Groups with infinite asymptotic dimension

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Coarse geometry

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Definition

Let X, Y be metric spaces. $f : X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with

$$\lim_{t \to \infty} \rho_-(t) = \infty$$

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and

$$\rho_-(d(a, b)) \leq d(f(a), f(b)) \leq \rho_+(d(a, b))$$

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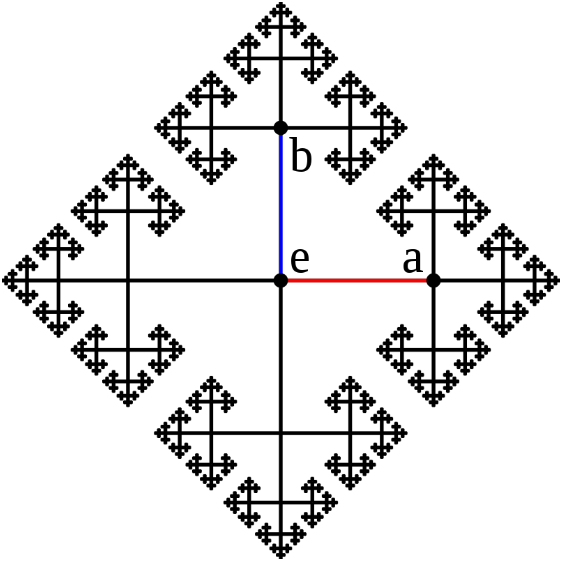
Any countable G carries a proper (bounded sets are finite), left-invariant metric.

Free group F_2

$F_2 = \langle a, a^{-1}, b, b^{-1} \rangle$



0-dimensional families

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If X is covered by an R-disjoint and uniformly bounded family, we consider X to be 0-dimensional at scale R.

Finite asymptotic dimension

Definition (M. Gromov (1993))

$\operatorname{asdim} X \le n$ if for any $R > 0$, X has a uniformly bounded cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ with each $\mathcal{U}_i$ R-disjoint.

For each $R > 0$, X decomposes into $n+1$ R-disjoint uniformly bounded families.

Examples

Definition

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- $\operatorname{asdim} \mathbb{Z} = 1$.
- $\operatorname{asdim} \mathbb{Z}^n = n$
- If G is a finitely generated abelian group of rank n, then $\operatorname{asdim} G = \operatorname{asdim} \mathbb{Z}^n = n$

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- If G is a finitely generated abelian group of rank n, then asdim $G =$ asdim $\mathbb{Z}^n = n$
- asdim $F_n = 1$
- Solvable groups
- Having finite asymptotic dimension is preserved by direct products and group extensions.
- Solvable groups whose factor groups are fin. gen. have finite asymptotic dimension

Examples

Theorem (J. Roe (2005))

If G is word hyperbolic, then $\operatorname{asdim} G < \infty$.

Theorem (N. Wright (2010))

Finite dimensional $CAT(0)$ cube complexes have finite asymptotic dimension.

Theorem (M. Bestvina, K. Bromberg, K. Fujiwara (2014))

If Σ is a compact orientable surface, then $\operatorname{asdim} MCG(\Sigma) < \infty$.

Motivation

The Coarse Baum-Connes Conjecture:
If X is a discrete metric space, we say X satisfies the Coarse Baum-Connes Conjecture if the assembly map

$$\mu : KX_*(X) \to K_*(C^*(X))$$

is an isomorphism.

Here $KX_*(X) = \lim_{d\to\infty} K_*(P_d(X))$, where $P_d(X)$ is the Rips complex of X for distance d, and K_* is the K-homology functor. $C^*(X)$ is a certain C^*-algebra which is a coarse invariant of X, and K_* is the K-theory functor.

The Coarse Baum-Connes Conjecture implies certain "topological rigidity" statements, notably the Novikov conjecture if $\pi_1(M)$ satisfies the CBCC.

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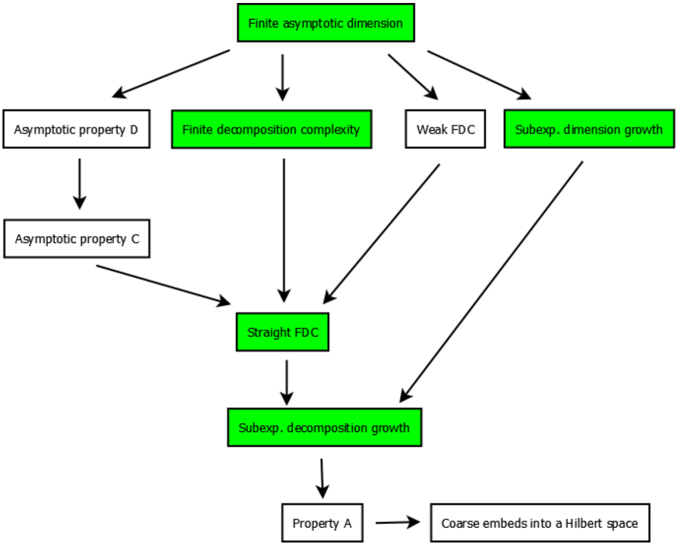
Theorem (Guenter, Tessera, Yu (2012))

Let M, N be compact, aspherical manifolds such that $\pi_1(M), \pi_1(N)$ have finite decomposition complexity. If M, N are homotopy equivalent, then for some n, $M \times \mathbb{R}^n$ and $N \times \mathbb{R}^n$ are homeomorphic.

Infinite asymptotic dimension



Decompositions of metric spaces

X is (R, n)*-decomposable over* $\mathcal{U}$ if $\exists R$-disjoint families $\mathcal{U}_1, \ldots, \mathcal{U}_n$ with

- $\bigcup_{i=1}^n \mathcal{U}_i$ a cover of X
- Each $\mathcal{U}_i \subset \mathcal{U}$.

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If $\mathcal{V}$ is a family of metric spaces and each $X \in \mathcal{V}$ has $X \xrightarrow{R,n} \mathcal{U}$, write

$$\mathcal{V} \xrightarrow{R,n} \mathcal{U}$$

Asymptotic dimension growth

Definition

Say *asdim* $X \leq n$ if for any $R \geq 0$ there is uniformly bounded $\mathcal{U}$ with $X \xrightarrow{R,n+1} \mathcal{U}$.

If the n needed for decompositions $X \xrightarrow{R,n} \mathcal{U}$ tends to infinity with R, $asdim(X) = \infty$.

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Definition (A. Dranishnikov (2006))

Let $ad_X(R) : \mathbb{N} \to \mathbb{N}$ be defined so $ad_X(R) = n$ is the least n such that $\exists$ uniformly bounded $\mathcal{U}$ with $X \xrightarrow{R,n} \mathcal{U}$.

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We say X has *subexponential asymptotic dimension growth* if $\lim_{R\to\infty} \sqrt[R]{ad_X(R)} = 1$.

Asymptotic dimension growth

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- $(\ldots((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \cdots \wr \mathbb{Z}) \wr \mathbb{Z}$ has polynomial dimension growth of degree $d-1$ (d $\mathbb{Z}$'s).
- Dimension growth of finitely generated G is bounded by the growth of $Vol_G(R) = |B_G(R, e)|$.
- Groups of subexponential volume growth have subexponential dimension growth (i.e. Grigorchuk's group). This is a subclass of amenable groups.
- Finite rank coarse median groups have subexponential dimension growth.

Asymptotic dimension growth

Problems with asymptotic dimension growth:

- Difficult to compute
- Subexponential dimension growth is likely not preserved by group extensions or direct unions
- It's just a quasi-isometry invariant, not a general coarse invariant.

Finite decomposition complexity

Decomposition game (Guenter, Tessera, Yu (2012))

Round 1: Player A picks an integer R_1. Player B $(R_1, 2)$-decomposes X over some family $\mathcal{U}_1$.

Round k: Player A picks an integer R_k. Player B $(R_k, 2)$-decomposes $\mathcal{U}_k$.

X has *finite decomposition complexity* if Player B can always decompose to a uniformly bounded family in finitely many rounds.

Finite decomposition complexity

- FDC is preserved by taking subgroups, group extensions, direct unions, and free prodcuts.
- Finite asymptotic dimension implies FDC.
- All elementary amenable groups have FDC
- All countable subgroups of $GL(n, R)$, where R is any commutative ring, have FDC.
- All countable subgroups of an almost connected Lie group have FDC.
- Does every countable amenable group have FDC? Open even for amenable groups of subexponential (but superpolynomial) volume growth.

Straight finite decomposition complexity

FDC can be weakened by requiring Player A to pick all the distances in advance.

Definition (A. Dranishnikov, M. Zarichnyi (2014))

X has *straight finite decomposition complexity (sFDC)* if for any sequence of positive integers $R_1 \leq R_2 \leq \ldots$ $\exists$ families $\mathcal{U}_i$, with $\mathcal{U}_n$ uniformly bounded for some n, and

$$X \xrightarrow{R_1,2} \mathcal{U}_1 \xrightarrow{R_2,2} \mathcal{U}_2 \xrightarrow{R_3,2} \ldots \xrightarrow{R_n,2} \mathcal{U}_n.$$

Decomposition complexity growth

Definition (D. (2019))

$f : \mathbb{N} \to \mathbb{N}$ is a *decomposition complexity growth function (dcg function)* for X if for any sequence of positive integers $R_1 \leq R_2 \leq \ldots$ $\exists$ families $\mathcal{U}_i$, with $\mathcal{U}_n$ uniformly bounded for some n, and

$$X \xrightarrow{R_1, g(R_1)} \mathcal{U}_1 \xrightarrow{R_2, g(R_2)} \mathcal{U}_2 \xrightarrow{R_3, g(R_3)} \ldots \xrightarrow{R_n, g(R_n)} \mathcal{U}_n.$$

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Definition

X has *subexponential decomposition complexity growth* if X has some subexponential function as a dcg function.

Subexponential decomposition complexity growth

Theorem

Let X be a metric space with bounded geometry. If X has subexponential decomposition growth, then X has property A.

Extension theorem

Theorem

If $1 \to K \to G \to H \to 1$ is a short exact sequence of countable groups such that K has sFDC and H has subexponential decomposition growth, then G has subexponential decomposition growth.

Decomposition growth of amenable groups?

- $(F_2 \wr \mathbb{Z}) \times G$
- All fin. gen. groups of subexponential volume growth have subexponential decomposition growth.
- All countable elementary amenable groups have subexponential decomposition growth.
- (Open question) What about amenable groups that are non-elementary amenable and have exponential volume growth?