(2) has nontrivial content. As a very special case, one can consider a set \( B \subset 2^\omega \times X \) with all vertical sections countable and \( E \) the identity on \( X \). Then (2) yields in particular a function uniformizing the set \( B \): for each \( y \) with \( B_y \) nonempty find the least \( \alpha = \alpha_y \) for which the vertical section \( (B_\alpha)_y \) is nonempty and then let \( f(y) = x \) for the unique \( x \in X \) such that \( \langle y, x \rangle \in B_\alpha \). This is the familiar countable-to-one uniformization.

**Theorem 9.3.2.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin poset which is balanced cofinally below \( \kappa \) and adequate below \( \kappa \). Let \( E \) be an analytic equivalence relation on a Polish space \( X \). Let \( W \) the symmetric Solovay model derived from \( \kappa \) and let \( G \subset P \) be a \( W \)-generic filter. In the model \( W[G] \), \( E \)-well-orderable uniformization holds.

**Proof.** We start with a claim.

**Claim 9.3.3.** Let \( \bar{p} \) be a balanced condition in the poset \( P \). Let \( E \) be a Borel equivalence relation on a Polish space \( X \). Let \( W \) be the symmetric Solovay model derived from \( \kappa \). In \( W \), let \( \tau \) be a \( P \)-name definable from ground model parameters such that \( P \Vdash \tau \subset X \) consists of well-orderably many \( E \)-classes. Then \( \bar{p} \Vdash \tau \) consists of realizations of virtual \( E \)-classes in \( V \).

**Proof.** Suppose not. Working in \( V \), in view of Corollary 8.1.3, there is a poset \( R_0 \) of size \( < \kappa \), an \( R_0 \)-name \( \eta \) for an element of \( X^\omega \) and an \( R_0 \)-name \( \sigma_0 \) for an element of \( P \) stronger than \( \bar{p} \) such that \( R \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash p \eta \) enumerates representatives of all \( E \)-classes in \( \tau \). Also there must be a poset \( R_1 \) of size \( < \kappa \), an \( R_1 \)-name \( \chi \) for an element of \( X \) which is forced not to realize any virtual \( E \)-class in \( V \), and an \( R_1 \)-name \( \sigma_1 \) for a condition in \( P \) stronger than \( \bar{p} \) such that \( R_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash \chi \in \tau \).

Working in \( W \), let \( H_0 \subset R_0 \) and \( H_1 \subset R_1 \) be filters mutually generic over \( V \), let \( y = \eta/H_0 \in X^\omega \), \( x = \chi/H_1 \in X \), \( p_0 = \sigma_0/H_0 \in P \) and \( p_1 = \sigma_1/H_1 \in P \). The balance of the condition \( \bar{p} \) implies that \( p_0, p_1 \) are compatible conditions in the poset \( P \) with a lower bound \( q \). The choice of the name \( \chi \) implies that \( x \) is not \( E \)-related to any element on the sequence \( y \). Since \( W \) is a symmetric Solovay extension of the model \( V[K][H_0] \), the forcing theorem applied in that model says that \( q \Vdash \tau = [\text{rng}(y)]_E \). Since \( W \) is a symmetric Solovay extension of the model \( V[K][H_1] \), the forcing theorem applied in that model says that \( q \Vdash \bar{x} \in \tau \). Since \( x \notin [\text{rng}(y)]_E \), this is a contradiction. \( \Box \)

To complete the argument for the theorem, suppose that \( E \) is a Borel equivalence relation on a Polish space \( X \). Let \( W \) be a symmetric Solovay extension derived from the cardinal \( \kappa \). In the model \( W \), let \( p \in P \) be a condition and \( \tau \) be a \( P \)-name such that \( p \Vdash \tau \subset 2^\omega \times X \) is a set whose vertical sections are well-orderable unions of \( E \)-classes. Let \( z \in 2^\omega \) be a parameter such that \( p, \tau \) are both definable from \( z \) and some additional ground model parameters. Let \( G \subset P \) be a generic filter meeting the condition \( p \) and let \( B = \tau/G \). We claim that every nonempty vertical section \( B_y \) consists of realizations of virtual \( E \)-classes of the model \( M_y = \text{HOD}_{V,y,z,G} \). Once this is proved, one can decompose the set \( B \) into the union \( B = \bigcup_\alpha B_\alpha \) by setting \( \langle y, x \rangle \in B_\alpha \) if \( \langle y, x \rangle \in B \) and \( x \) belongs
to the realization of $\alpha$-th virtual $E$-class in $M_y$ in the canonical well-ordering of the model $M_y$.

To show that the vertical section $B_y$ consists only of realizations of virtual $E$-classes in the model $M_y$, first use Claim 9.2.3 to see that the filter $G$ contains a realization of a balanced virtual condition in $M_y$. Then use Claim 9.3.3 in $M_y$ to conclude that $B_y$ must consist of realizations of virtual $E$-classes of $M_y$ as desired.

9.4 Saint Raymond uniformization

In this section, we will prove that a strong version of Saint Raymond uniformization holds in the generic extensions of the Solovay model by adequate forcing.

**Definition 9.4.1.** Let $I$ be a downward closed collection of closed subsets of a Polish space $X$. The $I$-Saint Raymond uniformization is the following statement: for every set $B \subseteq 2^\omega \times X$, the following are equivalent:

1. for every $y \in 2^\omega$, $B_y$ is a union of a well-ordered collection of sets in $I$;
2. $B = \bigcup_\alpha B_\alpha$ where for each $\alpha$ and each $y \in 2^\omega$, the vertical section $(B_\alpha)_y \subseteq X$ is closed and belongs to $I$.

As in the previous section, only the $(1) \rightarrow (2)$ implication has content. In the case of $I =$ the collection of singletons (plus the empty set), we can again derive the familiar countable-to-one uniformization as a very special case.

**Theorem 9.4.2.** Let $I$ be an analytic collection of closed subsets of a Polish space $X$. Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin forcing which is cofinally balanced in $\kappa$ and adequate below $\kappa$. Let $W$ be the symmetric Solovay model derived from $\kappa$ and let $G \subseteq P$ be a filter generic over $W$. Then $I$-Saint Raymond uniformization holds in $W[G]$.

**Proof.** We begin with a claim.

**Claim 9.4.3.** Let $\bar{p}$ be a balanced virtual condition in $P$. Let $W$ be the symmetric Solovay model and let $\tau \in W$ be a $P$-name for a subset of $X$ which is a union of well-ordered collection of elements in $I$. If $\tau$ is definable from ground model parameters, then $\bar{p} \Vdash \tau$ is a union of ground model coded elements of $I$.

**Proof.** In view of Corollary 8.1.2, $P \Vdash \tau$ is a countable union of elements of $I$. Thus, in $V$ there must be a poset $R_0$ and $R_0$-name $\eta$ for a countable sequence of elements of $I$ and an $R_0$-name $\sigma_0$ for a condition in $P$ stronger than $\bar{p}$ such that $R_0 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash \tau = \bigcup \text{rng}(\eta)$.

Suppose that the conclusion of the claim fails. Then, there must be a poset $R_1$ and $R_1$-name $\chi$ for an element of $X$ and an $R_1$-name $\sigma_1$ for a condition in $P$ stronger than $\bar{p}$ such that $R_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash \chi \in \tau$ and $\chi$ does not belong to any ground model element of $I$ which is a subset of $\tau$. 


Move to the model $W$. Let $H_0, H_1 \subset R_0, R_1$ be mutually generic filters, and let $p_0 = \sigma_0/H_0$ and $p_1 = \sigma_1/H_1$. By the balance of the condition $\bar{p}$, $p_0, p_1 \in P$ are compatible conditions with a lower bound $q \in P$. Let $x = \chi/H_1 \in X$. The contradiction is now reached by a split into cases.

**Case 1.** There is a condition $r \in H_1$ and a closed set $C \in \text{rng}(\eta/H_0)$ such that the ground model closed set $D = X \setminus \bigcup \{O : O \subset X \text{ is a basic open set such that } r \Vdash \chi \notin O\}$ is a subset of $C$. By the closure of the collection $I$ under subsets, $D \in I$ holds; by the definition of the set $D$, $x \in D$ holds. Now, $W$ is the symmetric Solovay extension of both models $V[H_0]$ and $V[H_1]$. The forcing theorem applied in $V[H_0]$ shows that $W \models q \Vdash D \subset C \subset \tau$. The forcing theorem applied in $V[H_1]$ shows that $W \models q \Vdash \bar{x} \in D$ and $\bar{x}$ belongs to no ground model closed set in $I$ which is a subset of $\tau$. Thus, the same condition $q$ forces two contradictory statements.

**Case 2.** Case 1 fails. Then, by the mutual genericity of the filters $H_0, H_1$ it must be the case that $x \notin \bigcup \text{rng}(\eta/H_0)$. Now, $W$ is the symmetric Solovay extension of both models $V[H_0]$ and $V[H_1]$. The forcing theorem applied in $V[H_0]$ shows that $W \models q \Vdash \tau = \bigcup \text{rng}(\eta/H_0)$. The forcing theorem applied in $V[H_1]$ shows that $W \models q \Vdash \bar{x} \in \tau \setminus \bigcup \text{rng}(\eta/H_0)$. So again, the same condition $q$ forces two contradictory statements. The proof of the claim is complete.

To complete the argument for the theorem, let $W$ be a symmetric Solovay extension derived from the cardinal $\kappa$. In the model $W$, let $p \in P$ be a condition and $\tau$ be a $P$-name such that $p \Vdash \tau \subset 2^\omega \times X$ is a set whose vertical sections are unions of well-orderable collection of sets in $I$. Let $z \in 2^\omega$ be a parameter such that $p, \tau$ are both definable from $z$ and some additional ground model parameters. Let $G \subset P$ be a generic filter meeting the condition $p$ and let $B = \tau/G$. We claim that every nonempty vertical section $B_\alpha$ is a union of sets in $I$ which are coded in the model $M_\eta = \text{HOD}_{V,y,z,G}$. Once this is proved, one can decompose the set $B$ into the union $B = \bigcup_\alpha B_\alpha$ by setting $\langle y, x \rangle \in B_\alpha$ if $\langle y, x \rangle \in B$ and $z$ belongs to the $\alpha$-th set in $I$ in the model $M_\eta$ in the canonical well-ordering of the model $M_\eta$, and this $\alpha$-th set is a subset of $B$.

To show that the vertical section $B_\alpha$ is a union of closed sets in $I$ in the model $M_\eta$, first use Claim 9.2.3 to see that the filter $G$ contains a realization of a balanced virtual condition in $M_\eta$. Then use Claim 9.4.3 in $M_\eta$ to conclude that $B_\eta$ must be a union of closed sets in $I$ coded in the model $M_\eta$ as desired.

### 9.5 Examples

In order to justify the extensive development in the earlier sections of this chapter, the class of adequate Suslin forcings better contain as many posets as possible. In this section, we show that this is indeed the case. The careful classification of balanced conditions in Chapter 5 will come handy over and over again.

**Proposition 9.5.1.** Let $K$ be a countable Borel flag complex on a Polish space $X$. Then the poset $P = P_K$ is adequate.
Proof. Let \( f : X \to X^{\aleph_0} \) be a Borel function witnessing the fact that \( \mathcal{K} \) is a countable flag complex. Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we will show that it is balanced.

For each \( x \in X \) let \( A_x = \{ p \in P : x \in f(p) \} \) and \( B_x = \{ p \in P : \exists y \in p \ x \in f(y) \lor y \in f(x) \} \). Note that \( \Sigma A_x, \Sigma B_x \) form a partition of the unit in the completion of the poset \( P \), for every point \( x \in X \). By the adequacy assumption, for each \( x \in X \), either \( Q \models \tau \leq \Sigma A_x \) or \( Q \models \tau \leq \Sigma B_x \). Let \( A \subseteq X \) be the set of all \( x \in X \) for which the former case occurs, and note that \( Q \models \tau \leq \tau_A \) where \( \tau_A \) is the balanced virtual condition described in Section 6.1. Thus the pair \( \langle Q, \tau \rangle \) is balanced, confirming the \( \aleph_0 \)-adequacy of the poset \( P \).

Proposition 9.5.2. Let \( \mathcal{K} \) be a modular Borel complex on a Polish space \( X \). Then the poset \( P = P_{\mathcal{K}} \) is adequate.

Proof. Let \( f : \mathcal{K} \to [Y]^{\aleph_0} \) be a Borel function witnessing the modularity of the complex \( \mathcal{K} \). Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we will show that it is a balanced pair.

For every point \( x \in X \) let \( A_x = \{ p \in P : x \in f(p) \} \) and for all \( y \in Y \) let \( B_y = \{ p \in P : y \in f(p) \} \). By the adequacy assumption, for each \( x \in X \) it must be the case that either \( Q \models \tau \leq \Sigma A_x \) or \( Q \models \tau \) is incompatible with \( \Sigma B_y \) in the completion of the poset \( P \). Let \( A \subseteq X \) be the collection of all \( x \in X \) for which the former case occurs. Similarly, for all \( y \in Y \) it must be the case that either \( Q \models \tau \leq \Sigma B_y \) or \( Q \models \tau \) is incompatible with \( \Sigma B_y \) in the completion of the poset \( P \). Let \( B \subseteq Y \) be the set of all \( y \in Y \) for which the former case occurs. In view of Theorem 6.2.19, it will be enough to show that \( B = f(A) \).

It is immediate that \( f(A) \subseteq B \) holds; the other impilcation is the crux of the proof. Let \( y \in B \) be an arbitrary point. The set \( D = \{ a \in \mathcal{K} : y \in f(a) \text{ and } a \text{ is inclusion minimal such} \} \) is analytic; let \( g : 2^\omega \to [X]^{<\aleph_0} \) be a continuous function whose image is \( D \). For each finite string \( t \in \omega^\omega \), let \( C_t = \{ p \in P : \exists z \in \omega^\omega \ t \subseteq z \text{ and } g(z) \subseteq a \} \). This is an analytic subset of the poset \( P \), and \( C_t = \bigcup_n C_{t \upharpoonright n} \).

By the adequacy assumption, for every finite string \( t \in T \), either \( Q \models \tau \leq \Sigma C_t \) or \( Q \models \tau \) is incompatible with \( \Sigma C_t \) in the completion of the poset \( P \). Let \( T \subseteq \omega^\omega \) be the set of all \( t \in \omega^\omega \) in which the former case occurs. Since \( C_0 = B_y, 0 \in T \). It is also immediate that \( T \) is a tree with no terminal nodes. Thus, there must be a branch \( z \in \omega^\omega \) through the tree \( T \). Let \( a = g(z) \). It will be enough to show that \( a \subseteq A \) since \( y \in f(a) \) is a part of the definition of the set \( D = \text{rng}(g) \).

Suppose towards contradiction that this fails. Then, in the \( Q \)-extension, \( \tau \) can be extended to a condition \( p \) such that \( y \in f(p) \), there is an inclusion minimal set \( a' \subseteq p \) such that \( y \in f(a') \), and \( a' \neq a \). There must be a point \( z' \in \omega^\omega \) such that \( f(z') = a' \). By the definition of the tree \( T \), \( z' \) is a branch of \( T \). By the continuity of the function \( g \), there must be initial segments \( t \subseteq z \) and \( t' \subseteq z' \) such that \( g''[t] \cap g''[t'] = \emptyset \). Since \( t, t' \) are both nodes in \( T \), \( p \) can be extended further to contain another inclusion minimal set \( a'' \subseteq p \) such that \( y \in f(a'') \) and such that there is \( z'' \in \omega^\omega \) extending \( t \) with \( a'' = g(z'') \).

Now, for the final push, note that \( g''[t] \cap g''[t'] \) implies that \( a' \neq a'' \). The set \( a' \cup a'' \) is a subset of \( p \) and therefore an element of the complex \( \mathcal{K} \). By the
This contradicts the assumptions that \( a', a'' \) were distinct inclusion minimal elements containing \( y \) in their \( f \)-image.

\[ \square \]

**Proposition 9.5.3.** Let \( K \) be a cofinally locally countable quotient Borel simplex on a Polish space \( X \). Then the poset \( P = P_K \) is adequate.

**Proof.** Let \( \lambda = \beth_\omega \). We will show that the poset \( P \) is \( \lambda \)-adequate. To this end, fix a Borel equivalence \( E \) on \( X \) witnessing the quotient feature of \( K \). Fix a Borel set \( B \subset K \) witnessing the cofinal feature, and fix a Borel function \( f : K \to [X]^{\aleph_0} \) witnessing the locally countable feature of the complex \( K \). Let \( \langle Q, \tau \rangle \) be a \( \lambda \)-adequate pair; we must argue that it is balanced.

Use the adequacy to show that for each \( x \in X^* \), \( Q \models \tau \) decides in \( P \) whether a realization of \( x \) belongs to \( \hat{A}_{\text{gen}} \) or not. Let \( A = \{ x \in X^* : Q \models \tau \models_P \text{ a realization of } x \text{ belongs to } \hat{A}_{\text{gen}} \} \). It will be enough to show that \( A \subset X^* \) is a maximal \( K^* \)-set. Then, by Theorem 6.4.17, the pair \( \langle \text{Coll}(\omega, \lambda), A \rangle \) is balanced; clearly \( \text{Coll}(\omega, \lambda) \times Q \models A \leq \tau \) in the separative quotient of the poset \( P \) and so \( \langle Q, \tau \rangle \) must be a balanced pair equivalent to \( \langle \text{Coll}(\omega, \lambda), A \rangle \).

To show that \( A \) is a maximal \( K^* \)-set, consider a point \( x \in X^* \setminus A \). In the \( Q \)-extension, there must be a strengthening \( \sigma \) of \( \tau \) which contains a finite set \( \dot{a} \) such that \( \{ x \} \cup \dot{a} \notin K \). By the properties of the set \( B \), we may assume that \( \dot{a} \in B \) is forced. The definitor properties of the function \( f \) imply that \( \{ x \} \cup \dot{a} \cap f(\{ x \}) \notin K \). Since the values of the function \( f \) are countable and \( x \) is a virtual \( E \)-class in \( V \), all points in \( f(\{ x \}) \) are realizations of virtual \( E \)-classes in \( V \). Thus, strengthening \( \sigma \) further, there must be a finite set \( b \) of virtual \( E \)-classes such that \( \{ x \} \cup b \notin K \) and \( \sigma \models \text{the realizations of virtual classes in } b \) are in \( \hat{A}_{\text{gen}} \). By the adequacy assumption, for each \( y \in b \) it must be the case that \( Q \models \tau \models_P \text{ a realization of } y \) belongs to \( \hat{A}_{\text{gen}} \) for all \( y \in b \). Thus, \( b \subset A \), and the maximality of \( A \) has been proved.

\[ \square \]

**Proposition 9.5.4.** The poset \( P \) of all infinite subsets of \( \omega \) ordered by inclusion is adequate.

**Proof.** Let \( \langle Q, \tau \rangle \) be an \( \aleph_0 \)-adequate pair; we will show that it is a balanced pair.

For every set \( x \subset \omega \) let \( A_x = \{ p \in P : p \subset x \} \). By the adequacy assumption, for each \( x \subset \omega \) it must be the case that either \( Q \models \tau \leq \Sigma A_x \) or \( Q \models \tau \) is incompatible with \( \Sigma A_x \) in the completion of the poset \( P \). Let \( U \) be the collection of those \( x \subset \omega \) for which the former alternative occurs. It is easy to see that \( U \) is a nonprincipal ultrafilter on \( \omega \). It is also clear that \( Q \models \tau \) is modulo finite included in every element of \( U \). The balance of \( \tau \) is now proved by a reference to Theorem 7.1.4.

\[ \square \]

**Proposition 9.5.5.** Let \( I \) be an \( F_\sigma \)-ideal on \( \omega \). The poset for adding a generic ultrafilter disjoint from \( I \) is adequate.

**Proposition 9.5.6.** Let \( (\Gamma, \cdot) \) be a countable semigroup. The poset \( P = P(\Gamma) \) of Section 7.3 is adequate.
Proof. Let \(<Q, \tau>\) be an \(\aleph_0\)-adequate pair; we will show that it is a balanced pair.

Let \(b \subset \Gamma\) be a set. Let \(A_b = \{p \in P: p \text{ accepts } b\}\). This is an analytic subset of \(P\). By the adequacy assumption, it must be the case that either \(Q \Vdash \tau \leq \Sigma A_b\) or \(Q \Vdash \tau\) is incompatible with \(\Sigma A_b\) in the completion of the poset \(P\). Let \(U\) be the collection of those \(b \subset \Gamma\) for which the former alternative occurs. It is easy to see that \(U\) is a nonprincipal ultrafilter on \(\Gamma\), and Proposition 7.3.5 shows that \(U\) is in fact an idempotent ultrafilter. Theorem 7.3.7 then implies that \(Q\) forces \(\tau\) to be smaller than the balanced condition derived from \(U\), showing that \(<Q, \tau>\) is balanced.

The collapse and uniformization posets of Definition 6.4.2 and 6.4.5 typically are not adequate and the countable-to-one uniformization will fail in the resulting extensions of the symmetric Solovay model. We treat two specific cases:

**Proposition 9.5.7.** Let \(X\) be a Polish space, \(E\) a countable Borel equivalence relation on \(X\), and let \(B \subset X \times X^\omega\) be a Borel \(E \times =^+\)-invariant Borel set such that for all \(<x, y> \in B\), \(\text{rng}(y) \subset [x]_E\). The uniformization poset \(P\) for \(E, =^+\), and \(B\) of Definition 6.4.5 is adequate.

Proof. Let \(<Q, \tau>\) be an \(\aleph_0\)-adequate pair for \(P\); we will show that it is a balanced pair.

For each pair \(x, x'\) of \(E\)-related points in \(X\), let \(A_{x,x'} = \{p \in P: \text{ for some pair } <x, y> \in p, x' \in \text{rng}(y)\}\). Then \(A_{x,x'} \subset P\) is a Borel set and so either \(Q \Vdash \tau \leq \Sigma A_{x,x'}\) or \(Q \Vdash \tau\) is incompatible with \(\Sigma A_{x,x'}\), by the adequacy of the pair \(<Q, \tau>\). Let \(g: X/E \to X^\omega/ =^+\) be the function assigning every class \([x]_E\) the class of all enumerations of the set \(\{x' \in x: Q \Vdash \tau \leq \Sigma A_{x,x'}\}\). The function \(g\) is a uniformization of the set \(A/E \times =^+\), and by Theorem 6.4.6, it generates a balanced virtual condition \(<\text{Coll}(\omega, X), \tau_g>\). It is immediate that \(Q \times \text{Coll}(\omega, X) \Vdash \tau \leq \tau_g\) and so the pair \(<Q, \tau>\) is balanced as required.

**Proposition 9.5.8.** Let \(E\) be a non-smooth Borel pinned equivalence relation on a Polish space \(X\). The collapse poset \(P\) of \(|E|\) to \(2^\omega\) of Definition 6.4.2 is not adequate. The countable-to-one uniformization fails in the resulting extension of the choiceless Solovay model.

Proof. We prove the last sentence. The poset \(P\) is balanced by the pinned assumption on \(E\) and Corollary 6.4.4. In the \(P\)-extension of the symmetric Solovay model, \(|E| \leq |2^\omega|\) holds by the definition of the poset \(P\). In addition, \(|\mathcal{E}_0| \leq |E|\) holds by the Glimm–Effros dichotomy as \(E\) is assumed to be non-smooth. In sum, \(|\mathcal{E}_0| \leq |2^\omega|\) holds, as witnessed by some function \(g: 2^\omega \to 2^\omega\). In addition, Theorem 10.1.28 shows that in the \(P\)-extension of the symmetric Solovay model, \(\mathcal{E}_0\) has no transversal, and therefore the function \(g\) has no left inverse. This feature of \(g\) stands witness to the failure of the countable-to-one uniformization.
We conclude this chapter by several questions whose solution clearly falls outside of the scope of this book. Is it possible to separate the various versions of uniformization present in this chapter?

**Question 9.5.9.** Let $E, F$ be analytic equivalence relations. Is it consistent with ZF+DC that $E$-countable-to-one uniformization holds while $F$-countable-to-one uniformization fails?

**Question 9.5.10.** Let $I, J$ be analytic downward closed collections of closed sets. Is it consistent with ZF+DC that $I$-uniformization holds while $J$-uniformization fails?
Chapter 10

Chromatic numbers in $W[G]$

Evaluation of the chromatic numbers of various analytic hypergraphs is a useful tool for distinguishing between various balanced extensions of the symmetric Solovay model and evaluating their theory, see [25]. In this chapter, we provide several tools for proving that various hypergraphs have uncountable chromatic number in such an extension.

10.1 Charms

Definition 10.1.1. Let $\Gamma$ be an analytic hypergraph on a Polish space $X$, let $P$ be a Suslin poset and $p \in P$. A $\Gamma$-charm below $p$ is a pair $\langle V[G], \bar{p} \rangle$ where $V[G]$ is a generic extension and in $V[G]$, $\bar{p} \leq p$ is a balanced virtual condition in $P$ such that for every partition $\langle A_n : n \in \omega \rangle$ of the space $X$ which is definable from $\bar{p}$ and the parameters in the ground model, there is an edge $e \in G$ all of whose vertices come from the same piece of the partition.

The notion of $\Gamma$-charm depends on the underlying poset $P$, which will always be understood from the context. The generic extension is of course always obtained from some poset in the ground model and in that sense the charm is an element of the ground model. We find the terminology with generic extensions more practical, if strictly speaking formally incorrect.

Definition 10.1.2. Let $\Gamma$ be an analytic hypergraph on a Polish space $X$, let $P$ be a Suslin poset and $\kappa$ be an inaccessible cardinal.

1. $P$ is $\Gamma$-charming if there is a $\Gamma$-charm below any condition in $P$;

2. $P$ is $\Gamma$-charming cofinally below $\kappa$ if $V_\kappa$ satisfies that every ordinal can be collapsed by a poset such that in the resulting extension, $P$ has $\Gamma$-charm.

Theorem 10.1.3. Suppose that $P$ is a Suslin forcing, $\Gamma$ is an analytic finitary hypergraph on a Polish space $X$ and $\kappa$ is an inaccessible cardinal such that $P$ is $\Gamma$-charming cofinally below $\kappa$. Let $W$ be the symmetric Solovay model derived
from $\kappa$ and $G \subset P$ be a filter generic over $W$. Then in $W[G]$, the chromatic number of the hypergraph $\Gamma$ is uncountable. If the forcing $P$ is $\sigma$-closed, then the finitary assumption on $\Gamma$ can be dropped.

Proof. Work in the model $W$. Towards contradiction assume that $p \in P$ is a condition and $\tau$ is a $P$-name for a function from $X$ to $\omega$ such that $p \Vdash \Gamma$ contains no $\tau$-monochromatic hyperedge. The condition $p$ as well as the name $\tau$ must be defined from some parameters from the ground model and some point $z \in 2^\omega$. The assumptions imply that there is an intermediate model $V[K]$ obtained by forcing of size $< \kappa$ such that $z \in V[K]$ and in the model $V[K]$ there is a $\Gamma$-charm below $p$. Let $\langle V[K][L], \bar{p} \rangle$ be the charm; it may be found in the model $W$.

Work in the model $V[K][L]$. For each number $n \in \omega$ let $X_n$ be the set of those points $x \in X$ such that $n$ is the smallest number for which there is a poset $R$ of size $< \kappa$ and an $R$-name $\sigma$ for a condition in $P$ below $\bar{p}$ such that $R \Vdash \text{Coll}(\omega, \kappa) \Vdash \sigma \Vdash_R \tau(x) = \bar{n}$. The collection $\langle X_n : n \in \omega \rangle$ is a partition definable from $\bar{p}$ and parameters from the ground model. Thus, there is an edge $e \in \Gamma$ all of whose vertices come from the same piece of the partition, say $X_n$.

Let $e = \{ x_m : m \in k \}$ be a listing of the vertices of the edge $e$. For each $m \in k$ let $R_m$ be a poset of size $< \kappa$ and $\sigma_m$ an $R_m$-name for an element of $P$ stronger than $\bar{p}$ such that $R_m \Vdash \text{Coll}(\omega, \kappa) \Vdash \sigma_m \Vdash_R \tau(x_m) = \bar{n}$. In $W$, find filters $H_m \subset R_m$ for $m \in k$ mutually generic over $V[K][L]$. Let $p_m = \sigma_m/H_m \in P$; we claim that the set $\{ p_m : m \in k \} \subset P$ has a common lower bound.

To find the bound, by induction on $m \in k$ build a descending sequence of conditions $q_m \in P$ such that $q_m \in V[K][L][H_m'] : m' \leq m$ and $q_m \leq p_m$. To start the induction, let $q_0 = p_0$. To perform the induction step, use the balance of the condition $\bar{p}$ with the mutually generic extensions $V[K][L][H_m'] : m' \leq m$ and $V[K][L][H_{m+1}]$ to conclude that the conditions $q_m, p_{m+1}$ are compatible in the poset $P$, and find their lower bound $q_{m+1}$ in the model $V[K][L][H_m'] : m' \leq m+1$. Once the induction is performed, the condition $q = q_{k-1}$ is the requested lower bound of the set $\{ p_m : m \in k \}$ in the poset $P$.

Finally, move back to the model $W$. Since $W$ is a symmetric Solovay extension of each of the models $V[K][L][H_m]$, the forcing theorem applied in each of these models shows that $q \Vdash_P \forall m \in k \tau(x_m) = \bar{n}$. Since $e = \{ x_m : m \in k \}$ is an edge in the hypergraph $\Gamma$, this contradicts the initial assumptions on the name $\tau$.

As an introduction to the art of constructing charms, we will show that certain balanced Suslin posets do not move chromatic numbers of analytic hypergraphs at all. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$ and let $P$ be the poset introducing a countable complete section to $E$ as isolated in Example 6.4.9. The poset $P$ can be presented as countable sets $p \subset X$ with the ordering defined by $q \leq p$ if $p \subset q$ and $p = q \cap [p]\E$. As in Example 6.4.9, the poset $P$ is balanced and its balanced virtual conditions are classified by sets $A \subset X$ which have nonempty intersection with each $E$-class.

**Theorem 10.1.4.** Let $E$ be a pinned Borel equivalence relation on a Polish space $X$. Let $P$ be the poset adding a countable complete $E$-section as in Ex-
ample 6.4.9. Let $\Gamma$ be any analytic hypergraph of uncountable Borel chromatic number. Then $P$ is $\Gamma$-charming.

Proof. By the dichotomy theorem for the Borel chromatic number of analytic hypergraphs [27], it is enough to verify this for a certain minimal hypergraph $\Gamma$. The domain of the hypergraph $\Gamma$ is the $G_\delta$-set of all points $z \in 2^{\omega}$ such that there are infinitely many $n$ such that $\forall i \in m \ z(i) \in z(m)$. To define the hypergraph, find finite strings $s_n \in \omega$ for each $n \in \omega$ such that the set $\{s_n : n \in \omega\}$ is dense in $\omega^{<\omega}$, and let $\langle z_i : i \in \omega \rangle \in \Gamma$ if the points $z_i$ agree on all entries except a single $n$, and then $s_n \subset z_i$ and $z_i(n) = i$ holds for all $i \in \omega$.

By Theorem 10.1.3, it is enough to produce a $\Gamma$-charm for $P$ below any condition $p$. The charm is very easy to describe: it is the pair $\langle V[z], \bar{p} \rangle$ where $z \in \omega^\omega$ is the Cohen generic point, and $\bar{p}$ is the balanced virtual condition in $V[z]$ corresponding to the set $A = \{x \in X : [x] E \cap p \neq 0 \rightarrow x \in p\}$. Note that the virtual condition $\bar{p}$ is definable in $V[z]$ from the ground model parameter $p$—this is the feature of the poset $P$ which distinguishes it from other posets.

To verify the uncountable definable chromatic number of $\Gamma$ in the model $V[z]$, move back to the ground model $V$. Suppose towards contradiction that $t \in \omega^{<\omega}$ is some condition in the Cohen forcing which forces $\dot{g} : \text{dom}(\Gamma) \rightarrow \omega$ to be a $\Gamma$-coloring definable by a certain specific formula with ground model parameters. Strengthening $t$ if necessary, we may assume that there is a number $m \in \omega$ such that $t \models \dot{g}(\dot{x}_{gen}) = m$ where $\dot{x}_{gen}$ is the Cohen name for the generic point in $\omega^\omega$. Find a number $n \in \omega$ such that $t \subset s_n$ where $s_n \in \omega^{<\omega}$ is one of the strings used in the definition of the hypergraph $\Gamma$.

Let $z \in \omega^\omega$ be a point Cohen generic over the ground model such that $s_n \subset z$. Use a genericity argument to see that $z$ belongs to the domain of the hypergraph $\Gamma$. For each $i \in \omega$ let $z_i \in \omega^\omega$ be a point obtained from $z$ by rewriting the value of $z(n)$ with $i$. Thus, the points $z_i \in \omega^\omega$ are still Cohen-generic over the ground model meeting the condition $t$ and they yield the same generic extension. The definition of the coloring must then yield the same map $g$, and by the forcing theorem it must be the case that $g(z_i) = m$ for all $i \in \omega$. However, $\langle z_i : i \in \omega \rangle$ is an edge in the hypergraph $\Gamma$, a contradiction.

Corollary 10.1.5. Let $E$ be a pinned Borel equivalence relation on a Polish space $X$ and let $\Gamma$ be an analytic hypergraph of uncountable Borel chromatic number.

1. Let $P$ be the poset adding a complete countable $E$-section as in Example 6.4.9. In the $P$-extension of the symmetric Solovay model, $\Gamma$ has uncountable chromatic number;

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, $E$ has a countable complete section, yet the chromatic number of $\Gamma$ is uncountable.

Proof. This is a conjunction of Theorems 10.1.3 and 10.1.4.
CHAPTER 10. CHROMATIC NUMBERS IN $W[G]$

The next two classes of examples concern a task common in descriptive set theory: selecting a structure for each $E$-equivalence class, where $E$ is a countable Borel equivalence relation on a Polish space $X$. To do this, we will always consider the equivalence relation $=^+$ on $X^\omega$, connecting points $y, z \in X^\omega$ if $\text{rng}(y) = \text{rng}(z)$. For a Borel set $A \subset X \times X^\omega$ which is $E \times =^+$-invariant, we will consider the uniformization poset $P_A$ as in Definition 6.4.5, uniformizing the set $A$ as a subset of the quotient product $X/E \times X^\omega/ =^+$. Several definitions are needed in order to state the results in their full generality.

**Definition 10.1.6.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $K$ be a Borel simplicial complex on $X$. We say that $K$ has finite supports in $E$ if there is a Borel function $\text{supp}: X \to [X]^{<\omega}$ such that:

1. for each finite set $a \subset X$, $a \in K$ if and only if $a \cap c \in K$ for each $E$-class $c$;
2. for all $x \in X$, $\text{supp}(x) \subset [x]_E$;
3. for each $a \in K$ write $\text{supp}(a) = \bigcup_{x \in a} \text{supp}(x)$. If $a, b \in K$ are sets with $\text{supp}(a) \cap \text{supp}(b) = 0$ then $a \cup b \in K$.

The following definition needs a simple preliminary observation. Let $K$ be a simplicial complex on a countable set $S$. The set of maximal $K$-subsets of $S$ is a $G_\delta$-subset of $P(S)$, and therefore Polish in the inherited topology. The inherited topology is generated by sets $O_a = \{A \subset S: A$ is a maximal $K$-set and $a \subset A\}$.

**Definition 10.1.7.** Let $E$ be a Borel equivalence relation on a Polish space $X$ and let $A \subset X \times X^{<\omega}$ be a Borel set. We say that $A$ has generic vertical sections in $E$ if

1. the set $A$ is $E \times =^+$-invariant;
2. there is a Borel simplicial complex $K$ on $X$ which has finite supports in $E$ such that for each $x \in X$, the vertical section $A_x$ consists of a dense $G_\delta$-subset of the space of maximal $K$-subsets of $[x]_E$.

In the following examples, the original equivalence relation $E$ on a Polish space $X$ must be extended to spaces such as $[X]^{<\omega}$ by letting any point $x \in X$ to be equivalent to all nonempty sets $a \subset [x]_E$ to get complexes with finite supports nad sets with generic vertical sections. The trivial adjustment is not spelled out.

**Example 10.1.8.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $A \subset (X^2)^\omega$ be the set of all pairs $(x, y)$ where $y$ is an (enumeration of an) acyclic connected graph on $[x]_E$. The set $A$ has $E$-generic sections. To see this, let $K$ be the complex of subsets of $E$ which are acyclic as unoriented graphs. It has finite supports in $E$, as witnessed by the function $\text{supp}$ which to an edge $e \in E$ assigns the set of the two vertices in $X$ that $e$ connects. For each point $x \in X$, the vertical section $A_x$ of the set $A$ is simply equal to all (enumerations of) maximal $K$-subsets of $E \upharpoonright [x]_E$. The poset $P_A$ is designed to add a treeing to the equivalence relation $E$.
Example 10.1.9. Let $E$ be a countable Borel equivalence relation on a Polish space $X$ with infinite classes. Let $A \subseteq X \times (X^2)\omega$ be the set of all pairs $(x, y)$ where $y$ is an (enumeration of) directed acyclic graph on $[x]_E$ isomorphic to the successor graph on $Z$. The set $A$ has $E$-generic sections. To show this, let $\mathcal{K}$ be the complex of acyclic subsets of $E$ in which every vertex has in-degree and out-degree at most one. The complex $\mathcal{K}$ has finite supports in $E$ as witnessed by the function $\text{supp}$ which assigns to each edge $e \in E$ the set of the two vertices in $X$ that $e$ connects. For each point $x \in X$, the vertical section $A_x$ is easily seen to consist of (all enumerations of) maximal $\mathcal{K}$-subsets of $[x]_E$ which are connected and in which every vertex has both in- and out-degree exactly one. This is a dense $G_\delta$-set of maximal $\mathcal{K}$-sets. The poset $P_A$ adds a linear ordering isomorphic to $\mathbb{Z}$ to each $E$-class.

Example 10.1.10. Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $A \subseteq X \times ([X]^{<\aleph_0})\omega$ be the set of all pairs $(x, y)$ where $y$ is an (enumeration of a) collection of finite subsets of $[x]_E$ in which any two sets are either disjoint or inclusion-comparable, such that any finite subset of $[x]_E$ is a subset of an element of $y$. The set $A$ has generic sections in $E$. To see this, let $\mathcal{K}$ be the simplicial complex on $[X]^{<\aleph_0}$ such that any set $a \in \mathcal{K}$ consists of sets of pairwise $E$-related elements and such that any two sets in $a$ are either disjoint or inclusion-comparable. $\mathcal{K}$ has finite supports in $E$ as witnessed by the function $\text{supp}$ assigning to each finite set $b \subseteq X$ consisting of pairwise $E$-related points of $X$ the set $b$ itself. For each point $x \in X$, the vertical section $A_x$ consists of (all enumerations of) maximal $\mathcal{K}$-subsets of $[x]^{<\aleph_0}$ such that every finite subset of $[x]_E$ is a subset of one element of the $\mathcal{K}$-set. This is easily seen to be a dense $G_\delta$-set of maximal $\mathcal{K}$-subsets of $[x]_E$. The poset $P_A$ expresses $E$ as an increasing union of countably many equivalence relations which have only finite classes.

Theorem 10.1.11. Let $E$ be a countable Borel equivalence relation on a Polish space $X$ and let $A \subseteq X \times X^{<\omega}$ be a Borel set with generic vertical sections in $E$. Let $\Gamma$ be an analytic hypergraph of fixed finite arity, with uncountable Borel chromatic number. Then $P = P_A$ is $\Gamma$-charming.

Proof. Fix a natural number $k \in \omega$. By the dichotomy theorem for the Borel chromatic number of analytic hypergraphs [27], it is enough to verify the conclusion of the theorem for a certain minimal hypergraph $\Gamma$ of arity $k$. To define it, choose sequences $s_n \in k^n$ for each $n \in \omega$ so that the set $\{s_n : n \in \omega\}$ is dense in $k^{<\omega}$ and let $\Gamma$ be the hypergraph of all $k$-tuples $(z_i : i \in k)$ of points in $k^{<\omega}$ which agree on all entries except for a single $n \in \omega$, and then $s_n \subseteq z_i$ and $z_i(n) = i$ for all $i \in k$. By Theorem 10.1.3, it is enough to produce a $\Gamma$-charm for $P$ below any condition $p$. To this end, we need to consider a suitable very Suslin forcing.

Let $\mathcal{K}$ be the Borel simplicial complex on $X$ which witnesses the fact that $A$ has generic sections in $E$. Let $\text{supp} : X \to [X]^{<\aleph_0}$ be a Borel function witnessing the fact that the complex $\mathcal{K}$ has finite supports. A straightforward Shoenfield absoluteness argument shows that $\mathcal{K}$ and $\text{supp}$ maintain their instrumental properties in all forcing extensions. Let $Q$ be the poset of finite $\mathcal{K}$-sets
ordered by inclusion. The following lemma is modeled after [5, Proposition 3], which was pointed out to us by Clinton Conley.

**Claim 10.1.12.** The poset $Q$ is very Suslin and Suslin $\sigma$-centered.

**Proof.** To see that the poset $Q$ is very Suslin, note that if $\{q_n : n \in \omega\}$ is a countable set of conditions, if there is a condition incompatible with them all, then there is one which consists only of subsets of the $E$-saturation of $\bigcup_n q_n$, since the conditions $q_n$ are inert outside of this set. Thus, checking whether the set $\{q_n : n \in \omega\} \subset Q$ is predense one has to consult only a countable set of conditions. This yields the very Suslin property.

To complete the proof of the claim, we will show that $Q = \bigcup_n Q_n$ such that each for each $n \in \omega$, $Q_n$ is a Borel centered set. To find the sets $Q_n$, use the Lusin–Novikov theorem to find Borel functions $\{f_m : m \in \omega\}$ from $X$ to $X$ such that for every $x \in X$, the set $\{f_m(x) : m \in \omega\}$ is exactly equal to the $E$-class of $x$. To find the integer parameter guaranteeing compatibility of conditions in $Q$, fix a condition $q \in Q$. A descriptor of $q$ is a tuple $d = \langle d_0, d_1, d_2, d_3 \rangle$ where

- $d_0$ is a collection of basic open subsets of $X$ such that each open set in $d_0$ contains exactly one element of $q \cup \text{supp}(q)$ and every element of $\text{supp}(q)$ belongs to a set in $d_0$. For each $O \in d_0$ write $x_O$ for the unique element of $q \cup \text{supp}(q)$ contained in $O$;
- $d_1 \subset d_0$ is the set of those elements of $d_0$ which contain elements of $q$;
- $d_2$ is the equivalence relation on $d_0$ connecting open sets $O_0, O_1 \in d$ if $x_{O_0} \leq x_{O_1}$;
- $d_3$ is a function from $d_2$ to $\omega$ which to a pair $\langle O_0, O_1 \rangle \in d_2$ assigns a number $k \in \omega$ such that $f_k(x_{O_0}) = x_{O_1}$.

A condition $q \in Q$ may have many descriptors. However, if conditions $q_0, q_1$ share the same descriptor $d$, then they are compatible. To see this, suppose that $x_0 \in q_0$ and $x_1 \in q_1$ are $E$-related points. If the sets $\text{supp}(q_0 \cap [x_0]_E)$ and $\text{supp}(q_1 \cap [x_1]_E)$ are disjoint then $(q_0 \cup q_1) \cap [x_0]_E \in \mathcal{K}$ by the properties of the function $\text{supp}$. If the sets $\text{supp}(q_0 \cap [x_0]_E)$ and $\text{supp}(q_1 \cap [x_1]_E)$ are not disjoint, containing some common point $x \in [x_0]_E$, then they can be recovered from $x$ using the functions $f_k$ by the process indicated by the descriptor $d$. Since the recovery is the same in both cases, the sets $\text{supp}(q_0 \cap [x_0]_E)$ and $\text{supp}(q_1 \cap [x_1]_E)$ must be equal and $(q_0 \cup q_1) \cap [x_0]_E \in \mathcal{K}$ again.

Finally, let $Q = \bigcup B_d$ where $d$ runs through all the countably many possible descriptors and $B_d = \{q \in Q : d \text{ is a descriptor of } q\}$. This concludes the proof of the claim. \qed

Let $z \in k^\omega$ be a Cohen generic point over $V$. In the model $V[z]$, let $R$ be the $\omega_1$-length iteration of the poset $Q$ with finite support. Let $H \subset R$ be a generic filter, and work in the model $V[z][H]$. We will produce a balanced condition $\bar{p}$ in $P_A$ such that there is no $\Gamma$-coloring with countably many colors which
is definable from $\bar{p}$ and parameters in the ground model. A minor variation of the construction produces a balanced virtual condition below any condition $p \in P \cap V$. To construct $\bar{p}$, recall that balanced conditions correspond to total uniformizations of the set $A$. For each ordinal $\alpha \in \omega_1$ let $H_\alpha \subset Q$ be the filter added on $\alpha$-th stage of the iteration and write $V_\alpha = V[z][H_\beta : \beta \in \alpha]$. Note that for each $x \in X \cap V_\alpha$, the set $(\bigcup H_\alpha) \cap [x]_E$ is a maximal $\aleph$-set, and by a genericity argument all its enumerations belong to the set $A_\alpha$. Finally, let $\bar{p}$ be the uniformization of the set $A$ in the quotient $X/E \times X^\omega/\equiv^+$ given by the following: for each $x \in X$ find the least ordinal $\alpha$ such that $x \in V_\alpha$ and let $\bar{p}(x)$ be the $\equiv^+$-class of all enumerations of $\bigcup H_\alpha \cap [x]_E$. Note also that the definition of $\bar{p}$ depends only on the model $V[z]$ and the filters $G_\alpha$ and not on the point $z$ itself. The function $\bar{p}$ is a uniformization of the set $A$ in the quotient and therefore a balanced condition.

To show that there is no countable coloring of $\Gamma$ by countably many colors definable from $\bar{p}$ and parameters in $V$, return to $V$. Choose a Cohen-$\bar{R}$-name $\sigma$ for the hyperfinitization constructed in the extension in the previous paragraph. Suppose towards contradiction that $(t, \dot{r})$ is a condition in the iteration Cohen-$\bar{R}$ and $\tau$ is a name in the iteration for a function from $k^\omega$ to $\omega$ which is a $\Gamma$-coloring definable using some parameters in the ground model and $\sigma$. Strengthening the condition if necessary, we may assume that there is a natural number $m \in \omega$ such that it forces $\tau(\dot{x}_{\text{gen}}) = \dot{m}$, where $\dot{x}_{\text{gen}}$ is the name for the Cohen generic element of $k^\omega$ added in the first step of the iteration, and that the condition identifies a specific definition of $\tau$ from $\sigma$. The condition $\dot{r}$ is forced to have a finite support bounded by some ordinal $\alpha$. The finite support iteration $R_\alpha$ of length $\alpha$ is a very Suslin, Suslin $\sigma$-centered forcing by Corollary 13.2.13 and Theorem 13.2.16. Strengthening the condition $(t, \dot{r})$ again if necessary, we may assume that there is an analytic set $A \subset R_\alpha$ which is centered and $t \Vdash \dot{r} \in \dot{A}$.

Find a number $n \in \omega$ and a string $s \in k^n$ extending $t$ which is used in the construction of the graph $\Gamma$. Let $z \in 2^\omega$ be a point Cohen generic over $V$, extending the string $s$. Let $z_i \in k^\omega$ be the points obtained from $z$ by rewriting the value $z(n)$ to $z_i(n) = i$. Thus, the points $z_i \in 2^\omega$ are Cohen generic over $V$, generating the same generic extension and moreover $(z_i : i \in k) \in \Gamma$. In the model $V[z_i]$, let $r_i = \dot{r}/z_i$ for all $i \in k$. These are conditions in the poset $R_\alpha$, all in the centered set $A$, and so have a common lower bound $r$. Let $H \subset R$ be a filter generic over the model $V[z]$, meeting the condition $r$, and move to the model $V[z][H]$.

Now, the cinch. The filters given by $z_i, H$ for $i \in k$ are all generic over $V$ for the iteration Cohen-$\bar{R}$. They all meet the initial condition $(t, \dot{r})$ and they all generate the same model. Moreover, the value of $\sigma/z_i, H$ is the same independently of $i \in k$ and then so must be the value of $\tau/z_i, H$. By the forcing theorem, $(\tau/z_i, H)(z_i) = m$ must hold for all $i \in k$. This is a contradiction though, since $(z_i : i \in k) \in \Gamma$. \qed

**Corollary 10.1.13.** Let $E$ be a countable Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the uniformization poset of Example 10.1.9. In the $P$-extension of
1. In the Solovay model, ZF+DC holds, countable-to-one uniformization holds, and the chromatic number of \( G_0 \) is uncountable.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, countable-to-one uniformization holds, \( E \) is generated as an orbit equivalence relation of a \( \mathbb{Z} \)-action, and yet the chromatic number of \( G_0 \) is uncountable.

Similar conclusions can be drawn for other structures on equivalence classes. As one possibility, considering Example 10.1.10, we get a model of ZF+DC+countable-to-one uniformization in which a given countable Borel equivalence relation \( E \) is an increasing union of countably many equivalence relations with all classes finite, and yet the chromatic number of \( G_0 \) is uncountable. As an important side effect, \( |E_0| > |2^\omega| \) must hold in these models since the countable-to-one uniformization together with \( |E_0| = |2^\omega| \) yields in ZF an \( E_0 \)-selector and therefore a partition of \( 2^\omega \) into two \( G_0 \)-anticliques.

The second class of uniformization posets uses a different type of vertical sections.

**Definition 10.1.14.** Let \( E \) be a countable Borel equivalence relation on a Polish space \( X \). A Borel set \( A \subset X \times X^\omega \) has compact sections in \( E \) if

1. \( A \) is invariant under the equivalence relation \( E \times E^+ \);
2. for every \( x \in X \) there is a compact set \( C_x \subset P([x]_E) \) such that the section \( A_x \) consists exactly of all enumerations of sets in \( C_x \).

**Example 10.1.15.** Let \( E \) be a countable Borel equivalence relation on a Polish space \( X \). Let \( G \subset E \) be a locally finite Borel graph. Let \( A \subset X \times (X^2)^\omega \) be the set of all pairs \( \langle x, y \rangle \) where \( x \in X \) and \( \text{rng}(y) \) is a perfect matching of the graph \( G \mid [x]_E \). Then the set \( A \subset X \times X^\omega \) has compact sections in \( E \). The uniformization poset \( P_A \) is designed to add a perfect matching of the graph \( G \), if it exists.

**Example 10.1.16.** Let \( E \) be a countable Borel equivalence relation on a Polish space \( X \). Let \( G \subset E \) be a Borel graph and let \( n \in \omega \) be a number. Let \( A \subset X \times (X \times n)^\omega \) be the set of all pairs \( \langle x, y \rangle \) where \( x \in X \) and \( \text{rng}(y) \) is an \( n \)-coloring of the graph \( G \). Then the set \( A \subset X \times X^\omega \) has compact sections in \( E \). The uniformization poset \( P_A \) is designed to add an \( n \)-coloring of the graph \( G \), if it exists.

Unlike the case of sets with generic vertical sections, the uniformization of sets with compact sections may require bringing the chromatic number down for certain graphs. For example, the \( G_0 \) graph has uncountable chromatic number, yet it contains no cycles of odd length. Therefore, for each point \( x \in 2^\omega \) the chromatic number of \( G_0 \mid [x]_{E_0} \) is simply two. The Borel sets of \( G_0 \)-colorings on \( E_0 \)-sections has compact vertical sections and its uniformization will make the chromatic number of \( G_0 \) equal to 2, while its Borel chromatic number is uncountable. Thus, the chromatic number preservation theorem for uniformization with compact sections must be more restrictive.
Theorem 10.1.17. Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $A \subset X \times X^\omega$ be a Borel set with compact sections in $E$. Then the poset $P = P_A$ is $\mathbb{E}_0$-charming.

Theorem 10.1.17 is a special case of a more general scheme described in the following definition and theorem.

Definition 10.1.18. Let $P$ be a Suslin forcing. Write $C$ for the set of equivalence classes of balanced pairs for $P$. We say that the poset $P$ is compactly balanced if there is a definable compact Hausdorff topology on the set $C$ such that for each condition $p \in P$, the set $\{ \bar{p} \in C : \bar{p} \leq p \} \subset C$ is closed. We allow $P$ (or the real parameter used to define $P$) to serve as a parameter in the definition of the topology.

As stated, the definition of compact balance is not absolute, since there is no clear limitation on the syntax of the definition of the topology. The definition also tacitly implies that the equivalence classes of balanced pairs form a set as opposed to proper class. These issues are resolved in a rather trivial way in all examples below.

Example 10.1.19. Let $P = \mathcal{P}(\omega)$ modulo finite. Then $P$ is compactly balanced. To see this, use Theorem 7.1.4 to conclude that the balanced classes are classified by non-principal ultrafilters on $\omega$. Viewing this set as a Stone–Čech remainder of $\omega$, we get the requested compact Hausdorff topology.

Example 10.1.20. Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $P$ be the linearization poset for the quotient $X/E$ as in Example 6.4.16. Then $P$ is compactly balanced. To see this, note that the balanced classes are classified by linear orderings of the virtual quotient space $X^{**}$ as proved in Theorem 6.4.17. The set of linear orderings can be viewed as a closed subset of $2^{(X^{**})^2}$ with the inherited topology. Note that in this example, the weight of the compact space is equal to $\lambda(E)$ which may be much larger than the continuum.

Example 10.1.21. Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $K$ be a locally finite $E$-quotient Borel simplicial complex and let $P = P_K$. Then $P$ is compactly balanced. To see this, use Theorem 6.4.13 to conclude that the balanced classes are classified by maximal $K^{**}$-sets. The set $C$ of maximal $K^{**}$-sets can be viewed as a subset of the space $D = \mathcal{P}(X^{**})$ with the inherited topology. The locally finite assumption now shows that $C \subset D$ is closed: if a set $A \subset X^{**}$ is not a $K$-set or not a maximal $K^{**}$-set, then there is a finite set $d \subset D$ such that no set $A' \subset X^{**}$ with $A' \cap d = A \cap d$ is a maximal $K^{**}$-set.

Example 10.1.22. Let $E$ be a countable Borel equivalence relation on a Polish space $X$. Let $A \subset X \times X^\omega$ be a Borel set which is $E \times =_+^\omega$-invariant, with compact vertical sections in $E$. Then the poset uniformizing $A$ as a subset of $X/E \times X^\omega/ =_+$ is compactly balanced. The balanced conditions are classified by functions which for each class $[x]_E$ select an element of the compact vertical section of $A$ associated with $[x]_E$. As the product of compact spaces is compact, the resulting topology on the space of (equivalence classes of) balanced conditions is compact Hausdorff.
Theorem 10.1.23. Suppose that $P$ is a Suslin poset which is compactly balanced in all forcing extensions. Then $P$ is $\mathcal{E}_0$-charming.

Proof. Fix a condition $p \in P$. Let $Q_0 = \mathcal{P}(\omega)$ modulo finite and let $Q_1$ be the Vitali forcing with its name $\dot{y}_{gen}$ for a generic element of $2^\omega$. Let $H_0 \subset Q_0$ and $H_1 \subset Q_1$ be mutually generic filters. We claim that in the generic extension $V[H_0][H_1]$, there is a balanced condition below $p$ in $P$ such that no countable coloring of $\mathcal{E}_0$ is definable from it and a ground model parameter. To this end, note that $Q_1$ adds no independent real by Fact 1.3.16 and so by a genericity argument, the filter $H_0$ generates an ultrafilter on $\omega$ in the model $V[H_0][H_1]$. In the model $V[H_0]$, let $\tau$ be any $Q_1$-name for a balanced condition below $p$. For every number $\eta \in \omega$, let $\dot{y}_n$ be the $Q_1$-name for the binary sequence obtained by replacing the first $n$ entries of $\dot{y}_{gen}$ with zeroes; note that $\dot{y}_n$ is also a $Q_1$-generic and generates the same model as $\dot{y}_{gen}$. By a compactness argument, the sequence $\langle \tau/\dot{y}_n : n \in \omega \rangle$ has a unique $H_0$-limit in the compact Hausdorff space of balanced conditions, and this limit will still be stronger than $p$. Let $\sigma$ be the $Q_1$-name for this limit. We will show that $V[H_0] \models Q_1 \models$ there is no coloring of $\mathcal{E}_0$ by countably many colors definable from $\sigma$ and parameters in $V[H_0]$. Work in $V[H_0]$ and suppose towards contradiction that a condition $q \in Q_1$ forces the opposite. Strengthening the condition $q$ if necessary, we may assume that it identifies a formula defining some $\mathcal{E}_0$-coloring $c : 2^\omega \to \omega$ with parameter $\sigma$ and further parameters from the ground model, and that there is also a number $n$ such that $q \models \check{c}(\dot{y}_{gen}) = \check{n}$. Use Fact 1.3.16 to find a nonzero element $\gamma$ of the rational Cantor group such that $\langle \gamma \cdot q \cap q \in Q_1 \rangle$. Now, step out of $V[H_0]$ and let $H_1 \subset Q_1$ be a filter generic over $V[H_0]$, containing the condition $\langle \gamma \cdot q \cap q \rangle$. Let $y \in 2^\omega$ be the point associated with $H_1$. Let $H_1' = \gamma^{-1} \cdot H_1$ and $y' = \gamma^{-1} \cdot y$. Then $H_1' \subset Q_1$ is a filter generic over $V[H_0]$ containing the condition $q$, $y' \in 2^\omega$ is its associated generic point, $y' \mathcal{E}_0 y$ and $V[H_0][H_1] = V[H_0][H_1']$. Observe that the sequences $\langle y_n : n \in \omega \rangle$ and $\langle y'_n : n \in \omega \rangle$ derived from $y$ and $y'$ as in the previous paragraph have the same tail, and so $\sigma/H_1 = \sigma/H_1'$. The definition of the coloring $c$ must then yield the same object in case of $H_1$ and $H_1'$, and by the forcing theorem, $c(y) = c(y') = n$. This means that $c$ is not a $\mathcal{E}_0$-coloring, a contradiction. \hfill $\Box$

Corollary 10.1.24. 1. Let $P$ be the poset $\mathcal{P}(\omega)$ modulo finite. In the $P$-extension of the symmetric Solovay model, $|\mathcal{E}_0| > |2^\omega|$ holds;

2. It is consistent relative to an inaccessible cardinal that $ZF + DC$ holds, there is a nonprincipal ultrafilter on $\omega$, and yet $|\mathcal{E}_0| > |2^\omega|$.

Proof. The poset $P$ is compactly balanced in ZFC by Example 10.1.19. Thus, Theorems 10.1.3 and 10.1.23 shows that in the extension of the symmetric Solovay model, $\mathcal{E}_0$ has uncountable chromatic number. In addition, the poset $P$ is adequate by Proposition ?? and so the countable-to-one uniformization holds in the extension, which means that $|\mathcal{E}_0| > |2^\omega|$ must hold there. This proves (1); (2) is an immediate conclusion. \hfill $\Box$

Corollary 10.1.25. Let $E$ be a Borel equivalence relation on a Polish space $X$. 
1. Let $P$ be the linearization poset for $E$ of Example 6.4.16. In the $P$-extension of the symmetric Solovay model, $|E_0| > 2^\omega$;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the quotient space $X/E$ can be linearly ordered, and yet $|E_0| > |2^\omega|$.

Proof. For (1) note first that in the $P$-extension of the symmetric Solovay model, the chromatic number of $E_0$ is uncountable. This follows from Theorem 10.1.3, Theorem 10.1.23 and the fact that the poset $P$ is compactly balanced in ZFC. Uncountability of the chromatic number of $E_0$ is in ZF equivalent to the nonexistence of $E_0$-selector. Now, the poset $P$ is adequate (Proposition 9.5.3 and therefore in the $P$-extension of the symmetric Solovay model, countable-to-one uniformization holds. The inequality $|E_0| > |2^\omega|$ follows.

(2) follows immediately from (1).  

Corollary 10.1.26. Let $\Gamma$ be a Borel locally finite graph on a Polish space $X$ such that every finite set $a \subset X$ has at least $|a|$-many $\Gamma$-neighbors.

1. Let $P$ be the uniformization poset selecting perfect matching for each $\Gamma$-connectedness class. In the $P$-extension of the symmetric Solovay model, $|E_0| > 2^\omega$.

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, $\Gamma$ has a perfect matching, and yet $|E_0| > |2^\omega|$.

Proof. Write $E$ for the $\Gamma$-connectedness equivalence relation on $X$; so $E$ has countable classes. Let $A \subset X \times (X^2)^\omega$ be the Borel set such that $A_x$ consists of all enumerations of perfect matchings for $\Gamma | [x]_E$. Since the graph $\Gamma$ satisfies the assumptions of Hall’s marriage theorem, the vertical sections of $A$ are nonempty; since the graph is locally finite, the set $A$ has compact sections in $E$ in the sense of Definition 10.1.14. The uniformization poset $P = P_A$ is compactly balanced in ZFC by Example 10.1.22. Thus, Theorem 10.1.3 and 10.1.23 together show that in the $P$-extension of the symmetric Solovay model, the chromatic number of $E_0$ is uncountable. A reference to the countable-to-one uniformization in the extension then yields $|E| > |2^\omega|$. The proof of (1) is complete; (2) follows.  

The comparison of the preservation theorems for sets with generic sections and compact sections brings up a curious question. Let $E$ be a countable Borel equivalence relation. The class of uniformization posets for Borel sets with generic vertical sections in $E$ is closed under countable support product in a natural sense; so is the class of uniformization posets for Borel sets with compact vertical sections in $E$. However, it seems to be difficult to combine the two. Hence the question:

Question 10.1.27. Can there be Polish spaces $X_0, X_1$ carrying the respective countable Borel equivalence relations $E_0, E_1$, and Borel sets $A_0 \subset X_0 \times X_0^\omega$ and $A_1 \subset X_1 \times X_1^\omega$, one of them with generic sections in $E_0$, and the other with compact sections in $E_1$, so that the existence of a uniformizing function for both at once implies in ZF+DC that there is a transversal for $E_0$?
We conclude this section with a rather surprising consistency result: it is consistent with ZF+DC to have $|E_0| \leq 2^\omega$ and at the same time to have the chromatic number of $E_0$ uncountable. Clearly, the countable-to-one uniformization has to fail in such circumstances.

**Theorem 10.1.28.** Let $E$ be a pinned Borel equivalence relation on a Polish space $X$ and let $P$ be its associated collapse poset of $|E|$ to $2^\omega$ as in Definition 6.4.2. $P$ is $E_0$-charming.

*Proof.* To find an $E_0$-charm for $P$ below any condition, consider the c.c.c. forcing designed to add an injection from $X/E$ to $2^\omega$ by finite approximations: $Q$ is the poset of all finite partial functions from $X$ whose range is a subset of $2^m$ for some number $m \in \omega$ and which assign the same output to $E$-related inputs. The ordering on $Q$ is defined by $q_1 \leq q_0$ if dom$(q_0) \subset$ dom$(q_1)$, and for all $x \in$ dom$(q_0)$, $q_0(x) \subseteq q_1(x)$. A straightforward genericity argument shows that $Q$ adds an injection from $X/E$ of the ground model to $2^{\omega}$ of the generic extension none of whose values comes from the ground model. We will show that in the generic extension given by the Cohen forcing followed by the finite support iteration of $\omega_1$-many copies of $Q$, there is an injection $\pi$: $X/E \to 2^\omega$ such that no countable coloring of $E_0$ is definable from $\pi$. In view of Theorem 6.4.3 (asserting that balanced conditions for $P$ are classified by injections from $X/E$ to $2^\omega$), this will complete the proof. The following claim is the heart of the proof.

**Claim 10.1.29.** The poset $Q$ is c.c.c. very Suslin, and it is Suslin Ramsey-centered.

*Proof.* It is immediate that the poset $Q$ is Suslin. To verify the very Suslin property, let $a \subset Q$ be a countable set and note that if there is a condition incompatible with all elements of $a$ then there is a condition like that whose domain is a subset of $\bigcup_{q \in a}$ dom$(q)$. Thus, to verify the predensity of $a$, one has to check only these countably many conditions, which is a Borel task.

To check the Suslin Ramsey-centered property, suppose that a number $r \in \omega$ is given. For all $m, n \in \omega$ write $A_{mn} = \{ q \in Q : |\text{dom}(q)| = n \text{ and } \text{rng}(q) \subset 2^m \}$; so $Q = \bigcup_{mn} A_{mn}$ is a partition into countably many Borel sets. To prove the Ramsey feature of this partition, fix numbers $m, n \in \omega$. Let $k$ be a number such that $k \to (\max(r, 3))^{mn^2+1}_2$; it will be enough to show that in the set $A_{mn}$, every collection of $k$ many conditions contains a subcollection of size $r$ with a common lower bound.

Suppose that $\{ q_i : i \in k \}$ are conditions in the set $A_{mn}$. Define a partition $\pi$ from $[k]^2$ to a set of size $mn^2 + 1$ as follows. If $i \in j \in k$ are numbers then the value $\pi(i, j)$ is “great” if $q_i, q_j$ are compatible. Otherwise, there must be disagreements between the conditions $q_i, q_j$ and the value of $\pi(i, j)$ will indicate one of them. Find $E$-related points $x_i \in \text{dom}(q_i)$ and $x_j \in \text{dom}(q_j)$ such that $q_i(x_i) \neq q_j(x_j)$, and let $\pi(i, j) = (s, t, u)$ where $x_i$ is the $s$-th element of dom$(q_i)$ in a fixed linear order of $X$, $x_j$ is the $t$-th element of dom$(q_j)$ in the linear order, and $u$ is some number such that $q_i(x_i)(u) \neq q_j(x_j)(u)$. Since $s, t \in n$ and $u \in m$, $|\text{rng}(\pi)| \leq mn^2 + 1$. 


By the Ramsey property of the number \( k \), there has to be a set of size \( \max(r,3) \) which is homogeneous for \( \pi \). If the homogeneous value is “great”, then great: we found a set of conditions of size at least \( r \) which are pairwise compatible, which in the poset \( Q \) means that the set has a common lower bound as desired. Thus, it will be enough to show that no other homogeneous value can occur. Suppose towards contradiction that it does, and let the value be \((s,t,u)\). Let \( i_0 < i_1 < i_2 \) be any three numbers in the homogeneous set. Let \( x_0 \) be the \( s \)-th element of \( \text{dom}(q_{i_0}) \), let \( x_1 \) be the \( s \)-th element of \( \text{dom}(q_{i_1}) \), let \( y_1 \) be the \( t \)-th element of \( \text{dom}(q_{i_1}) \) and let \( y_2 \) be the \( t \)-th element of \( \text{dom}(q_{i_2}) \). By the homogeneity, it must be the case that all these points come from the same \( E \)-class. By the homogeneity again, \( q_0(x_0)(u) \neq q_1(y_1)(u) \), \( q_1(x_1)(u) \neq q_2(y_2)(u) \) and \( q_0(x_0)(u) \neq q_2(y_2)(u) \). However, \( q_1 \) is a condition in the poset \( Q \) and so \( q_1(x_1)(u) = q_1(y_1)(u) \). That means that \( q_0(x_0)(u), q_1(x_1)(u) \) and \( q_2(y_2)(u) \) are three distinct bits in 2, overcrowding the poor 2 and yielding a contradiction. \( \square \)

Let \( z \in 2^\omega \) be a Cohen generic point over \( V \). In the model \( V[z] \), let \( R \) be the \( \omega_1 \)-length iteration of the poset \( Q \) with finite support. Let \( H \subset R \) be a generic filter. Work in \( V[z][H] \). For each ordinal \( \alpha \in \omega_1 \), let \( H_\alpha \subset Q \) be the filter added on \( \alpha \)-th stage of the iteration, write \( V_\alpha = V[z][H_\beta : \beta \in \alpha] \) and let \( \pi_\alpha : (X/E)^{V_\alpha} \to (2^\omega)^{V_{\alpha+1}} \) be the generic injection given by the filter \( H_\alpha \). Now let \( \pi \) be the function defined in \( V[z][H] \) as follows: for any \( x \in X \) find the smallest ordinal \( \alpha \in \omega_1 \) such that \( x \) has an \( E \)-relative \( x' \in V_\alpha \) and let \( \pi(x) = \pi_\alpha(x') \). This is a total injection from \( X/E \) to \( 2^\omega \) in the model \( V[z][H] \). Note also that the definition of \( \pi \) depends only on the model \( V[z] \) and the filters \( H_\alpha \) and not on the point \( z \) itself.

To confirm that there is no countable coloring of \( \mathbb{E}_0 \) by countably many colors definable from \( \pi \) and parameters in \( V \), return to \( V \). Find a Cohen+\( \hat{R} \)-name \( \sigma \) for the injection \( \pi \). Suppose toward contradiction that \( \langle t, \hat{r} \rangle \) is a condition in the iteration Cohen+\( \hat{R} \) and \( \tau \) is a name in the iteration for a function from \( 2^\omega \) to \( \omega \) which is a \( \mathbb{E}_0 \)-coloring definable using some parameters in the ground model and \( \sigma \). Strengthening the condition if necessary, we may assume that there is a natural number \( n \in \omega \) such that it forces \( \tau(\hat{x}_{\text{gen}}) = \hat{n} \), where \( \hat{x}_{\text{gen}} \) is the name for the Cohen generic element of \( 2^\omega \) added in the first step of the iteration, and that it identifies a specific definition of \( \tau \) from \( \sigma \). The condition \( \hat{r} \) is forced to have a finite support bounded by some countable ordinal \( \alpha \in \omega_1 \). The finite support iteration \( R_\alpha \) of length \( \alpha \) is a very Suslin, Suslin Ramsey-centered forcing by Corollary 13.2.13 and Theorem 13.2.21. Strengthening the condition \( \langle t, \hat{r} \rangle \) again if necessary, we may assume that there is an analytic set \( A \subset R_\alpha \) which is Ramsey-2-centered and \( t \vdash \hat{r} \in A \). Let \( k \in \omega \) be a natural number witnessing the Ramsey-2-centeredness of the set \( A \).

Let \( z \in 2^\omega \) be a point Cohen generic over \( V \), extending the string \( t \). Find points \( \{z_i : i \in k\} \in 2^\omega \) which are pairwise distinct, \( \mathbb{E}_0 \)-related to \( z \), and which extend the string \( t \). Thus, these points are all Cohen-generic over \( V \) and yield the same generic extension \( V[z] \). In the model \( V[z] \), let \( r_i = \hat{r}/z_i \) for all \( i \in k \). These are conditions in the poset \( R_\alpha \), both in the Ramsey-2-centered set \( A \), and so two of them, say \( r_{i_0}, r_{i_1} \), have a lower bound \( r \). Let \( H \subset R \) be a filter generic...
over the model $V[z]$, meeting the condition $r$, and move to the model $V[z][H]$.

Now, the cinch. The filters given by $z_{i_0}, H$ and $z_{i_1}, H$ are both generic over $V$ for the iteration Cohen$* \tilde{R}$, they both meet the initial condition $(t, \dot{r})$ and they both generate the same model. Moreover, $\sigma/z_{i_0}, H = \sigma/z_{i_1}, H$ and therefore $\tau/z_{i_0}, H = \tau/z_{i_1}, H$. By the forcing theorem, $(\tau/z_{i_0}, H)(z_{i_0}) = n = (\tau/z_{i_1}, H)(z_{i_1})$. This is a contradiction though, since $z_{i_0}, z_{i_1} \in 2^{\omega}$ are $E_0$-related points.

As a conjunction of Theorems 10.1.3 and 10.1.28, we now get:

**Corollary 10.1.30.** Let $E$ be a pinned Borel equivalence relation on a Polish space $X$.

1. Let $P$ be the collapse of $|E|$ to $|2^{\omega}|$ as in Definition 6.4.2. In the $P$-extension of the symmetric Solovay model, $E_0$ has no transversal;

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, $|E| \leq |2^{\omega}|$ and yet $E_0$ has no transversal.

### 10.2 Virtual charms

It occurs in a number of interesting cases that the hypergraph $\Gamma$ arrives with a Borel equivalence relation $E$ on the underlying Polish space $X$ such that the membership of an edge $e$ in $\Gamma$ depends only on the $E$-classes of the vertices of $e$. In such a case, it makes very good sense to consider the virtual version $\Gamma^{**}$ of the hypergraph on the virtual quotient space, and the corresponding virtual charms.

**Definition 10.2.1.** A quotient analytic hypergraph is a triple $\langle X, \Gamma, E \rangle$ where $X$ is a Polish space, $\Gamma$ is an analytic hypergraph on $X$, $E$ is a Borel equivalence relation on $X$, and for all finite sets $e, f \subset X$, if $[e]_E = [f]_E$ then $e \in \Gamma \leftrightarrow f \in \Gamma$.

**Definition 10.2.2.** Let $\langle X, \Gamma, E \rangle$ be a quotient analytic hypergraph, let $P$ be a Suslin poset. A virtual $\Gamma$-charm below a condition $p \in P$ is a pair $\langle V[G], \bar{p} \rangle$ where $V[G]$ is a generic extension $V[G]$ and $\bar{p} \leq p$ is a balanced virtual condition in $P$ in $V[G]$ such that for every partition $\langle A_n : n \in \omega \rangle$ of the space $X^{**}$ which is definable from $\bar{p}$ and the parameters in the ground model, there is an edge $e \in \Gamma^{**}$ all of whose vertices come from the same piece of the partition.

**Definition 10.2.3.** Let $\langle X, \Gamma, E \rangle$ be a quotient analytic hypergraph, let $P$ be a Suslin poset. The poset $P$ is $\Gamma$-charming if it has a $\Gamma$-charm below every condition $p \in P$.

The proof of the following theorem is essentially a repetition of the argument for Theorem 10.1.3, with two stars added to certain symbols:

**Theorem 10.2.4.** Suppose that $P$ is a Suslin forcing, $\langle X, \Gamma, E \rangle$ is a quotient analytic hypergraph and $\kappa$ is an inaccessible cardinal such that $P$ is $\Gamma$-charming
10.2. VIRTUAL CHARMS

Let \( W \) be the symmetric Solovay model derived from \( \kappa \) and \( G \subset P \) be a filter generic over \( W \). Then in \( W[G] \), the chromatic number of the hypergraph \( \Gamma \) is uncountable.

As a sample application, we study the chromatic numbers in the model \( W[U] \) where \( U \) is a Ramsey ultrafilter over the symmetric Solovay model \( W \). For the first one, let \( E \) be a Borel equivalence relation on a Polish space \( X \). The polarization graph for \( E \) is the graph \( \Gamma \) on \( X^2 \setminus E \) connecting \( \langle x_0, x_1 \rangle \) to \( \langle y_0, y_1 \rangle \) if \( x_0 \models E y_1 \) and \( x_1 \models E y_0 \). Clearly, this is a quotient analytic hypergraph.

**Theorem 10.2.5.** Let \( P \) be the poset of infinite subsets of \( \omega \) ordered by inclusion. Let \( \Gamma \) be the polarization graph for \( \mathbb{F}_2 \). \( P \) is \( \Gamma \)-charming.

**Proof.** We will prove the theorem by presenting a virtual \( \Gamma \)-charm below any condition in the poset \( P \). To simplify the notation slightly, we ignore the condition in the poset \( P \).

The charm is connected with a suitable forcing \( Q \) and interchangeable \( Q \)-names \( \check{x}_0, \check{x}_1 \) for virtual \( \mathbb{F}_2 \)-classes. Let \( Q \) be the countable support product of copies of Sacks forcing indexed by \( \omega_1 \times 2 \). Let \( \check{x}_0 \) be the \( Q \)-name for the set of Sacks reals added on the coordinates in \( \omega_1 \times \{0\} \), and let \( \check{x}_1 \) be the \( Q \)-name of the set of Sacks reals added at the coordinates in \( \omega_1 \times \{1\} \). As subsets of \( 2^\omega \), they are naturally identified with virtual \( \mathbb{F}_2 \)-classes by Theorem 2.3.4. The following automorphism claim is central:

**Claim 10.2.6.** For any conditions \( q_0, q_1 \in Q \) there is an automorphism \( \pi \) of \( Q \) such that \( \pi(q_0) \) is compatible with \( q_1 \) and \( \pi(\check{x}_0) = \check{x}_1 \) and \( \pi(\check{x}_1) = \check{x}_0 \). In particular, if \( \phi \) is any formula with ground model parameters, then \( Q \models \phi(\check{x}_0, \check{x}_1) \leftrightarrow \phi(\check{x}_1, \check{x}_0) \).

**Proof.** For the first sentence, let \( \alpha \in \omega_1 \) be a countable ordinal such that the supports of \( q_0, q_1 \) are both subsets of \( \alpha \times 2 \). Consider the permutation \( \pi \) of \( \omega_1 \times 2 \) which exchanges the pairs \( \langle \gamma, 0 \rangle \) and \( \langle \gamma, 1 \rangle \) if \( \gamma > \alpha + \alpha \), exchanges the pairs \( \langle \gamma, 0 \rangle \) and \( \langle \alpha + \gamma, 1 \rangle \) if \( \gamma \in \alpha \) and also exchanges the pairs \( \langle \gamma, 1 \rangle \) and \( \langle \alpha + \gamma, 0 \rangle \) if \( \gamma \in \alpha \). The permutation \( \pi \) naturally induces an automorphism of the poset \( Q \); it is easy to check that the induced automorphism satisfies the properties requested by the claim.

For the second sentence, suppose towards contradiction that \( q \in Q \) is a condition forcing \( \phi(\check{x}_0, \check{x}_1) \) and \( \neg \phi(\check{x}_1, \check{x}_0) \). Find an automorphism \( \pi \) such that \( q, \pi(q) \) are compatible conditions and let \( H \subset Q \) be a generic filter containing their common lower bound. Then \( V[H] = V[\pi''H] \) and by the forcing theorem applied to the two filters, the generic extension should satisfy both \( \phi(\check{x}_0/H, \check{x}_1/H) \) and \( \neg \phi(\check{x}_1/\pi''H, \check{x}_0/\pi''H) \). However, this is impossible as \( \check{x}_0/H = \pi(\check{x}_0)/\pi''H = \check{x}_1/\pi''H \) and \( \check{x}_1/H = \pi(\check{x}_1)/\pi''H = \check{x}_0/\pi''H \) by the choice of the automorphism \( \pi \).

The virtual \( \Gamma \)-charm is now easy to present. The charm is the pair \( (V[H, K], \bar{p}) \) where \( V[H, K] \) is the extension by the poset \( P \times Q \) and \( \bar{p} \) is the Coll(\( \omega, c \))-name for the set of all conditions \( p \in P \) which diagonalize the Ramsey filter added by
the left coordinate of the product. It is somewhat more difficult to verify the properties of the charm.

First of all, $\bar{p}$ is in fact a balanced virtual condition in the model $V[H,K]$. To see this, note that $H \subset \mathcal{P}(\omega)$ is in fact an ultrafilter in the model $V[H]$ by an elementary genericity argument. It is well-known that $Q$ adds no independent reals, and so by a genericity argument, $H$ generates an ultrafilter even in the $Q$ extension $V[H,K]$ of $V[H]$. The virtual condition $\bar{p}$ is clearly equivalent to the condition derived from this ultrafilter as in Theorem 7.1.4 and so it is balanced in $V[H,K]$ as required.

To verify the uncountable definable chromatic number of the virtual version of the graph $\Gamma$ in the model $V[H,K]$, consider the generic $Q$-objects $x_0,x_1$. It is enough to show that if $A$ is a set of $\mathcal{F}_2^{\omega}$ classes definable in the model $V[H,K]$ by some formula $\phi$ with parameters $\bar{p}$ and $z \in V$ and $\langle x_0,x_1 \rangle \in A$, then $\langle x_1,x_0 \rangle \in A$ since the two pairs together form a $\Gamma^{**}$-edge. However, this follows immediately from the second sentence of Claim 10.2.6 applied in the model $V[H]$. Note that $\bar{p}$ is definable in $V[H,K]$ from the filter $H$.

**Corollary 10.2.7.**

1. Let $P = \mathcal{P}(\omega)$ modulo finite. In the $P$-extension of the symmetric Solovay model, there is no tournament on the quotient space of $\mathcal{P}_2$;

2. it is consistent relative to an inaccessible cardinal that ZF+DC holds, there is a nonprincipal ultrafilter on $\omega$, and yet there is no tournament on the quotient space of $\mathcal{P}_2$.

**Proof.** This is a conjunction of Theorem 10.2.4 and 10.2.5. Note that in ZF, the existence of a tournament on a quotient space of a Borel equivalence relation is equivalent to the countable chromatic number of the associated polarization graph.

Another example uses a more specific variation of the polarization graph. Let $I$ be a Borel ideal on $\omega$. The *complement graph* $\Gamma$ associated with $I$ is the graph on $\mathcal{P}(\omega)$ connecting sets $x,y \subset \omega$ if $x = \omega \setminus y$ modulo $I$. It is immediate that $\Gamma$ is a quotient Borel graph on the space $\mathcal{P}(\omega)/I$.

**Theorem 10.2.8.** Let $P$ be the poset of infinite subsets of $\omega$ ordered by inclusion. Let $I$ be the summable ideal on $\omega$ and $\Gamma$ its associated complement graph. $P$ is $\Gamma$-charming.

**Proof.** We have to produce a $\Gamma$-charm below any condition $p \in P$. This time, the description of the $\Gamma$-charm is more involved. We have to come up with a poset $Q$ which adds no independent reals and has certain symmetry properties. This is not easy, and the concentration of measure phenomenon comes to the rescue. Let $\langle J_n : n \in \omega \rangle$ be a very fast sequence of successive intervals on $\omega$. The following is the concentration of measure type of demand on the intervals in question that we need. Let $\mu_n$ be the normalized counting measure on $2^{J_n}$. Let $d_n$ be the metric on $2^{J_n}$ defined by $d_n(x,y) = \Sigma \{ \frac{1}{m+1} : x(m) \neq y(m) \}$. We demand that the following holds for every $n \in \omega$:
(\ast) for every set \( a \subset 2^{J_n} \) of \( \mu_n \)-mass \( 1/n \), the \( 2^{-n} \)-neighborhood of \( a \) in \( 2^{J_n} \) in the sense of the metric \( d_n \) has \( \mu_n \)-mass greater than \( 1/2 \).

The concentration of measure computations in ??? show that such a fast sequence of intervals indeed exists. ???

Let \( T_{\text{ini}} \) be the tree of all finite sequences \( t \) such that for each \( n \in \text{dom}(t) \), \( t(n) \in 2^{J_n} \). Let \( Q \) be the partial order of all nonempty trees \( q \subset T_{\text{ini}} \) without endnodes such that for each \( m \in \omega \) there is \( n_m \in \omega \) such that for each \( t \in q \) of length \( n > n_m \), the set \( \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in q \} \) has \( \mu_n \)-mass \( \geq m/n \). The ordering on \( Q \) is that of inclusion.

There is a natural generic object for \( Q \). If \( K \subset Q \) is a generic filter, there is a unique inclusion maximal one and call it \( \pi \). Since the measures \( \mu_n \) are inclusion preserving injections from subsets of \( \omega \), the set \( \{ \pi(x) : x \in 2^{J_n} \} \) is that of inclusion.

Proof. The key part of the claim is the first sentence. Let \( t_0 \in q_0, t_1 \in q_1 \) be nodes of the same length such that for every node \( t \in q_0 \) of length \( n \geq |t_0| \), the set \( a_0(t) = \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in q_0 \} \) has \( \mu_n \)-mass at least \( 1/2n \), and for every node \( t \in q_1 \) of length \( n \geq |t_0| \), the set \( a_1(t) = \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in q_1 \} \) has \( \mu_n \)-mass at least \( 1/2n \) as well. We will produce a tree \( q_0' \leq q_0 \upharpoonright t_0 \) in \( Q \) and a level and order preserving injection \( \pi : q_0' \to q_1 \upharpoonright t_1 \) so that

\[ (\ast\ast) \text{ for each node } t \in q_0' \text{ and a number } n > |t_0| \text{ in } \text{dom}(t), d_n(t(n), 1 - \pi(t)(n)) < 2^{-n+1}. \]

Write \( q_1' = \pi'' q_0' \). Since the measures \( \mu_n \) are normalized counting measures, the map \( \pi \) naturally extends to an isomorphism \( \pi : Q \upharpoonright q_0' \to Q \upharpoonright q_1' \). The demand \( (\ast\ast) \) then implies that \( \pi \) indeed flips the names \( \hat{x}_0, \hat{x}_1 \) as required.

The map \( \pi \) is obtained by a hungry algorithm. Among all level and order preserving injections from subsets of \( q_0 \upharpoonright t_0 \) to \( q_1 \upharpoonright t_1 \) which satisfy \( (\ast\ast) \), select an inclusion maximal one and call it \( \pi \). It will be enough to show that \( q_0' = \text{dom}(\pi) \) belongs to \( Q \). To see this, first of all \( q_0' \) is closed under initial segment by an obvious maximality argument. To verify the branching condition in the definition of \( Q \) for \( q_0' \), let \( t \in q_0' \) of length some \( n \geq |t_0| \). It will be enough to argue that the set \( \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in q_0' \} \) has \( \mu_n \)-mass at least \( \min(\mu_n(a_0(t)), \mu_n(a_1(\pi(t)))) - 1/n \).

To do this, note that if both sets \( b_0 = \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in q_0' \setminus q_0 \} \) and \( b_1 = \{ x \in 2^{J_n} : t^\frown \langle x \rangle \in \pi(q_0') \setminus q_1 \} \) had \( \mu_n \)-mass greater than \( 1/n \), then by \( (\ast) \) the \( 1/2^n \)-neighborhood of \( b_0 \) and the set \( \{ 1 - x : x \in b_1 \} \) would both have \( \mu_n \)-mass greater than \( 1/2 \) and so they would intersect, making it possible to extend \( \pi \) while satisfying \( (\ast\ast) \) and contradicting the maximality of \( \pi \).
For the second sentence, suppose towards contradiction that $q \in Q$ is a condition forcing $\phi(x_0, x_1)$ and $\neg \phi(x_1, x_0)$. Find conditions $q_0, q_1 \leq q$ and an automorphism $\pi : Q \upharpoonright q_0 \to Q \upharpoonright q_1$ and let $H \subset Q$ be a generic filter containing $q_0$. Then $V[H] = V[\pi''H]$ and by the forcing theorem applied to the two filters, the generic extension should satisfy both $\phi(x_0/H, x_1/H)$ and $\neg \phi(x_1/H, x_0/H)$. However, this is impossible as $x_0/H = \pi(x_0)/\pi''H = x_1/H$ by the choice of the automorphism $\pi$.

The virtual $\Gamma$-charm is now easy to present. The charm is the pair $\langle V[H,K], \bar{p} \rangle$ where $V[H,K]$ is the extension by the poset $P \times Q$ and $\bar{p}$ is the Coll($\omega, c$)-name for the set of all conditions $p \in P$ which diagonalize the Ramsey filter added by the left coordinate of the product. It is somewhat more difficult to verify the properties of the charm.

First of all, $\bar{p}$ is in fact a balanced virtual condition in the model $V[H,K]$. To see this, note that $H \subset P(\omega)$ is in fact an ultrafilter in the model $V[H]$ by an elementary genericity argument. By [18, Theorem 4.4.8], $Q$ adds no independent reals and so by a genericity argument, $H$ generates an ultrafilter even in the $Q$ extension $V[H,K]$ of $V[H]$. The virtual condition $\bar{p}$ is clearly equivalent to the condition derived from this ultrafilter as in Theorem 7.1.4 and so it is balanced in $V[H,K]$ as required.

To verify the uncountable definable chromatic number of the virtual version of the graph $\Gamma$ in the model $V[H,K]$, consider the generic $Q$-objects $x_0, x_1$. It is enough to show that if $A$ is a set of $((-1)^2)^{**}$ classes definable in the model $V[H,K]$ by some formula $\phi$ with parameters $\bar{p}$ and $z \in V$ and $\langle x_0, x_1 \rangle \in A$, then $\langle x_1, x_0 \rangle \in A$ since the two pairs together form a $\Gamma^{**}$-edge. However, this follows immediately from the second sentence of Claim 10.2.9 applied in the model $V[H]$. Note that $\bar{p}$ is definable in $V[H,K]$ from the filter $H$.

**Corollary 10.2.10.**

1. Let $P = P(\omega)$ modulo finite. In the $P$-extension of the symmetric Solovay model, there is no ultrafilter on $\omega$ disjoint from the summable ideal;

2. It is consistent relative to an inaccessible cardinal that $ZF + DC$ holds, there is a nonprincipal ultrafilter on $\omega$, and yet there is no ultrafilter disjoint from the summable ideal.

**Proof.** This is a conjunction of Theorem 10.2.4 and 10.2.8. Note that if $I$ is an ideal on $\omega$ and $U$ is an ultrafilter disjoint from it, then the function $c : 2^\omega \to 2$ defined by $f(x) = 1 \leftrightarrow x^{-1}\{0\} \in U$ is a coloring of the complement graph derived from $I$ by two colors.

**10.3 Turbulent hypergraphs**

This section isolates a notion of turbulence in hypergraphs which parallels Hjorth’s notion of turbulence in actions to a great degree. The main feature
of turbulent hypergraphs is that their chromatic number in trim balanced extensions of the symmetric Solovay model is uncountable. The development of turbulent hypergraphs starts with the following recursive definition.

**Definition 10.3.1.** Let $X$ be a Polish space and $a \subset X$ be a finite set in some forcing extension of the ground model $V$. The set $a$ is *piecewise independent* over $V$ if either $|a| \leq 1$ or $a$ can be partitioned into two nonempty sets $a_0, a_1$ such that each is piecewise independent and $V[a_0] \cap V[a_1] = V$.

It is not difficult to argue by recursion on the size of piecewise independent subsets of $X$ that a nonempty subset of a piecewise independent set is again piecewise independent.

**Definition 10.3.2.** Let $\Gamma$ be a finitary analytic hypergraph on a Polish space $X$. $\Gamma$ is *turbulent* if for every nonempty open set $O \subset X$, in some forcing extension there is a finite set $a \subset O$ which is in $\Gamma$, is piecewise independent over $V$, and consists of points separately Cohen generic over $V$.

It is not difficult to use Proposition 3.5.5 to show that the notion of turbulence of analytic hypergraphs is absolute between generic extensions, and moreover, the piecewise independent sets witnessing the turbulence can always be added by a countable poset.

To formulate the main result of this section in its full strength, we need a definition of a hypergraph which occurs frequently in the construction of c.c.c. forcings.

**Definition 10.3.3.** A collection $A$ of finite sets is a $\Delta$-system if there is a finite set $b$ (the root of $A$) which is a subset of all elements of $A$ and the sets in the collection $\{a \setminus b : a \in A\}$ are pairwise disjoint. The letter $\Delta$ denotes the Borel hypergraph on $Y = \omega \times [2^\omega]^{<\omega_1}$ consisting of all sets $a \subset Y$ such that either the first coordinates of points in $a$ are not all the same, or the second coordinates do not form a $\Delta$-system.

**Theorem 10.3.4.** Let $\kappa$ be an inaccessible cardinal. Let $P$ be a Suslin poset which is trim balanced cofinally in $\kappa$. Let $\Gamma$ be a turbulent hypergraph on a Polish space $X$. Let $W$ be a symmetric Solovay model derived from $\kappa$ and $G \subset P$ be a filter generic over $W$. In the model $W[G]$, there is no homomorphism from $\Gamma$ to $\Delta$.

In particular, the hypergraph $\Gamma$ has uncountable chromatic number in $W[G]$.

**Proof.** Suppose towards contradiction that the conclusion fails. In the model $W$, there must be a condition $p \in P$ and a $P$-name $\tau : X \rightarrow Y$ is a homomorphism from $\Gamma$ to $\Delta$. The name $\tau$ as well as the condition $p$ must be definable from some $z \in 2^\omega$ and some parameters in the ground model $V$. The assumptions show that there is an intermediate model $V[K]$ which is a generic extension of $V$ by a poset of size $< \kappa$, $z \in V[K]$, and $P$ is trim balanced in $V[K]$.
Work in the model $V[K]$. Let $\bar{p} \leq p$ be a trim balanced virtual condition in the poset $P$. Let $P_X$ be the Cohen forcing on the Polish space $X$ and $\dot{x}_{gen}$ its name for a generic element of $X$. There must be a poset $R$ of size $< \kappa$, an $R \times P_X$-name $\sigma$ for a condition in the poset $P$ stronger than $\bar{p}$, an $R \times P_X$-name $\eta$ for a finite subset of $2^\omega$, a number $n \in \omega$, a finite set $b \subset 2^\omega$ and a nonempty open set $O \subset X$ such that $\langle 1, O \rangle \Vdash_{R \times P_X} \text{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau(\dot{x}_{gen}) = \langle \bar{n}, \eta \rangle$ and $\bar{c} = \eta \cap V[K]$.

Work in the model $W$. By the assumptions on the hypergraph $\Gamma$, there is a finite set $a \subset O$ which is piecewise independent over $V[K]$, consists of points separately Cohen generic over the ground model $V[K]$, and constitutes a $\Gamma$-edge. There are also filters $H_x \subset R$ for each $x \in a$ mutually generic over the model $V[K][a]$. For each $x \in a$ let $p_x = \sigma/H_x \times x \in P$ and $c_x = \eta/H_x \times x \in Y$. The following claim is central.

**Claim 10.3.5.** The set $\{p_x : x \in a\} \subset P$ has a common lower bound. The set $\{c_x : x \in a\}$ is a $\Delta$-system with root $c$.

**Proof.** By induction on the size of a nonempty subset $b \subset a$ argue that the set $\{p_x : x \in b\} \subset P$ has a lower bound and that the set $\{c_x : x \in b\}$ is a $\Delta$-system with root $c$. This is clear if $b$ is a singleton. To perform the induction step, partition a given set $b$ into two nonempty pieces $b_0, b_1$ such that $V[K][b_0] \cap V[K][b_1] = V$.

Use the mutual genericity of the filters $H_x \subset R$ for $x \in b$ to argue that $V[K][b_0][H_x : x \in b_0] \cap V[K][b_1][H_x : x \in b_1] = V[K]$. To verify the $\Delta$-system statement, use the induction hypothesis to note that $\{c_x : x \in b_0\}$ and $\{c_x : x \in b_1\}$ are two $\Delta$-systems with root $c$, and $V[K] \cap \bigcup_{x \in b_0} c_x = V[K] \cap \bigcup_{x \in b_1} c_x = c$.

To verify the compatibility statement, use the induction hypothesis to find a lower bound $q_0$ of $\{p_x : x \in b_0\}$ and a lower bound $q_1$ of $\{p_x : x \in b_1\}$.

Use the Mostowski absoluteness to find these lower bounds in the respective models $V[K][b_0][H_x : x \in b_0]$ and $V[K][b_1][H_x : x \in b_1]$. Use the trim balance of the condition $\bar{p}$ to conclude that $q_0, q_1$ are compatible in the poset $P$; their common lower bound is the sought bound of the set $\{p_x : x \in b\} \subset P$. \hfill $\Box$

Still work in the model $W$. Let $q$ be a common lower bound of the set $\{p_x : x \in a\} \subset P$. For each $x \in a$, $W$ is a $\text{Coll}(\omega, < \kappa)$-extension of the model $V[K][H_x, x]$ and therefore the forcing theorem applied in that model implies that $q \Vdash \dot{x} \in \text{dom}(\tau_n)$. Since the set $a \subset X$ forms a $\Gamma$-edge and the sets $\{c_x : x \in a\}$ form a $\Delta$-system with root $c$, this contradicts the assumption that $\tau$ is forced to be a homomorphism from $\Gamma$ to $\Delta$. \hfill $\Box$

The whole development of the notion of a turbulent hypergraph would be bare without examples. The whole point is that there are many useful turbulent hypergraphs which can be used to exclude many combinatorial objects from trim balanced extensions of the symmetric Solovay model as was done in [25].

**Example 10.3.6.** Let $\Gamma$ be the hypergraph on $2^\omega$ of arity 3, containing a triple $\langle x_0, x_1, x_2 \rangle$ for all but finitely many numbers $n \in \omega$, $\{x_0(n), x_1(n), x_2(n)\} = 2$. The hypergraph $\Gamma$ is turbulent.
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Proof. Let \( O \subset 2^\omega \) be a nonempty open set; we must produce a piecewise independent edge of \( \Gamma \) which consists of elements of \( U \) which are separately Cohen generic over the ground model. To simplify the notation, assume that \( O = 2^\omega \). Let \( X \subset (2^\omega)^3 \) be the closed set of all triples \( \langle x_0, x_1, x_2 \rangle \) such that for every number \( n \in \omega \), \( \{ x_0(n), x_1(n), x_2(n) \} = 2 \). Consider the poset \( P_X \) of nonempty relatively open subsets of \( X \). The poset adds a triple \( \dot{x}_0, \dot{x}_1, \dot{x}_2 \) of elements of \( 2^\omega \) which belongs to \( X \). It will be enough to show that for any \( i \in 3 \), the points \( \dot{x}_j : j \neq i \) are forced to be mutually Cohen-generic elements of \( 2^\omega \) and moreover, \( V[\dot{x}_1] \cap V[\dot{x}_j : j \neq i] = V \). This will mean that the generic triple is a piecewise independent \( \Gamma \)-edge consisting of separately Cohen generic points.

For definiteness let \( i = 2 \) and consider the maps \( f : X \to (2^\omega)^2 \) and \( g : X \to 2^\omega \) projecting \( X \) into the first two and the last coordinate respectively. It is easy to check that the two maps are continuous and open. Proposition 3.1.1 then shows that \( \dot{x}_0, \dot{x}_1 \) are forced to be mutually generic elements of \( 2^\omega \) and \( \dot{x}_2 \) is another generic element of \( 2^\omega \).

This leaves us with the last sentence. By Theorem 3.1.5, it is enough to show that the functions \( f, g \) are independent in the sense of Definition 3.1.4. Suppose that \( O \subset X \) is a relatively open set. Thinning \( O \) down if necessary, find \( n \in \omega \) and strings \( t_0, t_1, t_2 \in 2^n \) such that for every \( m \in n \), \( \{ t_0(m), t_1(m), t_2(m) \} = 2 \), and \( O = \{ \langle x_0, x_1, x_2 \rangle \in X : t_0 \subset x_0, t_1 \subset x_1, t_2 \subset x_2 \} \). Let \( P = \{ x \in 2^\omega : t_2 \subset x \} \). It will be enough to show that the set \( P \) witnesses the independence of \( g \) for the open set \( O \).

However, this is immediate. Let \( x, y \in P \) be any two points. Let \( x_0 = t_0 \uparrow 0^\omega \) and \( x_1 = t_1 \uparrow 1^\omega \). The two points \( \langle x_0, x_1, x \rangle, \langle x_0, x_1, y \rangle \) form an \( O \)-walk from \( x \) to \( y \) as desired. \( \square \)

Corollary 10.3.7. In trim balanced extensions of the symmetric Solovay model, there are no nonprincipal ultrafilters on \( \omega \).

Proof. If \( U \) is a nonprincipal ultrafilter on \( \omega \), then the function \( f : 2^\omega \to 2 \) defined by \( f(x) = 0 \leftrightarrow x^{-1}\{0\} \in U \) is a \( \Gamma \)-coloring into two colors. In trim balanced extensions of the symmetric Solovay model, \( \Gamma \) has an uncountable chromatic number by Theorem 10.3.4 and so there cannot be any nonprincipal ultrafilters on \( \omega \). \( \square \)

Example 10.3.8. Let \( n \in \omega \). The hypergraph \( \Gamma_n \) on \( 2^\omega \) consisting of \( n \)-tuples \( \langle x_i : i \in n \rangle \) such that for all but finitely many \( m \in \omega \), the set \( \{ i \in n : x_i(m) = 1 \} \) has size at most two, is turbulent.

Proof. Let \( X_n \) be the closed subspace of \( (2^\omega)^n \) consisting of \( n \)-tuples \( \langle x_i : i \in n \rangle \) such that for all \( m \in \omega \), the set \( \{ i \in n : x_i(m) = 1 \} \) has size at most two. Let \( f : X_{n+1} \to X_n \) and \( g : X_{n+1} \to 2^\omega \) be the projections in the first \( n \)-many and the last coordinate, respectively. It is easy to see that both functions are continuous and open.

We claim that \( f, g \) are independent functions in the sense of Definition 3.1.4. Suppose that \( O \subset X_{n+1} \) is a relatively open set. Thinning \( O \) down if necessary, find \( m \in \omega \) and strings \( t_i \in 2^n \) for \( i \in n + 1 \) such that for every \( k \in m \), the set
\{i \in n + 1: t_i(k) = 1\} has size at most 2 and \(O = \{(x_i: i \in n + 1) \in \mathbb{X}_{n+1}: \forall i \in n + 1 t_i \subset x_i\}\). Let \(P = \{x \in 2^\omega: t_n \subset x\}\). It will be enough to show that the set \(P\) witnesses the independence of \(g\) from \(f\) for the open set \(O\). To see this, let \(x, y \in P\) be any two points. Let \(z = (z_i: i \in n)\) be the point such that 
\(z_i = t_i^{-1}0^\omega\) for all \(i \in n\). Then the tuples \((z, x), (z, y)\) form an \(O\)-walk from \(x\) to \(y\) as desired.

It now follows by induction on \(n \in \omega\) that if \((x_i: i \in n)\) is a generic element of \(\mathbb{X}_n\), then each point \(x_i\) is a generic element of \(2^\omega\), for every \(m \in n\) the tuple \((x_i: i \in m)\) is a generic element of \(\mathbb{X}_m\), and by Theorem 3.1.5 \(V[x_i: i \in m] \cap V[x_m] = V\). It follows that the tuple is a piecewise independent set of Cohen generic points over \(V\). If \(t \in 2^{<\omega}\) is an arbitrary finite binary string, one can rewrite the initial segments of the points \(x_i\) with \(t\), obtaining a \(\Gamma\)-edge which is a piecewise independent set of Cohen generic points meeting the condition \(t\). The proof is complete. \(\square\)

**Corollary 10.3.9.** In trim balanced extensions of the Solovay model, there is no finitely additive probability measure on \(\omega\) which assigns mass zero to singletons.

**Proof.** Let \(\mu\) be such a measure on \(\omega\). Let \(\Lambda\) be the hypergraph on \(2^\omega\) of finite sets \(a \subset 2^\omega\) such that \(a\) contains subsets \(b_0, b_1\) each of size 10 such that for all but finitely many \(m \in \omega\), the sets \(\{x \in b_0: x(m) = 0\}\) and \(\{x \in b_1: x(m) = 1\}\) have both size at most 2. The measure \(\mu\) shows that the hypergraph \(\Lambda\) has chromatic number 2: just let \(c(x) = 0\) if \(\mu(\{m: x(m) = 1\}) \leq 1/3\) and \(c(x) = 1\) otherwise. The Fubini theorem shows that in an edge \(a \in \Delta\) as above, the set \(b_0\) must contain a point of color 0 while the set \(b_1\) must contain a point of color 1.

However, \(\Lambda\) is a turbulent hypergraph. The easiest way to see this is to force with a product of two forcings, each adding a piecewise independent \(\Gamma\)-edge, and flipping the digits of one of the \(\Gamma\)-edges. This yields a piecewise independent \(\Lambda\)-edge. In a trim balanced extension of the Solovay model, the turbulent hypergraph \(\Lambda\) cannot have a countable chromatic number by Theorem 10.3.4, completing the proof of the corollary. \(\square\)

**Example 10.3.10.** Let \(F\) be an uncountable Polish field. The hypergraph \(\Gamma\) on \(F\) consisting of quadruples \((x_i: i \in 4)\) of nonzero elements of \(F\) such that \(x_0x_2 + x_1x_3 = 1\) is turbulent.

**Proof.** Let \(X = \Gamma\), viewed as a closed subspace of \(F^4\). Consider the poset \(P_X\) of all nonempty open subsets of \(X\), ordered by inclusion. The poset adds a quadruple \((\hat{x}_i: i \in 4)\) which is forced to satisfy the equation \(\hat{x}_0\hat{x}_2 + \hat{x}_1\hat{x}_3 = 1\). The following two claims are central:

**Claim 10.3.11.** Whenever \(j \in 4\), the points \(\hat{x}_i\) for \(i \neq j\) are mutually Cohen generic elements of \(F\).

**Proof.** The projection map from \(X\) to the three coordinates different from \(j\) is a continuous open map, since \(x_j\) can be expressed as a continuous function of the remaining three coordinates. The claim now follows from Proposition 3.1.1. \(\square\)
Claim 10.3.12. \( P_X \) forces \( V[\check{x}_0, \check{x}_1] \cap V[\check{x}_2, \check{x}_3] = V \).

Proof. To simplify the notation, we view \( F^4 \) as \( F^2 \times F^2 \), denote the pairs in this product as \( u, v \), and view the equation from the definition of the set \( X \) as \( u \cdot v = 1 \). Let \( f_0(u, v) = u \) and \( f_1(u, v) = v \). It is easy to verify that \( f_0, f_1 \) are continuous open maps from \( X \) to \( F^2 \). In view of Theorem 3.1.5, it is enough to show that the maps \( f_0, f_1 \) are independent. To this end, let \( O \subset F^4 \) be an open set with nonempty intersection with \( X \), containing a point \( \langle u, v \rangle \).

Perturb the coordinates of \( u \) slightly to find a point \( w \in F_2 \) such that \( \langle w, v \rangle \in O \), \( w \cdot v = 1 \), and the determinant \( |u, w| \) is nonzero. The solutions to the system \( u \cdot z = 1, w \cdot z = 1 \) of linear equations with unknown \( z \in F^2 \) depend continuously on \( u, w \). As a result, one can find open sets \( P, Q, R \subset F_2 \) such that \( u \in P, w \in Q, v \in R, P \times R \subset O \), and for all \( u' \in P \) and \( v' \in Q \) there is \( v' \in R \) such that \( u' \cdot v' = 1 \) and \( u' \cdot v' = 1 \). We will argue that the set \( P \subset F^2 \) witnesses the independence of the maps \( f_0, f_1 \) in the sense of Definition 3.1.4.

For this, use the choice of the sets \( P, Q, R \) to see that for any two points \( u_0, u_1 \in P \) there are points \( w, v_0, v_1 \in Q \) and \( v_0, v_1 \in R \) such that \( u_0 \cdot v_0 = 1, w \cdot v_0 = 0, w \cdot v_1 = 1 \) and \( u_1 \cdot v_1 = 1 \). As a result, the sequence of points \( \langle u_0, v_0 \rangle, \langle w, v_0 \rangle, \langle w, v_1 \rangle, \langle u_1, v_1 \rangle \) is an \( O \)-walk from \( u_0 \) to \( u_1 \), confirming that the set \( P \) has the properties of Definition 3.1.2. The claim follows.

The statement of the example now easily follows from the two claims.

Corollary 10.3.13. In trim balanced extensions of the Solovay model, no uncountable Polish field has a transcendence basis over a countable subfield.

Proof. Let \( F \) be the uncountable Polish field and \( F_0 \) its countable subfield. Suppose that \( B \subset F \) is a transcendence basis over \( F_0 \). We will show that there is a homomorphism of \( \Gamma \) to \( \Delta \), where \( \Gamma \) is the hypergraph of Example 10.3.10 and \( \Delta \) is the hypergraph of Definition 10.3.3. This cannot occur in trim balanced extensions of the Solovay model by Theorem 10.3.4, proving the corollary.

To construct the partial homomorphisms witnessing \( \Gamma \leq \Delta \), let \( \leq \) be a Borel linear ordering of the field \( F \). Every element \( x \in F \) is algebraic over a finite subset \( b_x \subset B \), and there is an inclusion smallest set \( b_x \) like that, of size \( n_x \). There is an algebraic definition \( \phi_x \) from parameters in \( b \) which defines a finite set of which \( x \) is a member, and a number \( m_x \in \omega \) indicating the position of \( x \) in that set in the ordering \( \leq \). Let \( h(x) = \langle \langle n_x, \phi_x, m_x \rangle, b_x \rangle \). We will show that \( h \) is a homomorphism of \( \Gamma \) to \( \Delta \), where \( \Delta \) is defined on a product of a countable set and the set of finite subsets of \( F \), clearly naturally isomorphic to the original definition.

Let \( \{x_0, x_1, x_2, x_3\} \in [X]^4 \) be a \( \Gamma \)-edge. Suppose that the values of \( n_{x_i}, \phi_{x_i}, m_{x_i} \) do not depend on \( i \). Then \( b_{x_i}, i \in 4 \) must be pairwise distinct sets of the same size. By the definition of the graph \( \Gamma \), \( x_3 \) is definable from \( x_0, x_1, x_2 \) and so \( b_{x_3} \subseteq \bigcup_{i \in 3} b_{x_i} \). This cannot occur in a \( \Delta \)-system of finite sets, and so \( h \) is a homomorphism from \( \Gamma \) to \( \Delta \) as required.
Example 10.3.14. Let $Y$ be a Polish space and $\Lambda$ be a Polish group continuously acting on $Y$ in a generically turbulent way, inducing an orbit equivalence relation $E$. Let $X = Y \times Y$ and let $\Gamma$ be the graph on $X$ connecting pairs $\langle y_0, y_1 \rangle$ and $\langle y_2, y_3 \rangle$ if they consist of $E$-unrelated points and $y_0 E y_3$ and $y_1 E y_2$. The hypergraph $\Gamma$ is turbulent.

Proof. Let $Z$ be the closed subset of $\Lambda \times Y \times Y$ consisting of all triples $\langle \lambda, y_0, y_1 \rangle$ such that $\lambda \cdot y_0 = y_1$. The projections into the second and third coordinates are continuous open independent maps by Theorem 3.2.2. The product of these maps is independent as well by Theorem 3.1.10. By Theorem 3.1.5 applied to the product, for nonempty open set $O, P \subset X$ there are pairs $\langle y_0, y_1 \rangle, \langle y_0', y_1' \rangle \in P$ such that $y_0 E y_0'$ and $y_1 y_1'$ and $V[y_0, y_1] \cap V[y_0', y_1'] = V$. Setting $P =$ the flip of $O$, the edge $\langle \langle y_0, y_1 \rangle, \langle y_1', y_0' \rangle \rangle \in \Gamma$ witnesses the turbulence of the graph $\Gamma$. □

Corollary 10.3.15. Let $E$ be an orbit equivalence relation on a Polish space $Y$ induced by a generically turbulent Polish group action. In trim balanced extensions of the symmetric Solovay model, $Y/E$ is not a tournament cardinal.

Proof. Any tournament on the quotient $Y/E$ would yield a coloring of the graph $\Gamma$ with two colors. However, Example 10.3.14 together with Theorem 10.3.4 show that the chromatic number of $\Gamma$ is uncountable in all trim balanced extensions of the symmetric Solovay model.

Example 10.3.16. Let $X, Y$ be uncountable Polish spaces. Let $\Gamma$ be the hypergraph on $X \times Y$ consisting of products $a \times b$ where $a \subset X$ and $b \subset Y$ are sets of size 2. The hypergraph $\Gamma$ is not turbulent, and there is a trim balanced extension of the symmetric Solovay model in which the chromatic number of $\Gamma$ is countable.

As a side remark, in ZFC the hypergraph $\Gamma$ has countable chromatic number if and only if the Continuum Hypothesis holds. If CH holds, then the product $X \times Y$ can be covered by a countable union of graphs of functions from $X$ to $Y$ and inverses of functions from $Y$ to $X$ by an old result of Sierpinski, and neither of these graphs contains a $\Gamma$-edge. For the converse, one can invoke the standard partition result with $\kappa = \aleph_0$: every partition of $\kappa^{\kappa} \times \kappa^{\kappa}$ to $\kappa$ many classes contains arbitrarily large finite monochromatic rectangles, for every infinite cardinal $\kappa$. This implies that $\Gamma$ has uncountable chromatic number when $\kappa \geq \aleph_2$.

Proof. Choose any Borel dense ideal $I$ on $\omega$. Let $P$ be the poset of all functions $p$ whose domain is a countable subrectangle of $X \times Y$, whose range is a set in $I$, and such that there is no $p$-monochromatic $\Gamma$-edge which is a subset of $\text{dom}(p)$. The ordering is that of inclusion. It is easy to observe that $P$ is a Suslin forcing and that $P$ forces the union of the generic filter to be a $\Gamma$-coloring on the ground model version of $X \times Y$.

To prove that $P$ is cofinally trim balanced in every inaccessible cardinal $\kappa$, it is enough to show that under the Continuum Hypothesis, there is a virtual trim balanced condition $\bar{p}$ below any condition $p \in P$. To simplify the notation
somewhat, we will ignore \( p \). To produce the trim balanced condition \( \bar{p} \), use the continuum hypothesis assumption and a transfinite recursion argument to find a countable partition \( X \times Y = \bigcup_n A_n \cup \bigcup_m B_m \) such that each \( A_n \) is a graph of a partial function from \( X \) to \( Y \) and each \( B_m \) is a graph of a partial function from \( Y \) to \( X \). This makes it possible to find a map \( f : X \times Y \to \omega \) such that the range of \( f \) is an infinite set in \( I \) and the sets \( A_n, B_m \) are precisely the fibers of the function \( f \). Let \( \bar{p} \) be the \( \text{Coll}(\omega, X \times Y) \)-name for the analytic set of all conditions in \( P \) which contain \( f \) as a subset. We claim that \( \bar{p} \) is a trim balanced virtual condition.

It is immediate that \( \bar{p} \) is a virtual condition in \( P \). To see the trim balance of \( \bar{p} \), suppose that \( V[H_0], V[H_1] \) are generic extensions of the ground model such that \( V[H_0] \cap V[H_1] = V \) and let \( p_0 \in V[H_0], p_1 \in V[H_1] \) are conditions; we must produce a lower bound of \( p_0, p_1 \in P \). The assumption on the generic extensions implies that \( p_0 \cup p_1 \) is a function. By the definition of the poset \( P \), \( \text{rng}(p_0) \cup \text{rng}(p_1) \subset \omega \) is an \( I \)-small set. Let \( c \in I \) be an infinite set in \( I \) disjoint from both \( \text{rng}(p_0), \text{rng}(p_1) \). write \( \text{dom}(p_0) = a_0 \times b_0 \) and \( \text{dom}(p_1) = a_1 \times b_1 \). The lower bound \( q \) of \( p_0, p_1 \) has domain \( (a_0 \cup a_1) \times (b_0 \cup b_1) \), extends \( p_0, p_1 \), and on the set \( ((a_0 \cup a_1) \times (b_0 \cup b_1)) \setminus (\text{dom}(p_0) \cup \text{dom}(p_1)) \), it is an injection into the set \( c \). It is not difficult to verify that a function \( q \) with this description must be a lower bound of \( p_0, p_1 \) in \( P \).

Note that the poset in the previous example is not \( \sigma \)-closed. In the resulting extension of the symmetric Solovay model, the Axiom of Dependent Choices fails. This motivates the following question:

**Question 10.3.17.** (ZF+DC) Let \( \Gamma \) be the hypergraph of Example 10.3.16. Does \( \chi(\Gamma) = \aleph_0 \) imply the existence of an uncountable sequence of distinct reals?
Chapter 11

Other combinatorics in \( W[G] \)

The technology of balanced conditions can be applied to prove general theorems about lack of other combinatorial objects in the extensions under discourse. In this section, we include several theorems that are hopefully elegant axiomatizations and generalizations of earlier results in [25].

11.1 Maximal almost disjoint families

One rather surprising limitation of balanced extensions of the symmetric Solovay model is that they contain no maximal almost disjoint families of subsets of \( \omega \).

**Theorem 11.1.1.** In a balanced extension of the symmetric Solovay model there are no infinite maximal almost disjoint families of subsets of \( \omega \).

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that in \( V_\kappa \), every ordinal can be collapsed by a poset making \( P \) balanced. Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in \( W \). Suppose towards contradiction that there is a condition \( p \in P \) and a name \( \tau \) such that \( p \Vdash \tau \) is an infinite maximal almost disjoint family. The condition \( p \) as well as the name \( \tau \) must be definable from some parameter \( z \in 2^\omega \) and some parameters from the ground model. Use Fact 1.3.12 and the assumptions to find an intermediate generic extension \( V[K] \) such that \( z \in V[K] \) and \( P \) is balanced in \( V[K] \). Work in the model \( V[K] \).

Let \( q \leq p \) be a balanced condition. Let \( I \) be the set \( \{ a \subset \omega : \) for some poset \( R_a \) of size \( < \kappa \) and some \( R_a \)-name \( \sigma_a \) for a condition in \( P \) stronger than \( q \) such that \( R_a \Vdash \Coll(\omega, < \kappa) \Vdash \sigma \Vdash p, \dot{a} \in \tau \} \).

**Claim 11.1.2.** \( \omega \) cannot be covered by a finite set and finitely many elements of the set \( I \).
Proof. Suppose that \( J \subset I \) is a finite set such that \( \bigcup J \subset \omega \) is cofinite. Let 
\( H_a \subset R_a \) for \( a \in J \) be filters mutually generic over \( V[K] \) and let \( r_a = \sigma_a/H_a \in P \). By the balance of the condition \( q \), it follows that the set \( \{r_a: a \in J\} \subset P \) has a lower bound, denote it by \( \bar{p} \). Since the model \( W \) is a symmetric Solovay extension of each of the models \( V[K][H_a] \), the forcing theorem applied in the model \( V[K][H_a] \) shows that in \( W \), for each \( a \in J \), the condition \( \bar{p} \) forces \( J \subset \tau \). However, since \( \bigcup J \subset \omega \) is cofinite, it has to be the case that \( \bar{p} \forces \check{J} = \tau \), contradicting the initial assumptions on the name \( \tau \).

Let \( U \) be a nonprincipal ultrafilter on \( \omega \) disjoint from \( I \). There must be a poset \( R \) of size \( \kappa \), an \( R \)-name \( \eta \) for an infinite subset of \( \omega \) which is modulo finite included in all sets in \( U \), an \( R \)-name \( \chi \) for a subset of \( \omega \) and an \( R \)-name \( \sigma \) for an element of \( P \) stronger than \( q \) such that \( R \forces \chi \cap \eta \) is infinite and \( R \forces \text{Coll}(\omega, \kappa) \forces \sigma \forces \chi \in \tau \). This occurs since \( \tau \) is forced to be a maximal almost disjoint family and therefore must contain an element with infinite intersection with \( \eta \).

Now, let \( H_0, H_1 \subset R \) be filters mutually generic over the model \( V[K] \). Let 
\( a_0 = \chi/H_0, a_1 = \chi/H_1 \in \mathcal{P}(\omega) \), and \( r_0 = \sigma/H_0, r_1 = \sigma/H_1 \in P \). By the balance of the condition \( q \leq p \) it must be the case that \( r_0, r_1 \in P \) have a lower bound, denote it by \( \bar{p} \). Since \( W \) is the symmetric Solovay extension of each of the models \( V[K][H_0] \) and \( V[K][H_1] \), the forcing theorem applied in these models shows that \( W \) satisfies that \( \bar{p} \forces both \ a_0 \ and \ a_1 \ into \ \tau \). The proof will be complete if we show that \( a_0 \neq a_1 \) and \( a_0, a_1 \) have infinite intersection.

For \( a_0 \neq a_1 \), observe that neither \( a_0, a_1 \) can belong to the model \( V[K] \). If, say, \( a_0 \in V[K] \) then \( R \) witnesses the fact that \( a_0 \in I \), and consequently \( a_0 \) must have both finite and infinite intersection with \( \eta/H_0 \), a contradiction. Now, since \( a_0 \in V[K][H_0] \) and \( a_1 \in V[K][H_1] \), a mutual genericity argument shows that \( a_0 \neq a_1 \).

To establish that the set \( a_0 \cap a_1 \) is infinite, move back to the model \( V[K] \), let 
\( s_0, s_1 \) be a condition in the product \( R \times R \) and \( n \in \omega \) be a number; we must find a number \( m > n \) and conditions \( t_0 \leq s_0 \) and \( t_1 \leq s_1 \) such that \( t_0 \forces \check{m} \in \chi \) and \( t_1 \forces \check{m} \in \chi \). To this end, let \( b_0 = \{m \in \omega: \exists t \leq s_0 \ t \forces \check{m} \in \chi\} \) and \( b_1 = \{m \in \omega: \exists t \leq s_1 \ t \forces \check{m} \in \chi\} \). The sets \( b_0, b_1 \subset \omega \) are both forced to have infinite intersection with \( \eta \) and therefore must belong to the ultrafilter \( U \). This means that there is a natural number \( m > n \) in the intersection \( b_0 \cap b_1 \), and then the desired conditions \( t_0 \leq s_0 \) and \( t_1 \leq s_1 \) are found by the definition of the sets \( b_0 \) and \( b_1 \).

\[ \square \]

### 11.2 Linearly ordered sets

Another general obstruction for reaching ZF independence results in balanced extensions is encapsulated in the following theorem.

**Theorem 11.2.1.** Let \( \preceq \) be an \( F_\sigma \)-preorder on a Polish space \( X \) with no maximal element such that every countable linearly ordered set has an upper bound.
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In balanced \(\sigma\)-closed extensions of the symmetric Solovay model, \(\leq\) contains no unbounded linearly ordered sets.

A typical preorder satisfying the assumptions is the Turing reducibility preorder or the modulo finite domination ordering on \(\omega^\omega\).

**Proof.** We start with a simple claim which takes place in the context of ZFC. If \(R\) is a poset and \(\eta\) is an \(R\)-name for an element of \(X\), say that \(\eta\) is unbounded if \(R \forces \text{for every ground model element } x \in X, \eta \leq x\) fails.

**Claim 11.2.2.** Suppose that \(R_0, R_1\) are posets and \(\eta_0, \eta_1\) are respective unbounded names for elements of \(X\). Then \(R_0 \times R_1 \forces \eta_0, \eta_1\) are unbounded.

**Proof.** Let \(\leq = \bigcup_n F_n\) where for every \(n \in \omega\), \(F_n\) is closed. If the conclusion failed, there would have to be a pair \(\langle r_0, r_1 \rangle \in R_0 \times R_1\) forcing say \(\eta_0 \leq \eta_1\) and then strengthening the pair if necessary there would have to be a number \(n \in \omega\) such that \(\langle \eta_0, \eta_1 \rangle \in F_n\) is forced. Let \(M\) be a countable elementary submodel of a large structure containing \(R_1, r_1, \eta_1\), let \(g \subseteq R_1\) be a filter generic over the model \(M\) such that \(r_1 \in g\), and let \(x = \eta_1 / g\). Since \(\eta_0\) is an unbounded name, \(R_0 \forces \eta_0 \leq \bar{x}\) fails, and so there must be a condition \(r_0 ' \leq r_0\) and basic open sets \(O_0, O_1 \subseteq X\) such that \(\langle O_0 \times O_1 \rangle \leq \langle x \in O_1, r_1 \rangle\), and \(r_0 ' \forces \tau_0 \in O_0\).

By the forcing theorem, there must be a condition \(r_1 ' \in g\) below \(r_1\) such that \(r_1 ' \forces \tau_1 \in O_1\). Then the pair \(\langle r_0 ', r_1 ' \rangle \leq \langle r_0, r_1 \rangle\) forces \(\langle \eta_0, \eta_1 \rangle \in O_0 \times O_1\) and so \(\langle \eta_0, \eta_1 \rangle \notin F_n\), in contradiction with the initial assumptions. \(\square\)

Now, let \(\kappa\) be an inaccessible cardinal and \(P\) be a \(\sigma\)-closed Suslin forcing such that in \(V_\kappa\), every ordinal can be collapsed by a poset making \(P\) balanced. Let \(W\) be the symmetric Solovay model derived from \(\kappa\). Suppose towards contradiction that the conclusion of the theorem fails in the \(P\)-extension of \(W\). Then, in the model \(W\) there must be a condition \(p \in P\) and a \(P\)-name \(\tau\) such that \(p \forces \tau \subseteq X\) is an \(\leq\)-unbounded linearly ordered set. The condition \(p\) as well as the name \(\tau\) must be definable from some parameter \(z \in 2^\omega\) and some parameters in the ground model. Let \(V[K]\) be a generic extension of the ground model by a poset of size \(\leq \kappa\) such that \(p, z \in V[K]\) and \(P\) is balanced in \(V[K]\).

Work in the model \(V[K]\). Let \(\bar{p} \leq p\) be a balanced condition in the poset \(P\). Since \(\text{Coll}(\omega, \kappa) \forces p \forces \tau \subseteq X\) is an unbounded set in \(\leq\), for every element \(x \in X \cap V[K]\) the set \(\tau\) is forced to contain an element which is not \(\leq x\). Since \(\text{Coll}(\omega, \kappa) \forces p \forces X \cap V[K]\) is a countable set and DC holds, \(\tau\) is forced to contain a countable subset such that no element of \(X \cap V[K]\) is an upper bound of it. By the initial assumptions on the preorder \(\leq\), this countable set is forced to have an upper bound in \(\tau\), which is then an element of \(\tau\) which is not \(\leq x\) for any element \(x \in X \cap V[K]\). In total, in \(V[K]\) there must be a poset \(R\) of size \(\leq \kappa\), an \(R\)-name \(\sigma\) for a condition in \(P\) such that \(\sigma \leq \bar{p}\) and an unbounded \(R\)-name \(\eta\) for an element of \(X\) such that \(\bar{p} \forces \eta\) is not \(\leq\)-below any element of \(V[K]\) and \(R \forces \text{Coll}(\omega, \kappa) \forces \sigma \forces p \eta \in \tau\).

In the model \(W\), let \(H_0, H_1 \subseteq R\) be filters mutually generic over \(V[K]\). Let \(p_0 = \sigma / H_0 \in P\) and \(p_1 = \sigma / H_1 \in P\); by the balance of the condition \(\bar{p}\), the conditions \(p_0, p_1\) are compatible with some lower bound \(q \in P\). Let
$x_0 = \eta/H_0 \in X$ and $x_1 = \eta/H_1 \in X$; by Claim 11.2.2, these are $\leq$-incomparable elements of $X$. Since the model $W$ is a symmetric extension of both models $V[K][H_0]$ and $V[K][H_1]$, the forcing theorem applied in these models shows that in $W$, $q \Vdash \check{x}_0, \check{x}_1 \in \tau$. This contradicts the assumption that $\tau$ is forced to be linearly ordered by $\leq$. \hfill \Box

11.3 The arity obstruction

An important source of independence results for balanced extensions is the centeredness quality of certain forcing notions. The following definition captures one common feature of this kind.

**Definition 11.3.1.** Let $n \in m \in \omega$ be natural numbers. A poset $P$ is $m,n$-centered if whenever $a \subset P$ is a set of size $m$ such that any subset of $a$ of size $n$ has a common lower bound, then $a$ has a common lower bound.

Note that $m,n$-centeredness of Suslin posets is a $\Pi^1_2$ statement and therefore absolute among all forcing extensions. There is a natural technical weakening, which in many cases has similar consequences.

**Definition 11.3.2.** Let $P$ be a Suslin poset.

1. A virtual condition $\bar{p}$ in $P$ is $m,n$-balanced if in every generic extension, whenever $\{ H_i : i \in m \}$ are filters generic over $V$ such that for every set $a \subset m$ of size $n$ the filters $\{ H_i : i \in a \}$ are mutually generic over $V$, if $p_i \leq \bar{p}$ are conditions in $P$ in $V[H_i]$, then the set $\{ p_i : i \in m \}$ has a common lower bound in $P$.

2. The poset $P$ is $m,n$-balanced if for every condition $p \in P$ there is a $m,n$-balanced virtual condition $\bar{p} \leq p$.

**Example 11.3.3.** Let $E,F$ be Borel equivalence relations on Polish spaces. Let $P$ be the poset of countable injections from the $E$-quotient space to the $F$-quotient space, ordered by reverse inclusion. Then $P$ is 3,2-centered.

**Example 11.3.4.** Let $E$ be a Borel equivalence relation on a Polish space $X$. Let $P$ be the linearization poset for the quotient space $X/E$. The poset $P$ is 3,2-balanced but not 3,2-centered.

**Proof.** If $x_0,x_1,x_2 \in X$ are pairwise $E$-unrelated elements and $p_0 = \{ \langle x_0,x_1 \rangle \}$, $p_1 = \{ \langle x_1,x_2 \rangle \}$ and $p_2 = \{ \langle x_2,x_0 \rangle \}$, then $p_0,p_1,p_2 \in P$ are pairwise compatible conditions which do not have a lower bound, exemplifying the failure of the 3,2-centeredness.

To verify the 3,2-balance of the poset $P$, let $p \in P$ be an arbitrary condition, and let $\bar{p}$ be any linear ordering of the virtual $E$-quotient space extending $p$. It follows from Example 6.4.16 that $\bar{p}$ is a balanced virtual condition for $P$. We will check that $\bar{p}$ is 3,2-balanced, completing the proof. To this end, let $\{ V[H_i] : i \in 3 \}$ be pairwise mutually generic extensions of $V$, respectively containing some.
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conditions \( p_i \leq \bar{p} \) for each \( i \in 3 \). The pairwise mutual genericity implies that if \( i \neq j \) are distinct numbers smaller than 3, the intersection of the domains of \( p_i \) and \( p_j \) is the set of realizations of the virtual \( E \)-classes of \( V \), and on this set \( p_i, p_j \) agree and are equal to the linear ordering \( \bar{p} \). This means that the union \( \bigcup_i p_i \) can be extended to a linear ordering, which is a common lower bound of the conditions \( p_i \) for \( i \in 3 \).

Example 11.3.5. Let \( \Gamma \) be a Borel graph on a Polish space \( X \). Let \( P \) be the poset of all countable acyclic subsets of \( \Gamma \) ordered by reverse inclusion. The poset \( P \) is 3,2-balanced but not (as soon as \( \Gamma \) contains a cycle) 3,2-centered.

Proof. If \( \{ e_i : i \in n \} \subseteq \Gamma \) is a set of edges forming a cycle, then \( p_0 = \{ e_i : i \in n - 2 \} \), \( p_1 = \{ e_{n-2} \} \) and \( p_2 = \{ e_{n-1} \} \) is a collection of three conditions in the poset \( P \) which are pairwise compatible but have no common lower bound in the poset \( P \). Thus, 3,2-centeredness of \( P \) fails.

To verify the 3,2-balance of the poset \( P \), let \( p \in P \) be an arbitrary condition, and let \( \bar{p} \) be any maximal acyclic subgraph of \( \Gamma \). It follows from Theorem 6.2.21 that \( \bar{p} \) is a balanced virtual condition for \( P \). We will check that \( \bar{p} \) is 3,2-balanced, completing the proof. To this end, let \( \{ V[H_i] : i \in 3 \} \) be pairwise mutually generic extensions of \( V \), respectively containing some conditions \( p_i \leq \bar{p} \) for each \( i \in 3 \).

Claim 11.3.6. For each \( i \in 3 \) and vertices \( x_0, x_1 \in X \cap V \), the following three formulas are equivalent:

1. \( x_0, x_1 \) are connected by a path in \( \Gamma \cap V \);
2. \( x_0, x_1 \) are connected by a path in \( \bar{p} \);
3. \( x_0, x_1 \) are connected by a path in \( p_i \).

Proof. (1)→(2) is implied by the maximality of \( \bar{p} \). (2)→(3) follows from the fact that \( \bar{p} \subseteq p_i \), and the negation of (1) implies by the Mostowski absoluteness that \( x_0, x_1 \) are not connected by any path in the graph \( \Gamma \cap V[H_i] \), which is larger than \( p_i \) and therefore (3) has to fail.

The pairwise mutual genericity implies that if \( i \neq j \) are distinct numbers smaller than 3 and \( v_i, v_j \subseteq X \) are the sets of vertices mentioned in some edges in \( p_i, p_j \) respectively, then \( v_i \cap v_j \subseteq V \). This, together with the claim, means that the union \( \bigcup_i p_i \) cannot contain a cycle and so is the desired common lower bound of the conditions \( p_i \) for \( i \in 3 \).

The first theorem of this section shows that 3,2-balanced extensions of the symmetric Solovay model do not contain any discontinuous homomorphisms between Polish groups. This also means that they contain no Hamel basis for Polish vector spaces over countable fields (any such a basis yields a discontinuous homomorphism from the vector space to the reals) and no nonprincipal ultrafilter on \( \omega \) (any such ultrafilter yields a discontinuous homomorphism from \( \Gamma^\omega \) to \( \Gamma \) for any finite group \( \Gamma \) by computing an average of any element \( x \in \Gamma^\omega \)).
Theorem 11.3.7. In 3, 2-balanced extensions of the symmetric Solovay model, all homomorphisms between Polish groups are continuous.

Proof. Suppose that \( \kappa \) is an inaccessible cardinal. Suppose that \( P \) is a Suslin forcing such that in \( V_\kappa \), every cardinal can be collapsed to \( \aleph_0 \) by a forcing which makes \( P \) 3, 2-balanced. Let \( W \) be a symmetric Solovay model derived from \( \kappa \) and work in \( W \). Suppose towards contradiction that the conclusion of the theorem fails: thus, there are Polish groups \( \Gamma, \Delta \), a condition \( p \in P \) and a \( P \)-name \( \tau \) such that \( p \Vdash \tau : \Gamma \to \Delta \) is a discontinuous homomorphism. Both \( p \) and \( \tau \) must be definable in \( W \) from parameters in the ground model and some point \( z \in 2^\omega \). Let \( V[K] \) be some intermediate extension of \( V \) by a poset of size \( < \kappa \) such that \( z \in V[K] \) and \( P \) is 3, 2-balanced in \( V[K] \). Work in the model \( V[K] \).

Let \( \bar{p} \leq p \) be a 3, 2-balanced condition. Consider the poset \( P_\Gamma \) adding a generic element \( \bar{\gamma}_{\text{gen}} \) of the group \( \Gamma \). The following is a key claim:

Claim 11.3.8. \( P_\Gamma \) forces that there are disjoint basic open sets \( O_0, O_1 \subset \Delta \) such that \( \text{Coll}(\omega, < \kappa) \Vdash \exists p_0 \leq \bar{p} \exists p_0 \Vdash \tau(\bar{\gamma}_{\text{gen}}) \in O_0 \) and \( \exists p_1 \leq \bar{p} p_1 \Vdash \tau(\bar{\gamma}_{\text{gen}}) \in O_1 \).

Proof. Suppose towards contradiction that \( q \in P_\Gamma \) forces the opposite. Then, \( q \) forces that in the \( \text{Coll}(\omega, < \kappa) \)-extension there is a unique point \( \delta \in \Delta \) such that there is a condition \( p' \leq \bar{p} \) in the poset \( P \) forcing \( \tau(\bar{\gamma}_{\text{gen}}) = \delta \). Since the point \( \delta \) is in the symmetric Solovay model definable from \( \tau, \bar{p} \), and \( \bar{\gamma}_{\text{gen}} \), it must belong to the \( P_{\text{generic}} \)-extension, and there is a \( P_\Gamma \)-name \( \delta \) for it. Note that then, \( q \Vdash \text{Coll}(\omega, < \kappa) \Vdash \bar{p} \Vdash_P \delta = \tau(\bar{\gamma}_{\text{gen}}) \).

Now, pass to the symmetric Solovay model \( W \) and consider the set \( B = \{ \gamma \in \Gamma : \gamma \text{ is } P_\Gamma \text{-generic over } V[K] \text{ and } \gamma \in q \} \) and the function \( f : B \to \Delta \) assigning to each point \( \gamma \in B \) the point \( \delta/\gamma \). The function \( f \) is continuous on \( B \), and the choice of the name \( \delta \) implies that \( \bar{p} \Vdash f \in \tau \) holds. The set \( B \subset \Gamma \) is co-meager in \( q \). A standard result in the theory of Polish groups [22, Theorems 9.9 and 9.10] now says that a homomorphism from \( \Gamma \) to \( \Delta \) which is continuous on a set comeager in a nonempty open set is in fact continuous on the whole group \( \Gamma \). This contradicts the initial choice of the name \( \tau \).

Let \( X = \{ x \in \Gamma^3 : x(0) \cdot x(1) = x(2) \} \); this is a closed subset of \( \Gamma^3 \), equipped with the topology inherited from \( \Gamma^3 \). If \( i, j \in 3 \) are two distinct coordinates, it is clear that the remaining coordinate of a point \( x \in X \) is a continuous function of the coordinates \( x(i) \) and \( x(j) \), and therefore the projection from \( X \) to \( \Gamma^2 \) given by the \( i \)- and \( j \)-th coordinate is a continuous and open surjection. Let \( x \in X \) be a point \( P_X \) generic over \( V[K] \). Use Proposition 3.1.1 to see that whenever \( i, j \in 3 \) are distinct numbers then \( x(i), x(j) \in \Gamma \) are points \( P_\Gamma \)-mutually generic over \( V[K] \).

Working in the respective models \( V[K][x(0)] \) and \( V[K][x(1)] \), find posets \( R_0, R_1 \) of size \( < \kappa \) and respective names \( \sigma_0, \sigma_1 \) for conditions in the poset \( P \) stronger than \( \bar{p} \) and names \( \delta_0, \delta_1 \) for elements of the group \( \Delta \) such that \( V[K][x(0)] \Vdash R_0 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash_P \tau(x(0)) = \delta_0 \) and similarly for \( V[K][x(1)] \). In the model \( V[K][x(2)] \), find the disjoint basic open sets \( O_0, O_1 \subset
∆ as in Claim 11.3.8. Passing to a condition in the product \( R_0 \times R_1 \) and switching \( O_0, O_1 \) if necessary, we may assume that \( V[K][x(0), x(1)] \models R_0 \times R_1 \models \delta_0 \cdot \delta_1 \notin O_0 \). Use the choice of the set \( O_0 \) to find, in the model \( V[K][x(2)] \), a poset \( R_2 \) of size \( < \kappa \), \( \sigma_2 \) for an element of \( P \) stronger than \( \bar{p} \) and \( \delta_2 \) for an element of \( O_0 \subset \Delta \) such that \( R_2 \models \text{Coll} (\omega, < \kappa) \models \sigma_2 \models \text{P} (\check{\tau}(\check{x}(2)) = \check{\delta}_2) \).

Let \( H_0 \subset R_0, H_1 \subset R_1, H_2 \subset R_2 \) be filters mutually generic over the model \( V[K][x] \). For each \( i \in \{0, 1, 2\} \) write \( p_i = \sigma_i / H_i \in P \) and \( \delta_i = \delta_i / H_i \). Note that whenever \( i, j \in \{0, 1, 2\} \) are distinct numbers, the models \( V[K][x_i][H_i] \) and \( V[K][x_j][H_j] \) are mutually generic extensions of the model \( V[K] \). By the 3,2-balanced assumption on the balanced virtual condition \( \bar{p} \), the conditions \( p_0, p_1, p_2 \in P \) have a lower bound, call it \( q \in P \). Let \( W \) be a symmetric Solovay extension of the model \( V[K][x][H_0, H_1, H_2] \) and work in \( W \). The condition \( q \) forces that \( \tau(x(i)) = \delta_i \) for all \( i \in \{0, 1, 2\} \). Observe that \( x(0) \cdot x(1) = x(2) \) and \( \delta_0 \cdot \delta_1 \neq \delta_2 \) since \( \delta_0 \cdot \delta_1 \notin O_0 \) while \( \delta_2 \in O_0 \). This contradicts the assumption that \( \tau \) is forced to be a homomorphism from \( \Gamma \) to \( \Delta \).

The following two corollaries are just conjunctions of the initial examples of this section and Theorem 11.3.7.

**Corollary 11.3.9.** Let \( \Gamma \) be a Borel graph on a Polish space \( X \).

1. Write \( P \) for the poset adding a maximal acyclic subgraph of \( \Gamma \) as in Example 6.2.12. In the \( P \)-generic extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, \( \Gamma \) has a maximal acyclic subgraph and every homomorphism between Polish groups is continuous.

**Corollary 11.3.10.** Let \( E \) be a Borel equivalence relation on a Polish space \( X \).

1. Write \( P \) for the linearization poset for the quotient space \( X/E \) as in Example 6.4.16. In the \( P \)-generic extension of the symmetric Solovay model every homomorphism between Polish groups is continuous;

2. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the \( E \)-quotient space is linearly ordered and every homomorphism between Polish groups is continuous.

The more sophisticated results of this section investigate the fracture line between the 3,2-balance and 3,2-centered balance.

**Theorem 11.3.11.** Let \( E \) be a Borel unpinned equivalence relation on a Polish space \( X \). In 3,2-centered balanced extensions of the symmetric Solovay model, \( E \) has no spanning acyclic subgraph.

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a 3,2-centered Suslin forcing balanced below \( \kappa \). Let \( W \) be the derived Solovay model and work in \( W \). Suppose that \( p \in P \) is a condition and \( \tau \) is a name such that \( p \models \tau \) is a graph spanning...
E; we need to find a strengthening of \( p \) which forces an injective cycle in \( \tau \). The condition \( p \), the equivalence relation \( E \) as well as the name \( \tau \) must be definable from some parameter \( z \in 2^\omega \) and some parameters from the ground model. Use Fact 1.3.12 and the assumptions to find an intermediate generic extension \( V[K] \) such that \( z,p \in V[K] \) and the poset \( P \) is balanced in \( V[K] \). Work in the model \( V[K] \).

Let \( \bar{p} \leq p \) be a balanced condition in \( P \). By the unpinned assumption, there must be a poset \( R \) of size \( < \kappa \), an \( E \)-pinned \( R \)-name \( \eta \) for an element of \( X \) and a \( Q \)-name \( \sigma \) for a balanced virtual condition in \( P \) such that \( Q \) forces \( \eta \) not to be \( E \)-related to any element in \( V[K] \) and \( \sigma \leq \bar{p} \). Let \( H_i \subset Q \) for \( i \in 3 \) be filters mutually generic over \( V[K] \), and write \( x_i = \eta/H_i \) and \( p_i = p/H_i \) for \( i \in 3 \). Observe that the points \( x_i \) are \( E \)-related as the name \( \eta \) was \( E \)-pinned, and the conditions \( p_i \) are pairwise compatible by the balance of the condition \( \bar{p} \).

Fix numbers \( i \neq j \in 3 \) and work in the model \( V[K][H_i,H_j] \). There must be a poset \( R_{ij} \) of size \( < \kappa \), an \( R_{ij} \)-name \( \sigma_{ij} \) for a condition in \( P \) stronger than both \( p_i,p_j \), and an \( R_{ij} \)-name \( \chi_{ij} \) for a finite injective \( E \)-path between the points \( x_i,x_j \) such that \( R_{ij} \forces \text{Coll}(\omega,<\kappa) \forces \sigma_{ij} \forces p \chi_{ij} \subset \tau \). In the model \( W \), let \( L_{ij} \subset R_{ij} \) for all choices of \( i \neq j \in 3 \) be filters mutually generic over the model \( V[K][H_i : i \in 3] \), and let \( c_{ij} = \chi_{ij}/L_{ij} \) and \( p_{ij} = \sigma_{ij}/L_{ij} \). Observe that for any \( i \in 3 \), by the balance of the condition \( p_i \) in the model \( V[K][H_i] \) the conditions \( p_{ij} \) and \( p_{ik} \) for \( i \neq j \neq k \) are compatible in the poset \( P \). By the assumption on the poset \( P \), the conditions \( p_{ij} \) for all choices of \( i \neq j \in 3 \) have a common lower bound \( q \in P \). \( W \) is a symmetric Solovay extension of each of the models \( V[K][H_i,H_j][L_{ij}] \), and by the forcing theorem applied in each of these models, it must be the case that in the model \( W \), \( q \forces \bigcup_{ij} c_{ij} \subset \tau \).

Now, the cinch. The union \( \bigcup_{ij} c_{ij} \) forms a cycle in the graph \( E \); we must find an injective cycle contained in it. To find it, it is just necessary to verify that no vertex appears in all three paths \( c_{ij} \). Well, if \( y \in X \) was a vertex contained in all three paths, then for every \( i \neq j \neq k \), \( y \in V[K][H_i,H_j][L_{ij}] \) and \( y \in V[K][H_i,H_k][L_{ik}] \). By a mutual genericity argument over the model \( V[K][H_i] \), \( y \in V[K][H_i] \) for every \( i \in 3 \). By a final mutual genericity argument over the model \( V[K] \), \( y \in V[K] \). But then, \( x_i \models E y \) for every \( i \in 3 \), contradicting the assumption that \( V[K] \models Q \models \dot{x} \) is not \( E \)-related to any elements from \( V[K] \). In conclusion, \( q \models \) there is an injective cycle in \( \tau \), and the proof is complete.

**Theorem 11.3.12.** In 3, 2-centered, adequate, \( E_0 \)-charming forcing extensions of the symmetric Solovay model, \( E_0 \) is not linearly ordered.

In other words, to linearly order the \( E_0 \)-quotient space by a 3, 2-centered balanced Suslin forcing, it is necessary to add an injection from the quotient space to \( 2^\omega \). Note that the standard collapse of Definition 6.4.2 is indeed 3, 2-centered.

**Proof.** Let \( \kappa \) be an inaccessible cardinal. Let \( P \) be a Suslin forcing such that in \( V_\kappa \), \( P \) is 3, 2-centered, adequate, \( E_0 \)-charming and balanced in every forcing extensions. Let \( W \) be the symmetric Solovay model derived from \( \kappa \). Workin in the model \( W \). Towards contradiction, assume that \( p \in P \) is a condition
and $\tau$ is a $P$-name for a linear ordering of the $\mathbb{E}_0$-quotient space. There is a parameter $z \in 2^{\omega}$ such that all of $p$, $\tau$ is definable from $p$ and parameters in the ground model in the model $W$. Let $V[K]$ be an intermediate model obtained as a generic extension of the ground model by a poset of size $< \kappa$ such that $z \in V[K]$.

Let $\langle V[K][G], \bar{p} \rangle$ be an $\mathbb{E}_0$-charm for $P$ over $V[K]$ and work in $V[K][G]$. There must be an $\mathbb{E}_0$-class $c$ such that no element of $c$ is definable from $\bar{p}$, $c$ and parameters in $V[K]$; otherwise, one could compose a definable $\mathbb{E}_0$-selector and from it a $\mathbb{E}_0$ coloring. Consider the model of all sets hereditarily definable in $V[K][G]$ from $\bar{p}$ and parameters in $V[K]$. By general forcing theorems, this model of ZFC is a generic extension of $V[K]$, and we will denote it by $V[K][G_0]$. It is also the case that $V[K][G]$ is a generic extension of $V[K][G_0]$ by some forcing $Q$ and a filter $G_1 \subset Q$. Let $\chi, \eta \in V[K][G_0]$ be $Q$-names such that $\bar{p} = \chi/G_1$ and $c = \eta/G_1$. We spend some time analyzing the poset $Q$. The following claims all take place in the model $V[K][G_0]$.

**Claim 11.3.13.** There is a condition $q \in G_1$ which forces the following. For every condition $q' \leq q$ there is a filter $L \subset Q$ containing $q'$, generic over $V[K][G_0]$ such that the model $V[K][G_0][L]$ is equal to the whole generic extension and $\chi/L = \chi$ and $\eta/L = \eta$.

**Proof.** This follows abstractly from the definition of the model $V[K][G_0]$. Work in the model $V[K][G]$. Let $D = \{q' \in Q: \text{there is a filter } L \subset Q \text{ containing } q', \text{ generic over } V[K][G_0] \text{ such that } V[K][G_0][L] = V[K][G] \text{ and } \chi/L = \bar{p} \text{ and } \eta/L = c\}$. The set $D \subset Q$ is hereditarily definable from $\bar{p}, c$ and parameters in $V[K][G_0]$. By the definition of the model $V[K][G_0]$, $D \in V[K][G_0]$ holds. Also, $G_1 \subset D$ holds since $G_1$ itself witnesses the membership of any condition $q' \in G_1$ in $D$. By a genericity argument in $V[K][G_0]$, there must be a condition $q \in G_1$ such that for every $q' \leq q$, $q' \in D$ holds. By the forcing theorem in $V[K][G_0]$, strengthening the condition $q$ if necessary, we conclude that $q$ forces the desired statement.

**Claim 11.3.14.** There is a condition $q \in G_1$ such that $q \Vdash \eta$ has no $E$-equivalent in $V[K][G_0]$.

**Proof.** This follows from the forcing theorem and the fact that the $E$-class $c$ has no element in the model $V[K][G_0]$.

**Claim 11.3.15.** There is a virtual balanced condition $\bar{p}'$ in the poset $P$ and a condition $q \in G_1$ such that $q \Vdash \chi \leq \bar{p}'$.

**Proof.** This is the only place where the adequacy assumption on the poset $P$ is used. Let $\lambda < \kappa$ be a cardinal such that $P$ is $\lambda$-adequate in $V[K][G_0]$. By the balance of $\bar{p}$, every $P$-pin is either weaker than $\bar{p}$, or incompatible with $\bar{p}$. Let $A$ be the set of all $P$-pins in $V[K][G_0]$ realized on posets of size $< \lambda$ which are weaker than $\bar{p}$. By the definition of the model $V[K][G_0]$, the set $A$ belongs to the model $V[K][G_0]$. By the forcing theorem in the model $V[K][G_0]$, there is a
condition \( q \in G_1 \) forcing the statement “\( A \) is the set of all \( P \)-pins in \( V[K][G_0] \) realized on posets of size \( < \lambda \) which are weaker than \( \chi \)”.

In the model \( V[K][G_0] \), let \( \bar{R}, \bar{\sigma} \) be \( Q \)-names for a \( P \)-pin representing the balanced virtual condition \( \chi \). The pair \( \langle \bar{Q} \ast \bar{R}, \bar{\sigma} \rangle \) is \( \lambda \)-adequate by the definitions of the set \( A \), and by the assumptions on the poset \( P \) it is balanced. Use Theorem 5.2.6 to find a balanced virtual condition \( \bar{p}' \) equivalent to it. Finally, argue that \( \bar{p} \leq \bar{p}' \leq p \) as desired. \( \square \)

Trimming down the poset \( Q \) if necessary, we may assume that the conditions \( q \in G_1 \) found in the previous claims are all equal to the largest condition in \( Q \).

**Claim 11.3.16.** \( Q \times Q \models \text{Coll}(\omega, < \kappa) \models \) the following:

1. \( \chi_{\text{left}}, \chi_{\text{right}} \) are compatible elements of \( P \);

2. the conjunction of these two conditions does not decide the statement whether \( \langle \eta_{\text{left}}, \eta_{\text{right}} \rangle \in \tau \) or not.

**Proof.** The first item follows immediately from Claim 11.3.15: the conditions \( \chi_{\text{left}} \) and \( \chi_{\text{right}} \) both strengthen the balanced condition \( \bar{p}' \). For the second item, suppose towards contradiction that there are conditions \( q_0, q_1 \in Q \) such that (for definiteness) \( \langle q_0, q_1 \rangle \models \text{Coll}(\omega, < \kappa) \models \) the conjunction of \( \chi_{\text{left}}, \chi_{\text{right}} \) forces in \( P \) that \( \langle \eta_{\text{left}}, \eta_{\text{right}} \rangle \in \tau \). Let \( L_0, L_1 \subset Q \) be filters mutually generic over \( V[K][G_0] \) and containing \( q_0, q_1 \) respectively. By Claim 11.3.13, in the respective models \( V[K][G_0][L_0] \) and \( V[K][G_0][L_1] \), we can find filters \( L_1', L_0' \subset Q \) generic over \( V[K][G_0] \) such that

- \( q_1 \in L_1', V[K][G_0][L_0] = V[K][G_0][L_1'], \eta/L_0 = \eta/L_1' \) and \( \chi/L_0 = \chi/L_1' ; \)

- \( q_0 \in L_0', V[K][G_0][L_1] = V[K][G_0][L_0'], \eta/L_1 = \eta/L_0' \) and \( \chi/L_1 = \chi/L_0' \).

By Corollary 13.3.3, the filters \( L_0', L_1' \subset Q \) are in fact mutually generic over \( V[K][G_0] \). The forcing theorem applied in the (equal) models \( V[K][G_0][L_0, L_1] \) and \( V[K][G_0][L_0, L_1'] \) then shows that in the model \( W \), the conjunction of \( \chi/L_0 \) and \( \chi/L_1' \) forces \( \eta/L_0, \eta/L_1' \in \tau \), and the conjunction of \( \chi/L_0' \) and \( \chi/L_1 \) forces \( \eta/L_0', \eta/L_1 \in \tau \). However, the conjunctions are identical, and the two pairs of equivalence classes are just flips of one another. This contradicts the assumption that \( \tau \) is forced to be a tournament. \( \square \)

Now, let \( \langle L_i : i \in 3 \rangle \) be a collection of filters on \( Q \) mutually generic over \( V[K][G_0] \). For any number \( i \in 3 \), let \( s(i) = i + 1 \) modulo 3. In the model \( V[K][G_0][L_i, L_{s(i)}] \) let \( R_i \) be a poset of size \( < \kappa \) and \( \sigma_i \) be an \( R_i \)-name for an element of the poset \( P \) stronger than both \( \chi/L_i \) and \( \chi/L_{s(i)} \) such that \( R_i \models \text{Coll}(\omega, < \kappa) \models \sigma_i \models \langle \eta/L_i, \eta/L_{s(i)} \rangle \in \tau \). Let \( \langle H_i : i \in 3 \rangle \) be a collection of filters mutually generic over the model \( V[K][G_0][L_i : i \in 3] \), and let \( p_i = \sigma_i/H_i \).

**Claim 11.3.17.** The conditions \( p_i \in P \) for \( i \in 3 \) have a common lower bound.
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Proof. By the initial assumption on the poset \( P \), we need to verify only that each pair of conditions among the \( p_i \)’s has a lower bound. For definiteness, consider the case \( p_0, p_1 \); the other cases are the same. Note that the models \( V[K][G_0][L_0][L_1][H_0] \) and \( V[K][G_0][L_1][L_2][H_1] \) are mutually generic extensions of \( V[K][G_0][L_1] \), and the conditions \( p_0 \) and \( p_1 \) belong to these respective models and strengthen the virtual condition \( \chi/L_1 \). The compatibility then follows immediately from the balance of the virtual condition \( \chi/L_1 \) in the model \( V[K][G_0][L_1] \).

Finally, the common lower bound of the conditions \( p_i \) for \( i \in 3 \) forces the \( E_0 \)-classes \( \eta/L_0, \eta/L_1, \) and \( \eta/L_2 \) to form a cycle in \( \tau \). This contradicts the initial assumption that \( \tau \) is forced to be a linear ordering.

Corollary 11.3.18. Let \( E \) be a Borel equivalence relation on a Polish space \( X \). There is a balanced extension of the Solovay model in which \( X/E \) is a tournament cardinal, yet \( 2^{\omega}/E_0 \) is not linearly ordered.

Proof. Consider the Borel set \((X \times X) \setminus E\) and the simplicial complex \( K \) on it consisting of those sets \( a \) such that if \( (x_0, x_1) \) and \( (x'_0, x'_1) \) are both elements of \( a \) then one of \( x_0 E x'_1 \) and \( x_1 E x'_0 \) must fail. The simplicial complex \( K \) is easily verified to be a locally finite quotient Borel complex. As a result, the poset \( P_K \) satisfies the assumptions of Theorem 11.3.12: the \( E_0 \)-charm was constructed in the proof of Theorem 7.7, the adequacy is verified in Proposition 9.5.3, and 3, 2-centeredness is immediate from the fact that \( K \) is a flag complex. The poset \( P_K \) is designed to add a tournament on the quotient \( E \)-space as the set \( \{(c_0, c_1): c_0, c_1 \) are distinct \( E \)-classes such that for some (every) \( x_0 \in c_0, x_1 \in c_1 \), the pair \( (x_0, x_1) \) belongs to the generic \( K \)-set\}.
CHAPTER 11. OTHER COMBINATORICS IN $W[G]$
Chapter 12

Large cardinal considerations

12.1 Absoluteness

It turns out that the forcing extensions in this book have certain optimality features: independence results of certain syntactical form are provably easiest to achieve in them when compared with a large class of roughly similar models. To state the theorems asserting these features, we need the specification of the syntactical class of statements in question.

Definition 12.1.1. A sentence $\phi$ is $\Sigma^2_{1a}$ if there is a Borel equivalence relation $E$ on a Polish space $X$, a Borel quotient structure $\{R_n, f_n : n \in \omega\}$ on the quotient space $X/E$ possibly containing both relations and functions, and a $\Pi^1_1$ formula $\psi$ such that $\phi$ is of the form “there is a relation $S$ on $X/E$ such that $\langle X/E, R_n, f_n : n \in \omega, S \rangle \models \psi$”.

Example 12.1.2. The sentence “$X/E$ is linearly ordered” is $\Sigma^2_{1a}$.

Example 12.1.3. Let $\Gamma$ be a Borel quotient graph on the space $X/E$. The sentence “$\Gamma$ has countable chromatic number” is $\Sigma^2_{1a}$.

The expression power of $\Pi^1_1$ sentences grows dramatically when the language contains functional symbols. We can prove an absoluteness result in this stronger setting as well. However, the context must be limited to quotient spaces induced by pinned equivalence relations.

Definition 12.1.4. A sentence $\phi$ is $\Sigma^2_{1b}$ if there is a pinned Borel equivalence relation $E$ on a Polish space $X$, a Borel quotient structure $\{R_n, f_n : n \in \omega\}$ on the quotient space $X/E$ possibly containing both relations and functions, and a $\Pi^1_1$ formula $\psi$ such that $\phi$ is of the form “there is a relation $S$ and a function $g$ on $X/E$ such that $\langle X/E, R_n, f_n : n \in \omega, S, g \rangle \models \psi$”.

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Example 12.1.5. Let $K$ be a modular Borel simplicial complex. The statement "there is a maximal $K$-set" is $\Sigma^2_{1b}$.

Proof. In this case, $E$ is the identity on a suitable Polish space. Let $f : K \to [Y]^{\aleph_0}$ be a Borel modular function for $K$. The statement is in ZF equivalent to the following statement $\theta$. There is a set $S \subset X$ and a function $g : Y \to K$ such that $S \subset X$ is a $K$-set and for all $x \notin X$, if $x \notin S$ there is $y \in f(x)$ such that $g(y) \subset S$ and $y \in f(g(y))$. It is immediate that $\theta$ is a $\Sigma^2_{1b}$ statement. To see the ZF-equivalence between $\theta$ and the existence of a maximal $K$-set, first assume that $\theta$ holds as witnessed by $S,g$. Then clearly $S \subset X$ must be a maximal $K$-set, since for every $x \notin X$, for some $y \in f(x)$ $g(y) \subset S$ and $g(y) \subset \{x\} \notin K$ both hold by the modularity of the function $f$. On the other hand if $S \subset X$ is a maximal $K$-set, define a function $g : Y \to K$ by letting $g(y) =$the inclusion-minimal element $a \in [S]^{\aleph_0}$ such that $y \in f(a)$ if $y \in f(S)$, and $g(x,y) =$arbitrary if $y \notin f(S)$. The modularity of the function $f$ shows that if $y \in f(S)$ then there is a unique inclusion-minimal element $a \in [S]^{\aleph_0}$ such that $y \in f(a)$ and so the first alternative is well-defined. The pair $S,g$ witnesses the validity of the statement $\theta$. 

Example 12.1.6. Let $E,F$ be pinned Borel equivalence relations on Polish spaces $X,Y$. The statement $\{E\} \leq \{F\}$ is $\Sigma^2_{1b}$.

Example 12.1.7. Let $E,F$ be pinned Borel equivalence relations on Polish spaces $X,Y$. Let $B$ be a Borel subset of the quotient space $X/E \times Y/F$ with all vertical sections nonempty. The statement $\{B\}$ has a uniformization" is $\Sigma^2_{1b}$.

Finally, we are in a position to state an absoluteness theorem.

Theorem 12.1.8. Let $\phi$ be a $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentence. Let $P$ be a $\sigma$-closed Suslin forcing. Let $\kappa$ be an inaccessible cardinal such that

(*) in $V_\kappa$, for every ordinal $\alpha \in \kappa$ there is a forcing extension and in it a transitive set $A$ such that $V_\alpha(A) \models \text{ZF+DC}+\alpha = \aleph_0+P$ is balanced+$\neg\phi$.

Then in the $P$-extension of the Solovay model derived from $\kappa$, $\neg\phi$ holds.

In other words, the extensions of the Solovay model satisfy as few $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentences as possible given their general form. An important point of the theorem is that the balance of $P$ is a priori much weaker statement than the existence of a $P$-generic filter; see the examples below. The use of the DC assumption is rather minimal and in many specific cases can be eliminated.

Proof. Let $E$ be a Borel equivalence relation on a Polish space $X$ used in the sentence $\phi$. To cut down on the notational clutter, we will ignore the extra Borel structure on the quotient space $X/E$. Write $\phi$ as a sentence $\exists B \langle X/E,B \rangle \models \forall x \psi$ where $\psi$ is a formula with no quantifiers. If $\phi$ is $\Sigma^2_{1a}$ then $B$ is a relation, if $\phi$ is $\Sigma^2_{1b}$ then $B$ may be a function on the quotient space.

Let $W$ be the symmetric Solovay model derived from $\kappa$ and work in $W$. Suppose that the conclusion fails. Then there must be a condition $p \in P$ and...
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a $P$-name $\tau \in L(\mathbb{R})$ such that in $L(\mathbb{R})$, $p \Vdash \langle X/E, \tau \rangle \models \forall x \psi$. The name $\tau$ is definable from a point $z \in 2^\omega$ and some ground model parameters. Use the assumptions to find a transitive set $A$ which is countable in $W$ such that $z \in V(A)$ and $V(A) \models ZF+DC+P$ is balanced $+\neg \phi$. Note that $W$ is the symmetric extension of $V(A)$ by the poset $Coll(\omega, A) \times Coll(\omega, < \kappa)$.

Work in the model $V(A)$. Let $\bar{p} \leq p$ be a balanced condition. Let us first handle the case in which $\phi$ is a $\Sigma^2_{10}$ sentence; i.e. $\tau$ is a name for a relation on $X/E$. For simplicity assume that $\tau$ is a unary relation.

**Claim 12.1.9.** For every point $y \in X$, $Coll(\omega, A) \models Coll(\omega, < \kappa) \models \bar{p}$ decides the statement whether $[\bar{y}]_E \in \tau$ or not.

**Proof.** If this failed for some point $y \in X$, then there would be an ordinal $\lambda < \kappa$ and a $Coll(\omega, A) \times Coll(\omega, < \lambda)$-name $\sigma_0$ for a condition in the poset $P$ stronger than $\bar{p}$ such that $Coll(\omega, < \kappa) \models \sigma_0 \models [\bar{y}]_E \in \tau$, and also a name $\sigma_1$ for a condition in the poset $P$ stronger than $\bar{p}$ such that $Coll(\omega, < \kappa) \models [\bar{y}]_E \notin \tau$.

Move to the model $W$ and let $H_0, H_1 \subset Coll(\omega, A) \times Coll(\omega, < \lambda)$ be filters mutually generic over the model $V(A)$; let $p_0 = \sigma_0 / H_0$ and $p_1 = \sigma_1 / H_1$. The balance of the condition $\bar{p}$ implies that $p_0, p_1$ are compatible conditions in the poset $P$. By the forcing theorem applied in $V(A)[H_0]$ and $V(A)[H_1]$, their lower bound should force both $[\bar{y}]_E \in \tau$ and $[\bar{y}]_E \notin \tau$, an impossibility.

Still in the model $V(A)$, let $C = \{y \in X : Coll(\omega, A) \models Coll(\omega, < \kappa) \models \bar{p} \models [\bar{y}]_E \in \tau\}$. The claim shows that in the model $W$, $\bar{p} \Vdash \tau \cap V(A) = \bar{C}$. Now, let $G \subset P$ be a filter generic over the model $W$ and let $B = \tau / G$. The sentence $\forall x \psi$ holds in the model $\langle (X/E)^W, B \rangle$, and as a $\Pi^1_1$ sentence its validity trickles down to the smaller model $\langle (X/E)^V(\bar{A}), C \rangle$. This means precisely that $\phi$ holds in $V(A)$.

Let us now handle the case in which $\phi$ is a $\Sigma^2_{10}$-sentence. That is, the equivalence relation $E$ is pinned and $\tau$ is a name for a function. For simplicity assume that $\tau$ is a unary function.

**Claim 12.1.10.** For every point $y \in X$ there is a point $z \in X$ such that $Coll(\omega, A) \models Coll(\omega, < \kappa) \models \bar{p} \Vdash \tau([\bar{y}]_E) = [\bar{z}]_E$.

**Proof.** Fix a point $y \in X$. By the forcing theorem, there must be an ordinal $\lambda < \kappa$ and a $Coll(\omega, A) \times Coll(\omega, < \lambda)$-name $\sigma_0$ for a condition in the poset $P$ stronger than $\bar{p}$ and a name $\eta$ for an element of $X$ such that $Coll(\omega, < \kappa) \models \sigma_0 \models \tau([\bar{y}]_E) = [\eta]_E$. By the balance of the condition $\bar{p}$ it follows that the name $\eta$ must be $E$-pinned. By ??? (this is the only point where the DC assumption is used), the equivalence relation $E$ is pinned in the model $V(A)$ and so there must be a point $z \in X \cap V(A)$ such that $Coll(\omega, A) \models Coll(\omega, < \kappa) \models \eta E \bar{z}$. Another balance argument shows that $Coll(\omega, A) \models Coll(\omega, < \kappa) \models \bar{p} \Vdash \tau([\bar{y}]_E) = [\bar{z}]_E$ as desired.

Still in the model $V(A)$, let $f = \{\{y, z\} \in X : Coll(\omega, A) \models Coll(\omega, < \kappa) \models \bar{p} \Vdash \tau([\bar{y}]_E) = [\bar{z}]_E\}$. The claim shows that $f$ is a total function and in the model $W$, $\bar{p} \Vdash \tau \upharpoonright V(A) = \bar{f}$. Now, let $G \subset P$ be a filter generic over the model $W$ and
Example 12.1.11. Suppose that $\phi$ is a $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentence. Suppose that $\kappa$ is an inaccessible cardinal. If $\text{ZF+DC+there is a Hamel basis} + \neg \phi$ holds in many choiceless extensions below $\kappa$ then $\neg \phi$ holds in the $P$-extension of the Solovay extension where $P$ is the poset of all infinite subsets of $\omega$ ordered by inclusion.

**Proof.** It will be enough to show in $\text{ZF+DC}$ that the existence of an ultrafilter implies that $P$ is balanced. Indeed, let $p \in P$ be a condition. If there is an ultrafilter on $\omega$, then there must be an ultrafilter $\bar{p}$ containing $p$. Then, the pair $\langle \text{Coll}(\omega, \bar{p}), \tau \rangle$, where $\tau$ is the name for the set of all conditions diagonalizing $\bar{p}$, is a balanced condition in $P$ by Theorem 7.1.4.

Example 12.1.12. Suppose that $\phi$ is a $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentence. Suppose that $\kappa$ is an inaccessible cardinal. If $\text{ZF+DC+there is a Hamel basis} + \neg \phi$ holds in many choiceless extensions below $\kappa$ then $\neg \phi$ holds in the $P$-extension of the Solovay extension where $P$ is the poset of countable subsets of $\mathbb{R}$ linearly independent over $\mathbb{Q}$, ordered by reverse inclusion.

**Proof.** It will be enough to show in $\text{ZF+DC}$ that the existence of a Hamel basis implies that $P$ is balanced. To this end, suppose that $p \in P$ is a condition and $B \subseteq \mathbb{R}$ is a basis. Let $D \subseteq B$ be a countable set such that $p$ is a subset of the linear span of $C$. Find a basis $q \supseteq p$ of the linear span of $D$, and let $C = (B \setminus D) \cup q$. It is immediate that $C$ is a Hamel basis extending $p$. Then, the pair $\langle \text{Coll}(\omega, C), \tau \rangle$, where $\tau$ is the name for the set of all conditions extending $C$, is a balanced condition in $P$ below $p$ by Theorem 6.2.21.

Example 12.1.13. Suppose that $\phi$ is a $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentence. Suppose that $\kappa$ is an inaccessible cardinal. Suppose that $E, F$ are pinned Borel equivalence relations on respective Polish spaces $X, Y$. If $\text{ZF+DC+}\lvert E \rvert \leq \lvert F \rvert + \neg \phi$ holds in many choiceless extensions below $\kappa$ then $\neg \phi$ holds in the $P$-extension of the Solovay extension where $P$ is the collapse of $\lvert E \rvert$ to $\lvert F \rvert$ introduced in Definition 6.4.2.

**Proof.** Recall that $P$ is the set of all pairs $p = \langle a_p, b_p \rangle$ where $a_p$ is a countable injection from $X/E$ to $Y/F$ and $b_p$ is a countable subset of $Y/F$ ordered by coordinatewise reverse extension. It will be enough to show in $\text{ZF+DC}$ that $\lvert E \rvert \leq \lvert F \rvert$ implies that $P$ is balanced. To this end, suppose that $p = \langle a_p, b_p \rangle \in P$ is a condition and $h : X/E \to Y/F$ is an injection. Let $D_0 \subseteq X/E$ and $D_1 \subseteq Y/F$ be countable sets such that $\text{dom}(a_p) \subseteq D_0$ and $h''D_0 \cup \text{rng}(a_p) \cup b_p \subseteq D_1$. Let $a_p'$ be an extension of $a_p$ such that $\text{dom}(a_p') = D_0$ and $\text{rng}(a_p') \subseteq D_1 \setminus b_p$. Finally, let $h' = (h \upharpoonright (X/E \setminus D_0)) \cup a_p'$. Then, the pair $\langle \text{Coll}(\omega, X), \tau \rangle$, where $\tau$ is the name for the set of all conditions $q$ such that $h' \subseteq a_q$ and $V \setminus \text{rng}(a_q) \subseteq b_q$ is a balanced condition in $P$ below $p$ by Theorem 6.4.3.

Example 12.1.14. Suppose that $\phi$ is a $\Sigma^2_{1a}$ or $\Sigma^2_{1b}$ sentence. Suppose that $\kappa$ is an inaccessible cardinal. Suppose that $E$ is a pinned Borel equivalence relation
on a Polish spaces $X$. If $\text{ZF} + \text{DC} + \neg \phi$ holds in many choiceless extensions below $\kappa$ then $\neg \phi$ holds in the $P$-extension of the Solovay extension where $P$ is the poset forcing a complete countable section with countable approximations.

It follows that the existence of a countable complete section for, say, the equivalence relation $E = E_1$ is neither a $\Sigma^2_{1_0}$ nor $\Sigma^1_{1_0}$ sentence, since it fails in the model $V(\mathbb{R})$ in the extension of any ZFC model $V$ by the finite support product of uncountably many Cohen reals, but it holds in the $P$-extension of the symmetric Solovay model.

**Proof.** Recall that $P$ is the set of all countable subsets of $X$ ordered by $q \leq p$ if $p \subset q$ and $q \subset [p]_E = p$. The point is that $P$ is balanced in ZF. To see that, suppose that $p \in P$ is a condition. Consider the $\text{Coll}(\omega, X)$-name $\tau$ for the set of all conditions $q \in P$ such that $p \subset q$, $q \cap [p]_E = p$, and $\forall x \in V \setminus [p]_E$ $q \cap [x]_E = V \cap [x]_E$. It follows from Example 6.4.9 that the pair $(\text{Coll}(\omega, X), \tau)$ is a balanced virtual condition in $P$ below $p$ as required.\hfill $\Box$

### 12.2 Modular complexes

The modular complexes are unusual in the sense that not only it is relatively easy to characterize genericity, but it is also possible to produce generic filters over $L(\mathbb{R})$ in ZFC with just some large cardinal assumptions, independently of the structure of the continuum.

**Definition 12.2.1.** Let $\mathcal{K}$ be a modular Borel simplicial complex on a Polish space $X$.

1. A set $B \subset [X]^{\aleph_0}$ is **easy** if it is Borel, uncountable, consists of pairwise disjoint $\mathcal{K}$-sets and such that $p, q \in B$ implies $p \cup q$ is a $\mathcal{K}$-set;
2. A condition $p \in P_{\mathcal{K}}$ **excludes** $y \in Y$ if for every set $a \in \mathcal{K}$, either $p \cup a$ is not a $\mathcal{K}$-set or $y \notin f(a)$ holds;
3. a set $A \subset X$ is called **quasi-generic** if it is a maximal $\mathcal{K}$-set, for every easy set $B$ there is $p \in B$ such that $p \subset A$, and for every $y \in Y$, either $y \in f(A)$ or there is a condition $p \subset A$ which excludes $y$.

It should be noted that in important cases such as modular pre-geometries or the acyclic complex, the last demand in the last item is trivially satisfied for all maximal $\mathcal{K}$-sets. Quasi-generic sets are balanced as Definition 6.2.18 and a straightforward absoluteness argument make clear.

**Theorem 12.2.2.** Suppose that there is a weakly compact Woodin cardinal. Let $\mathcal{K}$ be a modular Borel simplicial complex on a Polish space $X$. A maximal $\mathcal{K}$-set $A \subset X$ is quasi-generic if and only if it is generic over $L(\mathbb{R})$ for the poset $P_{\mathcal{K}}$.

As elsewhere in the book, we did not attempt optimization of the large cardinal hypothesis used. The model $L(\mathbb{R})$ can be replaced by its larger cousins $L(\mathbb{R})(\Gamma)$ where $\Gamma$ is a pointclass of universally Baire sets, without any change to the argument.
Proof. Let \( f: \mathcal{K} \to [\mathcal{V}]^{\aleph_0} \) be the Borel modular function for \( \mathcal{K} \). The right-to-left implication is clear. Suppose that \( B \subseteq [X]^{\aleph_0} \) consists of \( \mathcal{K} \)-sets and is such that \( p, q \in A \) implies \( p \cap q = 0 \) and \( f(p) \cap f(q) = 0 \) if \( p \neq q \). It will be enough to show that \( \mu = \mu_B \) forces its generic set to contain an element of \( B \) as a subset. Indeed, if \( p \in \mu \) is a condition, then by a counting argument there must be a \( \mathcal{K} \)-set \( q \in B \) such that \( p \cap q = 0 \) and \( f(p) \cap f(q) = 0 \). By the modularity property of the function \( f \), \( p \cup q \) is a \( \mathcal{K} \)-set and a condition stronger than \( p \) which forces \( q \in B \) to be a subset of the generic set for \( \mu \). A genericity argument concludes the proof in this direction.

The left-to-right direction is the essence of the theorem. Suppose that \( A \subseteq X \) is a quasi-generic maximal \( \mathcal{K} \)-set. Let \( G \subseteq \mathcal{Q}_\kappa \) be a generic filter, let \( j: V \to M \) be the generic \( \mathcal{Q}_\kappa \)-ultrapower, and write \( W = V([\mathcal{R}]^{\mathcal{V}[G]}) \). We will show that \( j(\lambda) = P \)-generic over \( W \). It follows that \( j(\lambda) \) is also \( P \)-generic over \( L(\mathcal{R})^{\mathcal{V}[G]} = L(\mathcal{R})^M \). The elementarity of the embedding \( j \) then implies that \( A \) is \( P \)-generic over \( L(\mathcal{R}) \) as desired.

Suppose then that \( D \subseteq P \) is an open dense subset of \( P \) in the model \( W \); we must show that \( A \) contains a countable subset which is an element of \( D \). Since \( W \) is a symmetric Solovay extension of \( V \) derived from \( \kappa \) by Fact 1.3.15(2), \( D \) is definable in \( W \) from some element \( z \in 2^{\omega} \) and some ground model parameters. Use Fact 1.3.15(2) again to find a ground model Woodin cardinal \( \lambda < \kappa \) such that \( G_\lambda = G \cap \mathcal{Q}_\lambda \) is generic over \( V \) as a filter on \( \mathcal{Q}_\lambda \) and \( z \in V[G_\lambda] \). Let \( j_\lambda: V \to M_\lambda \) be the associated elementary embedding. Note that \( j_\lambda \) is a quotient of \( j \), in particular \( j_\lambda(A) \subseteq j(A) \).

Work in the model \( V[G_\lambda] \). By elementarity, \( M_\lambda \models j_\lambda(A) \) is quasi-generic, the model \( M_\lambda \) is closed under \( \omega \)-sequences by Fact 1.3.15(1) applied at \( \lambda \), therefore the set \( j_\lambda(A) \) is still quasi-generic in the model \( V[G_\lambda] \). By Theorem 6.2.19 it is balanced, corresponding to a balanced virtual condition \( \bar{p} \in P \). There must be a poset \( R \) of size \( < \kappa \), an \( R \)-name \( \sigma \) for a condition in \( P \) stronger than \( \bar{p} \) such that \( R \vDash \Coll(\omega, < \kappa) \vDash \sigma \in \bar{D} \).

Work in the model \( W \). Use Theorem 13.3.2 to find a continuous map \( y \mapsto H_y \) from \( \omega^2 \) to filters on \( R \) such that for distinct elements \( y_0, y_1 \in 2^\omega \), the filters \( H_{y_0}, H_{y_1} \subseteq R \) are mutually generic over \( V[G_\lambda] \). For each \( y \in 2^\omega \) let \( p_y = \sigma[H_y] \); the function \( y \mapsto p_y \) is Borel by Proposition 13.1.3. Note that for distinct elements \( y_0, y_1 \in 2^\omega \), the conditions \( p_{y_0}, p_{y_1} \in P \) are compatible and their intersection is exactly \( j_\lambda(A) \); in particular, the sets \( f(p_{y_0} \setminus j_\lambda(A)) \) and \( f(p_{y_1} \setminus j_\lambda(A)) \) must be pairwise disjoint. Let \( B = \{ p_y \setminus j_\lambda(A) : y \in 2^\omega \} \). This is a set which is a continuous injective image of \( 2^\omega \) and therefore Borel. We have just checked that the Borel set \( B \) is easy.

Work in the model \( V[G] \). The model \( M \) is closed under \( \omega \)-sequences, and therefore the set \( j(A) \) is a quasi-generic maximal subset of \( X \). Therefore, there exists an element \( y \in 2^\omega \) such that \( j(A) \) contains an element of the easy set \( B \), some \( p_y \setminus j_\lambda(A) \), as a subset. Since \( j_\lambda(A) \subseteq A \), it follows that \( p_y \subseteq A \). Since \( W \) is a symmetric Solovay extension of the model \( V[G_\lambda][H_y] \), it follows that \( p_y \in D \), and the proof is complete. \( \square \)
Theorem 12.2.3. Suppose that there is a weakly compact Woodin cardinal. Let $K$ be a modular Borel simplicial complex on a Polish space $X$ such that the values of its modular function are finite sets. There is a set $A \subset X$ which is generic over $L(\mathbb{R})$ for the poset $P_K$.

Proof. Let $f : K \to [Y]^{<\aleph_0}$ be the Borel modular function for $K$. The proof starts with a claim of independent interest.

Claim 12.2.4. Suppose that $A \subset X$ is a $K$-set of size $< c$ and $y \in Y$. Then either $A$ contains a countable subset which excludes $y$, or there is a finite set $a \subset X$ such that $y \in f(a)$ and $A \cup a$ is a $K$-set.

Proof. The argument uses a beautiful theorem on puncture sets [4]. Given a collection $C$ of nonempty sets, a set $b$ punctures $C$ if it has nonempty intersection with every element of $C$. [4, Theorem 12] shows that if $X$ is a Polish space and $C \subset X^\omega$ is an analytic set, then either there is a countable set $b \subset X$ puncturing the collection $\{\text{rng}(y) : y \in C\}$ or $C$ contains a perfect subset in which the ranges of any two distinct elements are disjoint.

Suppose that $A$ contains no countable set excluding $y$. Let $M$ be a countable elementary submodel of a large structure containing $A$. Since $A \cap M$ does not exclude $y$, there is a finite set $b \subset X$ such that $(A \cap M) \cup b$ is a $K$-set and $y \in f(b)$. We will produce a finite set $a \subset X$ such that $A \cup a$ is a $K$-set and $y \in f(a)$. If $b \subset M$ then $a = b$ would work by elementarity, so without loss of generality assume that $b \not\subset M$.

Let $c = b \cap M$ and $d = f(b) \cap M$. Since the values of the function $f$ are finite sets, $d \subset Y$ is an element of the model $M$. Let $B = \{a \in K : y \in f(a), c \subset a, c \neq a, \text{ and } d \subset f(a)\}$ and note that $b \in B$. Let $g : B \to [X \cup Y]^{<\aleph_0}$ be the function defined by $g(a) = (a \setminus c) \cup (f(a) \setminus d)$. The set $C = \{g(a) : a \in B\}$ is an analytic collection in $M$ consisting of finite subsets of $X \cup Y$. It cannot be punctured by a countable set, since such a countable set would be found in the model $M$ and it would have to intersect the set $g(b)$ in an element of $M$, contradicting the choices of the sets $c, d$. Thus, the set $C$ contains a subcollection of size continuum consisting of pairwise disjoint sets. The set $A \cup f(A)$ has size less than $c$, and so there has to be a set $a \in B$ such that $g(a) \cap (A \cup f(A)) = 0$. We claim that $A \cup a$ is a $K$-set; this will conclude the proof of the claim.

To do this, suppose that $c \subset A$ is a finite set containing $b \cap A \cap M$ as a subset and work to show that $f(a) \cap f(e) = f(a \cap e)$; the modularity of the function $f$ will then show that $a \cup e \in K$. To this end, fix a point $z \in f(a) \cap f(e)$. Since $z \in f(a) \cap f(A)$, it must be the case that $z \in d$ by the choice of $a$. Thus, $z \in M \cap f(A)$, and by the elementarity of $M$ there must be a finite set $e' \subset A \cap M$ containing $b \cap A \cap M$ as a subset such that $z \in f(e')$. Note also that $z \in f(b)$ since $z \in d$. Since $(A \cap M) \cup b$ is a $K$-set, the modularity of the function $f$ implies that $z \in f(e' \cap b) = f(b \cap A \cap M)$. The argument is complete.

Back to the proof of Theorem 12.2.3. By Theorem 12.2.2, it is enough to build a quasi-generic set. Let $\mu$ be the size of the continuum, let $\langle x_\alpha : \alpha \in \mu \rangle$ be the enumeration of all elements of $X$, let $\langle y_\alpha : \alpha \in \mu \rangle$ be an enumeration of
all elements of $Y$, and let $(B_{\alpha} : \alpha \in \mu)$ be an enumeration of all easy sets. By induction on $\alpha \in \mu$, build $K$-sets $A_{\alpha}$ such that

- the sets $A_{\alpha}$ form an inclusion increasing sequence with $A_0 = 0$ and $|A_{\alpha}| = \aleph_0 + |\alpha|$;
- $x_{\alpha} \in A_{\alpha+1}$ or $A_{\alpha+1} \cup \{x_{\alpha}\}$ is not a $K$-set;
- $y_{\alpha} \in f(A_{\alpha+1})$ or there is a countable subset of $A_{\alpha+1}$ which excludes $y_{\alpha}$;
- there is an element of $B_{\alpha}$ which is a subset of $A_{\alpha+1}$.

In the end, let $A = \bigcup_{\alpha} A_{\alpha}$; this will be the required quasi-generic set. To perform the induction step, first check if $A_{\alpha} \cup \{x_{\alpha}\}$ is a $K$-set or not. If yes, then let $C_0 = A_{\alpha} \cup \{x_{\alpha}\}$ and if not let $C_0 = A_{\alpha}$. Second, check if $C_0$ has a countable subset which excludes $y_{\alpha}$. If yes then let $C_1 = C_0$; if no use Claim 12.2.4 to find an extension $C_1$ of $C_0$ by a finite set such that $f(p) \cap f(C_1) = 0$. By the modularity property of the function $f$, it follows that $p \cup C_1$ is a $K$-set. Let $A_{\alpha+1} = p \cup C_1$; the induction step has just been completed.

Theorem 12.2.5. Suppose that there is a weakly compact Woodin cardinal. Let $K$ be the simplicial complex of free sets in a Borel modular pre-geometry. There is a set $A \subset X$ which is generic over $L(\mathbb{R})$ for the poset $P_K$.

Proof. By Theorem 12.2.2, it is enough to build a quasi-generic set. Let $\mu$ be the size of the continuum, let $\langle x_{\alpha} : \alpha \in \mu \rangle$ be the enumeration of all elements of $X$ and let $(B_{\alpha} : \alpha \in \mu)$ be an enumeration of all easy sets. By induction on $\alpha \in \mu$, build $K$-sets $A_{\alpha}$ such that

- the sets $A_{\alpha}$ form an inclusion increasing sequence with $A_0 = 0$ and $|A_{\alpha}| = \aleph_0 + |\alpha|$;
- $x_{\alpha} \in A_{\alpha+1}$ or $A_{\alpha+1} \cup \{x_{\alpha}\}$ is not a $K$-set;
- there is an element of $B_{\alpha}$ which is a subset of $A_{\alpha+1}$.

In the end, let $A = \bigcup_{\alpha} A_{\alpha}$; this will be the required quasi-generic set. To perform the induction step, first check if $A_{\alpha} \cup \{x_{\alpha}\}$ is a $K$-set or not. If yes, then let $C_0 = A_{\alpha} \cup \{x_{\alpha}\}$ and if not let $C_0 = A_{\alpha}$. The easy set $B_{\alpha}$ has size $\mathfrak{c}$ by the perfect set theorem, and so there must be a set $p \in B_{\alpha}$ such that $f(p) \cap f(C_1) = 0$. By the modularity property of the function $f$, it follows that $p \cup C_1$ is a $K$-set. Let $A_{\alpha+1} = p \cup C_1$; the induction step has just been completed.

The resolution of the following natural question is hindered by the lack of examples.

Question 12.2.6. Is it possible to drop the assumption on finiteness of the function $f$ altogether in Theorem 12.2.3?
12.3 Ultrafilters disjoint from an $F_\sigma$-ideal

Given any $F_\sigma$-ideal $I$, Section ??? considers the poset $P(I)$ of $I$-positive sets ordered by inclusion. The $P(I)$-generic filter can be identified with an ultrafilter disjoint from the ideal $I$. It turns out that $P(I)$ has a rather straightforward description of generic filters:

**Definition 12.3.1.** Let $I$ be an $F_\sigma$-ideal on $\omega$. An ultrafilter $U$ on $\omega$ is $I$-quasi-generic if it is a P-point disjoint from $I$, and for each closed set $C \subset \mathcal{P}(\omega)$, either $U \cap C \neq \emptyset$ or else there is a set $a \in U$ such that $\mathcal{P}(a) \cap C \subset I$.

It is a matter of an easy genericity argument to show that the $P(I)$ generic ultrafilter is $I$-quasi-generic. The following theorem contains the opposite implication:

**Theorem 12.3.2.** (with David Chudoumský) Let $\kappa$ be a weakly compact Woodin cardinal. Let $I$ be an $F_\sigma$-ideal on $\omega$. Every $I$-quasi-generic ultrafilter is generic over $L(\mathbb{R})$ for the poset $P(I)$.

**Proof.** We will need a couple of provisional definitions. Let $U$ be an arbitrary nonprincipal ultrafilter on $\omega$. Let $P_U$ be the standard poset for the diagonalization of $U$: a condition $p \in P$ is a pair $p = (a_p, b_p)$ where $a_p \subset \omega$ is finite and $b_p \in U$. The ordering is defined by $q \leq p$ if $a_p \subset a_q$, $b_q \subset b_p$, and $a_q \setminus a_p \subset b_p$. The poset $P_U$ adds a generic subset of $\omega$ defined as the union of the first coordinates of the conditions in the generic filter.

Now move to an arbitrary generic extension, and call a set $c \subset \omega$ $P_U$-quasi-generic if it is modulo finite included in every element of $U$, and (*) for every set $d \subset [\omega]^{<\aleph_0}$ in $V$, if $c$ contains no element of $d$ as a subset, then there is $b \in U$ such that $b$ contains no element of $d$ as a subset.

**Claim 12.3.3.** If $c \subset \omega$ is $P_U$-quasi-generic and $e \in 2^\omega$ is a Cohen generic point over $V[c]$, then $c \cap e^{-1}\{1\}$ is $P_U$-generic over the ground model.

**Proof.** Work in the model $V[c]$. For a binary string $t \in 2^{<\omega}$ write $c_t = t^{-1}\{1\}$. Let $O \subset P_U$ be an open dense set in $V$ and $t \in 2^{<\omega}$ be a condition in the Cohen forcing. By a genericity argument, it will be enough to find a condition $s \leq t$ in the Cohen forcing and a condition $\langle g, b \rangle \in O$ such that $g = c_s \cap c$ and $c \subset b \setminus \text{dom}(s)$.

To this end, let $d = \{f \in [\omega]^{<\aleph_0} : f \cap \text{dom}(t) = 0\}$ and there is a set $b \in U$ such that $\langle (c_t \cap c) \cup f, b \rangle \in O$. The set $d$ belongs to the ground model and by the density of the set $O$, every set $b \in U$ contains an element of $d$ as a subset. By the quasi-genericity of the set $c$, there is a set $f \in d$ such that $f \subset c$. Write $g = (c_t \cap c) \cup f$ and find a set $b \in U$ be such that $\langle g, b \rangle \in O$. Let $s \in 2^{<\omega}$ be an extension of $t$ such that $c_s \cap c = g$ and $c \subset b \setminus \text{dom}(s)$. Clearly, $s \leq t$ works as desired.

Now let $I$ be an $F_\sigma$-ideal on $\omega$. The point of the definition of a $I$-quasi-generic ultrafilter is found in the following claim.
Claim 12.3.4. If $U$ is $I$-quasi-generic ultrafilter and in some generic extension $c \subset \omega$ is an $I$-positive set diagonalizing $U$, then $c$ is $P_U$-quasi-generic.

Proof. Let $d \subset [\omega]^{<\omega}$ be a set in $V$; we need to check that (*) holds for $d$. Let $C = \{x \subset \omega : x$ contains no element of $d$ as a subset}; this is a closed subset of $\mathcal{P}(\omega)$ coded in the ground model. By the $I$-quasi-genericity of the ultrafilter $U$, there are two mutually exclusive options. Either $U \cap C \neq \emptyset$; in such a case the conclusion of (*) holds, and so (*) holds for $d$ again. Or, there is a set $b \in U$ such that $P(b) \cap C \subset I$; in this case, since $c \cap b \notin I$, the assumption of (*) fails and so (*) holds for $d$ again. 

Now, let $\kappa$ be a weakly compact Woodin cardinal, $G \subset \mathcal{Q}_\kappa$ be a generic filter over $V$, and $j : V \to M$ be the associated generic embedding. Write $W = V(\mathbb{R}^{[G]})$, thus, $W$ is the symmetric Solovay model derived from $\kappa$. It will be enough to show that $j(U)$ is $P(I)$-generic over the model $W$. To this end, suppose that $D \subset P(I)$ is an open dense set in $W$; we must show that $j(U) \cap D \neq 0$ holds.

The set $D$ is definable in $W$ from some element $z \in 2^\omega$ and some ground model parameters. By Fact 1.3.15(2) there is a Woodin cardinal $\lambda < \kappa$ such that $G_\lambda = G \cap \mathcal{Q}_\lambda$ is a filter generic over $V$ and $z \in V[G_\lambda]$. Let $j_\lambda : V \to M_\lambda$ be the associated generic embedding and observe that $j_\lambda(U)$ is an $I$-quasi-generic ultrafilter in $M_\lambda$ and therefore in $V[G_\lambda]$.

Claim 12.3.5. In $V[G_\lambda]$, $P_{j_\lambda(U)} \models \text{Coll}(\omega, < \kappa) \models \text{the } P_{j_\lambda(U)}$-generic set belongs to $D$.

Proof. By the forcing theorem, for any condition $(g, b)$ it will be enough to find, in the model $W$, a set $d \in D$ which is $P_{j_\lambda(U)}$-generic over $V[G_\lambda]$ and meets the condition $(g, b)$. First, find any $I$-positive set $c \subset b$ diagonalizing $j_\lambda(U)$. Use the density of the set $D$ and thin out $c$ if necessary to assert that $c \in D$. By Claim 12.3.4, $c$ is $P_{j_\lambda(U)}$-quasi-generic over $V[G_\lambda]$. Let $e \in 2^\omega$ be a point Cohen generic over $V[G_\lambda][c]$ and use Claim 12.3.3 to assert that the set $c \cap e^{-1}\{1\}$ is $P_{j_\lambda(U)}$-generic over $V[G_\lambda]$. The set $g \cup (c \cap e^{-1}\{1\})$ is as required.

Claim 12.3.6. $j(U)$ contains a set which is $P_{j_\lambda(U)}$-generic over $V[G_\lambda]$.

Proof. Argue in the model $V[G]$. First of all, $j(U)$ is a $P$-point ultrafilter disjoint from $I$ by elementarity of the embedding $j$. The set $j(U) \cap V[G_\lambda] = j_\lambda(U)$ is a countable subset of $j(U)$, and so $j(U)$ contains a set $c \subset \omega$ which diagonalizes $j_\lambda(U)$. The set $c$ is $P_{j_\lambda(U)}$-quasi-generic over $V[G_\lambda]$ by Claim 12.3.4. The model $V[G_\lambda][c]$ contains only countably many nowhere dense subsets of $\mathcal{P}(\omega)$, and the set of characteristic functions of sets in $j(U)$ is non-meager in $2^\omega$ (as is the case for any ultrafilter). Therefore, there is an element $b \in j(U)$ which is Cohen-generic over the model $V[G_\lambda][c]$. The set $b \cap c$ is still in $j(U)$, and it is $P_{j_\lambda(U)}$-generic over $V[G_\lambda]$ by Claim 12.3.3 applied over $V[G_\lambda]$.

The conjunction of Claims 12.3.5 and 12.3.6 shows that $j(U) \cap D \neq 0$ as desired.
Chapter 13

Appendix

13.1 Descriptive set theory of forcing

In this section, we will show that various operations which are the essence of
the forcing method are Borel in a suitable sense. The resulting lemmas are
perhaps more difficult to state properly than they are to prove. Nevertheless,
they are quite useful in many complexity computations. At the same time, they
are absent in all textbook treatments of the forcing method.

For the notation in this section, let \( X = 2^{\omega \times \omega} \) be the Polish space of all
binary relations on \( \omega \). Each element of \( X \) is understood as a model for a language
with a single binary relational symbol. The following lemma is standard.

**Proposition 13.1.1.** Suppose that \( f : 2^\omega \to X \) is a Borel function,
\( \phi \) is a
formula of the language with \( n \) free variables, and \( g_i : 2^\omega \to \omega \) are Borel functions
for every \( i \in n \). The set \( \{ x \in X : f(x) \models \phi(g_0(x), g_1(x), \ldots, g_{n-1}(x)) \} \) is Borel.

**Proof.** By induction on complexity of the formula \( \phi \). Left to the reader. \( \square \)

Now, if \( M \) is a countable model of ZF and \( P \in M \) is a poset, one may want
to produce a filter \( G \subset P \) generic over \( M \) and construct a forcing extension
\( M[G] \). This is a Borel procedure, as the next two lemmas show.

**Proposition 13.1.2.** Suppose \( M : 2^\omega \to X \) and \( P : 2^\omega \to \omega \) are Borel functions
such that for every \( y \in 2^\omega \), \( M(y) \) is a model of ZF and \( M(y) \models P(y) \) is a poset.
Then, there is a Borel function \( G : 2^\omega \to P(\omega) \) such that for every \( y \in 2^\omega \), \( G(y) \)
is a filter on \( P(y) \) which is generic over \( M(y) \).

**Proof.** By induction on \( n \in \omega \) define Borel functions \( f_n : 2^\omega \to \omega \) so that

- for every \( y \in 2^\omega \), \( M(y) \models f_n(y) \) is the largest element of the poset \( P(y) \);
- for every \( y \in 2^\omega \) and \( n \in \omega \), if \( M(y) \models n \) is an open dense subset of the
  poset \( P(y) \), then \( f_{n+1}(y) \) is the smallest number \( m \) such that \( M(y) \models m \)
is an element of \( n \) and it is smaller that \( f_n(y) \) in the poset \( P(y) \); otherwise,
  \( f_{n+1}(y) = f_n(y) \).
The functions defined in this way are Borel by Proposition 13.1.1. Let $G : 2^\omega \to P(\omega)$ be defined so that for every $y \in \omega$, $G(y)$ is the set of all $m$ such that $M(y) \models m$ is an element of the poset $P(y)$ and for some $n \in \omega$, $M(y) \models f_n(y)$ is smaller than $m$ in the poset $P(y)$. It is clear that the function $G$ works.

**Proposition 13.1.3.** Suppose that $M : 2^\omega \to X$, $P : 2^\omega \to \omega$, $G : 2^\omega \to P(\omega)$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a model of ZF, $M(y) \models P(y)$ is a poset, and $G(y) \subset P(y)$ is a filter generic over $M(y)$. Then, there are Borel functions $M[G] : 2^\omega \to X$ and $re : 2^\omega \times \omega \to \omega$ such that for every $y \in 2^\omega$, $M[G](y)$ is a generic extension of $M(y)$ by $G(y)$ and for every $n \in \omega$, $re(y)(n) = n/G(y)$ whenever $M(y) \models n$ is a $P(y)$-name.

**Proof.** By induction on $n \in \omega$ build Borel functions $f_n : 2^\omega \to \omega$ such that for every $y \in 2^\omega$, $f_n(y)$ is the smallest number $m$ such that $M(y) \models m$ is a $P(y)$-name, and for every $n' < n$ it is not the case that the filter $G(y)$ contains a condition $p$ such that $M(y) \models p \models P(y) f_{n'}(y) = f_n(y)$. These functions are Borel by Lemma 13.1.1. Let $M[G] : 2^\omega \to X$ be the Borel function defined by $M[G](y) = \langle n,m \rangle$ if the filter $G(y)$ contains a condition $p$ such that $M[y] \models p \models P(y) f_n(y) \in f_m(y)$. Let $re : 2^\omega \times \omega \to \omega$ be the function defined by $re(y,n) = n$ if there is a condition $p$ in the filter $G(y)$ such that $M(y) \models p \models P(y) f_m(x) = n$. These functions $M[G], re$ work by basic theorems on forcing applied in the models $M[y]$ for $y \in 2^\omega$.

If $M$ is a transitive countable model of set theory, $P \in M$ is a poset, $\tau \in M$ is a $P$-name for a transitive set, and $a$ is a transitive set, one may ask whether there is a filter $G \subset P$ which is generic over $M$ such that $a = \tau/G$, and attempt to produce such a filter if it exists. The following proposition shows that this is a Borel procedure. An important nontrivial case arises in applications where $P \models V(\tau)$ fails the axiom of choice. The proposition apparently only works for wellfounded models as opposed to arbitrary (perhaps illfounded) models.

**Proposition 13.1.4.** Suppose $M : 2^\omega \to X$, $P : 2^\omega \to \omega$ and $a : 2^\omega \to X$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a wellfounded model of ZFC, $M(y) \models P(y)$ is a poset, $\tau(y)$ is a $P(y)$-name for a transitive set. Then,

1. the set $B = \{ y \in Y : \text{there is a filter } G \subset P(y) \text{ generic over the model } M(y) \text{ such that } \langle \tau(y)/G, \epsilon \rangle \text{ is isomorphic to } a \}$ is Borel;

2. there is a Borel function $G : B \to P(\omega)$ such that for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over $M(y)$ such that $\langle \tau(y)/G(y), \epsilon \rangle$ is isomorphic to $\langle a(y), \epsilon \rangle$.

**Proof.** Use Proposition 13.1.1 to find Borel functions $P_0, \sigma, \nu, P_1, P_2 : 2^\omega \to \omega$ so that for every $y \in 2^\omega$, $M(y)$ satisfies the following rest of this paragraph. $\sigma(y)$ is some $P(y) * Coll(\omega, |\tau(y)|)$-name for an isomorph of $\langle \tau(y), \epsilon \rangle$ with domain $\omega$. Write $Q$ be the poset of nonempty open subsets of the infinite permutation group $S_\infty$, adding a Cohen-generic element $\pi \in S_\infty$. Then $P_0(y)$ is the three step iteration $P(y) * Coll(\omega, |\tau(y)|) * Q$, $\nu(y)$ is the $P_0(y)$-name for the binary
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relation \( \sigma \circ \tau \) on \( \omega \). \( P_1(y) \) is the complete Boolean algebra generated by the name \( \nu(y) \), a complete subalgebra of the completion of the poset \( P_0(y) \). Let \( \bar{P}_2(y) \) be the \( P_1 \)-name for the remainder poset \( P_0(y)/P_1(y) \).

Let \( D = \{ \langle y, \pi, G \rangle \in 2^\omega \times \mathcal{P}(\omega) : G \subset P_1(y) \} \) be a filter generic over the model \( M(y) \) such that \( \nu(y)/G = a(y) \circ \pi \). This is a Borel set by Proposition 13.1.3. The projection of \( D \) into the \( 2^\omega \) coordinate is the set \( B \) by definitions. Write \( C \subset 2^\omega \times S_\infty \) for the projection of \( D \) into the first two coordinates.

Claim 13.1.5. 1. The \( \mathcal{P}(\omega) \)-sections of \( D \) are either empty or else singletons.

2. The \( S_\infty \)-sections of \( C \) are either empty or else comeager in \( S_\infty \).

Proof. To simplify the notation, fix \( y \in 2^\omega \) and omit the argument \( y \) from the expressions like \( M(y), \sigma(y) \ldots \).

(1) uses the wellfoundedness of the model \( M \). Suppose that the \( \mathcal{P}(\omega) \)-section \( D_{y,\pi} \) is nonempty. As \( M \models \text{"}P_1 \text{"} \) is completely generated by the name \( \nu^\pi \), the filter \( G \) can be recovered by transfinite induction using infinitary Boolean expressions in \( M \) applied to \( a \circ \pi \), and therefore it is unique.

(2) is more difficult, and it is the heart of the proof. Suppose that the \( S_\infty \)-section \( C_y \) is nonempty. Thus, there is a filter \( G_0 \subset P \) be a filter generic over \( M \) such that \( \langle y/G_0, \pi/G_0 \rangle \) is isomorphic to the binary relation \( a \) on \( \omega \). Let \( G_1 \subset \text{Coll}(\omega, \tau/G_0) \) be a filter generic over \( M[G_0] \), and let \( z' = \sigma/(G_0*G_1) \). Thus, \( z, z' \) are isomorphic binary relations on \( \omega \), an there is a permutation \( \pi_0 \in S_\infty \) such that \( z = z' \circ \pi_0 \). Recall that \( Q \) is the poset of nonempty open subsets of \( S_\infty \). Let \( N \) be a countable elementary submodel of a large enough structure containing \( M, G_0, G_1, y_0 \). It will be enough to show that every element of \( S_\infty \) which is \( Q \)-generic over \( N \) belongs to the set \( C_y \), since there are comeagerly many such points. Let \( \pi \in S_\infty \) be a point \( Q \)-generic over \( N \). Since the meager ideal on \( S_\infty \) is translation invariant, even the point \( \pi_0^{-1} \pi \) is \( Q \)-generic over \( N \) and therefore over the smaller model \( M[G_0][G_1] \) as well. Let \( G_2 \subset Q \) be the filter generic over \( M[G_0][G_1] \) associated with \( \pi_0^{-1} \pi \). Now, \( \nu \circ \pi = (\nu \circ \pi_0 \circ \pi_0^{-1} \circ \pi) = (\nu \circ \pi_0^{-1}) \circ \pi \). Therefore, \( z \circ \pi \) is equal to the point \( \nu/(G_0*G_1*G_2) \pi \) in \( C_y \) as required.

Now, as one-to-one projections of Borel sets are Borel [22, Theorem 15.1], the set \( C \) is Borel by Claim 13.1.5(1). As the category quantifier yields Borel sets [22, Theorem 16.1], the set \( B \) as the projection of \( C \) into the first coordinate is Borel. Borel sets with nonmeager vertical sections allow Borel uniformizations [22, Theorem 18.6], and so there is a Borel uniformization \( f : B \to S_\infty \) of \( C \). As the set \( D \) has singleton vertical sections, it is itself its uniformization \( g : C \to \mathcal{P}(\omega) \).

Let \( G_1 : B \to \mathcal{P}(\omega) \) be the function defined by \( G_1(y) = g(y, f(y)) \). Thus, for every \( y \in B \), \( G_1(y) \subset P_1(y) \) is a filter generic over \( M(y) \) such that \( \nu/G_1(y) \) is isomorphic to \( a(y) \).

The argument is now in its final stage. Let \( P_2 : 2^\omega \to \omega \) be a Borel function such that for every \( y \in 2^\omega \), \( M(y) \models P_2(y) \) is a name for the quotient poset \( P_0(y)/P_1(y) \). Use Propositions 13.1.2 and 13.1.3 to find a Borel function \( G_2 : B \to \mathcal{P}(\omega) \) such that \( G_2(y) \subset P_2/G_1(y) \) is a filter generic over the model.
Let \( M(y)[G_1(y)] \). Let \( G_0: B \to \mathcal{P}(\omega) \) be a Borel function indicating a filter on \( P_0(y) \) which is the composition of \( G_1 \) and \( G_2 \). Let \( G: B \to \mathcal{P}(\omega) \) be the function which indicates the first coordinate of the filter \( G_0(y) \subset P_0(y) \). Recall that the poset \( P_0(y) \) is a three stage iteration of which the first stage is \( P(y) \), so for every \( y \in B \), \( G(y) \subset P(y) \) is a filter generic over the model \( M(y) \). The function \( G \) has the required properties.

**Corollary 13.1.6.** Suppose that \( \Gamma \) is a Polish group continuously acting on a Polish space \( X \), inducing an orbit equivalence relation \( E \). Let \( M \) be a countable transitive model of set theory containing a code for the action, \( P \in M \) and \( \tau \in M \) be a poset and a name for an element of the space \( X \). Then

1. the set \( B = \{ x \in X : \exists G \subset P \text{ generic over } M \text{ such that } \tau/G = x \} \) is Borel;

2. the equivalence relation \( E \restriction B \) is Borel.

**Proof.** (1) is immediate from the proposition. To see (2), let \( Q = P \times P \) and let \( \sigma \) be the \( Q \)-name for the pair \( \langle \tau, \gamma_{\text{gen}} \cdot \tau \rangle \). Now, the set \( C = \{ (x,y) \in X^2 : \exists G \times H \subset P \times P \text{ generic over } M \text{ such that } \langle x,y \rangle = \sigma/G \times H \} \) is Borel by the proposition again. Now, for \( x,z \in B \), \( x E z \) just in case the set \( A_{xz} = \{ \gamma \in \Gamma : \langle x, \gamma \cdot z \rangle \in C \} \) is comeager in \( \Gamma \). To see that, note that if \( x E z \) fails then the set \( A_{xz} \subset \Gamma \) is actually empty, and if \( x E z \) holds with \( \delta \cdot x = z \), then the \( A_{xz} \supset \{ \gamma \in \Gamma : \gamma \cdot \delta^{-1} \text{ is } P_1\text{-generic over } M[x] \} \) and the latter set is comeager since it is a shift of the comeager set of all points in \( \Gamma \) which are \( P_1\)-generic over the model \( M[x] \).

Finally, observe that the equivalence relation \( E \restriction B \) is obtained by an application of the category quantifier to the Borel set \( \{ (x,z,\gamma) \in B \times B \times \Gamma : \gamma \in A_{xz} \} \) and so is Borel. \( \square \)

### 13.2 Very Suslin posets

In this section, we look more carefully at Suslin c.c.c. forcings. Recall:

**Definition 13.2.1.** \([15]\) \( \langle P, \leq \rangle \) is a Suslin poset if there is an ambient Polish space \( X \) such that \( P \subset X \) is an analytic set, \( \leq \) is an analytic subset of \( X^2 \), and the incompatibility relation \( \perp \) on \( P \) is an analytic subset of \( X^2 \). To ease the notational clutter, in this section we only require that \( \leq \) is a transitive relation on \( P \) containing the diagonal. In addition, the poset \( P \) is required to have a largest element.

A given Suslin poset \( \langle P, \leq \rangle \) can be reinterpreted in every forcing extension. Note that the demands on the analytic definitions of \( \leq \) and \( \perp \) are \( \Pi^1_2 \) and so the reinterpretation is again a Suslin poset. Importantly, the c.c.c. property of Suslin posets persists to forcing extensions.

**Fact 13.2.2.** \([15]\) Let \( \langle P, \leq \rangle \) be a Suslin c.c.c. poset. The reinterpretation of \( P \) in any forcing extension is c.c.c.
The purpose of this section is to analyze the finite support iterations of c.c.c. Suslin forcings from descriptive point of view. The analysis is interesting in its own right. One has to enter the much more restrictive class of very Suslin forcings to guarantee that the iterations have the desirable descriptive properties.

**Definition 13.2.3.** Suppose that $P$ is a Suslin c.c.c. poset on an ambient Polish space $X$. The poset is very Suslin if the set \( \{ a \in P^\omega : \text{rng}(a) \text{ is a maximal antichain in } P \} \) is an analytic subset of $X^\omega$.

Note that being a very Suslin partial order is a $\Pi^1_2$ statement in the code for the analytic set of maximal antichains and therefore absolute among all forcing extensions. Note that if the poset $P$ and the ordering $\leq$ are Borel, which is the case for all applications in this book and elsewhere as well, then both the incompatibility relation and the set of maximal antichains are coanalytic and therefore Borel by the Suslin theorem.

**Example 13.2.4.** The Cohen forcing, random forcing, and the eventually different real forcing are very Suslin. The Hechler forcing is Suslin, but not very Suslin.

*Proof.* The Suslinity of the definition of the posets is elementary and left for the reader to check; we only verify the very Suslin property.

The Cohen forcing is the poset $P$ of finite binary strings ordered by reverse inclusion. A set $a \subseteq P$ is predense if every condition in $P$ is compatible with some element of $a$, which in view of the fact that $P$ is countable is a Borel condition. If the random forcing $P$ is realized as the collection of compact $\mu$-positive measure subsets of $X$ ordered by inclusion, where $\mu$ is some Borel probability measure on a Polish space $X$, then a countable set $a \subseteq P$ is pre-dense just in case $\mu(\bigcup a) = 1$, which is a Borel condition. The case of the eventually different real forcing is somewhat more involved, and addressed in [38, Proposition 3.8.12]. Hechler forcing is not very Suslin in view of Proposition 13.2.5 below, since it adds a dominating real.

The most critical distinction (and the original motivation of the definition of the very Suslin class) between Suslin and very Suslin c.c.c. posets is that the latter cannot add dominating reals:

**Proposition 13.2.5.** Let $P$ be a very Suslin c.c.c. forcing. Then $P$ does not add dominating reals.

*Proof.* Let $p \in P$ and $\tau$ be a $P$-name for an element of $\omega^\omega$. Consider the set $A = \{ T \subseteq \omega^{<\omega} : \exists q \leq p \models \dot{T} \text{ has no infinite branch modulo finite dominated by } \tau \}$. By the evaluation of the complexity of the forcing relation in Proposition 13.2.7 below, $A$ is an analytic set of trees. At the same time, if $p \models \tau$ is a dominating real, then $A$ is the set of all well-founded trees, which is coanalytic and not analytic. Since the conclusion is impossible, the assumption must be false as well. The proposition follows.
To proceed with the main task of this section, we need to record a statement about the complexity of the forcing relation of very Suslin posets.

**Proposition 13.2.7.** Suppose that $P$ is a very Suslin c.c.c. poset and $X$ is a Polish space.

1. A *nice $P$-name* for an element of $X$ is a tuple $\tau = \langle A_n, f_n : n \in \omega \rangle$ such that $A_n \subseteq P$ is a maximal antichain, $f_n$ is a map from $A_n$ to basic open subsets of $X$ of radius $< 2^{-n}$, and if $m < n$ then $A_m$ refines $A_n$ and if $p_n \in A_n$ and $p_m \in A_m$ are such that $p_m \leq p_n$, then the closure of $f_m(p_m)$ is a subset of $f_n(p_n)$.

2. $X^P$ is the set of all nice $P$-names for elements of $X$.

Note that the set $X^P$ is analytic in a suitable ambient space. The definition of $X^P$ appears to depend on the choice of the metric for $X$, a dependence that we will happily suppress. It is clear that for every name for an element of the space $X$ there is a nice name which is forced to be equal to the original, perhaps not so nice, name.

**Proposition 13.2.7.** Suppose that $P$ is a very Suslin c.c.c. poset, $X$ is a Polish space, and $A \subseteq X$ is an analytic set. The set $\{ \langle p, \tau \rangle : p \forces \tau \in A \}$ is an analytic subset of $P \times X^P$.

**Proof.** Let $\tau = \langle A_n, f_n : n \in \omega \rangle$. Let $h : \omega^\omega \to A$ be a continuous surjection. The statement $p \forces \tau \in A$ is equivalent to the existence of a system $\langle B_n, g_n : n \in \omega \rangle$ where for each $n \in \omega$, $B_n$ is a maximal antichain of $P$ of conditions which are either below $p$ or incompatible with $p$, $g_n : B_n \to \omega^{<\omega}$, $B_{n+1}$ refines $B_n$ and $A_n$, whenever $r \leq q$ are conditions in $B_{n+1}$ and $B_n$ respectively then $g_{n+1}(r)$ properly extends $g_n(q)$, and if $r \leq q$ is an element of $B_{n+1}$ below $p$ and an element of $A_n$ respectively then $[g_{n+1}(r)] \subseteq h^{-1}f_n(q)$. This is an analytic statement.

Now we are in a position to define the two-step iteration of very Suslin c.c.c. posets.

**Definition 13.2.8.** Let $P$ be a very Suslin poset on a Polish space $X$ and $Q$ be a very Suslin poset on a Polish space $Y$. Define $P * \dot{Q}$ to be the set of all ordered pairs $\langle p, \dot{q} \rangle$ where $p \in P$ and $q$ is a nice $P$-name for an element of $Y$ such that $p \forces \dot{q} \in \dot{Q}$. The ordering is defined by $\langle p_1, \dot{q}_1 \rangle \leq \langle p_0, \dot{q}_0 \rangle$ if $p_1 \leq p_0$ and $p_1 \forces \dot{q}_1 \leq \dot{q}_0$.

**Proposition 13.2.9.** $P * \dot{Q}$ is a very Suslin c.c.c. poset.

**Proof.** This is a routine complexity calculation. The underlying set $P * \dot{Q}$ is analytic by Proposition 13.2.7, since $p \forces \dot{q} \in \dot{Q}$ is an analytic statement. The ordering on $P * \dot{Q}$ is analytic for the same reason. To check the analyticity of the incompatibility relation, observe that conditions $\langle p_0, \dot{q}_0 \rangle$ and $\langle p_1, \dot{q}_1 \rangle$ are incompatible just in case there is a (countable) maximal antichain $A \subseteq P$ such
that for every $p \in A$, either $p$ is incompatible with $p_0$, or it is incompatible with $p_1$, or it is below both $p_0, p_1$, and in the latter case, $p \forces \check{q}_0, \check{q}_1$ are incompatible in the poset $Q$. This is an analytic statement by Proposition 13.2.7 and the assumption that the incompatibility relation on $Q$ is analytic.

The poset $P \ast Q$ is c.c.c. because $Q$ remains c.c.c. in the $P$-extension by Fact 13.2.2 and so $P \ast Q$ is an iteration of two c.c.c. forcings. Finally, we have to check that for a countable set $B = \{(p_n, \check{q}_n) : n \in \omega\} \subset P \ast Q$, the statement that $B$ is predense is analytic. To see this, let $Y_0$ be the Polish space resulting from adding an isolated point $0$ to $Y$, and consider the following formula $\phi$: there exists a name $\tau$ for an element of $(Y_0)^\omega$ and maximal antichains $A_n \subset P$ for $n \in \omega$ such that for each $n$, $A_n$ consists of elements which are either incompatible with $p_n$ or stronger than $p_n$, if they are stronger than $p_n$, then they force $\tau(n) = \check{q}_n$, if they are incompatible with $p_n$ then they force $\tau(n) = 0$, and $1_P$ forces $\text{rng}(\tau) \cap Y$ is predense in $\check{Q}$. Parsing the formula $\phi$, we see that it is analytic by Proposition 13.2.7, and that it says that $1_P \forces \{\check{q}_n : p_n \text{ belongs to the generic filter}\}$ is predense in $\check{Q}$, which is exactly equivalent to the set $B$ being predense in $P \ast Q$.

The next step in the analysis of the iteration is to show that a direct limit of a collection of very Suslin posets is again a very Suslin poset. The following definitions and a proposition solve this task.

**Definition 13.2.10.** Let $P, Q$ be posets. A *projection* of $Q$ to $P$ is a pair of order-preserving functions $\pi : Q \to P$ and $\xi : P \to Q$ such that

1. $\pi \circ \xi$ is the identity on $P$;
2. whenever $\pi(q) \leq p$ then $q \leq \xi(p)$;
3. whenever $p \leq \pi(q)$ then there is $q' \leq q$ such that $\pi(q') \leq p$.

As a good initial example, if $P, Q$ are very Suslin posets, then the function $\pi : P \ast Q \to P$ defined by $\pi(p, \check{q}) = p$ together with the function $\xi : P \to P \ast Q$ defined by $\xi(p) = (p, 1_Q)$ is a projection.

**Definition 13.2.11.** A *very Suslin system* is a tuple $\langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega\rangle$ where

1. each $P_n$ is a very Suslin c.c.c. forcing;
2. for each $m \leq n \in \omega$, the functions $\pi_{nm} : P_n \to P_m$ and $\xi_{mn} : P_m \to P_n$ are analytic and form a projection of $P_n$ to $P_m$, with $\pi_{nm}$ and $\xi_{mn}$ equal to the identity on $P_n$;
3. the functions $\pi_{nm}$ commute, as do the functions $\xi_{mn}$.

The *limit* of the system is the poset $P_\omega$ of all pairs $\langle p, n \rangle$ where $p \in P_n$, ordered by $\langle q, n \rangle \leq \langle p, m \rangle$ if $m \leq n$ and $\pi_{nm}(q) \leq p$ in $P_n$. The limit also includes the projections $\pi_{nm} : P_n \to P_m$ and $\xi_{nm} : P_m \to P_n$ defined by $\xi_{nm}(p) = \langle p, m \rangle$ and $\pi_{nm}(\langle p, n \rangle) = \pi_{nm}(p)$ if $n > m$ and $\pi_{nm}(\langle p, n \rangle) = \xi_{nm}(p)$ if $n \leq m$. 
Proposition 13.2.12. Suppose that \( \langle P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega \rangle \) is a very Suslin system. The limit \( P_\omega \) is a very Suslin c.c.c. forcing. The functions \( \pi_{\omega m} \) and \( \xi_{m\omega} \) form an analytic projection and commute with the projection maps of the very Suslin system.

Proof. This is a straightforward complexity computation. Let \( X_n \) be the ambient Polish space of each poset \( P_n \); the ambient space of the poset \( P_\omega \) is then the disjoint union \( \bigcup_n X_n \) in product with \( \omega \). The ordering of \( P_\omega \) is analytic by the definition. The incompatibility relation is analytic as well: suppose that \( \langle q, n \rangle \) and \( \langle p, m \rangle \) are conditions in \( P_\omega \), say with \( n \geq m \). These two conditions are incompatible just in case \( \pi_{nm}(q) \) is incompatible with \( p \) in \( P_m \) by the definitory properties of projections.

The poset \( P_\omega \) is c.c.c. since it is a direct limit of c.c.c. forcings. To check the very Suslin property of the poset \( P_\omega \), note that a countable set \( A \subset P_\omega \) is predense if and only if for every number \( n \in \omega \), there is a maximal antichain \( B_n \subset P_\omega \) of conditions such that for each \( p \in B_n \), either there is a condition \( \langle q, m \rangle \in A \) such that \( m \leq n \) and \( \pi_{nm}(p) \leq q \) in \( P_m \), or there is a condition \( \langle q, m \rangle \in A \) such that \( m \geq n \) and \( p \leq \pi_{mn}(q) \) in \( P_n \). This is an analytic statement as all the posets \( P_n \) are very Suslin.

Checking the projection properties of the functions \( \pi_{\omega m} \) and \( \xi_{m\omega} \) is routine and left to the reader.

The following corollary is the main fruit of the effort of this section; it is proved by a straightforward transfinite induction argument using Propositions 13.2.9 and 13.2.12. To ease on the notational clutter, we deal only with iterations of the same very Suslin poset reinterpreted in the relevant forcing extensions.

Corollary 13.2.13. Let \( \alpha \in \omega_1 \) be a countable ordinal and \( P \) a very Suslin c.c.c. poset. The finite support iteration of \( P \) of length \( \alpha \) is isomorphic to a very Suslin c.c.c. poset.

In several places in the book, we need preservation theorems for the finite support iterations of very Suslin posets which are descriptive in nature. These appear to be novel, even if not exactly shocking.

Definition 13.2.14. Let \( P \) be a Suslin forcing. Say that \( P \) is Suslin \( \sigma \)-centered if \( P = \bigcup_n A_n \) where for each \( n \in \omega \), \( A_n \subset P \) is an analytic set, and every finite subset of \( A_n \) has a lower bound in \( P \).

Note that the definitory properties of the cover \( P = \bigcup_n A_n \) are \( \Pi^1_2 \) and therefore persist to all forcing extensions.

Example 13.2.15. Consider the poset \( P \) of all finite partial functions \( p : 2^\omega \rightarrow \omega \) assigning distinct values to distinct \( \mathbb{E}_0 \)-related elements. The ordering is that of reverse inclusion. The poset \( P \) is c.c.c. very Suslin, \( \sigma \)-centered but not Suslin \( \sigma \)-centered.

Proof. It is immediate that \( P \) is Suslin. To verify the very Suslin condition, note that for every \( \mathbb{E}_0 \)-invariant set \( a \subset 2^\omega \), the poset \( P_a = \{ p \in P : \text{dom}(p) \subset a \} \) is a
regular subposet of $P$. Thus, a countable set $b \subset P$ is predense in $P$ just in case every condition $p$ such that $\text{dom}(p)$ is a subset of the countable $\mathbb{E}_0$-saturation of $\bigcup \{\text{dom}(q) : q \in b\}$ is compatible with some condition in $b$. This is an analytic condition.

To see the $\sigma$-centeredness of the poset, apply the first paragraph to see that the poset $P$ is just a finite product of $P_a$ as $a$ runs through all $\mathbb{E}_0$-classes. In the context of the Axiom of Choice, this means that $P$ is isomorphic to the product of $c$ many Cohen forcings, which is well-known to be $\sigma$-centered.

To exclude the Suslin $\sigma$-centeredness, suppose towards contradiction that $P = \bigcup_n A_n$ where $A_n$ is an analytic centered set. Let $B_n = \{x \in 2^\omega : \{(x,0)\} \in A_n\}$. The set $B_n \subset 2^\omega$ is analytic, and since the set $A_n$ is centered, it must be the case that $B_n$ is a $\mathbb{G}_0$-anticlique. Also, $2^\omega = \bigcup_n B_n$. Since by the first reflection theorem, each $B_n$ is contained in a Borel $\mathbb{G}_0$-anticlique, this contradicts the fact that $\mathbb{G}_0$ has uncountable Borel chromatic number.

**Theorem 13.2.16.** Let $P$ be a very Suslin c.c.c. forcing which is Suslin $\sigma$-centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of $P$ of length $\alpha$ is a Suslin $\sigma$-centered forcing.

**Proof.** The argument proceeds by a straightforward transfinite induction argument given the following two claims.

**Claim 13.2.17.** Let $P, Q$ be very Suslin c.c.c. forcings, both of which are Suslin $\sigma$-centered. Then $P \ast Q$ is Suslin $\sigma$-centered.

**Proof.** Let $P = \bigcup_n A_n$ and $Q = \bigcup_m B_m$ be the covers of $P, Q$ by analytic centered sets. Let $C_{nm} \subset P \ast Q$ be the set of all conditions $(p, q)$ such that there exists a condition $(p', q') \leq (p, q)$ such that $p' \in A_n$ and $p' \vDash q' \in B_m$. It is not difficult to check that $P \ast Q = \bigcup_{nm} C_{nm}$, the sets $C_{nm}$ are centered. Moreover, the sets $C_{nm}$ are analytic by Proposition 13.2.7.

**Claim 13.2.18.** Let $(P_n : n \in \omega, \pi_{nm}, \xi_{mn} : m \leq n \in \omega)$ be a very Suslin system consisting of Suslin $\sigma$-centered forcings. Then the limit is Suslin $\sigma$-centered.

**Proof.** Let $P_\omega$ be the limit of the system as described in Definition 13.2.11. Let $P_n = \bigcup_m A_{nm} : n \in \omega$ be a cover by analytic centered sets for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subset P_\omega$ be the set $\{\langle p, n \rangle \in P_\omega : p \in A_{nm}\}$. It is immediate that these are centered analytic sets covering $P_\omega$.

The theorem follows.

A sample application of the Suslin $\sigma$-centered concept is encapsulated in the following proposition:

**Proposition 13.2.19.** Let $P$ be a c.c.c. very Suslin, Suslin $\sigma$-centered forcing. Let $n \in \omega$ and let $H$ be an analytic hypergraph of arity $n$ on a Polish space $X$ with uncountable Borel chromatic number. Let $G \subset P$ be a generic filter. In $V[G]$, $X$ cannot be covered by countably many analytic $H$-anticliques.
Proof. By the hypergraph dichotomies of [27], it is enough to prove the proposition for the specific hypergraph $H$ on $X = n^\omega$ which consists of $n$-tuples $\langle x_i : i \in n \rangle$ of points in $X$ which agree on all entries except for one, say $m$, where $x_i(m) = i$. Suppose towards contradiction that $\{ A_n : n \in \omega \}$ are names for analytic $H$-antiques in $2^\omega$ and $p \in P$ is a condition such that $p \Vdash 2^\omega \cap V \subset \bigcup_n A_n$. Let $P = \bigcup_m B_m$ be a union of analytic $\alpha$-centered sets. For each $m, n \in \omega$ let $C_{mn} = \{ x \in X : \exists q \leq p \ q \in B_m \land q \Vdash x \in A_n \}$. By Proposition 13.2.7, each $C_{mn} \subset X$ is an analytic set, and $X = \bigcup_{mn} C_{mn}$. The sets $C_{mn}$ have the Baire property, and so one of them is comeager in some basic open subset of $X$. In this set $C_{mn}$, one can find an $H$-hyperedge $\langle x_i : i \in n \rangle$. By the centeredness of the set $B_m$, the conditions $q_i \in B_m$ for $i \in n$ forcing $\check{x}_i A_n$ have a common lower bound $q$. But then, $q \Vdash A_n$ is not an $H$-antique as witnessed by the hyperedge $\langle x_i : i \in n \rangle$, contradicting the initial assumptions. \hfill \Box

We conclude this section with a somewhat more permissive regularity property of Suslin forcing notions, which should be compared to [33, Definition 3].

**Definition 13.2.20.** Let $P$ be a Suslin forcing.

1. If $r \in \omega$, a set $A \subset P$ is **Ramsey $r$-centered** if there is a number $k \in \omega$ such that every subset of $A$ of size $k$ has a subset of size $r$ with a common lower bound.

2. $P$ is **Suslin Ramsey-centered** if for every number $r \in \omega$ there is a cover $P = \bigcup_n A_n$ by analytic Ramsey-$r$-centered sets.

Note that the definitory properties of the covers are $\Pi_2^1$ and therefore persist to all forcing extensions.

**Theorem 13.2.21.** Let $P$ be a very Suslin c.c.c. forcing which is Suslin Ramsey-centered. Let $\alpha \in \omega_1$ be a countable ordinal. Then the finite support iteration of $P$ of length $\alpha$ is a Suslin Ramsey-centered forcing.

**Proof.** The argument proceeds by a straightforward transfinite induction argument given the following two claims.

**Claim 13.2.22.** Let $P, Q$ be very Suslin c.c.c. forcings, both of which are Suslin Ramsey-centered. The $P \ast Q$ is Suslin Ramsey-centered.

**Proof.** Let $r \in \omega$. Find a cover $Q = \bigcup_m B_m$ by analytic Ramsey-$r$-centered sets with corresponding numbers $k_m$. For each $m \in \omega$ find a cover $P = \bigcup_n A_{nm}$ by analytic $k_m$-centered sets, with corresponding numbers $k_{mn}$. Let $C_{nm} \subset P \ast \dot{Q}$ be the set of all conditions $\langle p, \dot{q} \rangle$ such that there exists a condition $\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle$ such that $p' \in A_{nm}$ and $p' \Vdash \dot{q}' \in B_m$. It is not difficult to check that $P \ast \dot{Q} = \bigcup_{mn} C_{nm}$, the sets $C_{nm}$ are Ramsey-$r$-centered as witnessed by the numbers $k_{nm}$. Moreover, the sets $C_{nm}$ are analytic by Proposition 13.2.7. \hfill \Box

**Claim 13.2.23.** Let $\langle P_n : n \in \omega, \pi_{nm}, \zeta_{nm} : m \leq n \in \omega \rangle$ be a very Suslin system consisting of Suslin Ramsey-centered forcings. Then the limit is Suslin Ramsey-centered.
13.3. PRODUCT FORCING

Proof. Let $P_\omega$ be the limit of the system as described in Definition 13.2.11. Let $r \in \omega$ be a number. Let $P_n = \bigcup_m A_{nm} : n \in \omega$ be a cover by analytic Ramsey-$r$-centered sets for each $n \in \omega$. For each $n, m \in \omega$ let $B_{nm} \subseteq P_\omega$ be the set $\{\langle p, n \rangle \in P_\omega : p \in A_{nm}\}$. It is immediate that these are Ramsey-$r$-centered analytic sets covering $P_\omega$.

The theorem follows.

A sample preservation application of the Suslin Ramsey-centered concept:

Proposition 13.2.24. Let $P$ be a c.c.c. very Suslin, Suslin Ramsey-centered forcing. Let $G \subseteq P$ be a generic filter. In $V[G]$, $2^\omega \cap V$ cannot be covered by countably many analytic $E_0$-anticliques.

Proof. Suppose towards contradiction that $\{\dot{A}_n : n \in \omega\}$ are names for analytic $E_0$-anticliques in $2^\omega$ and $p \in P$ is a condition such that $p \Vdash 2^\omega \subseteq \bigcup_n A_n$. Let $P = \bigcup_m B_m$ be a union of analytic Ramsey-2-centered sets. For each $m \in \omega$ let $k_m \in \omega$ be a number witnessing the Ramsey property of the set $B_m$. For each $m, n \in \omega$ let $C_{mn} = \{x \in 2^\omega : \exists q \leq p q \in B_m \land q \Vdash \dot{x} \in A_n\}$. By Proposition 13.2.7, each $C_{mn} \subseteq 2^\omega$ is an analytic set, and $2^\omega = \bigcup_{mn} C_{mn}$. The set $C_{mn}$ have the Baire property, and so one of them is comeager in some basic open subset of $2^\omega$. In this set $C_{mn}$, one can find an $E_0$-clique of size $k_m$. By the Ramsey property of the set $B_m$, there must be distinct points $x_0, x_1 \in C_{mn}$ in this clique such that the conditions $q_0, q_1 \in B_m$ forcing them into $\dot{A}_n$ are compatible with some lower bound $q$. But then, $q \Vdash A_n$ is not an $E_0$-anticlique as witnessed by the points $\dot{x}_0$ and $\dot{x}_1$, contradicting the initial assumptions.

13.3 Product forcing

The book is loaded with product forcing notions. This section provides basic information on products.

Fact 13.3.1. Let $P, Q$ be posets and in some generic extension, let $G \subseteq P, H \subseteq Q$ be filters separately generic over the ground model. The following are equivalent:

1. $G \times H \subseteq P \times Q$ is a filter generic over the ground model;
2. $G \subseteq P$ is generic over the model $V[H]$.

In the affirmative case, $V[G] \cap V[H] = V$.

If $G \times H \subseteq P \times Q$ is a filter generic over the ground model, we say that the filters $G, H$ (or their generic extensions) are mutually generic.

The following theorem is not used explicitly anywhere in this book, but it greatly simplifies the methodology of product forcing. It says that mutual genericity of forcing extensions can be characterized without an appeal to the specific generic filters and posets that were used to obtain the extensions, and indeed without any appeal to forcing at all.
Theorem 13.3.2. Let \( n \in \omega \) be a number and \( \{ P_i : i \in n \} \) be posets. Let \( \{ G_i : i \in n \} \) be filters separately generic over the ground model \( V \) over the respective posets. The following are equivalent:

1. \( \prod_i G_i \in \prod_i P_i \) is a filter generic over \( V \);

2. whenever \( \{ a_i : i \in n \} \) are subsets of the ground model in the respective models \( V[G_i] \) such that \( \bigcap_i a_i = 0 \) then there are sets \( \{ a'_i : i \in n \} \) in the ground model \( V \) such that for all \( i \in n \) \( a_i \subseteq a'_i \) and \( \bigcap_i a'_i = 0 \) holds.

Proof. Suppose first that (1) holds. Move to the ground model \( V \). Suppose that \( \{ \dot{a}_i : i \in n \} \) are \( P_i \)-names for subsets of the ground model and \( \langle p_i : i \in n \rangle \in \prod_i P_i \) is a condition forcing that \( \bigcap_i a_i = 0 \). Let \( a'_i = \{ x \in V : \exists p \leq p_i \ p \Vdash \dot{x} \in \dot{a}_i \} \) for each \( i \in n \). It is immediate that \( \bigcap_i a'_i = 0 \) holds, and for all \( i \in n \) and \( p_i \Vdash \dot{a}_i \subseteq \dot{a}'_i \) holds. (2) then follows by the forcing theorem.

Suppose now that (2) holds. To confirm (1), suppose towards contradiction that it fails. There must be an open dense set \( D \cap \prod_i P_i \) in the ground model such that \( \prod_i G_i \cap D = 0 \). For every number \( i \in n \), in the model \( V[G_i] \) consider the set \( a_i = \{ p_i : i \in n \} \in D : p_i \in G_i \} \subseteq \prod_i P_i \). The contradictory assumption gives that \( \bigcap_i a_i = 0 \); the assumption (2) yields sets \( a'_i \subseteq \prod_i P_i \) in the ground model such that \( \forall i \in n \ a_i \subseteq a'_i \) and \( \bigcap_i a'_i = 0 \). Each model \( V[G_i] \), a genericity argument shows that there must be a condition \( \dot{r}_j \in \dot{G}_j \) such that every tuple \( \langle p_i : i \in n \rangle \in D \) with \( p_j \leq r_j \) must in fact belong to the set \( a'_i \). Use the density of the set \( D \) to find a tuple \( \langle p_i : i \in n \rangle \in D \) such that for all \( i \in n \), \( p_i \leq r_i \) holds. Then \( \langle p_i : i \in n \rangle \in \bigcup_i a'_i \), contradicting the choice of the sets \( a'_i \) for \( i \in n \).

The following corollary is easy to prove without the theorem, but its present proof is much more appealing:

Corollary 13.3.3. Let \( n \in \omega \) be a natural number, \( \{ P_i : i \in n \} \) are posets, \( \{ Q_i : i \in n \} \) are posets, \( \prod_i G_i \subseteq \prod_i P_i \) is a generic filter, and for each \( i \in n \) \( H_i \in V[G_i] \) is a filter generic over \( V \) on the poset \( Q_i \). Then \( \prod_i H_i \subseteq \prod_i Q_i \) is a filter generic over \( V \).

Proof. The criterion (2) of Theorem 13.3.2 is preserved when passing to smaller models, in particular from \( V[G_i] \) to \( V[H_i] \).

The final remark in this section provides a perfect set of generics over any countable model—a standard trick which comes handy in several places in the book.

Theorem 13.3.4. Let \( M \) be a countable transitive model of set theory and \( P \in M \) be a poset. Then there is a continuous map \( h : 2^\omega \to P(P) \) such that for every finite set \( a \subset 2^\omega \), the product \( \prod_{x \in a} h(x) \) is a filter on the poset \( P^{[a]} \) generic over the model \( M \).

Proof. Let \( \langle D_n : n \in \omega \rangle \) be an enumeration of all open dense subsets of various finite powers of \( P \) which appear in the model \( M \). By induction on \( m \in \omega \) build maps \( g_m : 2^m \to P \) satisfying the following demands:
• $g_0 : \{0\} \to P$ is arbitrary;

• for every $m \in \omega$, every $s \in 2^m$ and every $t \in 2^{m+1}$ such that $s \subseteq t$, $g_{m+1}(t) \leq g_m(s)$ holds;

• for every $m \in \omega$ and every $n \leq m$, if $D_n \subseteq P_k$ is a set for some $k \leq 2^{m+1}$ and $(t_i : i \in k)$ is a sequence of distinct elements of $2^{m+1}$, then $(g_{m+1}(t_i) : i \in k) \in D$ holds.

This is easy to arrange as the second item requires meeting only finitely many open dense sets. In the end, let $h : 2^\omega \to \mathcal{P}(\omega)$ be the continuous function defined by $h(x) =$ the filter generated by the set $\{g_m(x \upharpoonright m) : m \in \omega\}$, and check that the demands of the theorem are satisfied.

\section*{13.4 Results in ZF}

This section contains results in ZF that are complementary to the various independence results obtained in the earlier parts of the book.

\textbf{Theorem 13.4.1.} (ZF) Let $X$ be a set and $E$ an equivalence relation on $X$ with all classes countable. If $|X^{\aleph_0}| \leq |E|$ then $|\mathbb{H} C| \leq |E|$.

Already the case of $E$ being the identity on $X$ is interesting. Note that $|\mathbb{H} C^{\aleph_0}| = |\mathbb{H} C|$ and so the theorem says that $|\mathbb{H} C|$ is the smallest fixed point of the Friedman–Stanley jump. Compared with Theorem ???, if one wanted to, for example, collapse the size of $|F_2|$ to $|2^\omega|$, this would automatically involve the collapse of $|\mathbb{H} C|$ to $|2^\omega|$ even though $|F_2| < |\mathbb{H} C|$.

\textit{Proof.} Let $g$ be an injection from the set of all countable subsets of $X$ to $E$-classes. Define a function $h$ from the collection of hereditarily countable sets to $E$-classes by $\varepsilon$-recursion: $h(a) = g(\bigcup h^a a)$. By induction on the minimum of the rank $a$ and $b$ argue that $a \neq b$ implies that $h(a) \neq h(b)$. Thus the function $h$ is an injection and the statement of the theorem follows. \qed

The following theorems deal with an impact of a nonprincipal ultrafilter on chromatic numbers and quotient spaces.

\textbf{Theorem 13.4.2.} (ZF + DC) Suppose that there is a nonprincipal ultrafilter on $\omega$. The class of analytic equivalence relations whose quotient space can be linearly ordered is closed under the following operations:

1. Borel reducibility;
2. countable product;
3. increasing union.
Proof. The first two items do not use the ultrafilter assumption. We will argue for the third item. Let $X$ be a Polish space and $\langle E_n : n \in \omega \rangle$ be an increasing sequence of analytic equivalence relations on $X$, each with linearly orderable quotient space. Use the DC assumption to pick linear orders $\leq$ elements, then let $\{x\}_{U}$ them. Let $X$ sequence of analytic equivalence relations on $E$ orderability of the quotient space of $E$. Let $X\times E$ be a Polish space and $\langle a \rangle_{E}$ for the third item. Let $X$ Proof. The first two items do not use the ultrafilter assumption. We will argue over ZF+DC that the chromatic number of $E$ is 2, since they imply the linear orderability assumption. By the Glimm–Effros dichotomy, the linear ordering of the quotient space $X/E$ can be pulled back to a linear ordering $\leq$ of the $E_0$ quotient space. Then, one can define a coloring $c$ of the graph $\langle a \rangle_{E}$ by $c(x_0, x_1) = 0$ if $[x_0]_{E_0} \leq [x_1]_{E_0}$, and $c(x_0, x_1) = 1$ otherwise. This completes the proof. □

Theorem 13.4.3. (ZF+DC) Suppose that there is a nonsmooth Borel equivalence relation $E$ on a Polish space $X$ and a linear ordering of the quotient space $X/E$. Then the chromatic number of $G_0$ is equal to 2.

Proof. It is enough to show that there is a Borel graph $\Gamma$ on some Polish space $Y$ of uncountable Borel chromatic number but of chromatic number 2. Then, by [21], there must be a continuous function $h : 2^\omega \to Y$ which is a homomorphism of $G_0$ to $\Gamma$. The 2-coloring of $\Gamma$ then pulls back to a 2-coloring of $G_0$.

To define the Borel graph $\Gamma$, let $Y = 2^\omega \times 2^\omega$ and let $\Gamma$ connect pairs $\langle x_0, x_1 \rangle$ and $\langle y_0, y_1 \rangle$ if $x_0, x_1$ are $E_0$-unrelated points, and $x_0, y_1$ are $E_0$-related, and $x_1, y_0$ are $E_0$-related.

First, observe that the Borel chromatic number of $\Gamma$ is uncountable. To see this, by the Baire category theorem it is enough to show that every Borel nonmeager subset of $Y$ contains a $\Gamma$-edge. This is routine and left to the reader. Finally, it is time to use the linear orderability assumption. By the Glimm–Effros dichotomy, the linear ordering of the quotient space $X/E$ can be pulled back to a linear ordering $\leq$ of the $E_0$ quotient space. Then, one can define a coloring $c$ of the graph $\Gamma$ by $c(x_0, x_1) = 0$ if $[x_0]_{E_0} \leq [x_1]_{E_0}$, and $c(x_0, x_1) = 1$ otherwise. This completes the proof. □

As an obvious corollary, the existence of a nonprincipal Ramsey ultrafilter or a (non-Borel) reduction of $E_0$ to the identity on the Cantor space both imply over ZF+DC that the chromatic number of $G_0$ is 2, since they imply the linear orderability of the quotient space of $E_0$.

The following several theorems are devoted to the consequences of an existence of a Hamel basis for various Polish vector spaces.

Theorem 13.4.4. (ZF+DC) Suppose that $X$ is a Polish vector space over a countable field $F$. If there is a basis for $X$ then there is a $E_0$-selector.

Proof. Let $B \subset X$ be a basis. Choose a countable set $C \subset B$ such that the linear span of $C$ is dense in $X$. Let $\Gamma$ be the linear span of $C$, understood as a countable discrete group with addition, let $\Gamma$ act on $X$ by addition, and let $E$ be the orbit equivalence relation on $X$. The key point is that $E$ has a transversal. Indeed, since $B \subset X$ is a basis, for each $E$-class $d \subset X$ there is a unique point
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$x_d \in d$ which belongs to the linear span of $B \setminus C$. The set $D = \{x_d: d \text{ is an } E\text{-class}\}$ is a transversal for $E$.

The remainder of the proof pulls the $E$-transversal down to $E_0$. The following simple claim is instrumental in this respect:

**Claim 13.4.5.** $E$ is not smooth.

**Proof.** Suppose that $f: X \to 2^\omega$ is a Borel reduction of $E$ to the identity. If for every number $n \in \omega$ there is a bit $b_n \in 2$ such that the set $a_n = \{x \in X: f(x)(n) = b_n\}$ is comeager, then the set $c = \{x \in X: f(x) = \langle b_n: n \in \omega\}\}$ is comeager. As the function $f$ is constant on $c$, $c$ must consist of pairwise $E$-related elements and so $c$ is countable. This contradicts the Baire category theorem.

Thus, there is a number $n \in \omega$ such that the sets $a_n^0 = \{x \in X: f(x)(n) = 0\}$ and $\{x \in X: f(x)(n) = 1\}$ are nonmeager. By the density of the group $\Gamma$, there is a group element $\gamma \in \Gamma$ and points $x_0 \in a_n^0$ and $x_1 \in a_n^1$ such that $\gamma \cdot x_0 = x_1$. Then the points $x_0, x_1$ contradict the assumption that $f$ is a reduction. $\square$

By the Glimm–Effros dichotomy, there is a Borel map $h: 2^\omega \to X$ which is a reduction of $E_0$ to $E$. For each $x \in D$ pick the smallest element $\gamma_x \in \Gamma$ in some fixed enumeration such that $\gamma_x \cdot x \in \text{rng}(h)$ if it exists, and let $A = h^{-1}\{\gamma_x \cdot x: x \in D\}$. It is immediate that $A \subset 2^\omega$ is an $E_0$-transversal. $\square$

**Theorem 13.4.6.** (ZF+DC) Let $Y$ be a separable Banach space. If there is a discontinuous homomorphism $h: Y \to Y$ then there is an $E_0$-selector, in particular $|E_0| \leq |2^\omega|$.

**Proof.** The discontinuity of the homomorphism and the DC assumption yield a sequence $\langle y_n: n \in \omega\rangle$ of elements of $Y$ such that $|y_n| > \sum_{m>n}|y_m|$ and $|h(y_{n+1})| > 2|h(y_n)|$. Now, for every $x \in 2^\omega$ let $g(x) = \sum\{y_n: x(n) = 1\} \in Y$. Let $d \subset 2^\omega$ be any $E_0$-class. The homomorphism assumptions on $h$ show that the function $g \restriction d$ is injective and also, the norms of the points in $h \circ g(d)$ diverge to infinity. Thus, $d$ contains a finite subset of points whose $h \circ g$-images have the smallest possible norm, and one can let $x_d \in d$ be the lexicographically smallest point in $d$ the norm of whose $h \circ g$-image is as small as possible. The set $\{x_d: d \text{ is an } E_0\text{-class}\}$ is an $E_0$-selector. $\square$

The assumption that $Y$ be a Banach space cannot be dropped entirely. Consider the case of $Y = 2^\omega$ with coordinatewise binary addition as a vector space over the binary field. Any nonprincipal ultrafilter $U$ on $\omega$ yields a discontinuous homomorphism from $2^\omega$ to $2$ by setting $h(x) = 1$ if $\{n \in \omega: x(n) = 1\} \in U$. At the same time, an existence of an ultrafilter does not imply the collapse $|E_0| \leq |2^\omega|$.

**Theorem 13.4.7.** (ZF) Let $Y$ be a separable Banach space. If there is a basis for $Y$ over the rationals then there is a subset of $X = (2^\omega)^\omega$ which intersects each $E_1$-class in a nonempty countable set.

In particular, $|E_1| \leq |\mathcal{P}_2|$ must hold.
Proof. The space \( Y \) has a presentation definable from some code in \( 2^\omega \); with an eye on the case \( Y = \mathbb{R} \), we will neglect the code in the proof. The first claim of the proof identifies a canonical inner model associated with a basis. Let \( A \subseteq Y \) be a basis; define \( B = \{ \langle y, O_i : i \in k \rangle : y \in Y, O_i \subseteq Y \} \) be basic open sets and there are points \( y_i \in O_i \cap A \) such that for all \( \langle y_i : i \in k \rangle \) with nonzero rational coefficients\}. For each point \( x \in X \), consider the model \( L[B, x] \). The following is well-known or easily checked.

Claim 13.4.8. 1. \( L[B, x] \) is a model of \( ZFC \), \( A \cap L[B, x] \subseteq L[B, x] \) and \( A \cap L[B, x] \) is a basis for \( Y \) in \( L[B, x] \);

2. Whenever \( M \) is an inner model of \( ZF \) such that \( x \in M \) and \( A \cap M \subseteq M \) and \( A \cap M \) is a basis in \( M \), then \( L[B \cap M, x] = L[B, x] \);

3. \( L[B, x] \) satisfies the Continuum Hypothesis and its constructibility order orders its elements of \( Y \) in ordertype \( \omega^L_{\langle B, x \rangle} \).

The first two items say that \( L[B, x] \) is the smallest inner model of \( ZF \) containing \( x \) to which \( A \) is admissible and forms a basis in it.

Proof. For (1), just observe that whenever \( y \in L[B, x] \) then the countable set \( C_y = \{ \langle O_i : i \in k \rangle : O_i \subseteq Y \} \) is a basis open sets and there are points \( y_i \in O_i \cap A \) such that \( y \) is a linear combination of \( y_i : i \in k \) with nonzero rational coefficients\} belongs to the model \( L[B, x] \), and by Mostowski absoluteness between \( V \) and \( L[B, x] \) there must be a unique tuple \( c_y = \langle y_i : i \in k \rangle \in L[B, x] \) of elements of \( Y \) such that for all \( \langle y_i : i \in k \rangle \in C_y \) and all \( i \in k, y_i \in O_i \). It is immediate that the elements of \( c_y \) are elements of \( A \) and form a linear combination with nonzero rational coefficients whose outcome is \( y \). Thus \( A \cap L[B, x] = \bigcup_{y \in L[B, x]} \text{rng}(c_y) \) is an element of \( L[B, x] \) and is a basis there.

(2) is immediate. (3) follows from a standard condensation argument, using only the fact that \( B \) is a subset of a Polish space. \( \square \)

Adjusting the basis \( A \) slightly, we may assume that it consists of points of norm between \( 1/2 \) and \( 1 \). For every point \( x \in X \) and a number \( n \in \omega \), write \( x \upharpoonright n \) for the element of \( X \) which returns the zero binary sequence for every \( m < n \), and the binary sequence \( x(m) \) for \( m \geq n \). Define \( V_n(x) = L[B, x \upharpoonright n] \), writing \( V_n \) for \( V_n(x) \) if \( x \) is understood from the context. Note that the models \( V_n(x) \) decrease as \( n \) increases, and also the sequence \( \langle V_m(x) : m \geq n \rangle \) is a class in the model \( V_n(x) \).

Claim 13.4.9. Let \( x \in X \) be arbitrary. There is a number \( n \in \omega \) such that \( X \cap V_m(x) = X \cap V_n(x) \) for all \( m \geq n \).

Proof. Suppose not. Then the set \( a = \{ n \in \omega : X \cap V_{n+1} \neq X \cap V_n \} \subset \omega \) is infinite. For every \( n \in a \) the set \( A \cap V_n \setminus V_{n+1} \) is nonempty. To see this, note that \( A \cap V_n \) spans all of \( Y \setminus V_n \), which is a strictly larger set than \( Y \setminus V_{n+1} \). Let \( y_n \in A \cap V_n \setminus V_{n+1} \) be the first such element in the constructibility well-ordering of \( V_n \). Also, let \( z_n = \sum(2^{-m} : y_m : m \geq n, m \in a) \). It is clear that \( z_n \in V_n \).
holds. Note that if \( n \in a \), then also \( z_n \notin V_{n+1} \) holds, since \( y_n = 2^n \cdot (z_n - z_m) \), \( z_m \in V_{n+1} \), and \( y_n \notin V_{n+1} \) where \( m \in a \) is the smallest number greater than \( n \).

Let \( n_0 \in a \) be any number. The point \( z_{n_0} \in Y \) can be expressed as a linear combination \( \phi \) of elements of \( A \) belonging to \( V_{n_0} \). Let \( k \in \omega \) be a number so large that all elements of \( A \) used in \( \phi \) belonging to \( V_k \) belong to the intersection \( \bigcap_n V_n \); therefore \( \phi \) decomposes into two linear combinations \( \psi_0 + \psi_1 \), \( \psi_0 \) using only elements of \( A \setminus V_k \) and \( \psi_1 \) using only elements of \( A \setminus \bigcap_n V_n \). Let \( \langle n_j : j \leq i \rangle \) be an increasing enumeration of all numbers in the set \( a \) between \( n_0 \) and some \( n_i > k \). Then, \( \psi_0 + \psi_1 = z_{n_0} = \sum_{j<i} 2^{-n_j} \cdot y_{n_j} + z_{n_i} \). In other words, the point \( z_{n_i} \in V_{n_i} \) can be expressed as a linear combination of elements of \( A \setminus V_{n_i} \) and elements of \( A \cap V_{n_i+1} \). The first part of the combination must be empty since \( A \cap V_{n_i} \) is a basis in \( V_{n_i} \). This indicates that \( z_{n_i} \in V_{n_i+1} \), contradicting the last sentence of the previous paragraph.

It follows from the claim that for all but finitely many \( n \in \omega \) it is the case that \( x \setminus n \in \bigcap_m V_m(x) \). For natural numbers \( n \leq m \in \omega \) define \( \rho_{nm}(x) \) to be the index of \( x \setminus n \) in the constructibility well-ordering of \( V_m(x) \) if \( x \setminus n \in V_m(x) \), and let \( \rho_{nm}(x) = 0 \) otherwise. Let \( \alpha(x) = \limsup_n \sup_{m \geq n} \rho_{nm}(x) \). The definition of the ordinal \( \alpha(x) \) does not depend on the choice of \( x \) within its equivalence class. Also, \( \alpha(x) \) is a countable ordinal. To see this, work in the model \( V_0(x) \) and evaluate the ordinal \( \alpha(x) \) there. Since \( V_0(x) \) is a model of choice, its \( \omega_1 \) is regular and so \( \alpha(x) \in \omega_1^{V_0(x)} \). Since \( \omega_1^{V_0(x)} \leq \omega_1 \), the countability of \( \alpha(x) \) follows.

Finally, let \( C = \{ x \in X : \exists n \ x = x \setminus n \text{ and for all } m \geq n, x \setminus n \in V_m(x) \text{ and } \rho_{nm}(x) \leq \alpha(x) \} \). To see that the set \( C \) meets every \( E_1 \) class in a nonempty countable set, let \( z \in X \) be arbitrary. The definition of the ordinal \( \alpha(z) \) shows that there is a number \( n \in \omega \) such that for all \( m \geq n \), \( z \setminus n \in V_m(z) \) and is enumerated before \( \alpha(z) \) there; clearly, letting \( x = z \setminus n \) we get \( x \in C \cap [z]_{E_1} \). Also, note that \( C \cap [z]_{E_1} \subseteq \{ x \in X : \exists n \ x \in V_n(z) \text{ and the index of } x \text{ in the constructibility well-ordering of } V_n(z) \leq \alpha(z) \} \) and observe that the latter set is countable as the ordinal \( \alpha(z) \) is countable.
Bibliography


