FORCING PROPERTIES OF IDEALS OF CLOSED SETS

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Abstract. With every σ-ideal I on a Polish space we associate the σ-ideal generated by closed sets in I. We study the quotient forcings of Borel sets modulo the respective σ-ideals and find connections between forcing properties of the two forcing notions. To this end, we associate to a σ-ideal on a Polish space an ideal on a countable set and show how forcing properties of the quotient forcing depend on the combinatorial properties of the ideal. For σ-ideal generated by closed sets, we also study the degrees of reals added by the quotient forcing. Among corollaries of our results, we get necessary and sufficient conditions for a σ-ideal I generated by closed sets, under which every Borel function can be restricted to an I-positive Borel set on which it is either 1-1 or constant. In a futher application, we show when does a hypersmooth equivalence relation admit a Borel I-positive independent set.

1. Introduction

This paper is concerned with the study of σ-ideals I on Polish spaces and associated forcing notions P_I of I-positive Borel sets, ordered by inclusion. If I is a σ-ideal on X, then by I^* we denote the σ-ideal generated by the closed subsets of X which belong to I. Clearly, $I^* \subseteq I$ and $I^* = I$ if I is generated by closed sets.

There are natural examples when the forcing P_I is well understood, whereas little is known about P_{I^*}. For instance if I is the σ-ideal of Lebesgue null sets, then the forcing P_I is the random forcing and I^* is the σ-ideal $\mathcal{E}$. The latter has been studied by Bartoszyński and Shelah [2], [1] but from a slightly different point of view. On the other hand,

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most classical forcing notions, like Cohen, Sacks or Miller forcings fall under the category of P_I for I generated by closed sets.

Some general observations are right on the surface. By the results of [13, Section 4.1] we have that the forcing P_{I^*} is proper and *preserves Baire category* (for a definition see [13, Section 3.5]). In case when $I \neq I^*$ on Borel sets, the forcing P_{I^*} is not ω^ω-bounding by [13, Theorem 3.3.1], since any condition $B \in P_{I^*}$ with $B \in I$ has no closed I^*-positive subset. It is worth noting here that the forcing P_{I^*} depends not only on the σ-ideal I but also on the topology of the space X.

One of the motivations behind studying the idealized forcing notions P_I is the correspodence between Borel functions and reals added in generic extensions. The well-known property of the Sacks or Miller forcing is that all reals in the extension are either ground model reals, or have the same degree as the generic real. Similar arguments also show that the generic extensions are minimal, in the sense that there are no intermediate models. On the other hand, the Cohen forcing adds continuum many degrees and the structure of the generic extension is very far from minimality. In [13, Theorem 4.1.7] the second author showed that under some large cardinal assumptions the Cohen extension is the only intermediate model which can appear in the P_I generic extension when I is universally Baire σ-ideal generated by closed sets.

The commonly used notion of degree of reals in the generic extensions is quite vague, however, and in this paper we distinguish two instances.

Definition 1.1. Let $V \subseteq W$ be a generic extension. We say that two reals $x, y \in W$ are of the same *continuous degree* if there is a partial homeomorphism from ω^ω to ω^ω such that $f \in V$, $\mathrm{dom}(f), \mathrm{rng}(f)$ are $\mathbf{G}_\delta$ subsets of the reals and $f(x) = y$. We say that $x, y \in W$ are of the same *Borel degree* if there is a Borel automorphism h of ω^ω such that $h \in V$ and $h(x) = y$.

Following the common fashion, we say that a forcing notion P_I *adds one continuous* (or *Borel*) *degree* if for any P generic extension $V \subseteq W$ any real in W either belongs to V, or has the same continuous (or Borel) degree as the generic real.

The following results connect the forcing properties of P_I and P_{I^*}. In some cases we need to make some definability assumption, namely that I is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$. For a definition of this notion see [8, Section 29.E] or [13, Section 3.8]. Note that if I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then I^* is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ too, by [8, Theorem 35.38].

Theorem 1.2. *If the forcing P_I is proper and ω^ω-bounding, then the forcing P_{I^*} adds one continuous degree.*

Theorem 1.3. *If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and the forcing P_I is proper and does not add independent reals, then the forcing P_{I^*} does not add independent reals.*

Theorem 1.4. *If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and the forcing P_I is proper and preserves outer Lebesgue measure, then the forcing P_{I^*} preserves outer Lebesgue measure.*

The methods of this paper can be extended without much effort to other cases, for example to show that if P_I is proper and has the weak Laver property, then P_{I^*} inherits this property. As a consequence, by the results of [14, Theorem 1.4] we get (under some large cardinal assumptions) that if P_I proper and preserves P-points, then P_{I^*} preserves P-points as well.

Interestingly enough, in this way we will not obtain any new information about the Miller forcing.

Proposition 1.5. *If I is a σ-ideal such that $I \neq I^*$, then P_{I^*} is neither equivalent to the Miller nor to the Sacks forcing.*

To prove the above results we introduce a combinatorial tree forcing notion $Q(J)$ for J which is a hereditary family of subsets of ω. These are relatives of the Miller forcing. To determine forcing properties of $Q(J)$ we study the position of J in the Katětov ordering, a generalization of the Rudin–Keisler order on ultrafilters. We show that the forcing P_I gives rise to a natural ideal J_I on a countable set and we correlate forcing properties of $Q_(J_I)$ with the Katětov properies of J_I. Finally, we prove that the forcing P_{I^*} is, in the nontrivial case, equivalent to $Q(J_I)$. The conjunction of these results proves all the above theorems.

Next, motivated by the examples of the Sacks and the Miller forcing we prove the following.

Theorem 1.6. *Let I be a σ-ideal generated by closed sets on a Polish space X. Any real in a P_I-generic extension is either a ground model real, a Cohen real, or else has the same Borel degree as the generic real.*

From this we immediately get the following corollary.

Corollary. *Let I be a σ-ideal generated by closed sets on a Polish space X. The following are equivalent:*

- *P_I does not add Cohen reals,*
- *for any $B \in P_I$ and any continuous function $f : B \to \omega^\omega$ there is $C \subseteq B$, $C \in P_I$ such that f is* 1-1 *or constant on C.*

Now, if we identify a Borel function with a smooth equivalence relation, then one way to look at the above result is as at a theorem about selectors for smooth equivalence relations. A much wider class, studied quite extensively, e.g. in [6], is the class of hypersmooth equivalence relations. An equivalence relation E on a Polish space X is *hypersmooth* if there is a sequence of Borel functions f_n on X such that for each $x, y \in X$ we have xEy if and only if $f_n(x) = f_n(y)$ for some $n < \omega$.

Solecki and Spinas [12, Corollary 2.2] showed that for any analytic set $E \subseteq (\omega^\omega)^2$ either all vertical sections of E are σ-compact, or there is a superperfect set $S \subseteq \omega^\omega$ such that $S^2 \cap E = \emptyset$. Motivated by this result, we prove the following.

Theorem 1.7. *Let X be a Polish space and I be a σ-ideal on X generated by closed sets such that the forcing P_I does not add Cohen reals. If E is a Borel hypersmooth equivalence relation on X, then there exists a Borel I-positive set $B \subseteq X$ such that*

- *B is contained in one equvalence class,*
- *or B consists of E-independent elements.*

This paper is organized as follows. In Section 3 we introduce the tree forcing notions $Q(J)$ and relate their forcing properties with the Katětov properties of J. In Section 4 we show how to assciate an ideal J_I to a σ-ideal I and how forcing properties of P_I determine Katětov properties of J. In Section 6 we prove Proposition 1.5. In Sections 7 and 8 we prove Theorems 1.6 and 1.7.

2. Notation

The notation in this paper follows the set theoretic standard of [4]. Notation concerning idealized forcing follows [13].

For a poset P we write $\mathrm{ro}(P)$ for the Boolean algebra of regular open sets in P. For a Boolean algebra B we write $\mathrm{st}(B)$ for the Stone space of B. If λ is a cardinal, then $\mathrm{Coll}(\omega, \lambda)$ stands for the poset of finite partial functions from ω into λ, ordered by inclusion.

If $T \subseteq Y^{<\omega}$ is a tree and $t \in T$ is a node, then we write $T \restriction t$ for the tree $\{s \in T : s \subseteq t \ \vee \ t \subseteq s\}$. For $t \in T$ we denote by $\mathrm{succ}_T(t)$ the set $\{y \in Y : t^\frown y \in T\}$. We say that $t \in T$ is a *splitnode* if $|\mathrm{succ}_T(t)| > 1$. The set of all splitnodes of T is denoted by $\mathrm{split}(T)$.

3. Combinatorial tree forcings

In this section we assume that J is a family of subsets of a countable set $\mathrm{dom}(J)$. We assume that $\omega \notin J$ and that J is *hereditary*, i.e. if $a \subseteq b \subseteq \mathrm{dom}(J)$ and $b \in J$, then $a \in J$. Occasionally, we will require that J is an ideal. We say that $a \subseteq \mathrm{dom}(J)$ is *J-positive* if $a \notin J$. For a J-positive set a we write $J \restriction a$ for the family of all subsets of a which belong to J.

Definition 3.1. The poset $Q(J)$ consists of those trees $T \subseteq \mathrm{dom}(J)^{<\omega}$ for which every node $t \in T$ has an extension $s \in T$ satisfying $\mathrm{succ}_T(s) \notin J$. $Q(J)$ is ordered by inclusion.

Thus the Miller forcing is just $Q(J)$ when J is the Fréchet ideal on ω. $Q(J)$ is a forcing notion adding the generic branch in $\mathrm{dom}(J)^\omega$, which also determines the generic filter. We write $\dot{g}$ for the canonical name for the generic branch. Basic fusion arguments literally transfer from the Miller forcing case to show that $Q(J)$ is proper and preserves the Baire category.

Proposition 3.2. *The forcing $Q(J)$ is equivalent to a forcing P_I where I is a σ-ideal generated by closed sets.*

Proof. To simplify notation assume $\mathrm{dom}(J) = \omega$. Whenever $f : \omega^{<\omega} \to J$ is a function, let $A_f = \{x \in \omega^\omega : \forall n < \omega \;\; x(n) \in f(x \restriction n)\}$. Note that the sets A_f are closed. Let I_J be the σ-ideal generated by all sets of this form.

Lemma 3.3. *An analytic set $A \subseteq \omega^\omega$ is I_J-positive if and only if it contains all branches of a tree in $Q(J)$.*

Proof. For a set $C \subseteq \omega^\omega \times \omega^\omega$ we consider the game $G(C)$ between Players I and II in which at n-th round Player I plays a finite sequence $s_n \in \omega^{<\omega}$ and a number $m_n \in \omega$, and Player II answers with a set $a_n \in J$. The first element of the sequence s_{n+1} must not belong to the set a_n. In the end let x be the concatenation of s_n's and y be the concatenation of m_n's. Player I wins if $\langle x, y\rangle \in C$.

Claim. *Player II has a winning strategy in $G(C)$ if and only if* $\mathrm{proj}(C) \in I_J$. *If Player I has a winning strategy in $G(C)$, then* $\mathrm{proj}(C)$ *contains all branches of a tree in $Q(J)$.*

The proof of the above Claim is standard (cf. [8, Theorem 21.2]) and we omit it. Now, if $C \subseteq \omega^\omega \times \omega^\omega$ is closed such that $\mathrm{proj}(C) = A$, then determinacy of $G(C)$ gives the desired property of A. □

This shows that P_{I_J} has a dense subset isomorphic to $Q(J)$, so the two forcing notions are equivalent. □

If J is coanalytic, then the σ-ideal I_J associated with the poset $Q(J)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The further, finer forcing properties of $Q(J)$ depend on the position of J in the Katětov ordering.

Definition 3.4 ([5])**.** Let H and F be hereditary families of subsets of $\mathrm{dom}(H)$ and $\mathrm{dom}(F)$ respectively. H is *Katětov above* F, or $H \geq_K F$, if there is a function $f : \mathrm{dom}(H) \to \mathrm{dom}(F)$ such that $f^{-1}(a) \in H$ for each $a \in F$.

For a more detailed study of this order see [3]. It turns out that for many preservation-type forcing properties ϕ there is a critical hereditary family H_ϕ such that $\phi(Q(J))$ holds if and only if $J \restriction a \not\geq_K H_\phi$ for every $a \notin J$. This section collects several results of this kind.

Definition 3.5. We say that $a \subseteq 2^{<\omega}$ is *nowhere dense* if every finite binary sequence has an extension such that no further extension falls into a. $\mathtt{NWD}$ stands for the ideal of all nowhere dense subsets of $2^{<\omega}$.

Theorem 3.6. *$Q(J)$ does not add Cohen reals if and only if $J \restriction a \not\geq_K$ $\mathtt{NWD}$ for every J-positive set a.*

Proof. On one hand, suppose that there exists J-positive set a such that $J \restriction a \geq_K \mathtt{NWD}$ as witnessed by a function $f : a \to 2^{<\omega}$. Then, the tree $a^{<\omega}$ forces the concatenation of the f-images of numbers on the generic sequence to be a Cohen real.

On the other hand, suppose that $J \restriction a \not\geq_K \mathtt{NWD}$. Let $T \in Q(J)$ be a condition and $\dot{y}$ be a name for an infinite binary sequence. We must show that $\dot{y}$ is not a name for a Cohen real. That is, we must produce a condition $S \leq T$ and an open dense set $O \subseteq 2^\omega$ such that $S \Vdash \dot{y} \notin \check{O}$.

Strengthening the condition T if necessary we may assume that there is a continuous function $f : [T] \to 2^\omega$ such that $T \Vdash \dot{y} = \dot{f}(\dot{g})$. For every splitnode $t \in T$ and for every $n \in \mathrm{succ}_T(t)$ pick a branch $b_{t,n} \in [T]$ such that $t^\frown n \subseteq b_{t,n}$. Use the Katětov assumption to find a J-positive subset $a_t \subseteq \mathrm{succ}_T(t)$ such that the set $\{f(b_{t,n}) : n \in a_t\} \subseteq 2^\omega$ is nowhere dense.

Consider the countable poset P consisting of pairs $p = \langle s_p, O_p\rangle$ where s_p is a finite set of splitnodes of T, $O_p \subseteq 2^\omega$ is a clopen set, and $O_p \cap \{f(b_{t,n}) : t \in s_p, n \in a_t\} = 0$. The ordering is defined by $q \leq p$ if

- $s_p \subseteq s_q$ and $O_p \subseteq O_q$,
- if $t \in s_q \setminus s_p$, then $f(x) \notin O_p$ for each $x \in [T]$ such that $t \subseteq x$.

Choose $G \subseteq P$, a sufficiently generic filter, and define $O = \bigcup_{p\in G} O_p$ and $S \subseteq T$ to be the downward closure of $\bigcup_{p\in G} s_p$. Simple density arguments show that $O \subseteq 2^\omega$ is open dense and moreover, $S \in Q(J)$, since for every node $t \in \bigcup_{p\in G} s_p$ and every $n \in a_t$ we have $t^\frown n \in S$. The definitions show that $f''[S] \cap O = \emptyset$ as desired. $\square$

Definition 3.7. Let $0 < \varepsilon < 1$ be a real number. The ideal S_ε has as its domain all clopen subsets of 2^ω of Lebesgue measure less than ε, and it is generated by those sets a with $\bigcup a \neq 2^\omega$.

This ideal is closely connected with the Fubini property of ideals on countable sets, as shown below in a theorem of Solecki.

Definition 3.8. If $a \subseteq \mathrm{dom}(J)$ and $D \subseteq a \times 2^\omega$, then we write

$$\int_a D \, dJ = \{y \in 2^\omega : \{j \in a : \ \langle j, y\rangle \notin D\} \in J\}.$$

J has the *Fubini property* if for every real $\varepsilon > 0$, every J-positive set a and every Borel set $D \subseteq a \times 2^\omega$ with vertical sections of Lebesgue measure less than ε, the set $\int_a D \, dJ$ has outer measure at most ε.

Obviously, the ideals S_ε as well as all families them in the Katětov ordering fail to have the Fubini property. The following theorem implicitly appears in [11, Theorem 2.1], the formulation below is stated in [3, Theorem 3.13] and proved in [9, Theorem 3.7.1].

Theorem 3.9 (Solecki)**.** *Suppose F is an ideal on a countable set. Then either F has the Fubini property, or else for every (or equivalently, some) $\varepsilon > 0$ there is a F-positive set a such that $F \restriction a \geq_K S_\varepsilon$.*

By μ we denote the outer Lebesgue measure on 2^ω. For a definition of *preservation of outer Lebesgue measure* and further discussion on this property see [13, Section 3.6].

Theorem 3.10. *Suppose that J is a universally measurable ideal. $Q(J)$ preserves outer Lebesgue measure if and only if J has the Fubini property.*

Proof. Suppose on one hand that J fails to have the Fubini property. Find a sequence of J-positive sets $\langle b_n : n \in \omega\rangle$ such that $J \restriction b_n \geq_K S_{2^{-n}}$, as witnessed by functions f_n. Consider the tree T of all sequences $t \in \mathrm{dom}(J)^{<\omega}$ such that $t(n) \in b_n$ for each $n \in \mathrm{dom}(t)$. Let $\dot{B}$ be a name for the set $\{z \in 2^\omega : \exists^\infty n \ z \in f_n(\dot{g}(n))\}$. T forces that the set $\dot{B}$ has measure zero, and the definition of the ideals S_ε shows that every ground model point in 2^ω is forced to belong to $\dot{B}$. Thus $Q(J)$ fails to preserve Lebesgue outer measure at least below the condition T.

On the other hand, suppose that the ideal J does have the Fubini property. Suppose that $Z \subseteq 2^\omega$ is a set of outer Lebesgue measure δ, $\dot{O}$ is a $Q(J)$-name for an open set of measure less or equal to $\varepsilon < \delta$, and $T \in Q(J)$ is a condition. We must find a point $z \in Z$ and a condition $S \leq T$ forcing $\check{z} \notin \dot{O}$.

By a standard fusion argument, thinning out the tree T if necessary, we may assume that there is a function $h : \mathrm{split}(T) \to \mathcal{O}$ such that

$$T \Vdash \dot{O} = \bigcup\{h(\dot{g} \restriction n+1) : \ \dot{g} \restriction n \in \mathrm{split}(T)\}.$$

Moreover, we can make sure that if $t_n \in T$ is the n-th splitting node, then $T \restriction t_n$ decides a subset of $\dot{O}$ with measure greater than $\varepsilon/2^n$. Hence, if we write $f(t_n) = \varepsilon/2^n$, then for every splitnode $t \in T$ and every $n \in \mathrm{succ}_T(t)$ we have $\mu(h(t^\frown n)) < f(t)$.

Now, for every splitnode $t \in T$ let $D_t = \{\langle O, x\rangle : O \in \mathrm{succ}_T(t), x \in 2^\omega \ \wedge \ x \in h(t^\frown O)\}$. It follows from universal measurability of J that the set $\int_{\mathrm{succ}_T(t)} D_t \, dJ$ is measurable. It has mass not greater than $f(t)$, by the Fubini assumption. Since $\sum_{t\in\mathrm{split}(T)} f(t) < \delta$, we can find

$$z \in Z \setminus \bigcup_{t\in\mathrm{split}(T)} \int_{\mathrm{succ}_T(t)} D_t \, dJ.$$

Let $S \subseteq T$ be the downward closure of those nodes $t^\frown n$ such that $t \in T$ is a splitnode and $n \in \mathrm{succ}_T(t)$ is such that $z \notin h(t^\frown n)$. S belongs to $Q(J)$ by the choice of the point z and $S \Vdash \check{z} \notin \dot{O}$, as required. □

An *independent real* is a set x of natural numbers in a generic extension such that both x and the complement of x meet every infinite set of natural numbers from the ground model.

Definition 3.11. $\mathtt{SPL}$ is the family of nonsplitting subsets of $2^{<\omega}$, i.e. those $a \subseteq 2^{<\omega}$ for which there is an infinite set $c \subseteq \omega$ such that $t \restriction c$ is constant for every $t \in a$.

Obviously, $\mathtt{SPL}$ is an analytic set, but it is not clear whether it is also coanalytic. In the following theorem we show that in two quite general cases $\mathtt{SPL}$ is critical for the property of adding independent reals.

Note that if J is an ideal, H is hereditary and H' is the ideal generated by K, then $J \leq_K H$ if and only if $J \leq_K H'$. Therefore, in case J is an ideal, $J \geq_K \mathtt{SPL}$ is equivalent to J being Katětov above the ideal generated by $\mathtt{SPL}$. The latter is analytic, so in particular it has the Baire property.

Theorem 3.12. *Suppose that J is coanalytic or J is an ideal with the Baire property. $Q(J)$ does not add independent reals if and only if $J \restriction a \not\geq_K \mathtt{SPL}$ for every J-positive a.*

Proof. Again, the left to right direction is easy. If $J \restriction a \geq_K \mathtt{SPL}$ for some J-positive set a, as witnessed by a function f, then the condition $a^{<\omega} \in Q(J)$ forces that the concatenation of $\langle f(\dot{g}(n)) : n \in \omega\rangle$ is an independent real.

For the right to left direction, we will need two preliminary general facts. For a set $a \subseteq \omega$ by an *interval in* a we mean a set of the form $[k, l) \cap a$.

First, let $a \subseteq \omega$ be a J-positive set, and let Players I and II play a game $G(a)$, in which they alternate to post consequtive (pairwise disjoint) finite intervals $b_0, c_0, b_1, c_1, \dots$ in the set a. Player II wins if the union of his intervals $\bigcup_{n<\omega} c_n$ is J-positive.

Lemma 3.13. *Player II has a winning strategy in $G(a)$ for any $a \notin J$.*

Proof. In case J is an ideal with the Baire property, this follows immediately from the Talagrand theorem [1, Theorem 4.1.2]. Indeed, if $\{I_k : k < \omega\}$ is a partition of a into finite sets such that each $b \in J$ covers only finitely many of them, then the strategy for II is as follows: at round n pick c_n covering one of the I_k's.

Now we prove the lemma in case J is coanalytic. Consider a related game, more difficult for Player II. Fix a continuous function $f : \omega^\omega \to \mathcal{P}(a)$ such that its range consists exactly of all J-positive sets. The new game $G'(a)$ proceeds just as $G(a)$, except Player II is required to produce sequences $t_n \in \omega^{<\omega}$ of length and all entries at most n, and in the end, Player II wins if $y = \bigcup_{n<\omega} t_n \in \omega^\omega$ and $f(y) \subseteq \bigcup_{n<\omega} c_n$.

Clearly, the game $G'(a)$ is Borel and therefore determined. If Player II has a winning strategy in $G'(a)$, then she has a winning strategy in $G(a)$ and we are done. Thus, we only need to derive a contradiction from the assumption that Player I has a winning strategy in $G'(a)$.

Well, suppose σ is such a winning strategy. We construct a strategy for Player I in $G(a)$ as follows. The first move $b_0 = \sigma(\emptyset)$ does not change. Suppose Player I is going to make her move after the sets $b_0, c_0, \dots, b_n, c_n$ have been chosen. For each possible choice of the sequences t_m for $m < n$ consider a run of $G'(a)$ in which Player I plays according to σ and Player II plays the pairs (b'_m, t_m), where b'_m are the intervals b_m adjusted downward to the previous move of Player I. The next move of Player I is now the union of all finitely many moves the strategy σ dictates against such runs in $G'(a)$. It is not difficult to see that this is a winning strategy for Player I in the original game G. However, Player I cannot have a winning strategy in the game G since Player II could immediately steal it and win herself. $\square$

Second, consider the collection F of those subsets $a \subseteq \omega^{<\omega}$ such that there is no tree $T \in Q(J)$ whose splitnodes all fall into a.

Lemma 3.14. *The collection F is an ideal.*

Proof. The collection F is certainly hereditary. To prove the closure under unions, let $a = a_0 \cup a_1$ be a partition of the set of all splitnodes of a $Q(J)$ tree into two parts. We must show that one part contains all splitnodes of some $Q(J)$ tree. For $i \in 2$ build rank functions $\mathrm{rk}_i : a_i \to \mathrm{Ord} \cup \{\infty\}$ by setting $\mathrm{rk}_i \geq 0$ and $\mathrm{rk}_i(t) \geq \alpha + 1$ if the set $\{n \in \omega : t^\frown n$ has an extension s in a_i such that $\mathrm{rk}_i(s) \geq \alpha\}$ is J-positive. If the rank rk_i of any splitnode is ∞ then the nodes whose rank rk_i is ∞ form a set of splitnodes of a tree in $Q(J)$, contained in a_i. Thus, it is enough to derive a contradiction from the assumption that no node has rank ∞.

Observe that if $t \in a$ is a node with $\mathrm{rk}_i(t) < \infty$, then there is $n \in \omega$ such that a contains nodes extending $t^\frown n$, but all of them either have rank less than $\mathrm{rk}_i(t)$ or do not belong to a_i. Thus, one can build a finite sequence of nodes on which the rank decreases and the last one has no extension in the set a_i. Repeating this procedure twice, we will arrive at a node of the set a which belongs to neither of the sets a_0 or a_1, reaching a contradiction. □

Now suppose that $J \restriction a \not\geq_K \mathtt{SPL}$ for every J-positive set a. Let $T \in Q(J)$ be a condition and $\dot{y}$ be a $Q(J)$-name for a subset of ω. We must prove that $\dot{y}$ is not a name for an independent real. That is, we must find an infinite set $b \subseteq \omega$ as well as a condition $S \leq T$ forcing $\dot{y} \restriction \check{b}$ to be constant. The construction proceeds in several steps.

First, construct a tree $T' \subseteq T$ and an infinite set $b \subseteq \omega$ such that for every splitnode $t \in T'$ there is a bit $c_t \in 2$ such that for all but finitely many $n \in b$, for all but finitely many immediate successors s of t in T' we have

$$T' \restriction s \Vdash \dot{y}(n) = c_t.$$

To do this, enumerate $\omega^{<\omega}$ as $\langle t_i : i \in \omega \rangle$, respecting the initial segment relation, and by induction on $i \in \omega$ construct a descending sequence of trees $T_i \subseteq T$, sets $b_i \subseteq \omega$, and bits $c_{t_i} \in 2$ as follows:

- if t_i is not a splitnode of T_i, then do nothing and let $T_{i+1} = T_i$, $b_{i+1} = b_i$ and $c_{t_i} = 0$;
- if t_i is a splitnode of T_i, then for each $j \in \mathrm{succ}_{T_i}(t_i)$ find a tree $S_j \leq T_i \restriction t_i^\frown j$ deciding $\dot{y} \restriction j$, and use the Katětov assumption to find a J-positive set $a \subseteq \mathrm{succ}_{T_i}(t_i)$, a bit $c_{t_i} \in 2$, and an infinite set $b_{i+1} \subseteq b_i$ such that whenever $j \in a$ and $n \in b_{i+1} \cap j$ then $S_j \Vdash \dot{y}(n) = c_{t_i}$. Let $T_{i+1} = T_i$, except below t_i replace $T_i \restriction t_i$ with $\bigcup_{j \in a} S_j$.

In the end, let $T' = \bigcap_{i<\omega} T_i$ and let b be any diagonal intersection of the sets b_i.

The second step uses Lemma 3.14 to stabilize the bit c_t. Find a condition $T'' \subseteq T'$ such that for every splitnode $t \in T''$, c_t is the same value, say 0.

The last step contains a fusion argument. For every splitnode $t \in T''$ fix a winning strategy σ_t for Player II in the game $G(\mathrm{succ}_{T''}(t))$. By induction on $i \in \omega$ build sets $S_i \subseteq T''$, functions f_i on S_i, and numbers $n_i \in b$ so that

- $S_0 \subseteq S_1 \subseteq \dots$, and in fact S_{i+1} contains no initial segments of nodes in S_i that would not be included in S_i already. The final condition will be a tree S whose set of splitnodes is $\bigcup_{i<\omega} S_i$;
- for every node $s \in S_i$, the value $f_i(s)$ is a finite run of the game $G(\mathrm{succ}_{T''}(s))$ according to the strategy σ_s, in which the union of the moves of the second player equals $\{j \in \omega : \exists t \in S_i\ s^\frown j \subseteq t\}$. Moreover, $f_i(s) \subseteq f_{i+1}(s) \subseteq \dots$. This will ensure that every node in $\bigcup_{i<\omega} S_i$ in fact splits into J-positively many immediate successors in the tree S;
- whenever $s \in S_i$ and $j \in \omega$ is the least such that $s \in S_j$, then $T'' \restriction s \Vdash \forall k \in j\ \dot{y}(n_k) = 0$. This will ensure that in the end, $S \Vdash \forall i < \omega\ \ \dot{y}(n_i) = 0$.

The induction step is easy to perform. Suppose that S_i, f_i, n_j have been found for $j < i$. Let $n_i \in b$ be a number such that

$$\forall s \in S_i \quad \forall^\infty n \in \mathrm{succ}_{T''}(s) \quad T'' \restriction s^\frown n \Vdash \dot{y}(n_i) = 0.$$

For every node $s \in S_i$, let d_s be a finite set such that for all $n \in \mathrm{succ}_{T''}(s) \setminus d_s$ and for all $j \leq i$

- $T'' \restriction s^\frown n \Vdash \dot{y}(n_j) = 0$
- and $s^\frown n$ is not an initial segment of any node in S_i.

Extend the run $f_i(s)$ to $f_{i+1}(s)$ such that the new moves by Player II contain no numbers in the set d_s.

Put into S_{i+1} all nodes from S_i as well as every t which is the smallest splitnode of T'' above some $s^\frown j$ where j is one of the new numbers in the set answered by Player II in $f_{i+1}(s)$.

In the end put $S = \bigcup_{i<\omega} S_i$. It follows from the construction that $S \Vdash \forall i < \omega\ \ \dot{y}(n_i) = 0$, as desired.

□

We finish this section with an observation about continuous degrees of reals in $Q(J)$ generic extensions.

Definition 3.15. We say that J has the *discrete set property* if for every J-positive set a and every function $f : a \to X$ into a Polish space, there is a J-positive set $b \subseteq a$ such that the set $f''b$ is discrete.

Obviously, the discrete set property is equivalent to being not Katětov above the family of discrete subsets of $\mathbb{Q}$. It is not difficult to show that it also equivalent to being not above the ideal of those subsets of the ordinal ω^ω which do not contain a topological copy of the ordinal ω^ω.

Proposition 3.16. *Suppose J has the discrete set property. Then $Q(J)$ adds one continuous degree.*

Proof. Let T be a condition in $Q(J)$ and $f : [T] \to \omega^\omega$ a continuous function. It is enough to find a tree $S \in Q(J)$, $S \leq T$ such that on $[S]$ the function f is either constant, or is a topological embedding. Suppose that f is not constant on any such $[S]$. By an easy fusion argument we build $S \subseteq T$, $S \in Q(J)$ such that for any splitnode s of S there are pairwise disjoint open sets U_i for $i \in \mathrm{succ}_S(s)$ such that $f''[S \restriction s^\frown i] \subseteq U_i$ for each $i \in \mathrm{succ}_S(s)$. This implies that f is a topological embedding on $[S]$. □

4. Closure ideals

In this section X is a Polish space with a complete metric, I a σ-ideal on X and $\mathcal{O}$ a countable topology basis for the space X.

Definition 4.1. For a set $a \subseteq \mathcal{O}$, define

$$\mathrm{cl}(a) = \{x \in X : \forall \varepsilon > 0\ \exists O \in a \quad O \subseteq B_\varepsilon(x)\},$$

where $B_\varepsilon(x)$ stands for the ball centered at x with radius ε. We write

$$J_I = \{a \subseteq \mathcal{O} : \mathrm{cl}(a) \in I\}.$$

It is immediate that the collection J_I is an ideal and that J_I is *dense*[1], i.e. every infinite set in $\mathcal{O}$ contains an infinite subset in J_I. If the σ-ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then J_I is coanalytic. On the other hand, if X is compact and J_I is analytic, then it follows from the Kechris Louveau Woodin theorem [7, Theorem 11] that J_I is $\mathbf{F}_{\sigma\delta}$.

Definition 4.2. An ideal J on a countable set is *weakly selective* if for every J-positive set a, any function on a is either constant or injective on a positive subset of a.

Obviously, this is just a restatement of the fact that the ideal is not Katětov above the ideal on $\omega \times \omega$ generated by vertical lines and graphs of functions.

Proposition 4.3. *J_I is weakly selective.*

[1]some authors prefer the term *tall*

Proof. Take a J_I-positive set a and $f : a \to \omega$. Suppose that f is not constant on any J_I-positive subset of a. We must find $b \subseteq a$ such that f is 1-1 on b. Write Y for $\mathrm{cl}(a)$ shrunk by the union of all basic open sets U such that $\mathrm{cl}(a) \cap U \in I$. Enumerate all basic open sets which have nonempty intersection with Y into a sequence $\langle U_n : n < \omega \rangle$. Inductively pick a sequence $\langle O_n \in a : n < \omega \rangle$ such that $O_n \subseteq U_n$ and $f(O_n) \neq f(O_i)$ for $i < n$. Suppose that O_i are chosen for $i < n$. Let $Y_n = Y \cap U_n$. This is an I-positive set and hence $a_n = \{O \in a : O \subseteq U_n\}$ is J_I-positive. Note that f assumes infinitely many values on a_n since otherwise we could find J_I-positive $b \subseteq a_n$ on which f is constant. Pick any $O_n \in a_n$ such that $f(O_n) \notin \{f(O_i) : i < n\}$. Now, the set $b = \{O_n : n < \omega\}$ is J_I-positive since $\mathrm{cl}(b)$ contains Y. □

We will now verify several Katětov properties of the ideal J_I depending on the forcing properties of P_I.

Proposition 4.4. *Suppose that P_I is a proper and ω^ω-bounding notion of forcing. Then the ideal J_I has the discrete set property.*

Proof. Take a J_I-positive set a and a function $f : a \to \mathbb{Q}$. Let $B = \mathrm{cl}(a)$. Let $\langle \dot{O}_n : n \in \omega \rangle$ be a sequence of P_I-names for open sets in a such that $\dot{O}_n$ is forced to be wholly contained in the 2^{-n}-neighborhood of the P_I-generic point in B. Passing to a subsequence and a subset of a if necessary, we may assume that the sets $\dot{O}_n$ are pairwise distinct
Case 1. Assume the values $\{f(\dot{O}_n) : n \in \omega\}$ are forced not to have any point in the range of f as a limit point. Use the ω^ω-bounding property of the forcing P_I to find a condition $B' \subseteq B$, a sequence of finite sets $\langle a_n : n \in \omega \rangle$ and numbers $\varepsilon_n > 0$ such that

- $B' \Vdash \forall m < \omega\, \exists n < \omega\ \dot{O}_m \in \check{a}_n$;
- the collection $\{B_{\varepsilon_n}(f(O)) : O \in a_n, n \in \omega\}$ consists of pairwise disjoint open balls.

To see how this is possible, note that B forces that for every point $y \in f''a$ there is an $\varepsilon > 0$ such that all but finitely many points of the sequence $\langle f(O_m) : m \in \omega \rangle$ have distance greater than ε from y.

Now let $b = \bigcup_{n<\omega} a_n$. Let M be a countable elementary submodel of a large enough structure and let $B'' \subseteq B'$ be a Borel I-positive set consisting only of generic points over M. It is not difficult to observe that $B \subseteq \mathrm{cl}(b)$ and therefore the set b is as required.
Case 2. If the values $\{f(\dot{O}_n) : n \in \omega\}$ can be forced to have a point in the range of f as a limit point, then, possibly shrinking the set a we can force the sequence $\langle f(\dot{O}_n) : n \in \omega \rangle$ to be convergent and not eventually constant, hence discrete. Similarly as in Case 1, we find $b \subseteq a$ such that $f''b$ is discrete. □

Proposition 4.5. *Suppose that P_I is a proper and outer Lebesgue measure preserving notion of forcing. Then J_I has the Fubini property.*

Proof. Suppose that $\varepsilon > 0$ is a real number, $a \subseteq \mathcal{O}$ is a J_I-positive set, and $D \subseteq a \times 2^\omega$ is a Borel set with vertical sections of measure at most ε. Assume for contradiction that the outer measure of the set $\int_a D\, dJ$ is greater than ε. Let $B = \mathrm{cl}(a)$. This condition forces that there is a sequence $\langle \dot{O}_n : n \in \omega \rangle$ of sets in a such that O_n is wholly contained in in the 2^{-n}-neighborhood of the generic point. Let $\dot{C}$ be a name for the set $\{z \in 2^\omega : \exists^\infty n < \omega\ y \notin \dot{O}_n\}$. This is a Borel set of measure greater or equal to $1 - \varepsilon$. Since the forcing P_I preserves the outer Lebesgue measure, there must be a condition $B' \subseteq B$ and a point $z \in \int_a D dJ$ such that $B' \Vdash \check{z} \in \dot{C}$. Consider the set $b = \{O \in a : z \notin O\}$. The set $\mathrm{cl}(b)$ must be I-positive, since the condition B' forces the generic point to belong to it. This, however, contradicts the assumption that $z \in \int_a D\, dJ$. □

5. Tree representation and Cohen reals

In this section we show that under suitable assumptions the forcing P_{I^*} is equivalent to the tree forcing $Q(J_I)$.

Definition 5.1. Let J be an ideal on $\mathcal{O}$ and $T \in Q(J)$. We say that T is *Luzin* if the sets on the n-th level have diameter less than 2^{-n} and for each $t \in T$ the immediate successors of t in T are pairwise disjoint. If T is Luzin, then we write $\pi[T]$ for $\{\bigcap_{n<\omega} x(n) : x \in [T]\}$.

Proposition 5.2. *Let I be a σ-ideal on a Polish space X. If $T \in Q(J_I)$ is Luzin, then $\pi(T) \in P_{I^*}$.*

Proof. The set $\pi[T]$ is a 1-1 continuous image of $[T]$, which is a Polish space, hence $\pi[T]$ is Borel. To see that $\pi[T]$ is I^*-positive consider the function $\varphi : [T] \to X$ which assings to any $x \in [T]$ the single point in $\bigcap_{n<\omega} x(n)$. Note that φ is continuous since the diameters of open sets on T vanish to 0. Now if $\pi[T] \subseteq \bigcup_{n<\omega} E_n$ where each E_n is closed and belongs to I, then $\varphi^{-1}(E_n)$ are closed sets covering the space $[T]$. By the Baire category theorem, one of them must have nonempty interior. So there is $n < \omega$ and $t \in T$ such that every immediate successor of t in T belongs to $\varphi^{-1}(E_n)$. Now for each $u \in \mathrm{succ}_T(t)$ we have $u \cap E_n \neq \emptyset$, which implies that $\mathrm{cl}(\mathrm{succ}_T(t)) \subseteq E_n$ and contradicts the fact that $\mathrm{cl}(\mathrm{succ}_T(t))$ is I-positive. □

The following proposition, combined with the propositions proved in the previous section, gives Theorems 1.2, 1.3 and 1.4 from the introduction (recall that the Cohen forcing adds an independent real and does not preserve outer Lebesgue measure).

Proposition 5.3. *Suppose I is a σ-ideal on a Polish space X such that the poset P_I is proper and is not equivalent to the Cohen forcing under any condition. For any $B \in P_{I^*}$*

- *either I^* and I contain the same Borel sets below B,*
- *or there is $C \in P_{I^*}$ below B such that below C the forcing P_{I^*} is equivalent to $Q(J_I)$.*

Proof. Suppose that $B \subseteq X$ is a Borel set which belongs to I but not to I^*. Assume also that B forces that the generic point is not a Cohen real. By the Solecki theorem [10, Theorem 1], we may assume that B is a $\mathbf{G}_\delta$ set and for every open set $O \subseteq X$, if $B \cap O \neq \emptyset$, then $B \cap O \notin I^*$. Represent B as a decreasing intersection $\bigcap_{n<\omega} O_n$ of open sets.

We build a Luzin scheme T of basic open sets U_t for $t \in \omega^{<\omega}$ satisfying the following demands:

- $U_t \subseteq O_{|t|}$ and $U_t \cap B \neq \emptyset$,
- the sets in $\mathrm{succ}_T(t)$ have pairwise disjoint closures and are disjoint from $\mathrm{cl}(\mathrm{succ}_T(t))$, which is an I-positive set.

To see how this is done, suppose that U_t are built for $t \in \omega^{\leq n}$ and take any $t \in \omega^n$. The set $\mathrm{cl}(B \cap U_t)$ is I-positive, and since the P_I-generic real is not forced to be a Cohen real, there is a closed nowhere dense I-positive subset C of $\mathrm{cl}(B \cap U_t)$. Find a discrete set $D = \{d_n : n < \omega\}$ such that $D \subseteq B \cap U_t$ and $C \subseteq \mathrm{cl}(D)$. For each $n < \omega$ find a basic open neighborhood $V_n \subseteq U_t \cap O_{|t|+1}$ of d_n such that the closures of the sets V_n are pairwise disjoint, disjoint from C and $C \subseteq \mathrm{cl}(\{V_n : n < \omega\})$. Put $U_{t^\frown n} = V_n$.

Let $T \in Q(J)$ be the Luzin scheme constructed above. Clearly, T is Luzin, as well as each $S \in Q(J_I)$ such that $S \leq T$. For each $S \leq T$ the set $\pi(S) \subseteq \pi(T)$ is Borel and I^*-positive by Proposition 5.2. We will complete the proof by showing that the range of π is a dense subset of P_{I^*} below the condition $\pi(T)$.

For $C \subseteq B$ which is an I^*-positive set we must produce a tree $S \in Q(J), S \subseteq T$, such that $\pi[S] \subseteq C$. By the Solecki theorem we may assume that the set C is $\mathbf{G}_\delta$, a decreasing intersection $\bigcap_{n<\omega} W_n$ of open sets and for every open set $O \subseteq X$ if $O \cap C \neq \emptyset$, then $O \cap C \notin I$.

By tree induction build a tree $S \subseteq T$ such that for every sequence on n-th splitting level, the last set on the sequence is a subset of W_n, and still has nonempty intersection with the set C. In the end, the tree $S \subseteq T$ will be as required.

Now suppose that immediate successors of nodes on the n-th splitting level have been constructed. Let t be one of these successors. Find its extension $s \in T$ such that the last set O on it is a subset of W_{n+1} and still has nonempty intersection with C. Note that

$$\mathrm{cl}(\pi[T]) \subseteq \pi[T] \cup \bigcup_{u \in T} \mathrm{cl}(\mathrm{succ}_T(u)).$$

Since $\mathrm{cl}(C \cap O) \notin I$ and $\pi[T] \subseteq B \in I$, this means that there must be an extension u of s such that $\mathrm{cl}(C \cap O) \cap \mathrm{cl}(\mathrm{succ}_T(u)) \notin I$. This can only happen if the set $b = \{V \in a_u : V \cap C \neq 0\}$ is J-positive, since $\mathrm{cl}(C \cap O) \cap \mathrm{cl}(\mathrm{succ}_T(u)) \subseteq \mathrm{cl}(b)$. Put all nodes $\{u^\frown V : V \in b\}$ into the tree S and continue the construction. $\square$

6. The cases of Miller and Sacks

In this section we prove Proposition 1.5. This depends on a key property of the Miller and Sacks forcings.

Lemma 6.1. *Suppose X is a Polish space, $B \subseteq X$ is a $\mathbf{\Delta}^1_2$ set, T is a Miller or a Sacks tree and $\dot{x}$ is a Miller or Sacks name for an element of the set B. Then there is $S \subseteq T$ and a closed set $C \subseteq X$ such that $C \setminus B$ is countable, and $S \Vdash \dot{x} \in \check{C}$.*

Proof. For the Sacks forcing it is obvious and we can even require that $C \subseteq B$. Let us focus on the Miller case.

Strengthening the tree T if necessary, we may assume that there is a continuous function $f : [T] \to B$ such that $T \Vdash \dot{x} = f(\dot{g})$. The problem of course is that the set $f''[T]$ may not be closed, and its closure may contain many points which do not belong to the set B.

For every splitnode $t \in T$ and for every $n \in \mathrm{succ}_T(t)$ pick a branch $b_{t,n} \in [T]$ such that $t^\frown n \subseteq b_{t,n}$. Next, find an infinite set $a_t \subseteq \mathrm{succ}_T(t)$ such that the points $\{f(b_{t,n}) : n \in a_t\}$ form a discrete set with at most one accumulation point x_t. For $n \in a_t$ find numbers $m_{t,n} \in \omega$ and pairwise disjoint open sets $O_{t,n}$ such that $f''[T \restriction (b_{t,n} \restriction m_{t,n})] \subseteq O_{t,n}$. Find a subtree $S \subseteq T$ such that for every splitnode $t \in S$, if $t^\frown n \in S$, then $n \in a_t$ and the next splitnode of S past $t^\frown n$ extends the sequence $b_{t,n} \restriction m_{t,n}$.

It is not difficult to see that $\mathrm{cl}(f''[S]) \subseteq f''[S] \cup \{x_t : t \in \omega^{<\omega}\}$, and therefore the tree S and the closed set $C = \mathrm{cl}(f''[S])$ are as needed. $\square$

Proposition 1.5 now immediately follows.

Proof of Proposition 1.5. If the σ-ideal I does not contain the same Borel sets as I^*, then any condition $B \in I \setminus I^*$ forces in P_{I^*} the generic point into B but outside of every closed set in the σ-ideal I. However, by Lemma 6.1 we have that if the Miller or the Sacks forcing forces a point into a Borel set in a σ-ideal, then it forces that point into a closed set in that σ-ideal. Thus, P_{I^*} cannot be in the forcing sense equivalent neither to Miller nor to Sacks forcing in the case that $I \neq I^*$. $\square$

7. Borel degrees

In this section we prove Theorem 1.6. To this end, we need to learn how to turn Borel functions into functions which are continuous and open.

Lemma 7.1. *Suppose I is a σ-ideal on a Polish space X such that P_I is proper. Let B be Borel I-positive, and $f : B \to \omega^\omega$ be Borel. For any countable elementary submodel $M \prec H_\kappa$ the set*

$$f''\{x \in B : x \textit{ is } P_I\textit{-generic over } M\}$$

is Borel.

Proof. Without loss of generality assume that $B = X$. Let $\dot{y}$ be a P_I-name for $f(\dot{g})$, where $\dot{g}$ is the canonical name for the generic real for P_I. Take $R \subseteq \mathrm{ro}(P_I)$ the complete subalgebra generated by $\dot{y}$. Notice that for each $y \in \omega^\omega$ we have

$$y \in f''C \quad \text{iff} \quad y \text{ is } R\text{-generic over } M.$$

Hence, it is enough to prove that $C' = \{y \in \omega^\omega : y \text{ is } R\text{-generic over } M\}$ is Borel. C' is a 1-1 Borel image of the set of ultrafilters on $R \cap M$ which are generic over M. The latter set is $\mathbf{G}_\delta$, so C' is Borel. $\square$

Now we show that Borel functions can turned into continuous and open functions after restriction their domain and some extension of topology. If Y is a Polish space and I is a σ-ideal on Y, then we say that Y is *I-perfect* if I does not contain any nonempty open subset of Y.

Proposition 7.2. *Suppose I is a σ-ideal on a Polish space X such that P_I is proper. Let $B \subseteq X$ be I-positive, and $f : B \to \omega^\omega$ be Borel. There are Borel sets $Y \subseteq B$ and $Z \subseteq \omega^\omega$ such that Y is I-positive, $f''Y = Z$ and*

- *Y and Z carry Polish zero-dimensional topologies which extend the original ones, preserve the Borel structures and the topology on Y is I-perfect.*

- *the function $f \restriction Y : Y \to Z$ is continuous and open in the extended topologies.*

Proof. Fix κ big enough and let $M \prec H_\kappa$ be a countable elementary submodel coding B and f. Let $Y = \{x \in B : x \text{ is } P_I\text{-generic over } M\}$. By Lemma 7.1 we have that $Z = f''Y$ is Borel.

Note that if a $\mathbf{\Sigma}^0_\alpha$ set A is coded in M, then there are sets A_n coded in M, $A_n \in \mathbf{\Pi}^0_{<\alpha}$ such that $A = \bigcup_n A_n$. Therefore, we can perform the construction from [8, Theorem 13.1] and construct a Polish zero-dimensional topology on X which contains all Borel sets coded in M. Note that the Borel sets coded in M form a basis for this topology. Moreover, Y is homeomorphic to the set of ultrafilters in $\mathrm{st}(P_I \cap M)$ which are generic over M. So Y is a $\mathbf{G}_\delta$ set in the extended topology. Let τ be the restriction of this extended topology to Y. The fact that τ is I-perfect on Y follows directly from properness of P_I.

Let σ be the topology on Z generated by the sets $f''(Y \cap A)$ and their complements, for all $A \subseteq B$ which are Borel and coded in M.

Now we prove that $f \restriction Y$ is a continuous open from (Y, τ) to (Z, σ). The fact that f is open follows right from the definitions. Now we prove that f is continuous. Fix a cardinal λ greater than $2^{2^{|P_I|}}$ and a Borel set A coded in M.

Lemma 7.3. *Given $x \in Y$ we have*

- *$f(x) \in f''(A \cap Y)$ if and only if*

$$M[x] \models \mathrm{Coll}(\omega, \lambda) \Vdash \exists x'\, P_I\text{-}\textit{generic over } M\ [x' \in A\ \wedge\ f(x) = f(x')],$$

- *$f(x) \notin f''(A \cap Y)$ if and only if*

$$M[x] \models \mathrm{Coll}(\omega, \lambda) \Vdash \forall x'\, P_I\text{-}\textit{generic over } M\ [x' \in A\ \Rightarrow\ f(x) \neq f(x')].$$

Proof. We prove only the first part. Note that in M there is a surjection from λ onto the family of all dense sets in P_I as well as sujections from λ onto each dense set in P_I. Therefore, if $x \in Y$ and $g \subseteq \mathrm{Coll}(\omega, \lambda)$ is generic over $M[x]$, then in $M[x][g]$ the formula

$$\exists x'\, P_I\text{-generic over } M\ [x' \in A\ \wedge\ f(x) = f(x')]$$

is analytic with parameters A, f and a real which encodes the family $\{D \cap M : D \in M \text{ is dense in } P_I\}$ and therefore it is absolute between $M[x][g]$ and V. Hence

$$M[x][g] \models \exists x'\, P_I\text{-generic over } M\ [x' \in A\ \wedge\ f(x) = f(x')]$$

if and only if $f(x) \in f^{-1}(f''(A \cap Y))$. $\square$

Now it follows from from Lemma 7.3 and the forcing theorem that both sets $Y \cap f^{-1}(f''(A \cap Y))$ and $Y \cap f^{-1}(Z \setminus f''(A \cap Y))$ are in τ. This proves that f is continuous.

We need to prove that Z with the topology σ is Polish. Note that it is a second-countable Hausdorff zero-dimensional space, so in particular metrizable. As a continuous open image of a Polish space, Z is Polish by the Sierpiński theorem [8, Theorem 8.19].

The fact that σ has the same Borel structure as the original one follows directly from Lemma 7.1.

$\square$

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let $C \in P_I$ and $\dot{x}$ be a name for a real such that

$$C \Vdash \dot{x} \text{ is not a Cohen real and } \dot{x} \notin V.$$

Without loss of generality assume that $C = X$ and $C \Vdash \dot{x} = f(\dot{g})$ for some continuous function $f : X \to \omega^\omega$. We shall find $B \in P_I$ and a Borel automorphism h of ω^ω such that

$$B \Vdash h(f(\dot{g})) = \dot{g}.$$

Find Polish spaces $Y \subseteq X$ and $Z \subseteq \omega^\omega$ as in Proposition 7.2. Without loss of generality assume that $Y = X$ and the extended topologies are the original ones (note that I is still generated by closed sets in any extended topology).

Now we construct $T \in Q(J_I)$ and a Borel automorphism h of ω^ω. To this end we build two Luzin schemes $U_t \subseteq X$ and $C_t \subseteq \omega^\omega$ (for $t \in \omega^{<\omega}$), both with the vanishing diameter property and such that

- U_t is basic open and C_t is closed,
- $f''U_t \subseteq C_t$
- for each $t \in \omega^{<\omega}$ the set $\{U_{t^\frown k} : k < \omega\}$ is J_I-positive.

We put $U_\emptyset = X$ and $C_\emptyset = \omega^\omega$. Suppose U_t and C_t are built for all $t \in \omega^{<n}$. Pick $t \in \omega^{n-1}$. Now $f''U_t$ is an open set. Let K be the perfect kernel of $f''U_t$. K is nonempty since $\dot{x}$ is forced not to be in V. Hence K is a perfect Polish space and $U_t \Vdash \dot{x} \in K$. Note that there is a closed nowhere dense $N \subseteq K$ such that $f^{-1}(N)$ is I-positive, since otherwise

$$U_t \Vdash \dot{x} \text{ is a Cohen real in } K.$$

Pick such an N and let $M = f^{-1}(N)$. N is closed nowhere dense in $f''U_t$ too, so M is closed nowhere dense in U_t because f is continuous and open.

Enumerate all basic open sets in U_t having nonempty intersection with M into a sequence $\langle V_k : k < \omega\rangle$. Inductively pick clopen sets $W_k \subseteq U_t$ and $C_k \subseteq \omega^\omega$ such that

- $W_k \subseteq f^{-1}(C_k) \cap V_k$ is basic open,
- C_k are pairwise disjoint,
- $f^{-1}(C_k)$ are disjoint from M.

Do this as follows. Suppose that W_i and C_i are chosen for $i < k$. Since $f^{-1}(C_i)$ are disjoint from M and $V_k \cap M \neq \emptyset$, the set $V_k \setminus \bigcup_{i<k} f^{-1}(C_i)$ is a nonempty clopen set. Pick $x_k \in V_k \setminus \bigcup_{i<k} f^{-1}(C_i) \setminus M$. Since $f(x_k) \notin N \cup \bigcup_{i<k} C_i$, there is a clopen neighborhood C_k of $f(x_k)$ which is disjoint from $N \cup \bigcup_{i<k} C_i$. Let W_k be a basic neighborhood of x_k contained in $f^{-1}(C_k) \cap V_k$. Put $U_{t^\frown k} = W_k$ and $C_{t^\frown k} = C_k$. Since $M \subseteq \mathrm{cl}(\{W_k : k < \omega\})$, we have that $\{U_{t^\frown k} : k < \omega\}$ is J_I-positive.

This ends the construction of $T \in Q(J_I)$. It is routine now to define a Borel automorphism h of ω^ω out of the sets U_τ and C_τ so that $T \Vdash \dot{g} = h(f(\dot{g}))$. This ends the proof.

□

8. Hypersmooth equivalence relations

In this section we prove Theorem 1.7. If F is an equivalence relation on X and $A \subseteq X$, then by $[A]_F$ we denote the *F-saturation* of A, i.e. the set $\{x \in X : \exists y \in A \ \ xFy\}$.

Lemma 8.1. *Let I be a σ-ideal on a Polish space X and let F be an analytic equivalence relation on X. There is an I-positive Borel set $Y \subseteq X$ which carries a Polish zero-dimensional I-perfect topology preserving the original Borel structure such that the F-saturation of any basic clopen set is clopen.*

Proof. Let $M \prec H_\kappa$ be an elementary countable submodel coding the relation F. As in Proposition 7.2 we take Y to be the set of all P_I-generic points over M and take the topology generated by all Borel sets which are coded in M. Now if B is a Borel set coded in M, then $[B]_F \cap Y$ is equal to $[\![\exists x \in B \ \dot{g} F x]\!]_{\mathrm{ro}(P_I)} \cap Y$. □

Proof of Theorem 1.7. Let $E = \bigcup_{n<\omega} E_n$ where each E_n is a smooth Borel equivalence relation on X. By continuous reading of names for P_I we can assume that each E_n is given by a continuous function and consequently E_n is closed. If for some $n < \omega$ there is a Borel I-positive set consisting of E_n-independent elements, then we are done. Suppose this is not the case. Suppose also that all equivalence classes of E are in I. We will construct then an I-positive Borel set consisting of E-independent elements.

By Lemma 8.1 we may assume that for each basic clopen set U its saturation $[U]_E$ is clopen and that all closed sets from I are nowhere dense. Since each class of E_n is closed and contained in a class of E, which belongs to I, the equivalence classes of E_n are closed nowhere dense.

Lemma 8.2. *For each $n < \omega$ and each nonempty open set $U \subseteq X$ there is an I-positive closed nowhere dense set C which is contained in one class of E_i for every $i \leq n$.*

Proof. By Theorem 1.6 applied for E_n, for each I-positive Borel set $U \subseteq X$ there is an I-positive Borel set $C \subseteq U$ which is either contained in one equivalence class of E_n, or is E_n-independent. We excluded the second possibility in the beginning, so applying Theorem 1.6 n-many times we get the desired set. $\square$

We will construct $S \in Q(J_I)$ such that for each $x, y \in \pi[S]$ if $x \neq y$, then $\neg\, xEy$. Enumerate $\omega^{<\omega}$ into a sequence $\langle t_k : k < \omega\rangle$ respecting the lexicographical order. We write $s_k = t_k \restriction (|t_k| - 1)$. Let $T_k = \{t_i : i \leq k\}$ and let T'_k denote the set of terminal nodes of T_k.

Inductively we construct a sequence of finite Luzin schemes $S_k = \langle U^k_t : t \in T_k\rangle$ with the properties:

- $U^{k+1}_t \subseteq U^k_t$ if $t \in T_k$ and $U^{k+1}_t = U^k_t$ if $t \in T_k \setminus T'_k$
- if $s_{k+1} = t_l$, then for each $s \in T'_{k+1} \setminus \{t_{k+1}\}$ we have
$$[U^{k+1}_s]_{E_i} \cap [U^{k+1}_{t_{k+1}}]_{E_i} = \emptyset$$
for each $i \leq l$.

At the end let $U_t = \bigcap_{k<\omega} U^k_t$ and let S be the Luzin scheme $\langle U_t : t \in \omega^{<\omega}\rangle$. Note that $\pi[S]$ will then be E-independent. Indeed, take $x, y \in \pi[S]$ such that $x \neq y$ and let $n < \omega$. Find $k < \omega$ such that all terminal nodes of T_k are indexed with numbers greater than n and $x \restriction i$ and $y \restriction j$ are in T'_k for some $i, j \in \omega$ such that $x \restriction i \neq y \restriction j$. Then clearly $\neg\, x' E_n y'$ for every $x' \supseteq x \restriction (i+1)$ and $y' \supseteq y \restriction (j+1)$. Thus, all we need is to make sure that $S \in Q(J_I)$.

To this end, additionally, along the construction, we build I-positive nowhere dense sets C_t for $t \in T_k \setminus T'_k$. We ensure that if $t = t_l \in T_k \setminus T'_k$, then

- $C_t \subseteq U^k_t$,
- for each $i \leq l$ the set $[C_t]_{E_i}$ is nowhere dense,
- for each $s \in T'_k$ and each $i \leq l$ we have
$$[U^k_s]_{E_i} \cap [C_t]_{E_i} = \emptyset,$$
- $C_t \subseteq \mathrm{cl}(\mathrm{succ}_S(t))$

Each set C_t gets constructed at the first step k such that $t \in T_k \setminus T'_k$. At this step we also fix an enumeration $\langle V_t^l : l < \omega\rangle$ of all basic open subsets of U_t^k which have nonempty intersection with C_t and we futher make sure that $U_{t^\frown l} \subseteq V_t^l$ for each $l < \omega$. The latter will imply that $C_t \subseteq \mathrm{cl}(\mathrm{succ}_S(t))$ and consequently $S \in Q(J_I)$.

We begin the construction by setting $U_\emptyset^0 = X$. Now we make the induction step from k to $k+1$. Let $j < \omega$ be such that $s_{k+1} = t_j$. There are two cases.

Case 1. The node s_{k+1} is a terminal node in T_k. In this case we need to construct $C_{s_{k+1}}$. Using Lemma 8.2 we find $C_{s_{k+1}} \subseteq U_{s_{k+1}}^k$ such that $[C_{s_{k+1}}]_{E_i}$ is nowhere dense for each $i \leq j$. At this point, fix an enumeration $\langle V_{s_{k+1}}^l : l < \omega$ of all basic clopen subsets of $U_{s_{k+1}}^k$ which have nonempty intersection with $C_{s_{k+1}}$.

Next, for each $t \in T'_k$ we shrink U_t^k to U_t^{k+1} which is disjoint from $\bigcup_{i\leq j}[C_{s_{k+1}}]_{E_i}$. To construct $U_{t_{k+1}}^{k+1}$ notice that the set

$$W = V_{s_{k+1}}^0 \setminus \bigcup_{i\leq j} \bigcup_{t\in T'_k} [U_t^{k+1}]_{E_i}$$

is a clopen set and it is nonempty since $V_{s_{k+1}}^0 \cap C_{s_{k+1}} \neq \emptyset$. Since for each $l < \omega$ such that $t_l \in T_{k+1} \setminus T'_{k+1}$ the set $\bigcup_{i\leq l}[C_{t_l}]_{E_i}$ is nowhere dense, we can pick $U_{t_{k+1}}^{k+1} \subseteq W$ which is disjoint from all these sets.

Case 2. The node s_{k+1} is not a terminal node in T_k. In this case $C_{s_{k+1}}$ is already constructed. Let $m < \omega$ be maximal such that $s_{k+1}{}^\frown m \in T_k$. We put $U_t^{k+1} = U_t^k$ for each $t \in T'_k$ and we only construct $U_{t_{k+1}}^{k+1}$. The set

$$W = V_{s_{k+1}}^m \setminus \bigcup_{i<j} \bigcup_{t\in T'_k} [U_t^k]_{E_i}$$

is again a nonempty clopen set. As previously, let $U_{t_{k+1}}^{k+1}$ be a basic clopen subset of W which is disjoint from $\bigcup_{i\leq l}[C_{t_l}]_{E_i}$ for each $l < \omega$ such that $t_l \in T_k \setminus T'_k$.

This ends the construction of the Luzin scheme S and the whole proof.

□

9. Questions

Question 1. Suppose that P_I is proper and does not add Cohen reals. Is it true that P_{I^*} does not add Cohen reals either?

Question 2. Let I be a σ-ideal generated by closed sets such that P_I does not add Cohen reals. Does J_I necessarily have the discrete set property?

Question 3. Let J be a dense $\mathbf{F}_{\sigma\delta}$ and weakly selective ideal on ω. Does there exist a Polish space with a countable base $\mathcal{O}$ and a σ-ideal I on X such that under some identification of ω and $\mathcal{O}$ the ideal J becomes J_I?

Question 4. Is $\mathtt{SPL}$ a Borel set?

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