Coloring closed Noetherian graphs*

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June 3, 2023

Abstract

If Γ is a closed Noetherian graph on a σ-compact Polish space with no infinite cliques, it is consistent with the choiceless set theory ZF+DC that Γ is countably chromatic and there is no Vitali set.

1 Introduction

Chromatic numbers of algebraic and σ-algebraic graphs on Euclidean spaces have been studied extensively in both ZFC and choiceless ZF+DC context [3, 4, 6, 10]. In this paper, I show it consistent for many such graphs Γ that ZF+DC holds, chromatic number of Γ is countable, yet there is no Vitali set. The main feature of the graphs exploited here is omission of several simple subgraphs.

Definition 1.1. The *half graph* is the graph on the vertex set $\omega \times 2$, connecting vertices $\langle n, 0\rangle$ and $\langle m, 1\rangle$ if $m < n$ and containing no other edges. A *variation* of the half graph is a graph obtained from the half graph by making vertices $\langle n, 0\rangle$ for $n \in \omega$ either all pairwise connected or all pairwise disconnected, and similarly for vertices $\langle n, 1\rangle$ for $n \in \omega$. The *three-quarter graph* is the graph on the same vertex set, connecting vertices $\langle n, 0\rangle$ and $\langle m, 1\rangle$ if $m \neq n$ and containing no other edges. Variations of the three-quarter graph are defined in the same way.

Thus, the half graph and the three-quarter graph have four variations each.

Definition 1.2. A graph Γ on a set X is *Noetherian* if it does not contain a variation of the half graph or the three quarter graph as a vertex-induced subgraph.

Interesting examples of Noetherian graphs are included in Section 3. The precise preservation result obtained in this paper concerns Hamming graphs.

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*2000 AMS subject classification 03E35, 05C15, 14P99.

†The author was partially supported by grant EXPRO 20-31529X of GA ČR.

Definition 1.3. The *infinite breadth Hamming graph* $\mathbb{H}_\omega$ is the graph on ω^ω connecting two points if they differ in exactly one entry. The *diagonal Hamming graph* $\mathbb{H}_{<\omega}$ is the restriction of $\mathbb{H}_\omega$ to the diagonal set $\prod_n(n+1)$.

It is not difficult to see that the Hamming graphs are Noetherian. One way to argue is to note that in the Hamming graphs, any two distinct vertices which have at least three common neighbors are connected, while no variation of the half graph or the three-quarter graph satisfies this property.

Theorem 1.4. *Suppose that Γ is a closed Noetherian graph on a σ-compact Polish space X.*

1. *If Γ contains no infinite clique, then it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable while that of $\mathbb{H}_\omega$ is not;*
2. *if there is a number $n \in \omega$ such that the graph Γ contains no clique of cardinality n, then it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable while that of $\mathbb{H}_{<\omega}$ is not.*

The inaccessible cardinal assumption is necessary only to make the proof fit the set-up of geometric set theory [5] and probably can be dropped. Similarly, the σ-compact assumption, satisfied in the important algebraic examples, is only necessary to evaluate the complexity of the coloring poset in Proposition 4.4 and probably can be dropped.

In both cases, the conclusion excludes a Vitali set. To see this, let $\{\varepsilon_{n,m} \colon n, m \in \omega\}$ be a collection of pairwise distinct positive rationals with a finite sum, and let $h \colon \omega^\omega \to \mathbb{R}$ be the function defined by $h(x) = \Sigma_n \varepsilon_{n,x(n)}$. The function h is a homomorphism from either of the Hamming graphs to the Vitali equivalence relation, and if $A \subset \mathbb{R}$ were a Vitali set, then the h-preimages of A and its rational shifts would show that the chromatic numbers of the Hamming graphs are countable.

Theorem 1.4 is stated in a way which covers many special cases. To state a couple of more specific consequences, for $n \geq 1$ and a set a of positive real numbers let Γ_{na} be the graph on $\mathbb{R}^n$ consisting of pairs of points whose Euclidean distance belongs to a.

Corollary 1.5. *Let $n \geq 1$ be a number and let a be a countable bounded set of positive reals with 0 as the only accumulation point. Then it is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ_{na} is countable, yet there is no Vitali set.*

This stands in contradistinction with the case in which a is the set of all positive rationals, where the countable chromatic number of Γ_{na} yields a Vitali set by the definitions. To prove the corollary, first use Example 3.7 to show that the graph Γ_{na} is closed and Noetherian. Theorem 1.4(1) then proves consistency of ZF+DC plus the chromatic number of Γ_{na} is countable while there is no Vitali set.

For the next corollary, let $\langle \varepsilon_n : n \in \omega \rangle$ be a sequence of positive real numbers such that $\Sigma_n (n+1)\varepsilon_n < \infty$. Let $a = \{m\varepsilon_n : n \in \omega, m \in n+1\}$.

Corollary 1.6. *Let $n \geq 1$ and let Γ be an arbitrary algebraic graph on a Euclidean space, without a perfect clique. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable, yet the chromatic number of Γ_{1a} is uncountable.*

This shows that it is possible to color algebraic graphs in general without coloring even quite simple instances of the distance graphs which are σ-algebraic. To prove the corollary, first observe that the Hamming graph $\mathbb{H}_{<\omega}$ can be homomorphically embedded into Γ_{1a} by a function $h\colon \prod_n (n+1) \to \mathbb{R}$ defined by $h(x) = \Sigma_n x(n) \cdot \varepsilon_n$. Now, use Theorem 3.2 to see that there is a finite bound on the cardinality of Γ-cliques. Theorem 1.4(2) then shows the consistency of ZF+DC plus the chromatic number of Γ is countable while that of $\mathbb{H}_{<\omega}$ is not–which by the existence of the homomorphism means that the chromatic number of Γ_{1a} is uncountable as well.

The techniques of this paper provide much more detailed information about the models obtained than what fits into the statements of the main theorems. However, a number of questions remain open. An affirmative answer to the following question would be a natural strengthening of the main results of this paper.

Question 1.7. Let Γ be a closed Noetherian graph on a σ-compact Polish space without an infinite clique. Is it consistent with ZF+DC that Γ is countably chromatic, yet there is no linear ordering of the set of all Vitali classes?

To describe the architecture of the paper, in Section 2 I provide basic insights into closed Noetherian graphs without infinite cliques. In particular, they carry a canonical Noetherian topology, and they are countably chromatic in ZFC. Section 3 provides examples associated with Euclidean spaces, which are the most interesting from historical point of view. Many quite different examples will doubtless be found in the future. Section 4 analyzes a canonical coloring poset used over the Solovay model to add a coloring of closed Noetherian graphs without infinite cliques. Section 5 discusses the main technical tool to control the generic extension of the Solovay model, namely the finite condition coloring poset known from the work of Todorcevic and others. Finally, Section 6 wraps up the proofs using the technology of [5, Chapter 11].

The notation of the paper follows the set theoretic standard of [1]. A graph Γ on a Polish space X is closed if the relation $\{\langle x, y\rangle \in X^2 : x \; \Gamma \; y \text{ or } x = y\}$ is a closed subset of X^2. A topology $\mathcal{T}$ on a set X is Noetherian if there are no infinite sequences of $\mathcal{T}$-closed sets strictly decreasing with respect to inclusion, or equivalently, the intersection of any collection of $\mathcal{T}$-closed sets is equal to the intersection of a finite subcollection. The Vitali equivalence relation on $\mathbb{R}$ connects points x, y if $x - y$ is a rational number; a Vitali set is a subset of $\mathbb{R}$ which intersects each class of the Vitali equivalence relation in exactly one point.

2 Initial observations

This section contains basic definitions and facts about Noetherian graphs without infinite cliques. In particular, they carry a canonical Noetherian topology which will be used repeatedly in the paper. I will use the following notation regarding graph neighborhoods throughout.

Definition 2.1. Let Γ be a graph on a set X.

1. If $x \in X$ is a vertex, the symbol $\Gamma(x)$ denotes the set $\{y \in X : y = x \vee y \ \Gamma\ x\}$;
2. for a finite set $a \subset X$, $\Gamma(a)$ is the set $\bigcap_{x \in a} \Gamma(x)$;
3. the Γ-*topology* or the *graph topology* is the smallest topology on X in which all sets $\Gamma(x)$ for $x \in X$ are closed.

As is usual in similar situations, I will never be interested in separation axioms for the graph topology, or in its open sets. As is suggested by the terminology, the graph topology of Noetherian graphs is Noetherian. This feature will be used throughout the paper.

Theorem 2.2. *Let Γ be a graph on a set X. The following are equivalent:*

1. *Γ is Noetherian;*
2. *the Γ-topology is Noetherian.*

Proof. Failure of (1) immediately implies the failure of (2). Let $\pi : \omega \times 2 \to X$ be an isomorphism of a variation of the half graph to a subgraph of X. Let $a_n = \{\pi(m, 0) : m \in n\}$, and observe that the sets $\Gamma(a_n)$ all contain all but finitely many points in the set $\{\pi(m, 1) : m \in \omega\}$ while their intersection contains none of these points. An identical argument works for an embedding of a variation of the three-quarter graph.

Now, suppose that (1) holds and work to establish (2).

Claim 2.3. *There is no sequence $\langle a_n : n \in \omega \rangle$ of finite subsets of X such that the sets $\Gamma(a_n)$ strictly decrease with n.*

Proof. Suppose towards a contradiction that there is such a sequence. Without loss assume that the sets a_n increase with $n \in \omega$, and, erasing needless entries if necessary, assume that there are points x_n such that $a_{n+1} = a_n \cup \{x_n\}$. Note that the points x_n for $n \in \omega$ must be pairwise distinct. For each number $n \in \omega$, let $y_n \in \Gamma(a_n) \setminus \Gamma(x_n)$ be an arbitrary point; note that these points also have to be pairwise distinct and in addition $x_n \neq y_n$. Repeatedly using the Ramsey theorem, find an infinite set $b \subset \omega$ such that

- each of the sets $\{x_n : n \in b\}$ and $\{y_n : n \in b\}$ is either a Γ-clique or a Γ-anticlique, and they are disjoint;

- either for every pair $m < n$ of numbers in b, $y_m \Gamma x_n$ holds, or for every pair $m < n$ of numbers in b, $y_m \Gamma x_n$ fails.

Now, let $\pi\colon \omega \to b$ be the increasing enumeration. Define the injection $h\colon \omega \times 2 \to X$ by $h(n, 0) = y_{\pi(n)}$ and $h(n, 1) = x_{\pi(n)}$. If the "either" case in the second item above prevails, then h is an isomorphism of a variation of the three-quarter graph to a subgraph of Γ. If the "or" case prevails, then h is an isomorphism of a variation the half graph to a subgraph of Γ. Both cases are ruled out by (1). A contradiction. $\square$

Now, consider the collection $\mathcal{T}$ of all finite unions of sets $\Gamma(a)$ as a ranges over all finite subsets of X.

Claim 2.4. *There are no infinite strictly descending sequences of sets in $\mathcal{T}$.*

Proof. This is a standard argument. Towards a contradiction, assume that $\{C_n \colon n \in \omega\}$ is a inclusion-decreasing sequence of sets in $\mathcal{T}$ which does not stabilize. By recursion on $m \in \omega$ build numbers n_m and finite sets $a_m \subset X$ such that

- $n_0 \in n_1 \in \dots$;
- $\Gamma(a_m) \subseteq C_{n_m}$ and the sequence $\{C_n \cap \Gamma(a_m) \colon n \in \omega\}$ does not stabilize;
- $\Gamma(a_{m+1})$ is a strict subset of $\Gamma(a_m)$.

The base step is subsumed in the recursion step. For the recursion step, suppose that n_m, a_m have been constructed. Since the sequence $\{C_n \cap \Gamma(a_m) \colon n \in \omega\}$ does not stabilize, there must be a number $n_{m+1} > n_m$ such that $C_n \cap \Gamma(a_n) \neq \Gamma(a_n)$. Since the set $C_{n_{m+1}}$ is a finite union of sets of the form $\Gamma(a)$, there must be a finite set $a \subset X$ such that $\Gamma(a) \subseteq C_{n_{m+1}}$ and the sequence $\{C_n \cap \Gamma(a_m) \cap \Gamma(a) \colon n \in \omega\}$ does not stabilize. Set $a_{m+1} = a_n \cup a$ and observe that the recursion step has been successfully performed.

In the end, the sets $\Gamma(a_m)$ contradict the conclusion of Claim 2.3. Thus, $\mathcal{T}$ contains no infinite strictly decreasing sequences of sets. $\square$

Now, observe that $\mathcal{T}$ is closed under finite intersections and unions by its definition. It is also closed under arbitrary intersections: the nonexistence of infinite strictly decreasing sequences of sets in $\mathcal{T}$ implies that an intersection of arbitrary collection of sets in $\mathcal{T}$ is equal to an intersection of a finite subcollection. Thus, $\mathcal{T}$ is exactly the collection of closed sets in the Γ-topology. The Noetherian property of the topology follows immediately from Claim 2.4. $\square$

The next result quantifies the complexity of the graph topology from descriptive point of view.

Theorem 2.5. *Suppose that Γ is a closed Noetherian graph on a σ-compact Polish space X. Then the Γ-topology is analytic.*

Proof. This is to say [11] that in the usual topology on the space $F(X)$ of all closed subsets of X, the collection of all sets closed in the Γ-topology is analytic.

Since the space X is σ-compact, the intersection and union functions on $F(X)$ are both Borel, and if Y is a Polish space and $C \subset Y \times X$ is a closed set, the map $y \mapsto C_y$ is a Borel map from Y to $F(X)$ [2, Section 12.C]. It follows that for every $n, k \in \omega$, the map $\pi_{nk} \colon (X^n)^k \to F(X)$ given by $\pi_{nk}(y) = \bigcup_{i \in k} \bigcap_{j \in n} \Gamma(y(i)(j))$ is Borel. Theorem 2.2 shows exactly that the Γ-topology is the union of the ranges of all functions π_{nk}, and therefore analytic in $F(X)$. $\square$

Finally, I show that all closed Noetherian graphs without uncountable cliques are countably chromatic in ZFC.

Theorem 2.6. *Let Γ be a closed Noetherian graph on a Polish space X, without an uncountable clique. The chromatic number of Γ is countable.*

The proof uses a definition and a proposition which will be of use later.

Definition 2.7. Let Γ be a graph on a set X.

1. Let $a \subset X$ be a finite set. Then $\heartsuit(a)$ denotes the set $\{x \in X \colon \forall y \in a\ x = y$ or $x\ \Gamma\ y$, and $\forall z \in X\ \forall y \in a (z = y \lor z\ \Gamma\ y) \to (x = z \lor x\ \Gamma\ z)\}$;
2. a set $A \subset X$ is Γ-*good* if for every finite subset $a \subset A$, $\heartsuit(a) \subset A$.

It is obvious that $\heartsuit(a)$ is a Γ-clique and that Γ-goodness is a closure property. In particular, an increasing union of Γ-good sets is again Γ-good, and if no uncountable cliques exist in Γ then every infinite subset of X can be enclosed in a Γ-good set of the same cardinality.

Proposition 2.8. *Let Γ be a closed Noetherian graph on a Polish space X. Let $A \subset X$ be a Γ-good set. For every point $x \in X \setminus A$ there is a basic open set $O \subset X$ containing x and containing no elements of A which are Γ-connected with x.*

Proof. Suppose that $a \subset A$ is a set of points, all connected to a point $x \in X$ which is an accumulation point of a. It will be enough to show that $x \in A$ holds.

To this end, use Theorem 2.2 to find a finite set $b \subset a$ such that the set $\Gamma(b)$ is as small as possible. Note that $x \in \Gamma(b)$. Moreover, x is Γ-connected to every point $y \in \Gamma(b)$ distinct from x. To see this, use the choice of the set b to observe that the point y is Γ-connected to every element of a. Since the graph Γ is closed and x is an accumulation point of a, $x\ \Gamma\ y$ follows.

It follows that $x \in \heartsuit(b)$, therefore $x \in A$ holds by the goodness of the set A. The proof is complete. $\square$

Proof of Theorem 2.6. Call a partial Γ-coloring c *suitable* if its range consists of basic open subsets of X and for each $x \in \mathrm{dom}(c)$, $x \in c(x)$ holds. By transfinite induction on the cardinality of an infinite Γ-good set $A \subset X$ prove that if d is a function with domain A assigning to any point $x \in A$ its basic open neighborhood $d(x)$, then there is a suitable coloring c with domain A such that $c(x) \subset d(x)$ holds for every $x \in A$. This will prove the theorem: in the end, one can apply it to $A = X$.

The statement is clear for countable A as the coloring c can in such a case be selected as an injection. Now, suppose that A is a Γ-good set of uncountable cardinality, such that for all good sets of smaller cardinality the statement is known. Let d be a function with domain A such that for each $x \in A$ the value $d(x)$ is an open neighborhood of A. Express $A = \bigcup_{\beta\in\alpha} A_\beta$ as an increasing union of Γ-good sets of smaller cardinality. For each $\beta \in \alpha$ let d_β be a function with domain A_β such that for every $x \in A_\beta$, $d_\beta(x)$ is an open neighborhood of x which is a subset of $d(x)$ and if $x \notin \bigcup_{\gamma\in\beta} A_\gamma$, then $d_\beta(x)$ contains no elements of $\bigcup_{\gamma\in\beta} A_\gamma$ which are Γ-connected to x. This is possible by Proposition 2.8. By the induction hypothesis, find suitable colorings c_β with domain A_β such that for every point $x \in A_\beta$, $c_\beta(x) \subset d_\beta(x)$. Now, let c be the function with domain A defined by $c(x) = c_\beta(x)$ where $\beta \in \alpha$ is the smallest ordinal such that $x \in A_\beta$. It is not difficult to see that c is a suitable Γ-coloring of the set A verifying the induction step. $\square$

3 Initial examples

The main theorems of the paper need a supply of examples of Noetherian graphs to be meaningful. I concentrate on algebraic graphs and certain special type of σ-algebraic graphs on Euclidean spaces.

Definition 3.1. Let X be a Euclidean space of dimension $n \geq 1$. A graph Γ on X is *algebraic* if there is a polynomial $\phi(\bar{u}, \bar{v})$ of $2n$ free variables and real parameters such that for distinct points $x, y \in X$, $x \ \Gamma \ y$ if and only if $\phi(x, y) = 0$.

Theorem 3.2. *Let Γ be an algebraic graph on a Euclidean space X. Then Γ is Noetherian, and exactly one of the following occurs:*

1. *Γ contains a perfect clique;*
2. *there is a number $m \in \omega$ such that Γ contains no clique of cardinality greater than m.*

Proof. To prove the Noetherian property of the graph Γ, work to exclude a variation of the three quarter graph from it (the half graph is treated in the same way). Suppose towards a contradiction that $x_n, y_n \colon n \in \omega$ are vertices in X which induce a copy of a variation of the three quarter graph. Consider the intersection of sets $\Gamma(x_n)$ for $n \in \omega$. This is an intersection of an infinite collection of algebraic sets which contains no points y_n for $n \in \omega$ since x_n is disconnected with y_n. However, the intersection of any finite subcollection contains all but finitely many points y_n for $n \in \omega$. This contradicts the Hilbert basis theorem.

(1) clearly implies the failure of (2). To show that the failure of (1) implies a finite bound on the size of Γ-cliques, I will need a general claim.

Claim 3.3. *There is a number k such that for every finite set $a \subset X$ there is a set $b \subseteq a$ of cardinality at most k such that $\Gamma(b) = \Gamma(a)$.*

Proof. Write n for the dimension of X and l for the degree of the polynomial defining the graph Γ. As a quite inefficient estimate, $k = n(2^n l)^n$ will work. To see this, by tree recursion build a tree T and functions f, g on T so that

- $f(0) = X$ and for every $t \in T$, $f(t) \subseteq X$ is always an irreducible algebraic subset of X;
- for every node $t \in T$, $g(t)$ is some element of a such that $\Gamma(g(t)) \cap f(t) \neq f(t)$ if it exists, otherwise t is a terminal node of T and $g(t)$ is left undefined;
- for every node $t \in T$, if $g(t) \in a$ then the set $\{f(s) \colon s$ is an immediate successor of t in $T\}$ lists irreducible components of the algebraic set $\Gamma(g(t)) \cap f(t)$ without repetition.

It turns out that the cardinality of the tree T is at most k. To see this, work to estimate the depth of the tree and the rate at which it branches. Since the function f maps the tree ordering on T to strict inclusion of irreducible algebraic subsets of X, the depth of the tree is at most n. By induction on $|t|$, where $t \in T$, argue that the set $f(t) \subseteq X$ is defined by a polynomial of degree at most $2^{|t|} l$. To see that, note that if p is a polynomial defining $f(t)$, then for every immediate successor s of t, $f(s)$ is defined by an irreducible factor of $p^2 + q^2(g(t))$ where $q(g(t))$ is the polynomial defining $\Gamma(g(t))$, which is of degree at most l. Lastly, since the immediate successors of the node t are labeled with irreducible components of $\Gamma(g(t) \cap f(t)$ and these are given by irreducible factors of $p^2 + q^2(g(t))$, the node t can have at most $2^{|t|+1} l$ many immediate successors. Simple arithmetic then shows that $|T| \leq k$.

In the end, let $b = \mathrm{rng}(g)$ and observe that the set b works: the sets $\Gamma(a)$ and $\Gamma(b)$ are both equal to the union of $f(t)$ as t ranges over all terminal nodes of the tree T. $\square$

Now, suppose that Γ has no perfect clique. Let $k \in \omega$ be a number which works as in the claim. Consider the set $B \subset X^k \times X$ defined by $\langle x, y \rangle \in B$ if $y \in \heartsuit(\mathrm{rng}(x))$. The set B is semi-algebraic, and all of its vertical sections are semi-algebraic Γ-cliques. None of them are uncountable by the assumption on Γ, and since every semi-algebraic set is either finite or uncountable, all vertical sections of B are finite. Every semi-algebraic set with finite vertical sections enjoys a finite bound on the cardinality of vertical sections [9, Chapter 3, Lemma 1.7]. Let $m \in \omega$ be such a bound for the cardinality of vertical sections of B. I claim there are no Γ-cliques of cardinality greater than m.

To see this, suppose that $a \subset X$ is a Γ-clique. Let $b \subset a$ be a set of cardinality at most k such that $\Gamma(b) = \Gamma(a)$. Let $x \in X^k$ enumerate, with possible repetitions, all elements of b. Clearly, $a \subseteq \heartsuit(b)$ holds, therefore $|a| \leq m$ as desired. $\square$

Example 3.4. Let $C \subset \mathbb{R}^2$ be an irreducible algebraic curve containing the origin, and not equal to a line through the origin. Let Γ_C be the graph on $\mathbb{R}^2$ connecting distinct points x, y if $x - y \in C$ or $y - x \in C$. The graph Γ_C has no uncountable clique. In fact, there is a number $n \in \omega$ such that Γ_C does not contain the bipartite graph $K_{2,n}$ as a subgraph.

Proof. First argue that Γ_C does not contain $K_{2,\omega}$. Suppose towards a contradiction that $x_0, x_1 \in \mathbb{R}^2$ are distinct points and $a \subset \mathbb{R}^2$ is an infinite set of points all of which are connected to both x_0 and x_1. For definiteness, assume that the set $b = \{y \in a \colon y - x_0 \in C \text{ and } y - x_1 \in C\}$ is infinite. Then the set b is an infinite subset of $(C + x_0) \cap (C + x_1)$. Since the two algebraic sets in this intersection are irreducible, their intersection is either finite or equal to both. In conclusion, $C + x_0 = C + x_1$ holds, in other words $C + (x_0 - x_1) = C$. Since $0 \in C$, this means that $n(x_0 - x_1) \in C$ for any $n \in \omega$, and C has infinite intersection with the line through the origin of direction $x_0 - x_1$. An irreducibility argument again shows that C has to be equal to that line, contradicting the initial choice of C.

Now, consider the algebraic set $B = \{\langle z, y\rangle \in (\mathbb{R}^2)^2 \times \mathbb{R}^2 \colon$ the two entries of z are distinct and y is Γ_C-related to each$\}$. This is a semi-algebraic set with finite vertical sections by the previous paragraph. By [9, Chapter 3, Lemma 1.7], there is a number $n \in \omega$ such that all vertical sections of the set B have cardinality at most n. This completes the proof. $\square$

Definition 3.5. Let X be a Euclidean space of dimension $n \geq 1$. A graph Γ on X is *tight σ-algebraic* if there are algebraic graphs Γ_m for $m \in \omega$ on X such that $\Gamma = \bigcup_m \Gamma_m$ and the real numbers $\varepsilon_m = \sup\{d(x, y) \colon x \; \Gamma_m \; y\}$ tend to zero.

Theorem 3.6. *Let Γ be a tight σ-algebraic graph on a Euclidean space X. Then Γ is Noetherian and exactly one of the following occurs:*

1. *Γ contains a perfect clique;*
2. *Γ contains no infinite clique.*

Proof. Let $\Gamma = \bigcup_m \Gamma_m$ be a witness to the tight σ-algebraicity of Γ. To verify the Noetherian property of Γ, suppose towards a contradiction that $x_n, y_n \colon n \in \omega$ are vertices in X which induce a copy of a variation of the three quarter graph (the half graph is treated in the same way). Thinning out if necessary I may assume that the set $\{x_n, y_n \colon n \in \omega\}$ is discrete. It follows that for each $n \in \omega$ there is a number $m_n \in \omega$ such that the points x_n are $\bigcup_{m \in m_n} \Gamma_m$-connected to points y_k for all $k \neq n$. Consider the intersection of all sets $\bigcup_{m \in m_n} \Gamma_m(x_n)$ for $n \in \omega$. This is an intersection of an infinite collection of algebraic sets which contains no points y_n for $n \in \omega$. The intersection of any finite subcolection contains all but finitely many points y_n for $n \in \omega$. This contradicts the Hilbert basis theorem.

Clearly (1) implies the failure of (2). Now, suppose that (2) fails and work to confirm (1). Let $\{x_n \colon n \in \omega\}$ is an infinite Γ-clique. I will produce a number $m \in \omega$ such that an infinite subset of this clique forms a Γ_m-clique. Then, an application of Theorem 3.2 provides a perfect Γ_m-clique, therefore a perfect Γ-clique.

Thinning out the clique if necessary, assume that it is discrete. By recursion on $k \in \omega$ build numbers n_k, m_k, and infinite sets $b_k \subset \omega$ such that

- $n_{k+1} > n_k$, $b_{k+1} \subset b_k$;
- $n_k \in b_k$;
- $x_{n_k}\ \Gamma_{m_k}\ x_n$ for all $n \in b_{k+1}$.

To start, let $n_0 = 0$ and $b_0 = \omega$. Since x_0 is an isolated point of the clique, the tightness condition on the graph Γ implies that there are only finitely many numbers m such that $x_0 \Gamma_m x$ holds for some $x \neq x_0$ in the clique. It follows that there must be a number m_0 such that the set $b_1 = \{n \in \omega \colon x_0\ \Gamma_{m_0}\ x_n\}$ is infinite. The recursion step is performed in a similar way.

Now, observe that the set $\{m_k \colon k \in \omega\}$ must be finite. Otherwise, consider the intersection $\bigcap_k \Gamma_{m_k}(x_{n_k})$. This is an intersection of of an infinite collection of algebraic sets. It contains no elements of the set $\{x_{n_k} \colon k \in \omega\}$ since all elements of this set are isolated in it and the numbers $m_k \in \omega$ grow arbitrarily large. Intersection of any finite subcollection always contains all but finitely many of the set $\{x_{n_k} \colon k \in \omega\}$. This contradicts the Hilbert basis theorem.

It is then possible to find a number $m \in \omega$ such that the set $c = \{k \in \omega \colon m_k = m\}$ is infinite. The set $\{x_{n_k} \colon k \in c\}$ is an infinite Γ_m-clique as desired. $\square$

Example 3.7. If $a \subset \mathbb{R}$ is a bounded countable set of positive reals with 0 as the only accumulation point, the graph Γ on a Euclidean space X connecting points whose distance belongs to the set a is tight σ-algebraic. It contains no infinite clique.

Proof. Suppose towards a contradiction that $\{x_n \colon n \in \omega\}$ is an infinite Γ-clique. Thinning out if necessary, assume that the clique is discrete. For every number $n \in \omega$, there must be a finite set $b_n \subset a$ such that the distance of the point x_n from any point x_m for $m \neq n$ belongs to the set b_n. Let $A_n \subset X$ be the algebraic set of all points in X whose distance from x_n belongs to the finite set b_n. The intersection of all sets A_n for $n \in \omega$ contains no elements of the clique since $x_n \notin A_n$. On the other hand, intersection of any finite subcollection contains all but finitely many elements of the clique. This contradicts the Hilbert basis theorem. $\square$

4 A balanced coloring poset

This section contains a description of a canonical poset adding a coloring to a closed Noetherian graph without an uncountable clique. The poset is balanced in the sense of [5, Chapter 5]. Its further preservation properties will be proved in Section 6.

Definition 4.1. Let Γ be a closed Noetherian graph on a σ-compact Polish space X without an uncountable clique. The *coloring poset* P_Γ consists of countable partial Γ-colorings p such that

1. $\mathrm{dom}(p)$ is a countable Γ-good subset of X;
2. $\mathrm{rng}(p)$ consists of basic open subsets of X and for each $x \in \mathrm{dom}(p)$, $x \in p(x)$ holds.

The ordering is defined by $q \leq p$ if $p \subseteq q$ and for every point $x \in \mathrm{dom}(q \setminus p)$, the set $q(x)$ contains no elements of $\mathrm{dom}(p)$ which are Γ-connected to x.

Verification of the key properties of the poset P_Γ proceeds by a series of propositions.

Proposition 4.2. *P_Γ is a σ-closed transitive relation.*

Proof. The transitivity is clear. If $\langle p_i : i \in \omega \rangle$ is a descending sequence of conditions in P_Γ, then $\bigcup_i p_i$ is their common lower bound. $\square$

Proposition 4.3. *Suppose that $a \subset P_\Gamma$ is a finite set. The following are equivalent:*

1. *a has a common lower bound in P;*
2. *for every point $x \in X$, a has a common lower bound in P which contains x in its domain;*
3. *$\bigcup a$ is a function and for any two distinct conditions $p_0, p_1 \in a$ and any two Γ-connected points $x_0 \in \mathrm{dom}(p_0 \setminus p_1)$ and $x_1 \in \mathrm{dom}(p_1 \setminus p_0)$, the sets $p_0(x_0)$ and $p_1(x_1)$ do not contain x_1 and x_0 respectively.*

In particular, a finite subset of P_Γ has a common lower bound if and only if it consists of pairwise compatible conditions.

Proof. (2) implies (1) which implies (3) by the definition of the ordering. To show that (3) implies (1), fix the finite set $a \subset P_\Gamma$ and a point $x \in X$, assume that (3) holds, and work to find a common lower bound of a which contains x in its domain.

Let $b \subset X$ be a Γ-good countable set containing $\mathrm{dom}(p)$ for every $p \in a$ and the point x as well. I will produce a lower bound q of a such that $b = \mathrm{supp}(q)$. To this end, let $d = b \setminus \bigcup_{p \in a} \mathrm{dom}(p)$. For every point $y \in d$, use Proposition 2.8 to find an open neighborhood $O_y \subset X$ of y such that O_y contains no point in $\bigcup_{p \in a} \mathrm{dom}(p)$ which is Γ-connected to y. Then, find an injection r from d to basic open subsets of X such that for every point $y \in d$, $r(y) \subset O_y$ and $y \in r(y)$ holds. It will be enough to show that $q = r \cup \bigcup_{p \in a} p$ is a common lower bound of the set a.

First of all, argue that q is a Γ-coloring. To see this, suppose that $x_0, x_1 \in \mathrm{dom}(q)$ are distinct Γ-connected points. There are several cases.

Case 1. $x_0, x_1 \in d$. In this case, $q(x_0) \neq q(x_1)$ since $q \restriction d$ is an injection.
Case 2. Exactly one point among x_0, x_1, say x_0 belongs to d. By the choice of the set O_{x_0}, $x_1 \notin q(x_0)$ holds. At the same time, $x_1 \in q(x_1)$ holds and therefore $q(x_0) \neq q(x_1)$ as desired.
Case 3. Neither x_0 nor x_1 belongs to d. Pick conditions $p_0, p_1 \in a$ such that $x_0 \in \mathrm{dom}(p_0)$ and $x_1 \in \mathrm{dom}(p_1)$ holds. If either $x_0 \in \mathrm{dom}(p_1)$ or $x_1 \in \mathrm{dom}(p_0)$ holds, then x_0, x_1 receive distinct colors since each p_0, p_1 is separately a Γ coloring. Otherwise, it must be the case that $x_0 \in \mathrm{dom}(p_0 \setminus p_1)$ and $x_1 \in \mathrm{dom}(p_1 \setminus p_0)$ holds, and then $q(x_0) \neq q(x_1)$ holds by (3) and the definition of the ordering P_Γ.

Second, show that for each $p \in a$, $q \leq p$ holds. To this end, let $y \in \mathrm{dom}(q \setminus p)$ be an arbitrary point; the value $q(y)$ must not contain any point $z \in \mathrm{dom}(p)$ which is Γ-connected to y. This is clear if y belongs to the domain of some other condition in a by (3). Otherwise, $y \in d$ holds and then $z \notin O_y$ and $z \notin q(y)$ holds by the choice of the set O_y. The proof is complete. □

Proposition 4.4. *P_Γ is a Suslin forcing.*

Proof. The complexity calculation starts with an easy initial observation.

Claim 4.5. *For every $n \in \omega$ the set $B_n \subset X^n \times X$ of all pairs $\langle y, x\rangle$ such that $x \in \heartsuit(\mathrm{rng}(y))$ is Borel.*

Proof. To show that the set B_n is Borel, for every relatively open set $O \subset X$ define $C_O = \{y \in X^n \colon \exists x \in O\ \forall z \in \mathrm{rng}(y)\ x\ \Gamma\ z \lor x = z\}$. Since the set $O \subset X$ is K_σ, a compactness argument shows that $C_O \subset X^n$ is K_σ as well. Then $\langle y, x\rangle \in B_n$ if $\forall z \in \mathrm{rng}(y)\ x\ \Gamma\ z \lor x = z$ and for every pair O_0, O_1 of disjoint basic open subsets of X such that $O_0 \times O_1 \cap \Gamma = 0$, either $x \notin O_0$ or $y \notin C_{O_1}$. This presents B_n as a Borel set. □

Now, since vertical sections of the set B_n are Γ-cliques, they are countable. By the Lusin–Novikov theorem, each set B_n is a union of graphs of countably many Borel functions. It is clear then that conditions are exactly those Γ-colorings p with countable domain such that the domain is closed under all said Borel functions, and for each $x \in \mathrm{dom}(p)$, $p(x) \subset X$ is a basic open neighborhood of x. This shows that the set of conditions in P_Γ is Borel.

The ordering on P_Γ is obviously a Borel set. Proposition 4.3 provides a Borel characterization of compatibility of conditions in P_Γ, completing the proof. □

Proposition 4.6. *P_Γ forces the union of the generic filter to be a total Γ-coloring.*

Proof. It is only necessary to show that for every condition $p \in P_\Gamma$ and every $x \in X$ there is a condition $q \leq p$ such that $x \in \mathrm{dom}(q)$. This follows from Proposition 4.3 applied to the set $a = \{p\}$. □

It is now time to prove the instrumental amalgamation property of the coloring poset P_Γ. Recall [11] the following notions.

Definition 4.7. 1. Let $\mathcal{T}$ be an analytic Noetherian topology on a σ-compact Polish space X. If M is a transitive model of ZFC containing the code for $\mathcal{T}$ and $A \subset X$ is a set, the symbol $C(M, A)$ denotes the smallest $\mathcal{T}$-closed set coded in M which contains A as a subset.

2. Generic extensions $V[H_0], V[H_1]$ are *mutually Noetherian* if for every analytic Noetherian topology $\mathcal{T}$ on a K_σ Polish space X coded in the ground model V and for every set $A_1 \subset X$ in $V[H_1]$, $C(V, A_1) = C(V[H_0], A_1)$ holds, and vice versa: for every set $A_0 \subset X$ in $V[H_0]$, $C(V, A_0) = C(V[H_1], A_0)$ holds.

For example, mutually generic extensions are mutually Noetherian.

Definition 4.8. Let P be a Suslin poset.

1. A pair $\langle Q, \tau\rangle$ is *Noetherian balanced* if $Q \Vdash \tau \in P$ and for every mutually Noetherian pair $V[H_0]$ $V[H_1]$ of generic extensions, all filters $G_0 \subset Q$ in $V[H_0]$ and $G_1 \subset Q$ in $V[H_1]$ generic over V, and conditions $p_0 \leq \tau/G_0$ in $V[H_0]$ and $p_1 \leq \tau/G_1$ in $V[H_1]$, the conditions $p_0, p_1 \in P$ have a common lower bound.
2. The poset P is Noetherian balanced if for every condition $p \in P$ there is a Noetherian balanced pair $\langle Q, \tau\rangle$ such that $Q \Vdash \tau \leq \check{p}$.

Thus, Noetherian balance is a strengthening of the usual balance of Suslin posets [5, Chapter 5]. It is an amalgamation tool which for example rules out nonprincipal ultrafilters on ω in P-extensions of the choiceless Solovay model [11]. It is irrelevant for the main theorems of this paper, but it will be used in future work.

Proposition 4.9. *The poset P_Γ is Noetherian balanced.*

Proof. Let $p \in P_\Gamma$ be a condition. I will show that there is a total Γ-coloring c on X such that $p \subset c$ and for every $x \in \mathrm{dom}(c \setminus p)$, the value $c(x)$ is a basic open subset of X which contains no elements of $\mathrm{dom}(p)$ Γ-connected to x. Then I will show that the pair $\langle \mathrm{Coll}(\omega, \mathbb{R}), \check{c}\rangle$ is a Noetherian balanced pair. This will prove the proposition.

To find c, first for every point $x \in X \setminus \mathrm{dom}(p)$ find an open neighborhood $d(x) \subset X$ containing x and no points of $\mathrm{dom}(p)$ which are Γ-connected to x. This is possible by Proposition 2.8. By the proof of Theorem 2.6, there is a total Γ-coloring e which to each point $x \in X$ assigns a basic open subset $e(x) \subset d(x)$ containing x. Then define c by $c(x) = p(x)$ if $x \in \mathrm{dom}(p)$ and $c(x) = d(x)$ if $x \notin \mathrm{dom}(p)$; this coloring c is as required.

It is clear that $\mathrm{Coll}(\omega, \mathbb{R}) \Vdash \check{c} \leq \check{p}$. Suppose now that $V[H_0], V[H_1]$ are mutually Noetherian extensions of V, each containing respective condition $p_0 \leq c$ and $p_1 \leq c$; I must show that p_0, p_1 are compatible. This means that item (3) of Proposition 4.3 must be verified for $a = \{p_0, p_1\}$. Suppose that $x_0 \in \mathrm{dom}(p_0 \setminus c)$ and $x_1 \in \mathrm{dom}(p_1 \setminus c)$ are Γ-connected points, and towards a contradiction assume that (e.g.) $x_1 \in p_0(x_0)$. The set $A_1 = \{y \in X : y = x_0 \vee y \,\Gamma\, x_0\}$ is closed in the Γ-topology, coded in $V[H_0]$, and contains x_1. By the Noetherian assumption, the smallest closed in the Γ-topology set B_1 coded in V containing x_1 is a subset of A_1. By Mostowski absoluteness between V and $V[H_1]$, B_1 contains a ground model point y in the basic open set $p(x_0)$. However, this contradicts the assumption that $p_0 \leq c$ holds. $\square$

Corollary 4.10. *Let Γ be a closed Noetherian graph on a σ-compact Polish space without an uncountable clique. In the P_Γ-extension of the Solovay model,*

1. *there are no discontinuous homomorphisms between Polish groups;*
2. *there is no nonprincipal ultrafilter on ω;*
3. *the Lebesgue null ideal is closed under well-ordered unions.*

Proof. Item (1) follows from the balance and the 3, 2-centeredness of the poset P_Γ as proved in [5, Theorem 13.2.1]. (2) follows from (1), since a nonprincipal ultrafilter U on ω yields a discontinuous homomorphism from the Cantor group 2^ω to 2 assigning to any point $x \in 2^\omega$ its prevailing value. However, (2) also follows from the Noetherian balance of the poset P_Γ by [11]. (3) follows from the Noetherian balance of P_Γ again [11]. $\square$

Properties of the P_Γ extension of the Solovay model pertaining to countable Borel equivalence relations must be checked by the methods of the following sections.

5 The control poset

This section discussed the main technical tool used to control the generic extension of the Solovay model by the balanced coloring poset P_Γ obtained in Section 4.

Definition 5.1. Let Γ be a closed graph on a Polish space X. The Γ-*control poset* Q_Γ is the poset of all finite partial Γ-colorings $q : X \to \omega$ ordered by reverse inclusion.

I will be interested in c.c.c. of the control posets. The first observation in this direction will not be needed below.

Proposition 5.2. *Let Γ be a closed graph on a Polish space X. Exactly one of the following occurs:*

1. *Q_Γ is c.c.c.;*
2. *Γ contains a perfect clique.*

Proof. Clearly, (2) implies negation of (1): if $C \subset X$ is a perfect clique then the set $\{\{\langle x, 0\rangle\} : x \in C\}$ forms an uncountable antichain in Q_Γ. To see that negation of (1) implies (2), consider the Polish space $Y = X \times \omega$ and the graph Δ on Y connecting $\langle x, i\rangle$ with $\langle y, j\rangle$ if either $x \Gamma y$ and $i = j$ holds, or $x = y$ or $i \neq j$ holds. The graph Δ is closed. It is clear that the poset Q_Γ is exactly the poset of all finite Δ-anticliques ordered by inclusion. Now assume that (1) fails. By a classical result of Todorcevic [8, Proposition 1] applied to Δ, Δ contains an uncountable clique. Thinning down the uncountable anticlique if necessary, we may assume that the second coordinates of its points are all equal, so the first coordinates form an uncountable clique in Γ. Since Γ is closed, the closure of this Γ-clique is again a Γ-clique, which by the Cantor–Bendixson theorem contains a perfect subclique, confirming (2). $\square$

I need to precisely quantify the nature of c.c.c. of the control posets in case that the graph Γ is closed, Noetherian, and contains no infinite cliques. The following common parlance will be useful throughout.

Definition 5.3. Let Γ be a closed graph on a Polish space X.

1. A *location* is a pair $\langle a, f\rangle$ where a is a finite collection of pairwise disjoint basic open subsets of X and $f : a \to \omega$ is a function such that for any two distinct open sets $O_0, O_1 \in a$, either $(O_0 \times O_1) \cap \Gamma = 0$ or $f(O_0) \neq f(O_1)$ holds;
2. a function q such that $\mathrm{dom}(q)$ is a selector in a and for every $x \in \mathrm{dom}(q)$ $q(x) = f(O)$ for the unique point $O \in a$ containing x is *at location* $\langle a, f\rangle$;
3. if a location $\langle a, f\rangle$ is given, for any condition q at this location and any $O \in a$ the symbol $q(O)$ denotes the unique point in the domain of q which belongs to O.

Note that every function as in (2) is a condition in the control poset Q_Γ. The first theorem evaluates the descriptive complexity of the control poset.

Definition 5.4. A c.c.c. poset Q is *very Suslin* if

1. Q is Suslin. That is, in an ambient Polish space the set Q is analytic, and the relations of ordering and incompatibility are analytic as well;
2. the set $\{d \in Q^\omega : \mathrm{rng}(d) \subset Q \text{ is a predense set}\}$ is analytic.

Very Suslin c.c.c. forcings do not add dominating reals [7], and their complexity is preserved under finite support iterations of countable length.

Theorem 5.5. *If Γ is a closed Noetherian graph on a Polish σ-compact space X without an infinite clique, then the poset Q_Γ is very Suslin.*

Proof. Suslinness of Q_Γ follows immediately from the definitions. To start, use Kuratowski–Ryll-Nardzewski theorem [2, Theorem 12.13] to find Borel functions $f_n : F(X) \to X$ for $n \in \omega$ such that for every nonempty closed set $C \subset X$, the set $\{f_n(C)\colon n \in \omega\}$ is a dense subset of C. Now, let $d \in \mathbb{Q}_\Gamma^\omega$ be a sequence of conditions. Write $b = \bigcup\{\mathrm{dom}(q)\colon q \in \mathrm{rng}(d)\}$ and $c = b \cup \bigcup\{f_n(\Gamma(a))\colon n \in \omega, a \in [b]^{<\aleph_0}\}$. I claim that $\mathrm{rng}(d) \subset Q_\Gamma$ is predense if and only if every condition whose domain is a subset of c is compatible with some element of $\mathrm{rng}(d)$.

First, note that this will show that the set $\{d \in Q^\omega\colon \mathrm{rng}(d) \subset Q$ is a predense set$\}$ is analytic. The universal quantifier on the right side of the equivalence is restricted to a countable subset of Q_Γ which is obtained in a Borel way from d.

Now, the left-to-right implication is immediate. The right-to-left implication is proved in the contrapositive. Suppose that the set $\mathrm{rng}(d)$ is not predense and work to find a condition which is incompatible to all elements of $\mathrm{rng}(d)$, whose domain is a subset of c. Let $q \in Q_\Gamma$ be any condition incompatible with all elements of $\mathrm{rng}(d)$. For each point $x \in \mathrm{dom}(q) \setminus b$, let $b' \subset b$ be a finite set of points Γ-connected to x and such that the set $C_x = \Gamma(b')$ is as small as possible. Such a set is possible to find by the Noetherian assumption and Theorem 2.2.

Now, let $\langle a, f\rangle$ be any location of the condition q. Define a condition r at the same location using the following description.

- if $O \in a$ is such that $q(O) \in b$ then let $r(O) = q(O)$;
- if $O \in a$ is such that $x = q(O) \notin b$ and there is a point $y \in C_x \cap O \cap b$ which is Γ-disconnected from x, then let $r(O) = y$;
- finally, suppose that $O \in a$ is such that $x = q(O) \notin b$ and all points of $C_x \cap O \cap b$ are Γ-connected to x. In such a case, $C_x \cap O \cap b$ is a Γ-clique by the minimal choice of C_x. By the initial assumptions on the graph Γ, this clique is finite. Note that $x \in C_x$ holds and x does not belong to the finite set $C_x \cap O \cap b$, simply because $x \notin b$ holds. Since the set $c \cap C_x$ is dense in C_x, there is a point $y \in c \cap C_x \cap O$ which does not contain any elements of $C_x \cap O \cap b$. Let $r(O) = y$.

To complete the proof, argue that r is incompatible with all conditions in $\mathrm{rng}(d)$. To do that, suppose that $s \in \mathrm{rng}(d)$ is a condition. Then s, q are incompatible, and there must be points $x \in \mathrm{dom}(q)$ and $z \in \mathrm{dom}(s)$ such that either $x = z$ and $q(x) \neq s(z)$, or $x \;\Gamma\; z$ and $q(x) = s(z)$. The discussion now splits into cases.

If $x \in b$ then s, r are incompatible by virtue of the same points x, z since $q \restriction b \subseteq r \restriction b$ by the first item above. If $x \notin b$, then let $O \in a$ be the unique open set which contains it, and let $y = r(O)$. No matter whether the second or third item above occurred, the minimality of C_x, the fact that $y \in C_x$, and the fact that $z \in b$ guarantee that $z = y$ or $z \;\Gamma\; y$ holds. It will be enough to rule out the possibility that $z = y$ holds, since then s, r are incompatible by virtue of the points y, z. Now, if the second item above occurred for O, $z = y$ is impossible since $y \;\Gamma\; x$ fails in that case while $z \;\Gamma\; x$ holds. If the third item above occurred for O, then $y \notin b$ and y must again be distinct from z since $z \in b$ holds. The proof is complete. $\square$

Perhaps more importantly, the Noetherian property of the graph Γ has implications for the centeredness properties of the control poset Q_Γ. Fine distinctions are important here, and we restate the definitions of [5, Chapter 11]:

Definition 5.6. Let Q be a poset.

1. A set $A \subset Q$ is *liminf-centered* if for every infinite set $a \subset A$ there is a condition $q \in Q$ which forces the generic filter to contain infinitely many elements of a.
2. If Q is a Suslin poset, it is *Suslin-σ-liminf-centered* if there are analytic liminf-centered sets $A_n \subset Q$ for $n \in \omega$ such that $Q = \bigcup_n A_n$.

Note that a liminf centered set cannot contain an infinite antichain, so σ-liminf centeredness implies c.c.c.

Theorem 5.7. *Suppose that Γ is a closed Noetherian graph on a Polish space X, without an infinite clique. Then the control poset Q_Γ is Suslin-σ-liminf centered.*

Proof. Let $\langle a, f\rangle$ be a location. It is enough to show that the set A of all conditions at that location is liminf-centered. To this end, the following abstract claim will be useful.

Claim 5.8. *Let $b \subset X$ be an infinite set. There is an infinite set $c \subset b$ such that each element of X is either Γ-connected with only finitely many elements of c or with all elements of c.*

Proof. First use the Ramsey theorem to shrink the set b if necessary to a Γ-anticlique. Use Theorem 2.2 to find an inclusion-minimal set $C \subset X$ in the Γ-graph topology such that $c = C \cap b$ is infinite. I claim that the set c works.

Indeed, suppose that $x \in X$ is a point. If $x \in c$, then x is Γ-disconnected from all other elements of c since c is a Γ-anticlique. If $x \notin c$ and x is Γ-connected to infinitely many elements of c, then $C = C \cap \Gamma(x)$ by the minimal choice of C and therefore x is Γ-connected to all elements of c. The claim follows. $\square$

Now, let $A = \{q_n : n \in \omega\}$ be an infinite collection of conditions at location $\langle a, f\rangle$. Shrinking the collection if necessary, it is possible to find a partition $a = a_0 \cup a_1$ such that for every $O \in a_0$ the points $q_n(O)$ for $n \in \omega$ are all the same, while for every set $O \in a_1$ the points $q_n(O)$ for $n \in \omega$ are pairwise distinct. Using the claim repeatedly, it is possible to further shrink the set A so that for every $O \in a_1$, every point in X is Γ-connected to only finitely many points among $\{q_n(O) : n \in \omega\}$ or to all of them. It will be enough to show that q_0 forces that the generic filter contains infinitely many conditions in the set A.

To see this, suppose that $r \leq q_0$ is a condition; it will be enough to show that r is compatible with all but finitely many conditions in A. Indeed, suppose that $n \in \omega$ is large enough so that for every point $x \in \mathrm{dom}(r)$ and every $O \in a_1$, $q_n(O) \neq x$ and if x is Γ-connected to only finitely many elements of $q_n(O)$, then it is not Γ-connected to $q_n(O)$. It will be enough to show that r is compatible with q_n. For this, note that $q_n \cup r$ is a function by the choice of the number n. To show that $q_n \cup r$ is a Γ-coloring, suppose that $x \in \mathrm{dom}(r)$ and $O \in a_1$ is a set and write $y = q_n(O)$. I must show that $x \;\Gamma\; y$ implies $r(x) \neq q_n(y)$. To see this, note that $x \, \Gamma \, y$ implies that x is Γ-connected with all points in the set $\{q_i(O) \colon i \in \omega\}$ by the choice of the number n. In particular, x is Γ-connected to $z = q_0(O)$, and since $r \leq q_0$ holds, $r(x) \neq q_0(z)$. Now, the conditions q_0 and q_n are at the same location and therefore $q_0(z) = q_n(y)$. It follows that the values $r(x)$ and $q_n(y)$ are distinct as required. $\square$

Definition 5.9. Let Q be a poset.

1. A set $A \subset Q$ is *Ramsey centered* if for every $m \in \omega$ there is $n \in \omega$ such that every collection of n many elements of A contains a subcollection of cardinality m which has a common lower bound.

2. If Q is a Suslin poset, it is *Suslin-σ-Ramsey-centered* if there are analytic Ramsey-centered sets $A_n \subset Q$ for $n \in \omega$ such that $Q = \bigcup_n A_n$.

Note that a Ramsey-centered set cannot contain an infinite antichain, so σ-Ramsey centeredness implies c.c.c. again.

Theorem 5.10. *Suppose that $m \in \omega$ is a number and Γ is a closed graph on a Polish space X, containing no cliques of cardinality m. Then the control poset Q_Γ is Suslin-σ-Ramsey centered.*

Note that this theorem does not refer to any Noetherian assumption; indeed, its proof is much easier than that of Theorem 5.7.

Proof. Let $\langle a, f \rangle$ be a location. It is enough to show that the set A of all conditions at that location is Ramsey-centered. To this end, let $m \in \omega$ be a number; increasing m if necessary assume that Γ contains no cliques of cardinality m. Writing s for the cardinality of the set A, any natural number k such that $k \to (m)^2_{s+1}$ witnesses Ramsey centeredness of the set A for the number m. To see this, let $q_i \colon i \in k\rangle$ be a collection of conditions in A. Define a map c with domain $[k]^2$ by setting $c(i, j) = O$ if the conditions q_i, q_j are incompatible and this fact is witnessed by the open set $O \in a$ in the sense that the unique elements x_i, x_j in $\mathrm{dom}(q_i) \cap O$ and $\mathrm{dom}(q_j) \cap O$ are Γ-connected. Define $c(i, j) =!$ if the conditions q_i, q_j are compatible.

By the choice of the number k there is a set $b \subset k$ of cardinality m homogeneous for the partition c. The homogeneous color cannot be a set $O \in a$, since then the unique points in $\mathrm{dom}(q_i) \cap O$ for $i \in b$ would form a Γ-clique of cardinality m, contradicting the initial choice of m. Thus, the homogeneous color is ! and the set $\{q_i \colon i \in b\}$ consists of pairwise compatible conditions. Then $\bigcup_{i \in b} q_i$ is a common lower bound of this set, confirming Ramsey-centeredness of the set A. $\square$

6 Wrapping up

The theorems in the introduction are now obtained via the technologies of [5, Chapter 11].

Definition 6.1. Let P be a Suslin poset. Say that P has *liminf control* if provably in ZFC, there is a definable map which assigns to each condition $p \in P$ objects R, π, Q, τ so that

1. R is a forcing and Q, τ are R-names;
2. $\pi \colon R \to \mathrm{Ord}$ is a function such that preimages of singletons are liminf-centered;
3. $R \Vdash \langle Q, \tau \rangle$ is a balanced pair in P and $Q \Vdash \tau \leq \check{p}$.

In other words, there may not be a definable balanced pair for P, but there is a definable and suitably centered way of forcing such a pair. The following is a complementary concept from descriptive set theory.

Definition 6.2. [5, Definition 11.1.5] An analytic graph on a Polish space X has Borel σ-finite clique number if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that $\bigcup_n B_n = X$ and no set B_n contains an infinite clique.

A good example of a Borel graph with uncountable Borel σ-finite clique number is the Hamming graph $\mathbb{H}_\omega$ of infinite breadth from Definition 1.3. The following is proved in [5, Theorem 11.7.5].

Fact 6.3. *Suppose that P is balanced and has liminf control. Suppose that Δ is an analytic graph with uncountable Borel σ-finite clique number. Let κ be an inaccessible cardinal and W be the derived choiceless Solovay model. In the P-extension of W, Δ has uncountable chromatic number.*

Proposition 6.4. *Let Γ be a closed Noetherian graph on a σ-compact Polish space X without infinite cliques. Then the coloring poset P_Γ has liminf control.*

Proof. Let $p \in P_\Gamma$ be a condition. Let R be the finite support iteration of the control poset Q_Γ of Definition 5.1 of length ω_1. Since the control poset is Suslin-σ-liminf-centered by Theorem 5.7, the iteration R is definable and definably liminf-centered as in Definition 6.1 as proved in [5, Proposition 11.7.3] I will produce a definition of an R-name for a total coloring of X stronger than p. This will prove the proposition, since by the (proof of) Proposition 4.9 such a coloring defines a balanced pair.

First, note that the iteration R produces a transfinite sequence $\langle M_\alpha \colon \alpha \in \omega_1 \rangle$ of forcing extensions, where we put M_0 to be the ground model. By a c.c.c. argument, the space X in the R-extension is a subset of the union $\bigcup_\alpha M_\alpha$. The iteration R also produces a transfinite sequence $\langle c_\alpha \colon \alpha \in \omega_1 \rangle$ of partial Γ-colorings. Namely, each coloring c_α is the union of the generic filter on the α-th iterand of R, and as such its domain is exactly $X \cap M_\alpha$. The colorings do not cohere in any way. It is necessary to stitch them together in a definable way to produce a total coloring of the space X in the R-extension.

Let B be a basis of the space X, and let $B = \bigcup_n B_n$ be a partition of B into countably many subbases. Define the coloring c by the following description. If $x \in \mathrm{dom}(p)$ then $c(x) = p(x)$. If $x \notin \mathrm{dom}(p)$, then let $\alpha \in \omega_1$ be the smallest ordinal such that $x \in M_\alpha$ holds, and let $c(x)$ be the first (in some fixed enumeration) element of the basis $B_{c_\alpha(x)}$ which contains no points in the set $\mathrm{dom}(p) \cup \bigcup_{\beta\in\alpha} M_\beta$ which are Γ-connected to x. Such a set exists by Proposition 2.8. It is important to check the assumptions of that proposition, i.e. the set $C_\alpha = X \cap (\mathrm{dom}(p) \cup \bigcup_{\beta\in\alpha} M_\beta)$ should be Γ-good. To see this, if $\alpha = 0$ it follows from the fact that $\mathrm{dom}(p)$ is Γ-good. If α is a successor of some ordinal β, then it follows from a Mostowski absoluteness argument for the model M_β. Finally, if the ordinal α is limit then C_α is the increasing union of Γ-good sets C_β for $\beta \in \alpha$ and therefore Γ-good as well.

Now, work to verify that the map c is a Γ-coloring. To see that, suppose first that $x \neq y$ are Γ-connected points in the domain of c, and work to show that they receive different colors. Let α_x, α_y be the smallest ordinals such that $x \in M_{\alpha_x}$ and $y \in M_{\alpha_y}$. If the two ordinals are distinct (say $\alpha_y \in \alpha_x$) then the color $c(y)$ is an open set containing y, while the color $c(x)$ is an open set containing no elements of the model M_{α_y} connected to x, in particular $y \notin c(x)$ and $c(x) \neq c(y)$. If the two ordinals α_x, α_y are equal to some α then $c_\alpha(x) \neq c_\alpha(y)$ and therefore $c(x) \neq c(y)$. As a special case, if $\alpha = 0$ and one of the points (say y) belongs to $\mathrm{dom}(p)$, then either $x \in \mathrm{dom}(p)$ and $c(x) = p(x) \neq p(y) = c(y)$, or $x \notin \mathrm{dom}(p)$ and then $c(x)$ is an open set containing no elements of $\mathrm{dom}(p)$ connected to x, in particular $y \notin c(x)$ and $c(x) \neq c(y)$ again.

Finally, it is clear that the collapse of the continuum forces $c \leq p$ by the definition of the map c. The proof of the proposition is complete. □

The story is entirely parallel in the case of Ramsey-σ-centered posets, except the eventual conclusion is stronger.

Definition 6.5. Let P be a Suslin poset. Say that P has *Ramsey control* if provably in ZFC, there is a definable map which assigns to each condition $p \in P$ objects R, π, Q, τ so that

1. R is a forcing and Q, τ are R-names;
2. $\pi \colon R \to \mathrm{Ord}$ is a function such that preimages of singletons are Ramsey-centered;
3. $R \Vdash \langle Q, \tau\rangle$ is a balanced pair in P and $Q \Vdash \tau \leq \check{p}$.

In other words, there may not be a definable balanced pair for P, but there is a definable and suitably centered way of forcing such a pair. The following is a complementary concept from descriptive set theory.

Definition 6.6. An analytic graph Δ on a Polish space X has *countable Borel σ-bounded chromatic number* if there are Borel sets $B_n \subset X$ for $n \in \omega$ such that $\bigcup_n B_n = X$ and Δ restricted to every finite subset of B_n has chromatic number at most $n + 2$.

A good example of a Borel graph with uncountable Borel σ-bounded chromatic number is the diagonal Hamming graph $\mathbb{H}_{<\omega}$ of Definition 1.3. The following is proved in [5, Theorem 11.6.5]:

Fact 6.7. *Suppose that P has Ramsey control. Let Δ be an analytic graph on a Polish space with uncountable Borel σ-bounded chromatic number. Let κ be an inaccessible cardinal and W be the derived choiceless Solovay model. In the P-extension of W, Δ has uncountable chromatic number.*

Proposition 6.8. *Let Γ be a closed Noetherian graph on a σ-compact Polish space X. Suppose that there is a number m such that Γ contains no cliques of cardinality m. Then the coloring poset P_Γ has Ramsey control.*

The proof is a verbatim repetition of the proof of Proposition 6.4 with references to Theorem 5.7 and [5, Proposition 11.7.3] replaced with Theorem 5.10 and [5, Proposition 11.6.3]. It should be noted that while the Suslin-σ-Ramsey-centeredness of the control poset in this case does not need the Noetherian assumption, the proof of Proposition 6.8 does need it at the place where the transfinite sequence of colorings is stitched into a single one.

Finally, it is possible to state the proofs of the main theorems of the introduction in their entirety. For Theorem 1.4(1), let Γ be a closed Noetherian graph on a σ-compact Polish space. Let κ be an inaccessible cardinal, let W be the choiceless Solovay model associated with κ, and let $W[G]$ be a generic extension of W obtained by the coloring poset P_Γ of Definition 4.1. The poset is balanced and it has liminf-centered control by Proposition 6.4. By Fact 6.3, in the model $W[G]$, the chromatic number of the Hamming graph $\mathbb{H}_\omega$ of infinite breadth is uncountable, and Theorem 1.4(1) follows. If there is a finite bound on the size of Γ-cliques, then the coloring poset P_Γ has Ramsey control by Proposition 6.8. By Fact 6.7, in the model $W[G]$, the chromatic number of the diagonal Hamming graph $\mathbb{H}_{<\omega}$ of infinite breadth is uncountable, proving Theorem 1.4(2).

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