Imaginary sorts and strong conceptual completeness for $L_{\omega_1\omega}$

Ronnie Chen

University of Illinois at Urbana–Champaign

SEALS, March 1, 2020

$\mathcal{L}_{\omega_1\omega}$

This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$

This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$ = extension of first-order logic w/ countable $\bigwedge, \bigvee$.

$\mathcal{L}_{\omega_1\omega}$

This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$ = extension of first-order logic w/ countable $\bigwedge, \bigvee$.

For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\mathrm{Mod}(\mathcal{T}) := \text{standard Borel space of models of } \mathcal{T} \text{ on } 0, 1, 2, \ldots, \mathbb{N},$$

$\mathcal{L}_{\omega_1\omega}$

This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$ = extension of first-order logic w/ countable $\bigwedge, \bigvee$.

For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\begin{aligned}\mathrm{Mod}(\mathcal{T}) :=\ & \text{standard Borel space of models of } \mathcal{T} \text{ on } 0, 1, 2, \dots, \mathbb{N}, \\ & \text{together w/ isomorphisms between them,}\end{aligned}$$

$\mathcal{L}_{\omega_1\omega}$

This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$ = extension of first-order logic w/ countable $\bigwedge, \bigvee$.

For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\begin{aligned}\mathrm{Mod}(\mathcal{T}) := &\text{ standard Borel space of models of } \mathcal{T} \text{ on } 0, 1, 2, \dots, \mathbb{N}, \\ &\text{ together w/ isomorphisms between them,}\end{aligned}$$

the **standard Borel groupoid of countable models of** $\mathcal{T}$.

$\mathcal{L}_{\omega_1\omega}$

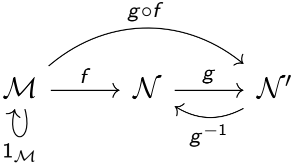
This talk is about $\mathcal{L}_{\omega_1\omega}$ and Polish/Borel groupoids.

$\mathcal{L}_{\omega_1\omega}$ = extension of first-order logic w/ countable $\bigwedge, \bigvee$.

For a countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\begin{aligned}\mathrm{Mod}(\mathcal{T}) :=\ & \text{standard Borel space of models of } \mathcal{T} \text{ on } 0, 1, 2, \ldots, \mathbb{N}, \\ & \text{together w/ isomorphisms between them,}\end{aligned}$$

the **standard Borel groupoid of countable models of $\mathcal{T}$**.

(**Groupoid**: 

$\cdots$)

Strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

Theorem (C. 2019)

A countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ may be canonically recovered, up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability, from $\mathrm{Mod}(\mathcal{T})$.

Strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

Theorem (C. 2019)

A countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ may be canonically recovered, up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability, from $\mathrm{Mod}(\mathcal{T})$.

Harrison-Trainor–Miller–Montalbán (2018): a countable structure $\mathcal{M}$ is determined up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability by $\mathrm{Aut}(\mathcal{M})$.

Strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

Theorem (C. 2019)

A countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ may be canonically recovered, up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability, from $\mathrm{Mod}(\mathcal{T})$.

Harrison-Trainor–Miller–Montalbán (2018): a countable structure $\mathcal{M}$ is determined up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability by $\mathrm{Aut}(\mathcal{M})$.

Note: $\mathrm{Mod}(\mathcal{T})$ is *not* equipped with underlying set functor!

Strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

Theorem (C. 2019)

A countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ may be canonically recovered, up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability, from Mod($\mathcal{T}$).

Harrison-Trainor–Miller–Montalbán (2018): a countable structure $\mathcal{M}$ is determined up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability by Aut($\mathcal{M}$).

Note: Mod($\mathcal{T}$) is *not* equipped with underlying set functor!

This is a **strong conceptual completeness** theorem for $\mathcal{L}_{\omega_1\omega}$, in the sense of Makkai (1988) who proved that an $\mathcal{L}_{\omega\omega}$-theory can be recovered from its category of models equipped w/ ultraproducts.

Strong conceptual completeness for $\mathcal{L}_{\omega_1\omega}$

Theorem (C. 2019)

A countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ may be canonically recovered, up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability, from $\mathrm{Mod}(\mathcal{T})$.

Harrison-Trainor–Miller–Montalbán (2018): a countable structure $\mathcal{M}$ is determined up to $\mathcal{L}_{\omega_1\omega}$-bi-interpretability by $\mathrm{Aut}(\mathcal{M})$.

Note: $\mathrm{Mod}(\mathcal{T})$ is *not* equipped with underlying set functor!

This is a **strong conceptual completeness** theorem for $\mathcal{L}_{\omega_1\omega}$, in the sense of Makkai (1988) who proved that an $\mathcal{L}_{\omega\omega}$-theory can be recovered from its category of models equipped w/ ultraproducts.

Main ingredient: $\mathcal{L}_{\omega_1\omega}$-**imaginary sorts** $\leftrightarrow$ $\mathrm{Mod}(\mathcal{T})$-**actions**.

Fiberwise countable Borel actions

Fix $\mathcal{T}$. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(\vec{x})$,

- countable model $\mathcal{M} \longmapsto$ countable set $\phi^{\mathcal{M}}$

Fiberwise countable Borel actions

Fix $\mathcal{T}$. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(\vec{x})$,

- countable model $\mathcal{M} \longmapsto$ countable set $\phi^{\mathcal{M}}$
- isomorphism $\mathcal{M} \cong \mathcal{N} \longmapsto$ bijection $\phi^{\mathcal{M}} \cong \phi^{\mathcal{N}}$

Fiberwise countable Borel actions

Fix $\mathcal{T}$. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(\vec{x})$,

- countable model $\mathcal{M} \longmapsto$ countable set $\phi^{\mathcal{M}}$
- isomorphism $\mathcal{M} \cong \mathcal{N} \longmapsto$ bijection $\phi^{\mathcal{M}} \cong \phi^{\mathcal{N}}$

Moreover, these assignments are "Borel".

Fiberwise countable Borel actions

Fix $\mathcal{T}$. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(\vec{x})$,

- countable model $\mathcal{M} \longmapsto$ countable set $\phi^{\mathcal{M}}$
- isomorphism $\mathcal{M} \cong \mathcal{N} \longmapsto$ bijection $\phi^{\mathcal{M}} \cong \phi^{\mathcal{N}}$

Moreover, these assignments are "Borel".

A **fiberwise countable Borel action** of $\mathrm{Mod}(\mathcal{T})$ is:

- a Borel functor $\mathrm{Mod}(\mathcal{T}) \to \{0, 1, 2, \ldots, \mathbb{N}\}$; or equivalently

Fiberwise countable Borel actions

Fix $\mathcal{T}$. For an $\mathcal{L}_{\omega_1\omega}$-formula $\phi(\vec{x})$,

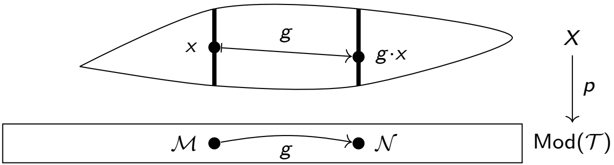
- countable model $\mathcal{M} \longmapsto$ countable set $\phi^{\mathcal{M}}$
- isomorphism $\mathcal{M} \cong \mathcal{N} \longmapsto$ bijection $\phi^{\mathcal{M}} \cong \phi^{\mathcal{N}}$

Moreover, these assignments are "Borel".

A **fiberwise countable Borel action** of $\mathrm{Mod}(\mathcal{T})$ is:

- a Borel functor $\mathrm{Mod}(\mathcal{T}) \to \{0, 1, 2, \dots, \mathbb{N}\}$; or equivalently
- a countable-to-1 Borel $p : X \to \mathrm{Mod}(\mathcal{T})$, together with Borel

$$(g : \mathcal{M} \cong \mathcal{N},\, x \in p^{-1}(\mathcal{M})) \longmapsto g \cdot x \in p^{-1}(\mathcal{N}).$$



Imaginary sorts

- So $\mathcal{L}_{\omega_1\omega}$-formula $\phi \longmapsto$ fbwise ctble Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}})_{\mathcal{M}}$.

Imaginary sorts

- So $\mathcal{L}_{\omega_1\omega}$-formula $\phi \longmapsto$ fbwise ctble Borel $\mathsf{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}})_{\mathcal{M}}$.
- Countably many formulas $\phi_0, \phi_1, \ldots \longmapsto \mathsf{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}})_{\mathcal{M}}$.

Imaginary sorts

- So $\mathcal{L}_{\omega_1\omega}$-formula $\phi \longmapsto$ fbwise ctble Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}})_{\mathcal{M}}$.
- Countably many formulas $\phi_0, \phi_1, \ldots \longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}})_{\mathcal{M}}$.
- Formulas $\phi(\vec{x}), \varepsilon(\vec{x}, \vec{y})$ such that

$$\mathcal{T} \models \text{“}\varepsilon \text{ is an equivalence relation on } \phi\text{”}$$

$\longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}}/\varepsilon^{\mathcal{M}})_{\mathcal{M}}$.

Imaginary sorts

- So $\mathcal{L}_{\omega_1\omega}$-formula $\phi \longmapsto$ fbwise ctble Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}})_{\mathcal{M}}$.
- Countably many formulas $\phi_0, \phi_1, \ldots \longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}})_{\mathcal{M}}$.
- Formulas $\phi(\vec{x}), \varepsilon(\vec{x}, \vec{y})$ such that

$$\mathcal{T} \models \text{“}\varepsilon \text{ is an equivalence relation on } \phi\text{”}$$

$\longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}}/\varepsilon^{\mathcal{M}})_{\mathcal{M}}$.

- More generally, countably many formulas ϕ_i, ε_{ij} such that

$$\mathcal{T} \models \text{“}\bigsqcup_{i,j} \varepsilon_{ij} \text{ is an equivalence relation on } \bigsqcup_i \phi_i\text{”}$$

$\longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}} / \bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}})_{\mathcal{M}}$.

Imaginary sorts

- So $\mathcal{L}_{\omega_1\omega}$-formula $\phi \longmapsto$ fbwise ctble Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}})_{\mathcal{M}}$.
- Countably many formulas $\phi_0, \phi_1, \ldots \longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}})_{\mathcal{M}}$.
- Formulas $\phi(\vec{x}), \varepsilon(\vec{x}, \vec{y})$ such that

$$\mathcal{T} \models \text{“}\varepsilon \text{ is an equivalence relation on } \phi\text{”}$$

$\longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\phi^{\mathcal{M}}/\varepsilon^{\mathcal{M}})_{\mathcal{M}}$.

- More generally, countably many formulas ϕ_i, ε_{ij} such that

$$\mathcal{T} \models \text{“}\bigsqcup_{i,j} \varepsilon_{ij} \text{ is an equivalence relation on } \bigsqcup_i \phi_i\text{”}$$

$\longmapsto \mathrm{Mod}(\mathcal{T}) \curvearrowright (\bigsqcup_i \phi_i^{\mathcal{M}} / \bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}})_{\mathcal{M}}$.

An $\mathcal{L}_{\omega_1\omega}$-**imaginary sort** of $\mathcal{T}$ is such a countable family of $\mathcal{L}_{\omega_1\omega}$-formulas $(\phi_i, \varepsilon_{ij})_{i,j}$.

Representing actions via imaginary sorts

Main lemma

For any countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ and fiberwise countable Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$*, there is an $\mathcal{L}_{\omega_1\omega}$-imaginary sort $(\phi_i, \varepsilon_{ij})_{i,j}$ which defines an action isomorphic to X.*

Representing actions via imaginary sorts

Main lemma

For any countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$ and fiberwise countable Borel $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$, *there is an $\mathcal{L}_{\omega_1\omega}$-imaginary sort $(\phi_i, \varepsilon_{ij})_{i,j}$ which defines an action isomorphic to X.*

Note: X has *a priori* no connection with underlying sets of models!

Syntactic σ-pretoposes

The **syntactic σ-pretopos** $\langle\mathcal{T}\rangle$ of $\mathcal{T}$ is the category with:

- objects: imaginary sorts;
- morphisms: definable functions ($\mathcal{T} \models$ "ϕ is graph of a fn").

Syntactic σ-pretoposes

The **syntactic σ-pretopos** $\langle \mathcal{T} \rangle$ of $\mathcal{T}$ is the category with:

- objects: imaginary sorts;
- morphisms: definable functions ($\mathcal{T} \models$ "ϕ is graph of a fn").

$\{\sigma$-coherent functors $\langle \mathcal{T}_1 \rangle \to \langle \mathcal{T}_2 \rangle\} \simeq \{\mathcal{L}_{\omega_1\omega}$-interpretations $\mathcal{T}_1 \to \mathcal{T}_2\}$
Hence, $\mathcal{T}_1$ bi-interpretable w/ $\mathcal{T}_2 \iff \langle \mathcal{T}_1 \rangle \simeq \langle \mathcal{T}_2 \rangle$.

Syntactic σ-pretoposes

The **syntactic σ-pretopos** $\langle \mathcal{T} \rangle$ of $\mathcal{T}$ is the category with:

- objects: imaginary sorts;
- morphisms: definable functions ($\mathcal{T} \models$ "ϕ is graph of a fn").

$\{\sigma\text{-coherent functors } \langle \mathcal{T}_1 \rangle \to \langle \mathcal{T}_2 \rangle\} \simeq \{\mathcal{L}_{\omega_1\omega}\text{-interpretations } \mathcal{T}_1 \to \mathcal{T}_2\}$
Hence, $\mathcal{T}_1$ bi-interpretable w/ $\mathcal{T}_2 \iff \langle \mathcal{T}_1 \rangle \simeq \langle \mathcal{T}_2 \rangle$.

Main theorem

For any countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\langle \mathcal{T} \rangle \simeq \{\textit{fbwise ctble Borel actions of } \mathrm{Mod}(\mathcal{T})\}.$$

Syntactic σ-pretoposes

The **syntactic σ-pretopos** $\langle \mathcal{T} \rangle$ of $\mathcal{T}$ is the category with:

- objects: imaginary sorts;
- morphisms: definable functions ($\mathcal{T} \models$ "ϕ is graph of a fn").

$\{\sigma\text{-coherent functors } \langle \mathcal{T}_1 \rangle \to \langle \mathcal{T}_2 \rangle\} \simeq \{\mathcal{L}_{\omega_1\omega}\text{-interpretations } \mathcal{T}_1 \to \mathcal{T}_2\}$
Hence, $\mathcal{T}_1$ bi-interpretable w/ $\mathcal{T}_2 \iff \langle \mathcal{T}_1 \rangle \simeq \langle \mathcal{T}_2 \rangle$.

Main theorem

For any countable $\mathcal{L}_{\omega_1\omega}$-theory $\mathcal{T}$,

$$\langle \mathcal{T} \rangle \simeq \{\textit{fbwise ctble Borel actions of } \mathrm{Mod}(\mathcal{T})\}.$$

- completeness theorem for $\mathcal{L}_{\omega_1\omega}$ (conservative)
- Lopez-Escobar theorem (full-on-subobjects)
- main lemma (essentially surjective)

Proof idea of main lemma

Main lemma

Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ *is isomorphic to that defined by an* $\mathcal{L}_{\omega_1\omega}$*-imaginary sort* $(\phi_i, \varepsilon_{ij})_{i,j}$*.*

Proof idea of main lemma

Main lemma

Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ *is isomorphic to that defined by an* $\mathcal{L}_{\omega_1\omega}$*-imaginary sort* $(\phi_i, \varepsilon_{ij})_{i,j}$.

1. Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ ("Borel $\cong$-equivariant assignment of countable sets") can be turned into a **countable étale action** ("continuous $\cong$-equivariant assignment of countable sets").

Proof idea of main lemma

Main lemma

Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ *is isomorphic to that defined by an* $\mathcal{L}_{\omega_1\omega}$*-imaginary sort* $(\phi_i, \varepsilon_{ij})_{i,j}$.

1. Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ ("Borel $\cong$-equivariant assignment of countable sets") can be turned into a **countable étale action** ("continuous $\cong$-equivariant assignment of countable sets").

This uses the groupoid version of the **Becker–Kechris topological realization theorem** of Lupini (2017), adapted to étale setting.

Proof idea of main lemma

Main lemma

Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ *is isomorphic to that defined by an* $\mathcal{L}_{\omega_1\omega}$*-imaginary sort* $(\phi_i, \varepsilon_{ij})_{i,j}$.

1. Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ ("Borel $\cong$-equivariant assignment of countable sets") can be turned into a **countable étale action** ("continuous $\cong$-equivariant assignment of countable sets").

This uses the groupoid version of the **Becker–Kechris topological realization theorem** of Lupini (2017), adapted to étale setting.

2. Every countable étale action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ is isomorphic to that defined by a Σ_1**-imaginary sort** $(\phi_i, \varepsilon_{ij})_{i,j}$.

Proof idea of main lemma

Main lemma

Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ *is isomorphic to that defined by an* $\mathcal{L}_{\omega_1\omega}$*-imaginary sort* $(\phi_i, \varepsilon_{ij})_{i,j}$.

1. Every fiberwise countable Borel action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ ("Borel $\cong$-equivariant assignment of countable sets") can be turned into a **countable étale action** ("continuous $\cong$-equivariant assignment of countable sets").

This uses the groupoid version of the **Becker–Kechris topological realization theorem** of Lupini (2017), adapted to étale setting.

2. Every countable étale action $\mathrm{Mod}(\mathcal{T}) \curvearrowright X$ is isomorphic to that defined by a Σ_1-**imaginary sort** $(\phi_i, \varepsilon_{ij})_{i,j}$.

This follows from the **Joyal–Tierney representation theorem** (1984) for Grothendieck toposes via localic groupoids, which is an analogous result for $\mathcal{L}_{\infty\omega}$.

Thank you