Fraenkel–Mostowski models revisited*

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Abstract

I provide several natural properties of group actions which translate into fragments of axiom of choice in the associated permutation models of choiceless set theory.

1 Introduction

The field of symmetric and permutation models of choiceless set theory is notoriously chaotic. One needs to only look at the encyclopedic form of [1] to understand where that impression comes from. The purpose of this paper is to provide several natural properties of group actions which translate into fragments of axiom of choice in the associated permutation models. Very many results quoted in [1] follow essentially immediately, and every now and then a novel conclusion about a known model appears. The main point is that the properties of group actions considered have intrinsic interest, perhaps justifying the study of permutation models in the eyes of a non-specialist. Evaluation of these properties in very natural cases is often challenging, and it leads to natural questions in model theory, combinatorics, geometric topology, and other fields.

The sections are ordered in decreasing strength of the fragments of axiom of choice they deal with. In Section 2, I fix the terminology. In Section 3, I consider the axiom of well-ordered choice, and provide a dynamical equivalent to it, cofinal orbits–Theorem 3.3. The ideal of nowhere dense subsets of the rationals has cofinal orbits. For any interesting topological space, the status of cofinal orbits seems to be a challenging open problem. In Section 3.1, I consider the axiom of dependent choices. The dynamical criterion here is a certain infinite game; a gameless version has been considered previously by Karagila and Schilhan [4]. I show that the DC game is determined in many useful cases, Theorem 3.15. The good player has a winning strategy in the interesting case of the nowhere dense ideal on a Euclidean space, Example 3.20. In Section 4, I consider the axiom of countable choice and its dynamical equivalent, σ-closure.

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I show that many dynamical ideals have this property, such as the ideal of countable closed subsets of the real line, or the well-ordered subsets of rationals. In Section 4.1, I consider the statement that unions of well-orderable collections of well-orderable sets are well-orderable. There is a neat dynamical criterion similar to topological simplicity of groups. Naturally enough, abelian group actions are never simple except for trivial cases, Theorem 4.25. One simple ideal is the ideal of finite sets on many limit Fraisse structures, Theorem 4.12. Finally, in Section 4.2, I show how to rule out amorphous or infinite, Dedekind-finite sets from permutation models. The criterion uses a stratification of the ideal into an increasing union which exhibits a degree of σ-closure.

2 Notation and terminology

In this section, I show how to obtain a permutation model (a Fraenkel–Mostowski model in the terminology of [1]) of the theory ZF with atoms from an object called a dynamical ideal. The construction is well-known; the section serves mostly to introduce suitable terminology.

Definition 2.1. ZFA, the set theory with atoms, is the theory with the following description.

1. Its language contains a binary relational symbol $\in$ and a unary relational symbol $\mathbb{A}$ for atoms;
2. its axioms include all usual axioms of ZF, except for the axiom of extensionality, which is stated only for sets which are not atoms;
3. there are two additional axioms: $\forall x\ \mathbb{A}(x) \to \forall y\ y \notin x$ and $\exists y\ \forall x\ y \in x \leftrightarrow \mathbb{A}(x)$ (atoms form a set).

ZFCA is ZFA plus the axiom of choice.

The main issue in ZFA and ZFCA (about which its axioms say nothing) is the structure of its set of atoms. It is not difficult to build a model of ZFCA with a prescribed set of atoms.

Definition 2.2. Let X be a set. There is up to class isomorphism unique class model M of ZFCA satisfying the following demands:

1. $X = \mathbb{A}^M$;
2. the membership relation of M is well-founded;
3. for every $m \in M$ the collection $\{n \in M : n \in^M m\}$ is a set as opposed to proper class;
4. for every set $B \subset M$ there is an element $m \in M$ such that $B = \{n \in M : n \in^M m\}$.

The model M will be denoted by $V[[X]]$.

By an abuse of notation, the membership relation of $V[[X]]$ will be denoted by $\in$. By another abuse of notation, I will identify X with the element of the model $V[[X]]$ which contains exactly all the $V[[X]]$-atoms.

Definition 2.3. A set $A \in V[[X]]$ is *pure* if $V[[X]] \models$ the transitive closure of A contains no atoms. The class $\{A \in V[[X]] \colon A \text{ is pure}\}$ is called the *pure part* of $V[[X]]$ and denoted by V.

It is immediate that the pure part of $V[[X]]$ is definably isomorphic to the set-theoretic universe in which the model $V[[X]]$ is built. In the construction of inner models of ZFA, group actions play central role. The following definition records key notational elements.

Definition 2.4. Suppose that $\Gamma \curvearrowright X$ is a group action.

1. if $x \in X$ then $\operatorname{stab}(x)$, the stabilizer of x, is the set $\{\gamma \in \Gamma \colon \gamma \cdot x = x\}$;
2. if $a \subset X$ then $\operatorname{pstab}(a)$, the *pointwise stabilizer of* x, is the set $\{\gamma \in \Gamma \colon \forall x \in a\ \gamma \cdot x = x\} = \bigcap_{x \in a} \operatorname{stab}(x)$;
3. $\Gamma \curvearrowright V[[X]]$ is the unique action extending the original one and satisfying $\gamma \cdot A = \{\gamma \cdot B \colon b \in A\}$ for all $\gamma \in \Gamma$ and $A \in V[[X]]$.

Stabilizers and pointwise stabilizers are subgroups of Γ. The extension of the action to all of $V[[X]]$ is an action of Γ by $\in$-automorphisms; it is not difficult to show by $\in$-recursion that the action $\Gamma \curvearrowright V[[X]]$ fixes all pure elements. To construct an inner model of ZFA, an additional piece of data is needed:

Definition 2.5. A *dynamical ideal* is a tuple $\Gamma \curvearrowright X, I$ where Γ is a group acting on a set X and I is an ideal on the set X containing all singletons, invariant under the action. That is to say, for every $a \in I$ and every $\gamma \in \Gamma$, the set $\gamma \cdot a = \{\gamma \cdot x \colon x \in a\}$ belongs to I.

One point in this paper is that the class of dynamical ideals is much broader than the examples normally discussed in connection with permutation models, and that there are tools to investigate even the more "exotic" examples.

Example 2.6. Let $\mathcal{S}$ be a structure with countable universe X. Let Γ be the group of automorphisms of $\mathcal{S}$, acting on X by application. Let I be the ideal of finite subsets of X. Then $\Gamma \curvearrowright X, I$ is a dynamical ideal.

Example 2.7. Let X be a topological space, let Γ be the homeomorphism group acting on X by application, and let I be an ideal defined from the topology only:

1. the ideal of nowhere dense sets;
2. the ideal of sets with countable closure;
3. the ideal of sets with zero-dimensional closure.

Then $\Gamma \curvearrowright X, I$ is a dynamical ideal.

Definition 2.8. Let $\Gamma \curvearrowright X, I$ be a dynamical ideal. The *associated permutation model* $W[[X]]$ is the transitive part of the class $\{A \in V[[X]] \colon \exists b \in I \colon \mathrm{pstab}(b) \subseteq \mathrm{stab}(A)\}$.

Note that the notation $W[[X]]$ abstracts away from the group action and the ideal for the sake of brevity. This should not cause any confusion. The following is well-known [3, Chapter 4].

Fact 2.9. $W[[X]]$ *is a model of Zermelo–Fraenkel set theory with atoms.*

In the rest of this section, I derive several properties of the model $W[[X]]$ which hold irrespective of the dynamical ideal used.

Proposition 2.10. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal.*

1. *X and I both belong to the associated permutation model;*
2. *$V \subset W[[X]]$;*
3. *(support invariance) the relation $\{\langle b, A\rangle \colon b \in I$ and $\mathrm{pstab}(b) \subset \mathrm{stab}(A)\}$ is invariant under the group action and as such belongs to the permutation model;*
4. *the permutation model is invariant under the group action.*

Proof. For (1), note that $\{x\} \in I$ and $\mathrm{pstab}(\{x\}) = \mathrm{stab}(x)$; in conclusion $X \subset W[[X]]$. Since X is invariant under the action, $X \in W[[X]]$. For every set $a \in I$, $\mathrm{pstab}(a) \subset \mathrm{stab}(a)$, so $a \in W[[X]]$. Finally, since the ideal I is invariant under the action, $\Gamma = \mathrm{stab}(I)$ holds and $I \in W[[X]]$ as desired. (2) is clear as all pure sets are fixed by the group action.

For (3), suppose that $\mathrm{pstab}(b) \subseteq \mathrm{stab}(A)$ and $\gamma \in \Gamma$ is an element. To show that $\mathrm{pstab}(\gamma \cdot b) \subseteq \mathrm{stab}(\gamma \cdot A)$, let $\delta \in \mathrm{pstab}(\gamma \cdot b)$ be an arbitrary element. Then $\gamma^{-1}\delta\gamma \in \mathrm{pstab}(b)$, so $\gamma^{-1}\delta\gamma \cdot A = A$, and multiplying both sides by γ, get $\delta \cdot (\gamma \cdot A) = \gamma \cdot A$. (4) is an immediate corollary of the definition of the permutation model and (3). $\square$

In the permutation model, we view well-orderable sets as trivial: every structure on a well-orderable set is a copy of a structure in V. It will be useful to have a characterization of well-orderable sets.

Proposition 2.11. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal. The following are equivalent for every set $A \in W[[X]]$:*

1. *A is well-orderable in $W[[X]]$;*
2. *there is a set $b \in I$ such that $\mathrm{pstab}(b) \subseteq \mathrm{pstab}(A)$;*
3. *$\mathcal{PP}(A) \cap W[[X]] = \mathcal{PP}(A) \cap V[[X]]$;*

4. *$\mathcal{PP}(A)$ is well-orderable in $W[[X]]$.*

The last sentence of the theorem is exactly the feature of permutation models which sets them apart from most symmetric models of ZF obtained as submodels of generic extensions.

Proof. For (1) implies (2), let R be a well-ordering of A in the permutation model, let $b \in I$ be such that $\mathrm{pstab}(b) \subseteq \mathrm{stab}(R)$, and by transfinite induction on R prove that $\mathrm{pstab}(b) \subseteq \mathrm{stab}(B)$ for every $B \in A$. For (2) implies (3), note that $\mathrm{pstab}(b) \subset \mathrm{pstab}(B)$ for every $B \subset A$ and then $\mathrm{pstab}(b) \subset \mathrm{pstab}(C)$ for every $C \subset \mathcal{P}(A) \cap V[[X]]$. For (3) implies (1), use the axiom of choice in $V[[X]]$ to find a well-ordering on A and code it as the set of its initial segments (an element of $\mathcal{PP}(A)$) to transfer it to $W[[X]]$.

Finally, (2) implies (4) since $\mathrm{pstab}(b) \subseteq \mathrm{pstab}(\mathcal{PP}(A))$ and then $\mathrm{pstab}(b) \subset \mathrm{stab}(R)$ for any relation R on $\mathcal{PP}(A)$ in $V[[X]]$, in particular for a well-ordering R on $\mathcal{PP}(A)$ obtained in $V[[X]]$ using the axiom of choice there. (4) implies (1) trivially: the map $B \mapsto \{\{B\}\}$ is an injection from A to $\mathcal{PP}(A)$ in $W[[X]]$. $\square$

Finally, I discuss a natural closure-type property of a dynamical ideal which is necessary for the statement of certain characterization theorems.

Definition 2.12. Let $\Gamma \curvearrowright X, I$ be a dynamical ideal.

1. a set $a \subset X$ is *definably closed* if for every $x \notin a$ there is $\gamma \in \mathrm{pstab}(a)$ such that $\gamma \cdot x \neq x$;
2. the dynamical ideal is *definably closed* if every set in I is a subset of a definably cosed set in I.

It is not hard to see that for every set $a \subset X$ there is the smallest set $b \subset X$ which is definably closed and $a \subset b$, namely $b = \{x \in X \colon \mathrm{pstab}(a) \subset \mathrm{stab}(x)\}$; I will call b the *definable closure of* a. A brief diagram-chasing argument shows that the definable closure operator is invariant under the group action. For every dynamical ideal I there is also the smallest dynamical ideal J which is definably closed and $I \subset J$, namely the ideal of all sets which are subsets of definable closures of sets in I. It is easy to check that the ideal J thus defined is invariant under the group action. I will call J the *definable closure of* I. The following is nearly trivial.

Proposition 2.13. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal, and let $\Gamma \curvearrowright X, J$ be its definable closure. The two dynamical ideals generate the same permutation model.*

Example 2.14. Let X be a topological space, let Γ be the group of self-homeomorphisms of X acting on X by application, and let $a \subset X$ be a set. The topological closure is a subset of the definable closure of a. This is the reason why in this situation I consider only ideals generated by closed sets.

3 Axiom of well-ordered choice

The first common fragment of axiom of choice under consideration in this paper is the following.

Definition 3.1. [1, Form 40] The *axiom of well-ordered choice* is the statement that every well-ordered family of non-empty sets has a choice function.

The dynamical counterpart to the well-ordered axiom of choice is the following.

Definition 3.2. Let $\Gamma \curvearrowright X, I$ be a dynamical ideal. The ideal has *cofinal orbits* if for every $a \in I$ there is $b \in I$ which is *a-large*: for every $c \in I$ there is $\gamma \in \operatorname{pstab}(a)$ such that $c \subseteq \gamma \cdot b$.

Theorem 3.3. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal.*

1. *if the ideal has cofinal orbits then the associated permutation model satisfies the axiom of well-ordered choice;*
2. *If the ideal I is definably closed and the associated permutation model satisfies the axiom of well-ordered choice, then the ideal has cofinal orbits.*

Proof. For (1), let $W[[X]]$ be the permutation model, and suppose that A is a well-orderable set of nonempty sets in $W[[X]]$. Use Proposition 2.11 to find a set $a \in I$ such that $\operatorname{pstab}(a) \subset \operatorname{pstab}(A)$. Let $b \in I$ be a set such that the $\operatorname{pstab}(a)$-orbit of b is cofinal in I. Now, we claim that every set $B \in A$ contains a set $C \in B$ such that $\operatorname{pstab}(b) \subseteq \operatorname{stab}(C)$. To see this, let $D \in B$ be an arbitrary set, and let $d \in I$ be such that $\operatorname{pstab}(d) \subseteq \operatorname{stab}(D)$. Find a group element $\gamma \in \operatorname{pstab}(a)$ such that $d \subset \gamma \cdot b$ and let $C = \gamma^{-1} \cdot D$; we claim that the set C works as required. First of all, it is clear that $C \in B$ holds since γ (and γ^{-1}) fixes every element of A; in particular, it fixes B. Second, a diagram chasing argument shows that for every element $\delta \in \operatorname{pstab}(b)$, $\gamma\delta\gamma^{-1} \in \operatorname{pstab}(d)$, so $\gamma\delta\gamma^{-1} \cdot D = D$ and $\delta \cdot (\gamma^{-1} \cdot D) = \gamma^{-1} \cdot D$ as required.

Now, let f be any selector on A such that for every $B \in A$, $\operatorname{pstab}(b) \subseteq \operatorname{stab}(f(A))$ holds; such a selector exists by the previous paragraph. It is immediate that $\operatorname{pstab}(b) \subseteq \operatorname{stab}(f)$, so $f \in W[[X]]$ and (1) follows.

For (2), suppose that the dynamical ideal is solid and $W[[X]]$ satisfies well-ordered choice. Let $a \in I$ be an arbitrary set. In $W[[X]]$, consider the set A of all well-orderings whose domain belongs to I. Note that for every set $b \in I$, every relation R on b in $V[[X]]$ belongs to $W[[X]]$ since $\operatorname{pstab}(b) \subset \operatorname{stab}(R)$; in particular, b carries a well-ordering in $W[[X]]$. Let E be the $\operatorname{pstab}(a)$-orbit equivalence relation on A, and let B be the set A/E. Clearly, $\operatorname{pstab}(a) \subset \operatorname{stab}(A, E, B)$, so $A, E, B \in W[[X]]$ holds. In addition, for every E-class $C \subset A$, $\operatorname{pstab}(a) \subset \operatorname{stab}(C)$, so B is even well-ordered in $W[[X]]$. By the axiom of well-ordered choice in $W[[X]]$, there is a selector f on B.

Let $b \in I$ be such that $\operatorname{pstab}(b) \subset \operatorname{stab}(f)$ and $a \subset b$. Use the solidity assumption on the ideal to icrease b if necessary so that $\forall x \in X \setminus b\ \exists \gamma \in \operatorname{pstab}(b)\ \gamma \cdot x \neq x$. I claim that b is a-large. Indeed, suppose that $c \in I$ is a set.

Let $\leq$ be a well-ordering on c, let $C \in B$ be the pstab(a)-orbit of $\langle c, \leq \rangle$, and consider the value $f(C)$. Since $\text{pstab}(b) \subset \text{stab}(f, C)$, $\text{pstab}(b) \subset \text{stab}(f(C))$ holds. Now, $f(C) = \langle \gamma \cdot c, \gamma \cdot \leq \rangle$ for some $\gamma \in \text{pstab}(a)$; since $\gamma \cdot \leq$ is a well-ordering on $\gamma \cdot c$, a transfinite induction argument along it shows that $\forall x \in \gamma \cdot c \ \text{pstab}(b) \subset \text{stab}(x)$. By the solidity assumption on the set b, $\gamma \cdot c \subset b$. This concludes the proof that b is a-large; (2) follows. $\square$

The concept of cofinal orbits is worthless without examples.

Example 3.4. Let X be any Euclidean space, let Γ be the Polish group of self-homeomorphisms of X, and let I be the ideal of bounded subsets of X. Then I has cofinal orbits. In dimension one, this is [1, Model N47], in which apparently the status of well-ordered choice has not been known.

Example 3.5. [1, Model N33] Let $X = \mathbb{Q}$ with its ordering, let Γ be the Polish group of all automorphisms of X, and let I be the ideal of bounded subsets of X. Then I has cofinal orbits.

Example 3.6. Let $X = \mathbb{Q}$ with its ordering, let Γ be the Polish group of all automorphisms of X, and let I be the ideal of nowhere dense subsets of X. Then I has cofinal orbits.

Proof. The argument consists of several simple back and forth constructions encapsulated in the following claims.

Claim 3.7. *Let $d_0, d_1 \subset \mathbb{Q}$ be two sets such that both they and their complements are dense in $\mathbb{Q}$. Then there is an order-preserving permutation π of $\mathbb{Q}$ such that $\pi'' d_0 = d_1$.*

Claim 3.8. *Let $a \subset \mathbb{Q}$ be a nowhere dense set. Then a can be extended to a nowhere dense set which is in addition closed, coinitial and cofinal in $\mathbb{Q}$, and such that no point of it is isolated from left or right.*

Nowhere dense sets satisfying the properties spelled out in the last claim will be called *good*.

Claim 3.9. *Let $d_0, d_1 \subset \mathbb{Q}$ be two nowhere dense good sets. Then there is an order-preserving permutation π of $\mathbb{Q}$ such that $\pi'' d_0 = d_1$.*

Proof. Let Y_0 be the linearly ordered set whose universe consists of elements of d_0 and inclusion-maximal intervals of $\mathbb{Q}$ disjoint from d_0 and the ordering is the natural one inherited from $\mathbb{Q}$; the same for subscript 1. The goodness of the set d_0 implies that the linear ordering on Y_0 is dense without endpoints, $d_0 \subset Y_0$ is dense, and so is $Y_0 \setminus d_0 \subset Y_0$. Claim 3.7 shows that there is an order-preserving bijection $\theta \colon Y_0 \to Y_1$ such that $\theta'' d_0 = d_1$. For each interval $i \in Y_0 \setminus d_0$, let $\pi_i \colon i \to \theta(i)$ be an order-preserving bijection. In the end, $\pi = (\theta \restriction d_0) \cup \bigcup \{\pi_i \colon i \in Y_0 \setminus d_0\}$ is the desired permutation of $\mathbb{Q}$. $\square$

Finally, let $a \subset X$ be a nowhere dense set; I must produce an a-large nowhere dense set b. Enlarging a if necessary, I may assume that a is good. Write J for the set of all inclusion-maximal intervals of X disjoint from a. Let $b \subset X$ be a good nowhere dense set such that $a \subset b$ and in each $i \in J$ the set b is both coinitial and cofinal. I will show that b is a-large.

To prove this, let $c \subset X$ be a nowhere dense set. Enlarging c if necessary, I may assume that it contains a, it is good, and in each interval $i \in J$ it is coinitial and cofinal. For each interval $i \in J$ use Claim 3.9 to find an order preserving permutation θ_i of i such that $\theta_i''(b \cap i) = c \cap i$. Then $\pi = (\mathrm{id} \restriction a) \cup \bigcup\{\theta_i : i \in J\}$ is an order-preserving permutation of X which is constant on a and moves b to c. $\square$

Example 3.10. [1, Model N12(κ)] Let κ be an uncountable cardinal, let X be a set of cardinality at least κ, let Γ be the group of all permutations of X acting by application, and let I_κ be the ideal of subsets of X of cardinality smaller than κ. Then $\Gamma \curvearrowright X, I_\kappa$ has cofinal orbits iff κ is a successor cardinal.

Proof. If κ is a successor cardinal, $\kappa = \lambda^+$, then for any set $a \in I_\kappa$ any set b such that $a \subset b$ and $|b \setminus a| = \lambda$ has a cofinal orbit under the action of $\mathrm{pstab}(a)$. On the other hand, if κ is a limit cardinal, then given a set $b \in I_\kappa$, its orbit consists of sets of cardinality $|b|$ only, but there are sets of cardinality greater than $|b|$ in the ideal I_κ. Thus, in that case I_κ fails to have cofinal orbits. $\square$

Example 3.11. Let X be a set and let I be an ideal on it. Say that I is *uniform* if there is an infinite cardinal κ such that I is generated by sets of cardinality κ, and for every set $a \in I$ there is a set $b \in I$ of cardinality κ which is disjoint from a. If I is a uniform ideal on X and Γ is the group of all permutations of X with support in I acting on X by application, then $\Gamma \curvearrowright X, I$ has cofinal orbits.

Proof. Let $a \in I$ be an arbitrary set, and let $b \in I$ be a set disjoint from I of the uniform cardinality κ. It will be enough to show that the $\mathrm{supp}(a)$-orbit of $a \cup b$ is cofinal. To this end, let $c \in I$ be an arbitrary set; enlarging c if necessary, we may assume that $a \cup b \subset c$. Let $\pi \colon b \to c \setminus a$ be any bijection. Let $d = c \setminus b$. The uniformity assumption shows that there are pairwise disjoint sets d_n for $n \in \omega$ such that $d_0 = d$, all d_n for $n > 0$ are disjoint from c and have cardinality $|d|$, and $\bigcup_n d_n \in I$. For each $n \in \omega$ let $\pi_n \colon d_n \to d_{n+1}$ be a bijection. Then $\pi \cup \bigcup_n \pi_n$ is a permutation of a set in I disjoint from a. Let γ be the permutation of X extending $\pi \cup \bigcup_n \pi_n$ by the identity outside of its domain. Then $\gamma \in \mathrm{pstab}(a)$ and $c \subset \gamma \cdot (a \cup b)$ as desired. $\square$

Example 3.12. Let X be an uncountable completely metrizable space, let Γ be its self-homeomorphism group acting by application, and let I be the ideal generated by countable closed subsets of X. The ideal does not have cofinal orbits: if $b \in I$ is a closed set, it has countable Cantor–Bendixson rank α, and its orbit consists of sets of rank α only. However, there are sets in I of higher Cantor–Bendixson rank.

Now suppose that the space X is the Euclidean space. The axiom of well-ordered (in fact, ω_1) choice fails in the associated permutation model. Consider the sequence $\langle C_\alpha \colon \alpha \in \omega_1\rangle$ where C_α is the set of closed countable subsets of X of Cantor–Bendixson rank α. It is clear that the sequence is invariant under the group action and therefore belongs to the permutation model. I claim that the sequence has no selector in the permutation model. Suppose towards a contradiction that there is such a selector $s = \langle a_\alpha \colon \alpha \in \omega_1\rangle$, and let $b \subset X$ be a countable closed set such that $\mathrm{pstab}(b) \subset \mathrm{stab}(s)$. Select a countable ordinal α greater than the Cantor–Bendixson rank of b, and a point $x \in a_\alpha \setminus b$. The homogeneity properties of the Euclidean space imply that the $\mathrm{pstab}(b)$-orbit of x contains a nonempty open set; in particular, it is uncountable. However, it should be a subset of the countable set a_α, a contradiction.

3.1 The axiom of dependent choices

The axiom of dependent choices is one of the most commonly considered fragments of AC, partly because it allows an uneventful development of mathematical analysis and descriptive set theory. It is implied by well-ordered choice [2] and it implies countable choice.

Definition 3.13. [1, Form 43] The *axiom of dependent choices*, DC, is the statement that in every partial order there is either a minimal element or an infinite strictly descending sequence.

I define a dynamical game which is related to DC. A gameless version of this criterion has been considered by Karagila and Schilhan [4].

Definition 3.14. Let Γ act on X, let I be an invariant ideal on X. Consider the *DC game*: Players I and II alternate, Player I plays sets $a_n \in I$, Player II answers with $\gamma_n \in \Gamma$. The only rule for Player II is that $\gamma_0 = 1$ and each γ_n must fix all elements of $\bigcup_{m\in n} \gamma_m \cdot a_m$. Player II wins if $\bigcup_{n\in\omega} \gamma_n \cdot a_n \in I$.

Theorem 3.15. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal.*

1. *If $W[[X]]$ fails the axiom of dependent choices then Player I has a winning strategy in the DC game.*
2. *if X is a Polish space and I is generated by an analytic set of closed subsets of X, then the DC game is determined.*

Proof. For (1), suppose that $\langle P, \leq\rangle$ is a partially ordered set in the permutation model such that for every $p \in P$ there is $q \in P$ with $q < p$ and there is no infinite descending sequence in the permutation model. Let Player I on the side choose an element $p_0 \in P$, and play $a_0 \in I$ such that $\mathrm{pstab}(a_0) \subset \mathrm{stab}(P, \leq, p)$. Let $\gamma_0 \in \Gamma$ be Player II's answer; to simplify the notation, assume that $\gamma_0 = 1$. Now, as the game proceeds, at round $n+1$ Player I will on the side choose elements $p_{n+1} \leq \gamma_n \cdot p_n$, and play a set $a_{n+1} \in I$ such that $\mathrm{pstab}(a_{n+1}) \subseteq \mathrm{stab}(p_{n+1})$. By induction on $n \in \omega$ one can then prove that, setting $q_n = \gamma_n \cdot p_n$, $\mathrm{pstab}(\gamma_n \cdot a_n) \subset \mathrm{stab}(q_n)$ and $q_{n+1} \leq q_n$. This strategy must end in Player I's victory, since if the set $b = \bigcup_n \gamma_n \cdot a_n$ belonged to I, then the elements of $\mathrm{pstab}(b)$ fix all conditions

q_n, so even their sequence $\langle q_n : n \in \omega \rangle$. Thus $\langle q_n : n \in \omega \rangle$ would belong to the permutation model, contradicting the assumption that P witnesses the failure of DC there.

For (2), let $\mathcal{B}$ be a countable base of the topology on X, let $f : \omega^\omega \to \mathcal{P}(\mathcal{B})$ be a continuous function such that I is generated by the sets $X \setminus \bigcup\{O \in \mathcal{B} : y(O) = 1\}$ as y ranges over all elements of ω^ω. Consider the unraveled version of the DC game in which Player II produces in addition an element $y \in \omega^\omega$ such that

- for every $n \in \omega$, $\gamma_n \cdot a_n \cap \bigcup\{O \in \mathcal{B} : y(O) = 1\} = 0$;
- for every $m \in \omega$, $y(m)$ can be played only at some round n where $n > m, y(m)$.

It is not difficult to see that the payoff set for Player II is Borel in the (large) space of all possible plays: the set of plays in which Player II specifies all entries of y is G_δ, and the failure of the first item is relatively open in this G_δ set by the continuity of the function f. Thus, the unraveled version of the DC game is determined by Borel determinacy [5]. Clearly, the unraveled version is more difficult for Player II. Thus, to prove the determinacy of the original game, it will be enough to show that if Player I has a winning strategy σ in the unraveled game, then he has a winning strategy in the original game.

To this and, note that during the first n moves of the unraveled game, Player II has only finitely many options for playing the entries of the point $y \in \omega^\omega$ due to the second item above. Thus, at round n of the original game, Player I can consult the strategy σ, take all the finitely many sets σ advises him to play against the counterplay $\langle \gamma_i : i \in n \rangle$ augmented with all possible combinations of entries for y in the unraveled game, and play the union a_n of all such sets, which remains in the ideal I. In this way, at every round n the following will be satisfied:

- for every sequence of entries t for y before round n obeying the second item above, $u = \langle \gamma_i : i \in n, t \rangle$ is a legal counterplay of Player II against the strategy σ, against which σ produces sets $\langle a_i^u : i \in n \rangle$ and for each $i \in n$, $a_i^u \subseteq a_i$.

I claim that with this strategy in the original game, Player I must win. Suppose towards a contradiction that $\langle \gamma_n : n \in \omega \rangle$ is a counterplay against this strategy in which Player II won; so $\bigcup_n A_n \in I$. Then there must be a point $y \in \omega^\omega$ such that $\bigcup_n A_n$ is a set disjoint from $\bigcup\{O \in \mathcal{B} : f(y)(n) = 1\}$. Consider the counterplay against σ in the unraveled game in which Player II plays $\langle \gamma_n : n \in \omega \rangle$ and on the side produces the point y in any way consistent with the second item above. The outcome set in this play will be smaller that $\bigcup_n A_n$, so this play would result in a victory for Player II, contradicting the assumption that σ was a winning strategy for Player I. $\square$

The concept of DC game calls for an army of examples.

Example 3.16. If $I = \bigcup_n I_n$ where each collection I_n is invariant, closed under subset, and not equal to I, then Player I has a winning strategy in the DC game. Player I in this case can completely ignore moves of Player II and just play sets $a_n \in I \setminus I_n$.

Such will be the case for the ideal of finite sets regardless of the action or the underlying infinite set.

Example 3.17. Let $X = [0, 1]$, let Γ be the group of orientation-preserving self-homeomorphisms of X acting by application, and let I be the ideal of countable closed sets. Then Player I has a winning strategy in the DC game: he can play finite sets $a_n \subset X$ such that between any two elements of the finite set $\bigcup_{m \in n} \gamma_m \cdot a_m$ there is an element of the set a_n. The outcome of the play will be a subset of $[0, 1]$ which is dense in itself, and such sets have uncountable closure. The failure of DC in the permutation model is again immediate, as there X will be a dense linear order which contains no order-isomorphic copy of the rationals.

Examples 3.12, 4.4, and 3.17 show that in the permutation model of the above example axiom of countable choice holds, but both DC and axiom of well-ordered choice fail.

Example 3.18. If I is a σ-ideal then Player II has a winning strategy in the DC game. In this case, Player II can ignore Player I entirely and play any group elements whatsoever.

Definition 3.19. Let X be a Polish space.

1. X is *K-elastic* if it is locally compact and for every nonempty open set $O \subset X$ with compact closure and every compact set $K \subset X$, there is a self-homeomorphism $h \colon X \to X$ such that $K \subset h''O$, and the set $\{x \in X : x \neq h(x)\}$ has compact closure.
2. X is *locally K-elastic* if it has a basis consisting of K-elastic sets.

For example, every open ball in a Euclidean space is K-elastic, therefore the Euclidean space itself is locally K-elastic.

Example 3.20. Let X be a Polish locally K-elastic space without isolated points. Let Γ be the group of (compactly supported) self-homeomorphisms of X acting by application, and let I be the ideal of nowhere dense subsets of X. Player II has a winning strategy in the DC game.

Proof. Let $\{R_n : n \in \omega\}$ enumerate a basis of the space X. Before round n of the game, Player I will have created nowhere dense subsets $C_m \subseteq X$ and Player II will have indicated selfhomeomorphisms $h_m \colon X \to X$ such that h_m fixes the set $\bigcup_{k \in m} h_k'' C_k$ pointwise. In addition, before round n of the game, Player II will have created, for all $m \in n$, open sets $P_{mn} \subset X$ with compact closure and open elastic sets Q_{mn} such that

- the closure of P_{mn} is a subset of Q_{mn};
- if $m \in n_0 \in n_1$ then $Q_{mn_1} \subseteq Q_{mn_0}$ and $P_{mn_0} \subseteq P_{mn_1}$;
- the sets Q_{mn} for $m \in n$ are pairwise disjoint, and each is disjoint from $\bigcup_{m\in n} h''_m C_m$;
- R_n has nonempty intersection with $\bigcup_{m\leq n} P_{mn}$.

If this is done, then in the end the set $\bigcup_{mn} P_{mn}$ is open dense, and it is disjoint from the set $\bigcup_m h''_m C_m$. This will confirm the victory of Player II in the DC game.

To show how Player II handles the round $n \in \omega$, let Player I choose a nowhere dense set $C_n \subset X$. For every number $m \in n$ such that $R_n \cap Q_{mn} \neq 0$, expand P_{mn} to $P_{m,n+1}$ which is still an open subset of Q_{mn} with compact closure and has nonempty intersection with R_n; for other numbers $m \in n$ let $P_{m,n+1} = P_{mn}$. For each $m \in n$, use the elasticity assumption to find elastic open sets $O_m \subset Q_m n$ with compact closure disjoint from the set C_n. Find self-homeomorphisms $k_m\colon Q_{mn} \to Q_{mn}$ such that $P_{m,n+1} \subseteq k''_m O_m$ and k_m is equal to identity off a compact subset of Q_{mn}. Let $Q_{m,n+1} = k''_m O_m$ (note that this is an elastic set because it is homeomorphic to the elastic set O_m). Extend each k_m to a self-homeomorphism of X by defining it to be an identity off Q_{mn} (note that these are indeed self-homeomorphisms of X as k_m is the identity off a compact subset of Q_{mn}) and let h_n be the composition of k_m for $m \in n$ (note that these homeomorphisms have pairwise disjoint supports, therefore commute and it does not matter in which order one takes the composition).This part of the construction is trivial when $n = 0$; thus, $h_n = 0$.

Finally, find a set $Q_{n,n+1}$ which is elastic and disjoint from $\bigcup_{m\in n} Q_{m,n+1}$ and $\bigcup_{m\leq n} h''_m C_m$; if the set R_n was disjoint from $\bigcup_{m\in n} Q_{mn}$, take care to choose $Q_{n,n+1}$ as a subset of R_n. Let $P_{n,n+1} \subset Q_{n,n+1}$ be any open set whose closure is compact and a subset of $Q_{n,n+1}$. The induction hypotheses are satisfied. □

The final class of examples is obtained by the cofinal orbits.

Theorem 3.21. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal. If the ideal has cofinal orbits then Player II has a winning strategy in the DC game.*

Proof. The proof depends on two simple properties of cofinal orbits:

Claim 3.22. *The set $\{\langle a, b\rangle\colon a, b \in I$ and b is a-large$\}$ is Γ-invariant.*

Proof. Let b be a-large and $\delta \in \Gamma$. To show that $\delta \cdot b$ is $\delta \cdot a$-large, let $c \in I$ be arbitrary. Since b is a-large, there is $\gamma \in \mathrm{pstab}(a)$ such that $\gamma \cdot \delta^{-1} \cdot c \subset b$. Multiplying by δ on both sides, we get $\delta\gamma\delta^{-1} \cdot c \subset \delta \cdot b$. Since $\gamma \in \mathrm{pstab}(a)$, $\delta\gamma\delta^{-1} \in \mathrm{pstab}(\delta \cdot a)$ holds and the claim follows. □

Claim 3.23. *Let b be a-large and $c \in I$. Then there is $\gamma \in \mathrm{pstab}(a)$ such that $\gamma \cdot c \subseteq b$ and b is $\gamma \cdot c$-large.*

Proof. Let $d \in I$ be c-large; without loss $c \subset d$. Use the largeness assumption on b to find a group element $\gamma \in \text{pstab}(a)$ such that $\gamma \cdot d \subset b$. By Claim 3.22, $\gamma \cdot d$ is $\gamma \cdot c$-large; since b is a superset of $\gamma \cdot d$, it follows that b is $\gamma \cdot c$-large as desired. □

To describe the winning strategy for Player II, let Player I indicate a_0; Player II answers with $\gamma_0 = 1$. Find a set $b \in I$ which is a_0-large. The strategy will play in such a way that, writing $c_n = \bigcup_{m \in n} \gamma_m \cdot a_m$, we will have that $c_n \subset b$ and b is d_n-large. Such a strategy will lead to Player II's victory, since in the end, $\bigcup_n \gamma_n \cdot a_n \subseteq b$, so the infinite union is in the ideal I.

To see how round $n \geq 1$ is played, suppose that Player I indicates a set a_n. Adding the set c_n to it if necessary, without loss we may assume that $c_n \subseteq a_n$ holds. By Claim 3.23, there is an element $\gamma_n \in \text{pstab}(c_n)$ such that $\gamma_n \cdot a_n \subset b$ and b is $\gamma_n \cdot a_n$-large. That will be Player II's move; the play proceeds to the next round. □

Example 3.24. [4] Let X be a countable dense linear ordering without endpoints, let Γ be the group of all its automorphisms acting by application, and let I be the ideal of nowhere dense sets. Then in the associated permutation model, DC holds–in fact, by Example 3.6, the dynamical ideal has cofinal orbits, so the axiom of well-ordered choice holds in the permutation model.

4 The axiom of countable choice

One obvious and commonly used weakening of well-ordered choice is the axiom of countable choice.

Definition 4.1. [1, Form 8] The *axiom of countable choice* is the statement that every countable family of nonempty sets has a choice function.

This fragment of AC has a clean counterpart among properties of dynamical ideals:

Definition 4.2. Let $\Gamma \curvearrowright X, I$ be a dynamical ideal. Say that the dynamical ideal is *σ-complete* if for every set $a \in I$ and every countable sequence $\langle b_n : n \in \omega \rangle$ of sets in I there are group elements $\gamma_n \in \text{pstab}(a)$ such that $\bigcup_n \gamma_n \cdot b_n \in I$.

Theorem 4.3. *Let $\Gamma \curvearrowright X, I$ be a dynamical ideal.*

1. *If the dynamical ideal is σ-complete then the associated permutation model satisfies the countable axiom of choice;*

2. *if the dynamical ideal is definably closed and the associated permutation model satisfies the countable axiom of choice, then the dynamical ideal is σ-complete.*

Proof. For (1), assume that $A = \langle A_n \colon n \in \omega \rangle$ is a countable sequence of nonempty sets in the permutation model. Let $a \in I$ be such that $\text{pstab}(a) \subset \text{stab}(A)$. Outside of the permutation model, for each $n \in \omega$ pick a set $B_n \in A_n$ and a set $b_n \in I$ such that $\text{pstab}(b_n) \subset \text{stab}(B_n)$. Use the countable completeness assumption on the dynamical ideal to find elements $\gamma_n \in \text{pstab}(a)$ such that $c = \bigcup_n \gamma_n \cdot b_n \in I$. Now, consider the sequence $C = \langle \gamma_n \cdot B_n \colon n \in \omega \rangle$. Since each γ_n belongs to $\text{pstab}(a)$, it is still the case that $\gamma_n \cdot B_n \in A_n$. At the same time, for each $n \in \omega$ $\text{pstab}(\gamma \cdot b_n) \subset \text{stab}(\gamma_n \cdot B)$ holds by Proposition 2.10, so $\text{pstab}(c) \subset \text{stab}(C)$, C belongs to the permutation model, and it witnesses the axiom of countable choice instance for the collection A.

For (2), suppose that $a, b_n \colon n \in \omega$ witness the failure of σ-closure. Let A_n be the set of all well-orderings on sets of the form $\gamma \cdot b_n$ where γ ranges over all elements of $\text{pstab}(a)$, and let $A = \langle A_n \colon n \in \omega \rangle$. Note that for every set of the form $\gamma \cdot b_n \in I$, clearly $\text{pstab}(\gamma \cdot b_n)$ fixes it pointwise, and therefore all relations on $\gamma \cdot b_n$ in $V[[X]]$, including well-orderings, belong to $W[[X]]$. Since $\text{pstab}(a) \subset \text{stab}(A_n)$ holds for every $n \in \omega$, it must be the case that $A \in W[[X]]$. It will be enough to show that A has no selector in $W[[X]]$.

Assume towards a contradiction that $C = \langle \gamma_n \cdot b_n, \leq_n \colon n \in \omega \rangle$ is a selector in $W[[X]]$, and let $c \in I$ be a set such that $\text{pstab}(c) \subset \text{stab}(C)$. By the solidity assumption on the ideal, I may assume that c is solid. By the initial assumptions on the sets b_n, it must be the case that $\bigcup_n \gamma_n b_n \notin I$. In particular, there is a number $n \in \omega$ and a point $x \in \gamma_n \cdot b_n$ which does not belong to c. Use the solidity of c to find an element $\delta \in \text{pstab}(c)$ such that $\delta \cdot x \neq x$. Now note that δ fixes $\gamma_n \cdot b_n$ as well as the well-ordering $\leq_n$. By a transfinite induction along $\leq_n$, it is easy to show that δ fixes the set $\gamma_n \cdot b_n$ pointwise. This contradicts the assertion that δ moves the point x. $\square$

Example 4.4. Let X be the closed unit interval $[0, 1]$, let Γ be the group of orientation preserving self-homeomorphisms of X acting by application, and let I be the ideal of closed countable subsets of X. Then the dynamical ideal $\Gamma \curvearrowright X, I$ is σ-complete.

Compare this with Example 3.12 and 3.17: the associated permutation model satisfies the axiom of countable choice, but not the axiom of well-ordered choice or dependent choice. This example was improved by Justin Young who showed that its conclusion holds for X an arbitrary Euclidean space.

Proof. Let $a, b_n \colon n \in \omega$ be countable closed subsets of X. Let J be the set of all inclusion-maximal open intervals disjoint from a. For each interval $j \in J$ and every $n \in \omega$, find an open interval $j(n) \subset j$ which is disjoint from b_n and its endpoints are still in j, and find an oreintation-preserving self-homemorphism $h_{jn} \colon j \to j$ such that $j \setminus h_{jn}(j(n))$ consists of one interval at each end of j, both of length less than 2^{-n}. For each number $n \in \omega$, let $\gamma_n = \text{id} \restriction a \cup \bigcup_{j \in J} h_{jn}$; I will show that these group elements work as required.

It will be enough to show that $c = a \cup \bigcup_n \gamma_n \cdot b_n$ is a closed set, because it is clearly countable. Suppose then that $\langle x_k \colon k \in \omega \rangle$ is a converging sequence of points in c with limit x, and work to show that $x \in c$. There are two cases.

Case 1. No interval $j \in J$ contains a tail of the sequence. In this case, the limit x must be in the set a, completing the proof.
Case 2. Case 1 fails. Let $j \in J$ be the unique interval containing a tail of the sequence. If the sequence has nonempty intersection with only fiinitely many sets $\gamma_n \cdot b_n$, then the limit must belong to one of those finitely many sets, completig the proof. If, on the other hand, the sequence has nonempty intersection with infinitely many sets $\gamma_n \cdot b_n$, then by the choice of the homeomorphisms h_{jn} it must be the case that the limit is equal to one of the endpoints of j. Both of these endpoints are in the set a. The example has been demonstrated. □

Example 4.5. [1, Model N23] Let $\langle X, \leq \rangle$ be a countable dense linear order without endpoints, let Γ be its automorphism group, and let I be the ideal of those subsets of X which are well-ordered by $\leq$. The dynamical ideal $\Gamma \curvearrowright X, I$ is σ-complete.

Proof. Let $a \in I$ and b_n for $n \in \omega$ be sets in the ideal; I must produce automorphisms $\gamma_n \in \mathrm{pstab}(a)$ for all $n \in \omega$ such that $a \cup \bigcup_n \gamma_n \cdot b_n \in I$ holds. Let U be the set of all open intervals of X which are disjoint from a and maximal such. Inside each interval $i \in U$ pick points x_{in} for $n \in \omega$ such that they form an increasing sequence cofinal in i, and automorphisms γ_{in} equal to identity on $X \setminus i$ such that $\gamma_{in}(\min(b_n \cap i)) \geq x_{in}$ whenever $b_n \cap i \neq 0$. Let γ_n be the composition of all γ_{in} for $i \in U$; this makes sense since the automorphisms γ_{in} for $i \in U$ have pairwise disjoint supports. I claim that the set $c = a \cup \bigcup_n \gamma_n \cdot b_n$ is well-ordered as desired.

To see this, suppose towards a contradiction that $\langle x_m : m \in \omega \rangle$ is a strictly decreasing sequence in c. By the well-ordering assumption on a, the points $y_m = \min\{z \in a : z \geq x_m\}$ have to stabilize at some point; the tail of the sequence then belongs to the same open interval $i \in U$ which borders on its right end at the eventually stable value of the points y_m. By the cofinal choice of the points x_{in} in i, there is an $n \in \omega$ such that a tail of the sequence consists of elements of i which are smaller than x_{in}. However, this tail is then a subset of the set $\bigcup_{k \in n} \gamma_k \cdot b_k$. Since this is a finite union of well-ordered sets, it is itself well-ordered, and it cannot contain an infinite strictly decreasing sequence. A contradiction. □

Example 4.6. [1, Model N10] Let $\langle X, \leq \rangle$ be a countable dense linear order without endpoints, let Γ be its automorphism group, and let I be the ideal of those subsets of X which are well-ordered by $\leq$, bounded in X, and bounded below every element of X. The dynamical ideal $\Gamma \curvearrowright X, I$ is σ-complete. The status of countable choice has apparently not been known in the associated permutation model.

Proof. Let $a \in I$ and b_n for $n \in \omega$ be sets in the ideal; I must produce automorphisms $\gamma_n \in \mathrm{pstab}(a)$ for all $n \in \omega$ such that $a \cup \bigcup_n \gamma_n \cdot b_n \in I$ holds. Let U be the set of all open intervals of X which are disjoint from a and maximal such. Note that the right endpoint of each interval $i \in u$ belongs to a, except for the rightmost, unbounded interval in U; in any case, the sets b_n are all bounded in

i. Inside each interval $i \in U$ pick points x_{in} for $n \in \omega$ such that they form an increasing sequence which is bounded and does not have a supremum in X, pick points $y_{in} \in i$ such that $b_n \cap i$ is bounded by y_{in}, and pick automorphisms γ_{in} equal to identity on $X \setminus i$ such that $\gamma_{in}(\min(b_n \cap i)) \geq x_{in}$ whenever $b_n \cap i \neq 0$ and $\gamma_{in}(y_{in}) = x_{in+1}$. Let γ_n be the composition of all γ_{in} for $i \in U$; this makes sense since the automorphisms γ_{in} for $i \in U$ have pairwise disjoint supports. I claim that the set $c = a \cup \bigcup_n \gamma_n \cdot b_n$ is well-ordered and bounded below each point as desired.

For the well-ordering, suppose towards a contradiction that $\langle x_m \colon m \in \omega \rangle$ is a strictly decreasing sequence in c. By the well-ordering assumption on a, the points $y_m = \min\{z \in a \colon z \geq x_m\}$ have to stabilize at some point; the tail of the sequence then belongs to the same open interval $i \in U$ which borders on its right end at the eventually stable value of the points y_m. By the choice of the points x_{in} in i, there is an $n \in \omega$ such that a tail of the sequence consists of elements of i which are smaller than x_{in}. However, this tail is then a subset of the set $\bigcup_{k \in n} \gamma_k \cdot b_k$. Since this is a finite union of well-ordered sets, it is itself well-ordered, and it cannot contain an infinite strictly decreasing sequence. A contradiction.

For the boundedness, let $z \in X$ be an arbitrary point. Since a is bounded below z, z is an internal point or the right-hand endpoint of some interval $i \in U$. If z is greater than all points x_{in} for $n \in \omega$, then as these points do not have a supremum in X, the sequence $\langle x_{in} \colon n \in \omega \rangle$ is bounded below z, and so is the set c. If there is a number n such that $z \leq x_{in}$, then $c \cap i$ below z is the union of the sets $\gamma_k \cdot b_k$ for $k \in n$, each of which is bounded below z, so c is bounded below z again. The proof is complete. $\square$

Example 4.7. [1, Model N21] Let κ be an uncountable regular cardinal. Let X be a κ-branching tree, let Γ be the group of all automorphisms of X acting by application, and let I be the ideal generated by well-founded subtrees of X of cardinality smaller than κ. Then $\Gamma \curvearrowright X, I$ is a σ-complete dynamical ideal. In the permutation model, X witnesses the failure of DC, while the axiom of countable choice holds.

Proof. Let a, b_n for $n \in \omega$ be sets in the ideal I; increasing them if necessary, I may assume they are all well-founded trees. For every node $x \in a$ and every $n \in \omega$, write $c(n, x) = \{y \in b_n \setminus a \colon y \text{ is an immediate successor of } x\}$. For each $n \in \omega$ choose $\gamma_n \in \operatorname{pstab}(a)$ to be an automorphism of X such that for every node $x \in a$, the sets $\gamma \cdot c(n, x)$ for $n \in \omega$ are pairwise disjoint. This is possible since the sets $c(n, x)$ have cardinality smaller than κ while the set of immediate successors of x has cardinality κ. I claim that the tree $a \cup \bigcup_n \gamma_n \cdot b_n \subset X$ is well-founded. Indeed, any putative infinite branch v through it either must be a subset of a (an impossibility by the well-foundedness assumption on a) or the set $v \cap a$ is finite and contains a smallest element, call it x. If y is the immediate successor of x in v, then there is a unique number $n \in \omega$ such that $y \in \gamma_n \cdot b_n$ by the choice of the automorphisms γ_n. This means that the branch v must be a subset of $\gamma_n \cdot b_n$, contradicting the assumed well-foundedness of b_n. $\square$

4.1 Simplicity

One fragment of axiom of choice for which an attractive dynamical criterion can be found is well-orderable closure.

Definition 4.8. [1, Form 231] *Well-orderable closure* is the statement that for every well-orderable set A consisting of well-orderable sets, $\bigcup A$ is well-orderable.

This is often verified in permutation models as a consequence of well-ordered axiom of choice (Section 3). In situations where this is not available, the following elegant criterion on dynamical ideals is helpful.

Definition 4.9. A dynamical ideal $\Gamma \curvearrowright X, I$ is *simple* if for every $a \subseteq b \in I$, the only normal subgroup of $\mathrm{pstab}(a)$ containing $\mathrm{pstab}(b)$ is $\mathrm{pstab}(a)$ itself.

Theorem 4.10. *Suppose that $\Gamma \curvearrowright X, I$ is a simple dynamical ideal. Then in the associated permutation model, a union of well-orderable set of well-orderable sets is well-orderable.*

Proof. Suppose that $\Gamma \curvearrowright X, I$ is simple. Let A be a well-orderable set consisting of well-orderable sets in the permutation model. By Proposition 2.11, there is a set $a \in I$ such that $\mathrm{pstab}(a) \subset \mathrm{pstab}(A)$. It will be enough to show that $\mathrm{pstab}(a) \subset \mathrm{pstab}(\bigcup A)$. In other words, given a set $B \in A$ and $C \in B$, I must prove that $\mathrm{pstab}(a) \subset \mathrm{stab}(C)$. Let $\delta \in \mathrm{pstab}(a)$ be any element and work to show that $\delta \cdot C = C$.

To prove this, use Proposition 2.11 again to find a set $b \in I$ such that $a \subseteq b$ and $\mathrm{pstab}(b) \subset \mathrm{pstab}(B)$. By the simplicity assumption, there is a number $n \in \omega$ and elements $\gamma_m \in \mathrm{pstab}(a)$ and $\delta_m \in \mathrm{pstab}(b)$ for $m \in n$ such that $\delta = \prod_{m\in n} \gamma_m^{-1}\delta_m\gamma_m$. It will be enough to show that for every $m \in n$, $\gamma_m^{-1}\delta_m\gamma_m \cdot C = C$. Towards this, observe that δ_m fixes all elements of B' in particular, it fixes $\gamma_m \cdot C$. The proof is complete. $\square$

The simplicity is an attractive concept with an attractive conclusion. However, as in the case of simplicity of groups, it is not always easy to check. The first class comes from model theory.

Definition 4.11. Let $\mathcal{F}$ be a Fraisse class with disjoint amalgamation. Say that $\mathcal{F}$ has *hereditary canonical amalgamation* if there is a function C which to each ordered pair of $\mathcal{F}$-structures $\langle A, B\rangle$ in amalgamation position assigns a minimal disjoint amalgamation so that

1. C is invariant under isomorphism in both variables: if $\phi\colon A \to A'$ and $\psi\colon B \to B'$ are isomorphisms which agree on $\mathrm{dom}(A) \cap \mathrm{dom}(B)$, then there is an isomorphism of $C(A, B)$ to $C(A', B')$ extending $\phi \cup \psi$;
2. C is hereditary for substructures in the left variable: if A' is an induced substructure of A such that $\mathrm{dom}(A) \cap \mathrm{dom}(B) \subset \mathrm{dom}(A')$, then there is an isomorphism between$C(A', B)$ and the algebraic closure of $\mathrm{dom}(A') \cup \mathrm{dom}(B)$ in $C(A, B)$ which is the identity on $\mathrm{dom}(A') \cup \mathrm{dom}(B)$.

Compared to the canonical amalgamation of Paolini and Shelah [6], the hereditary clause is added.

Theorem 4.12. *Let $\mathcal{F}$ be a Fraisse class with hereditary canonical disjoint amalgamation, let X be its limit structure, let Γ be the group of all automorphisms of X acting by application, and let I be the ideal of finite sets on X. Then $\Gamma \curvearrowright X, I$ is a simple dynamical ideal.*

Proof. As a matter of terminology, say that a finite set $a \subset X$ is *algebraically closed* if it is closed under all functions of X; $\mathrm{acl}(a)$ is the smallest algebraically closed subset of X containing a. Let C be the canonical amalgamation function on $\mathcal{F}$. Consider Γ as a Polish group with the usual automorphism group topology. Let $a \subseteq b$ be finite subsets of X; I must prove that within $\mathrm{pstab}(a)$, the normal subgroup Δ generated by $\mathrm{pstab}(b)$ is equal to $\mathrm{pstab}(a)$ itself. I may assume that the sets a, b are both algebraically closed. Since Δ is a group generated by an open subset of $\mathrm{pstab}(a)$, it is open, therefore closed. Thus, it is enough to show that Δ is dense in $\mathrm{pstab}(a)$.

To this end, let $\pi\colon c \to d$ be any morphism between two finite subsets of X, both including a as a subset, both algebraically closed, and such that $\pi \restriction a = \mathrm{id}$. I must find an element of Δ extending π. To do this, let $e = \mathrm{acl}(b \cup c \cup d)$. By the saturation properties of X, there must be a finite set $f \subset X$ such that

- $f \cap e = a$;
- there is an isomorphism $\theta\colon e \to f$ which is identity on a;
- $X \restriction \mathrm{acl}(e \cup f)$ is isomorphic to $C(X \restriction e, X \restriction f)$.

Let $\delta \in \mathrm{pstab}(a)$ be an automorphism extending θ. By the heredity and invariance properties of C, there is a morphism between the algebraic closures of $c \cup f$ and $d \cup f$ extending the function $\pi \cup \mathrm{id}_f$. Let $\gamma \in \mathrm{pstab}(f)$ be any automorphism of X extending it. Now, $\gamma \in \mathrm{pstab}(f)$ implies that $\delta^{-1}\gamma\delta \in \mathrm{pstab}(e) \subseteq \mathrm{pstab}(b)$. It follows that $\gamma = \delta(\delta^{-1}\gamma\delta)\delta^{-1}$ is an element of Δ extending π as desired. $\square$

Nearly all permutation models associated with limits of Fraisse structures allow of a much more detailed analysis than just simplicity; this will appear in forthcoming work. For now, I will list only two very special examples.

Example 4.13. Let $\mathcal{F}$ be the class of structures with no relations and no functions; hereditary canonical amalgamation obviously holds. Let X be its limit, the countable set with no structure. The associated model is [1, Model N1]. In this model, $[X]^{<\aleph_0}$ is a set of well-orderable sets without a selector, showing that the conclusion of Theorem 4.10 cannot be strengthened.

Example 4.14. The class of rational ultrametric spaces has hereditary canonical amalgamation. Given any two finite rational ultrametric spaces A, B in the amalgamation position with nonempty intersection, with their metrics denoted by d_A and d_B respectively, define the metric d on $\mathrm{dom}(A) \cup \mathrm{dom}(B)$ by letting $d_A \cup d_B \subset d$, and for points $x \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$ and $y \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$, the distance $d(x, y)$ is defined in two cases. If there is a point $z \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$ such that $d_A(x, z) \neq d_B(z, y)$, then set $d(x, y) = \{d_A(x, z), d_B(z, y)\}$, and if there is no such z, then let $d(x, y) = \min\{d_A(z, y) \colon z \in \mathrm{dom}(A) \cap \mathrm{dom}(B)\}$. It must be verified that the definition is sound and yields an ultrametric space. It is clear that such amalgamation is hereditary and canonical.

Example 4.15. The class of vector spaces over a fixed finite field has hereditary canonical amalgamation: $C(A, B)$ will be the vector space with the underlying set $\mathrm{dom}(A) \times \mathrm{dom}(B)$ modulo the equivalence relation E defined by $\langle x, y\rangle\ E\ \langle x', y, \rangle$ if $x - x'$ and $y' - y$ are identical elements of $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. Addition and scalar multiplication is defined coordinatewise.

Example 4.16. Let $\mathcal{F}$ be the set of finite structures with a function (coded as a relation) which from each unordered quadruple selects an unordered pair. Then $\mathcal{F}$ does not have canonical amalgamation: there is no isomorphism-invariant way to amalgamate disjoint sets A and B such that A has three elements and B has one.

Finally, the collection of relational Fraisse classes with hereditary canonical disjoint amalgamation is closed under superposition, which yields interesting classes such as linearly ordered rational ultrametric spaces and the like.

Let me now turn to more set-theoretic examples of simple dynamical ideals.

Example 4.17. Let X be an ultrahomogeneous linear order , let Γ be the group of all automorphisms of X acting by application, and let I be the ideal of nowhere dense sets. Then $\Gamma \curvearrowright X, I$ is a simple dynamical ideal.

Proof. The argument uses the following key claim:

Claim 4.18. *Let $a \subset X$ be a nowhere dense set, and let $\gamma \in \Gamma$ be an automorphism whose orbits are both cofinal and coinitial in X. Then there are $\gamma_i \in \mathrm{pstab}(a)$ and $\delta_i \in \Gamma$ for $i \in 4$ such that $\gamma = \prod_{i \in 4} \delta_i^{-1} \gamma_i \delta_i$.*

Proof. Without loss of generality assume that for all $x \in X$, $\gamma \cdot x > x$; the proof for $\gamma \cdot x < x$ is symmetric. Let $\langle x_n \colon n \in \mathbb{Z}\rangle$ be an increasing enumeration of one of the orbits. The ultrahomogeneity assumption shows that there are automorphisms $\alpha_0, \alpha_1 \in \Gamma$ such that $a_0 = \alpha_0 \cdot a \subset \bigcup_n (x_{4n-1}, x_{4n+1})$ and $a_1 = \alpha_1 \cdot a \subset \bigcup_n (x_{4n+1}, x_{4n+3})$. I will express γ as a composition of four automorphisms in $\mathrm{pstab}(a_0) \cup \mathrm{pstab}(a_1)$. Since $\mathrm{pstab}(a_0) = \alpha_0 \mathrm{pstab}(a) \alpha_0^{-1}$ and $\mathrm{pstab}(a_1) = \alpha_1 \mathrm{pstab}(a) \alpha_1^{-1}$, this will conclude the proof.

First, use the ultrahomogeneity assumption on X to argue that there is an automorphism $\delta_0 \in \mathrm{pstab}(a_0)$ fixing x_{4n} and such that $\delta_0(x_{4n+1}) = x_{4n+2}$ and $\delta_0(x_{4n+2}) = x_{4n+3}$ for all $n \in \mathbb{Z}$. Similarly, let $\delta_1 \in \mathrm{pstab}(a_1)$ be an automorphism fixing x_{4n+2} and x_{4n+3} and such that $\delta_1(\delta_0(x_{4n+3})) = x_{4n+4}$ and $\delta_1(x_{4n}) = x_{4n+1}$ for all $n \in \omega$. Let $\beta = \delta_1 \delta_0$ and observe that $\{x_n \colon n \in \mathbb{Z}\}$ is an orbit of β. Now, let $\delta_2 \in \mathrm{pstab}(a_0)$ be an automorphism which fixes the interval (x_{4n-1}, x_{4n+1}) pointwise and on (x_{4n+1}, x_{4n+3}) it is equal to $\gamma\beta^{-1}$, this for all $n \in \mathbb{Z}$. Similarly, let $\delta_3 \in \mathrm{pstab}(a_1)$ be an automorphism which fixes the interval (x_{4n+1}, x_{4n+3}) pointwise and on (x_{4n-1}, x_{4n+1}) it is equal to $\gamma\beta^{-1}$, this for all $n \in \mathbb{Z}$. Note that δ_2 and δ_3 have disjoint supports, therefore they commute and $\delta_2\delta_3 = \delta_3\delta_2 = \gamma\beta^{-1}$. In consequence, $\gamma = \delta_3\delta_2\delta_1\delta_0$ as desired. $\square$

Now, suppose that $a \subset b$ are sets in the ideal I, let $\gamma \in \mathrm{pstab}(a)$, and work to show that γ belongs to the normal subgroup of $\mathrm{pstab}(a)$ generated by $\mathrm{pstab}(b)$. Let U be the set of all intervals of X in which γ has an orbit which is both coinitial and cofinal in U. Note that such intervals do not have smallest or largest elements, therefore they are ultrahomogeneous themselves, and every orbit is both coinitial and cofinal in them. Note also that $a \subset X \setminus \bigcup U$ and γ fixes all elements of $X \setminus \bigcup U$. Now, note that each interval $u \in U$ is order-isomorphic to X; write Γ_u for its group of automorphisms. Apply the claim with u and Γ_u instead of X and Γ, and with the nowhere dense set $a \cap u \subset u$ to find automorphisms $\gamma_{iu} \in \mathrm{pstab}(a \cap u)$ in Γ_u and $\delta_{iu} \in \Gamma_u$ for $i \in 4$ such that $\gamma \restriction u = \prod_{i\in 4} \delta_{iu}^{-1}\gamma_{iu}\delta_{iu}$. Finally, for each $i \in 4$ let $\gamma_i \in \Gamma$ be the union of all γ_{iu} for $u \in U$ together with the identity on $X \setminus \bigcup U$, and similarly for δ_i. It is clear that $\gamma = \prod_{i\in 4} \delta_i^{-1}\gamma_i\delta_i$, so $\gamma \in \Delta$ as desired. $\square$

Example 4.19. [1, Model N16] Let κ be an uncountable cardinal, let X be a set of cardinality at least κ, let Γ be the group of all permutations of X acting by application, and let I_κ be the ideal of subsets of X of cardinality smaller than κ. Then $\Gamma \curvearrowright X, I_\kappa$ is a simple dynamical ideal. The status of well-orderable closure for κ of countable cofinality has apparently not been known.

Proof. Let $a \subseteq b$ be sets in I_κ, let Δ be the normal subgroup of $\mathrm{pstab}(a)$ generated by $\mathrm{pstab}(b)$, let $\gamma \in \mathrm{pstab}(a)$ be an arbitrary group element, and work to show that $\gamma \in \Delta$. Let $c \in I$ be a set containing b and such that c is closed under γ. Let $\beta \in \mathrm{pstab}(a)$ be a group element such that $\beta \cdot b \cap c = a$. Then $(\gamma \restriction c) \cup (\mathrm{id} \restriction \delta \cdot b)$ is a function defined on a set of cardinality smaller than κ; let $\alpha \in \mathrm{pstab}(a)$ be any permutation of X extending it.

Now, $\alpha \in \mathrm{pstab}(\beta \cdot b)$ holds, so $\delta^{-1}\alpha\delta \in \mathrm{pstab}(b)$ holds, and $\alpha = \delta(\delta^{-1}\alpha\delta)\delta^{-1}$ is an element of Δ. At the same time, $\alpha = \gamma$ on the set c, so $\alpha^{-1}\gamma \in \mathrm{pstab}(c) \subset \mathrm{pstab}(b)$ must hold. It follows that $\gamma = \alpha(\alpha^{-1}\gamma)$ belongs to Δ as required. $\square$

Example 4.20. [1, Model N21] Let κ be an uncountable regular cardinal. Let X be a κ-branching tree, let Γ be the group of all automorphisms of X acting by application, and let I be the ideal generated by well-founded subtrees of X of cardinality smaller than κ. Then $\Gamma \curvearrowright X, I$ is a simple dynamical ideal. The status of well-orderable closure in the associated permutation model apparently has not been known.

Proof. Let $a \subseteq b$ be sets in the ideal I. Since the definable closure of any subset of X is a subtree of X, I may assume that both a and b are in fact well-founded trees. Let Δ be the normal subgroup of $\mathrm{pstab}(a)$ generated by $\mathrm{pstab}(b)$ and let $\gamma \in \mathrm{pstab}(a)$ be an arbitrary automorphism of X; I must show that $\gamma \in \Delta$ holds.

Let $c \subset X$ be any tree of cardinality smaller than κ which contains b and which is closed under γ and its inverse. Let $\delta \in \mathrm{pstab}(a)$ be an automorphism such that $c \cap \delta \cdot b = a$. Then the map $(\gamma \restriction c) \cup (\mathrm{id} \restriction \delta \cdot b)$ is an automorphism of a subtree of X of cardinality smaller than κ. As such, it can be extended to a full automorphism $\alpha \in \mathrm{pstab}(a)$. Then, $\alpha \in \mathrm{pstab}(\delta \cdot b)$, so $\delta^{-1}\alpha\delta \in \mathrm{pstab}(b)$, and $\alpha = \delta(\delta^{-1}\alpha\delta)\delta^{-1} \in \Delta$. At the same time, $\alpha = \gamma$ on the set c, so $\alpha^{-1}\gamma \in \mathrm{pstab}(c) \subset \mathrm{pstab}(b)$ must hold. It follows that $\gamma = \alpha(\alpha^{-1}\gamma)$ belongs to Δ as required. □

Last, but not least, the concept of simple dynamical ideal has been formulated in such a way that it survives many common operations on ideals. I will state only the most obvious:

Example 4.21. A subideal of a simple dynamical ideal is simple.

In view of Example 4.22, this includes such dynamical ideals as the following:

Example 4.22. [1, Model N23] Let X be an ultrahomogeneous linear order, let Γ be the group of all automorphisms of X acting by application, and let I be the ideal of sets which are well-ordered by the linear order of X. Then $\Gamma \curvearrowright X, I$ is a simple dynamical ideal. The status of well-orderable closure in this model has apparently not been known.

Example 4.23. An increasing union of simple dynamical ideals is again simple.

Finally, a brief nonexample shows that cofinal orbits do not imply simplicity.

Example 4.24. Let $X = \mathbb{R}^2$, let Γ be the homeomorphism group of X acting by application, and let I be the ideal of bounded sets. Then I has cofinal orbits by Example 3.4. However, it is not simple: let a be the set containing only the origin, and let b_ε be the closed disc of radius ε. It is not difficult to check that the normal subgroup of $\mathrm{pstab}(a)$ generated by $\mathrm{pstab}(b_1)$ is equal to $\bigcup_{\varepsilon>0} \mathrm{pstab}(b_\varepsilon)$. Any nontrivial rotation of $\mathbb{R}^2$ shows that this subgroup is not equal to $\mathrm{pstab}(a)$.

On the opposite end of the spectrum lurk the dynamical ideals associated with abelian groups. This is the content of the following theorem.

Theorem 4.25. *Let $\Gamma \curvearrowright X, I$ be an abelian dynamical ideal. Then, in the associated permutation model,*

1. *every set is a union of a well-orderable collection of well-orderable sets;*
2. *the axiom of choice for families of finite sets implies the full axiom of choice.*

Proof. For (1), let $A \in W[[X]]$ be a set, and let $a \in I$ be a set such that $\mathrm{pstab}(a) \subset \mathrm{stab}(A)$. I need to produce a well-orderable set B such that B is well-orderable, every element of B is, and $\bigcup B = A$. To this end, let B be the set of all $\mathrm{pstab}(a)$-orbits of elements of A. Every element of B is $\mathrm{pstab}(a)$-invariant, so B belongs to the permutation model and it is well-orderable there by Proposition 2.11. Now, let $C \in B$ be an arbitrary set, let $D \in C$ be arbitrary, and let $d \in I$ be such that $a \subset d$ and $\mathrm{pstab}(d) \subset \mathrm{stab}(D)$. It will be enough to show that pstab(d) fixes all elements of C–then, C is well-orderable in the permutation model by Proposition 2.11 again.

Thus, suppose that $\gamma \in \text{pstab}(d)$ is arbitrary. Let $\delta \in \text{pstab}(a)$ be arbitrary; I must show that $\gamma\delta \cdot D = \delta \cdot D$. To this end, use the commutativity of the group Γ to conclude that the left-hand side of the equation is equal to $\delta\gamma \cdot D$ which is equal to $\delta \cdot D$ as $\gamma \in \text{stab}(D)$. (1) follows.

For (2), in $W[[X]]$, consider the set A which contains every nonempty finite subset of X and every unordered pair of nonempty disjoint subsets of X. It will be enough to show that whenever f is a selector function on A and $b \in I$ is a set such that $\text{pstab}(b) \subset \text{stab}(f)$, then $\text{pstab}(f) = \text{pstab}(X)$. To do this, fix f and b as assumed, and towards a contradiction assume that there are elements $\gamma \in \text{pstab}(b)$ and $x \in X$ such that $\gamma \cdot x \neq x$. Write c for the orbit of x under γ and γ^{-1}. There are two cases.

Case 1. The set $c \subset X$ is finite. Then $c \in \text{dom}(f)$, both f and c are fixed by γ, but no element of c is. This must include the element $f(c) \in c$, contradicting the fact that the action by γ is an $\in$-automorphism of the permutation model.

Case 2. The set $c \subset X$ is infinite. In such a case, define an equivalence relation E on X connecting elements $y, z \in X$ if there is an even integer n such that $y = \gamma^n \cdot z$. The equivalence relation E is invariant under the group action: whenever $\delta \in \Gamma$ is arbitrary and $y = \gamma^n \cdot z$ then $\delta \cdot y = \delta\gamma^n \cdot z = \gamma^n\delta \cdot z$, where the last equality follows from commutativity of the group Γ. It follows that E belongs to the permutation model. Consider the set d of the two E-equivalence classes represented in the set c. It is clear that $d \in \text{dom}(f)$, γ fixes both f and d. It is also clear that γ flips the two elements of d, so it moves $f(d)$. This again contradicts the fact that the action by γ is an $\in$-automorphism of the permutation model. $\square$

Example 4.26. [1, Model N2(LO)] Let $X = \omega \times \mathbb{Z}$, let $\Gamma = \mathbb{Z}^\omega$ be the abelian group acting on X by $\gamma \cdot \langle n, z\rangle = \langle n, \gamma(n) + z\rangle$, let I be the ideal generated by the vertical sections of X. Axiom of choice fails in the associated permutation model; thus, there must be a collection of finite sets for which there is no choice function. To find it, for each $n \in \omega$ let E_n be the equivalence relation on n-th vertical section of X defined by $\langle n, z_0\rangle E_n \langle n, z_1\rangle$ if $z_0 - z_1$ is an even integer. Let Y_n be the set of the two E_n-equivalence classes. It is not difficult to see that in the permutation model, $\langle Y_n : n \in \omega\rangle$ is a sequence of two-element sets without a choice function.

4.2 Dedekind finite and amorphous sets

In the broad field of definitions of finiteness, the following definitions stand out:

Definition 4.27. A set is *Dedekind finite* if it does not contain an injective image of ω. A set is *amorphous* if it cannot be partitioned into two infinite sets.

Non-existence of amorphous or infinite, Dedekind finite sets is one of the more common fragments of axiom of choice [1, Form 9]. The purpose of this section is to provide a dynamical criterion related to the previous concepts which implies it.

Definition 4.28. A dynamical ideal $\Gamma \curvearrowright X, I$ is said to be *stratified* if there are Γ-invariant ideals I_n on X for $n \in \omega$ such that

1. I is the increasing union of all I_n's;
2. for every $n \in \omega$ and all sets $a, b_m \colon m \in \omega$ in I_n there are elements $\gamma_m \in \text{pstab}(a)$ such that $\bigcup_m \gamma_m \cdot b_m \in I_{n+1}$.

Theorem 4.29. *Let $\Gamma \curvearrowright X, I$ be a layered dynamical ideal. In the associated permutation model, every set is either a countable union of finite sets or it contains an injective image of ω.*

In particular, the permutation model contains no amorphous sets. A countable union of finite sets can certainly be an infinite, Dedekind finite set. However, such an eventuality can be excluded if, for example, the ideals in the stratification of I are simple. Then, even the union is simple (Example 4.23), therefore in the permutation model a countable union of finite sets must be countable by Theorem 4.10.

Proof. Let $\{I_n \colon n \in \omega\}$ be the stratification of I. Let A be a set in the permutation model. Let $a \in I$ be a set such that $\text{pstab}(a) \subseteq \text{stab}(A)$ holds, and pick $n \in \omega$ such that $a \in I_n$ holds. For every number $m \in \omega$ let $A_m = \{B \in A \colon \exists b \in I_m \ \text{pstab}(b) \subseteq \text{stab}(B)\}$. By the invariance of supports (Proposition 2.10), the sequence $\langle A_m \colon m \in \omega \rangle$ belongs to the permutation model, and A is the increasing union of the sets A_m. It will be enough to show that if one of the sets A_m is infinite, then it contains an injective image of ω.

Suppose then that $m \geq n$ is such that A_m is infinite. Let B_k for $k \in \omega$ be some injective k-tuples of elements of A_m. Note that since I_m is an ideal, for each $k \in \omega$ there is a set $b_k \in I_m$ such that $\text{pstab}(b_k) \subseteq \text{stab}(B_k)$. Use the stratification assumption to find group elements $\gamma_k \in \text{pstab}(a)$ such that $c = \bigcup_{k \in \omega} \gamma_k \cdot b_k \in I_{m+1}$. Consider the sequence $C = \langle \gamma_k \cdot B_k \colon k \in \omega \rangle$. By the invariance of supports (Proposition 2.10), $\text{pstab}(c) \subseteq \text{stab}(C)$, so C belongs to the permutation model. Since each γ_k fixes the set A, the sequence C consists of k-tuples of elements of A for each k. Clearly, the set $\bigcup_k \text{rng}(\gamma_k \cdot B_k)$ is an infinite countable subset of A as required. $\square$

Example 4.30. Let X be a countable dense linear ordering without endpoints. Let Γ be its automorphism group acting on X by application. Let $\alpha > \omega$ be a countable ordinal closed under ordinal addition, and let I_α be the ideal of subsets of X which are well-ordered of ordertype less than α. If α is closed under ordinal multiplication, then $\Gamma \curvearrowright X, I_\alpha$ is a stratified dynamical ideal. The ideal is also simple by Example 4.22; in conclusion, in the associated permutation model every infinite set contains an injective image of ω.

Proof. Let $\langle \beta_n : n \in \omega \rangle$ be an increasing sequence of ordinals closed under ordinal addition, converging to α, and such that $\beta_{n+1} > (\omega \cdot \beta_n) \cdot \beta_n$. Then I_α is the increasing union of I_{β_n} for $n \in \omega$. In addition, an inspection of the proof of Example 4.5 shows that Definition 4.28 holds for this stratification of I_α. $\square$

Example 4.31. Let κ be an uncountable cardinal, let X be a set of cardinality at least κ, let Γ be the group of all permutations of X acting by application, and let I_κ be the ideal of subsets of X of cardinality smaller than κ. If κ has countable cofinality, then $\Gamma \curvearrowright X, I$ is a stratified dynamical ideal. The ideal is simple by Example 4.19; in conclusion, in the associated permutation model every infinite set contains an injective image of ω.

Proof. Just let $\langle \lambda_n : n \in \omega \rangle$ be an increasing sequence of cardinals converging to κ. Then I_κ is an increasing union of ideals $I_{\lambda_n^+}$ for $n \in \omega$. Each of these dynamical ideals is σ-complete, and in fact has cofinal orbits by Example 3.10; thus, Definition 4.28 holds for this stratification of I_κ. $\square$

Example 4.32. In the fairly popular situation where I is generated by a countable increasing sequence of invariant sets, $\Gamma \curvearrowright X, I$ is a stratified dynamical ideal. For example, when $\mathbb{Z}$ acts on a countable set X and I is the ideal of finite sets, then there are two cases. Either there is an infinite orbit. In such a case, for any $x \in X$ in the infinite orbit, $\mathrm{stab}(x) = \{0\}$ and $W[[X]] = V[[X]]$; in this trivial case, the full axiom of choice holds in the associated permutation model. Or, all orbits are finite, and then X is an increasing union of countably many finite invariant sets.

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