Definition 0.1. $E_1$ is the equivalence relation on the space $Y = (2^\omega)^\omega$ connecting points $y_0, y_1 \in Y$ if there is $n \in \omega$ such that for all $m > n$, $y_0(n) = y_1(n)$.

Theorem 0.2. $E_1$ is not Borel reducible to any orbit equivalence relation.

Proof. The argument needs several auxiliary general claims which have nothing to do with $E_1$.

Claim 0.3. (Mostowski absoluteness) Let $M$ be a transitive model of a large fragment of ZFC. Let $X$ be a Polish space and $A \subset X$ be an analytic set, both with a definition in $M$. Let $x \in X \cap M$ be a point. Then $x \in A$ if and only if $M \models x \in A$.

Proof. We show it for the special case of the space $X$ of all trees on $\omega$ and the analytic set $A$ of all illfounded trees. Since $A$ is a universal analytic set, this will be enough. So let $x \in M$ be a tree on $\omega$. If on one hand $M \models x \in A$ then there is $b$ such that $M \models b$ is an infinite branch through $x$. Since being a branch of a given tree is a bounded formula, it follows that $b$ is truly an infinite branch of $x$ and so $x \in A$ as desired. If on the other hand $M \not\models x \notin A$ then by the axiom of choice in $M$, there is an $f$ such that $M \models f$ is an order-preserving map from $x$ to the ordinals. Since being an ordinal and an order-preserving function are bounded formulas, $f$ is truly an order preserving map from $x$ to the ordinals, and so $x \notin A$.

Now, let $\Gamma$ be a Polish group acting on a Polish space $X$, inducing an orbit equivalence relation $F$. Consider the poset $P_{\Gamma}$ of all nonempty open subsets of $\Gamma$ ordered by inclusion. Note that basic open sets are dense in $P_{\Gamma}$. The poset $P_{\Gamma}$ adds a single element of the group $\Gamma$ which belongs to all sets in the generic filter. The important point is that shifts of a generic point are themselves generic, as captured by the following claim:

Claim 0.4. Let $M$ be a transitive model of set theory and $\gamma \in \Gamma$ be a point $P_{\Gamma}$-generic over the model $M$. Let $\delta \in \Gamma$ be a point in $M$. Then $\gamma \cdot \delta$ is $P_{\Gamma}$-generic over $M$ as well.

Proof. Working in the model $M$, we see that the right multiplication by $\delta$ is a self-homeomorphism of $\Gamma$, and therefore it extends to a natural automorphism of the poset $P_{\Gamma}$, shifting each open set by right multiplication from the right. Finally, an automorphic image of a generic filter is a generic filter.

The last auxiliary claim compares generic extensions of different models. It is the key tool.

Claim 0.5. Let $M_0, M_1$ be transitive models of set theory containing $F$-related points $x_0, x_1 \in X$ respectively. Let $\gamma \in \Gamma$ be a point $P_{\Gamma}$-generic over some model containing both $M_0, M_1$. Then $M_0[\gamma] \cap M_1 \subseteq M_0$. 


Proof. For simplicity, we will argue that the inclusion holds for sets of ordinals. Suppose towards contradiction that it fails. By the forcing theorem, there would have to be a condition \( p \in P_\tau \), a \( P_\tau \)-name \( \tau \in M_0 \) and a set \( s \in M_1 \setminus M_0 \) such that \( p \Vdash \tau = s \). This means that \( M_0 \models p \Vdash \tau \notin M_0 \). Now, working in the model \( M_0 \), observe that there must be an ordinal \( \alpha \) such that \( p \) does not decide the statement \( p \Vdash \alpha \in \tau \): if the condition \( p \) decided all statements \( \alpha \in \tau \), then one could form in \( M_0 \) the set \( t = \{ \alpha : p \Vdash \alpha \in \tau \} \), conclude that \( p \Vdash \tau = t \) and so \( s = t \), contradicting the assumption that \( s \notin M_0 \). \( \square \)

Now, suppose towards contradiction that \( h: Y \rightarrow X \) is a Borel reduction of \( E_1 \) to \( F \). Let \( M \) be a countable transitive model of a large fragment of ZFC containing the definition of \( h \). Let \( \langle z_n : n \in \omega \rangle \) be a sequence in the space \( Y \) which is generic over the model \( M \) for the modulo finite product of infinitely many copies of the Cohen forcing. For each number \( m \in \omega \), let \( y_m \in Y \) be the sequence which returns zero at the entries \( n < m \), and for \( n \geq m \) \( y_m(n) = z_n \) holds. Thus, the points \( z_m \) for \( m \in \omega \) are \( E_1 \)-equivalent. Let \( M_m = M[y_m] \) and \( x_m = h(y_m) \in X \). Note that the point \( x_m \) belongs to the model \( M[y_m] \) for every \( m \in \omega \). Let \( \gamma \in \Gamma \) be a point generic over the model \( M_0 \), let \( x_\omega = \gamma \cdot x_0 \) and consider the model \( M_\omega = M[x_\omega] \).

There must be a point \( y_\omega \in M_\omega \) such that \( h(y_\omega) \) \( \gamma \cdot x_0 \): the set \( \{ x \in X : \exists y h(y) F x \} \) is analytic, contains \( x_\omega \) (as witnessed by any point \( y_m \) for \( m \in \omega \)) and so by Claim 0.3 it must be the case that \( M \models \exists y h(y) F x \). Since \( h \) is a Borel reduction of \( E_1 \) to \( F \), it must be the case that \( y_\omega \) is \( E_1 \)-related to all the points \( y_m \) for \( m \in \omega \). In particular, there must be a number \( n \in \omega \) such that \( z_n = y_\omega(n) \). We will reach a contradiction by showing that \( z_n \notin M_\omega \).

Let \( \delta \in \Gamma \) be a point in the model \( M_0 \) such that \( \delta x_{n+1} = x_0 \). Then the point \( \gamma \delta \) is \( P_\tau \)-generic over \( M_0 \) by Claim 0.4. The point \( \gamma \delta \) must be also \( P_\tau \)-generic over the model \( M_n \) since \( M_n \subset M_0 \) (\( M_n \) contains fewer open dense subsets of \( P_\tau \) than \( M_0 \)). Look at the model \( M_{n+1}[\gamma \delta] \) and observe that \( x_\omega \in M_{n+1}[\gamma \delta] \) (since \( x_\omega = \omega = \gamma \delta \cdot h(x_{n+1}) \)) and \( y_\omega \in M_{n+1}[\gamma \delta] \) (since \( y_\omega \in M[x_\omega] \)) and \( z_n \in M_{n+1}[\gamma \delta] \). At the same time, by Claim 0.5 applied to the models \( M_n \) and \( M_{n+1} \), it must be the case that \( M_{n+1}[\gamma \delta] \cap M_n \subset M_{n+1} \). This, however, is impossible since \( z_n \in M_n \) and \( z_n \notin M_{n+1} \). \( \square \)

As a last remark, consider the group \( \Gamma = (2^\omega)^\omega \): \( 2^\omega \) is equipped by coordinatewise Boolean addition (the Cantor group structure) and \( (2^\omega)^\omega \) is just the product group equipped with the product topology. Consider the subgroup \( \Delta \subset \Gamma \) consisting of those elements \( \gamma \in \Gamma \) such that for all but finitely many \( n \), \( \gamma(n) \) is the zero element of \( 2^\omega \); this is an \( F_\tau \)-subgroup. Then \( \Delta \) (as any subgroup) acts on the whole group \( \Gamma \) by left multiplication. The orbit equivalence relation of the action is exactly \( E_1 \). Comparing with the theorem, we get

**Corollary 0.6.** \( \Delta \) is not a Polish group.