Definition 0.1. $\mathbb{E}_1$ is the equivalence relation on the space $Y = (2^\omega)^\omega$ connecting points $y_0, y_1 \in Y$ if there is $n \in \omega$ such that for all $m > n$, $y_0(n) = y_1(n)$.

Theorem 0.2. *$\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation.*

Proof. The argument needs several auxiliary general claims which have nothing to do with $\mathbb{E}_1$.

Claim 0.3. (Mostowski absoluteness) *Let M be a transitive model of a large fragment of ZFC. Let X be a Polish space and $A \subset X$ be an analytic set, both with a definition in M. Let $x \in X \cap M$ be a point. Then $x \in A$ if and only if $M \models x \in A$.*

Proof. We show it for the special case of the space X of all trees on ω and the analytic set A of all illfounded trees. Since A is a universal analytic set, this will be enough. So let $x \in M$ be a tree on ω. If on one hand $M \models x \in A$ then there is b such that $M \models b$ is an infinite branch through x. Since being a branch of a given tree is a bounded formula, it follows that b is truly an infinite branch of x and so $x \in A$ as desired. If on the other hand $M \models x \notin A$ then by the axiom of choice in M, there is an f such that $M \models f$ is an order-preserving map from x to the ordinals. Since being an ordinal and an order-preserving function are bounded formulas, f is truly an order preserving map from x to the ordinals, and so $x \notin A$. □

Now, let Γ be a Polish group acting on a Polish space X, inducing an orbit equivalence relation F. Consider the poset P_Γ of all nonempty open subsets of Γ ordered by inclusion. Note that basic open sets are dense in P_Γ. The poset P_Γ adds a single element of the group Γ which belongs to all sets in the generic filter. The important point is that shifts of a generic point are themselves generic, as captured by the following claim:

Claim 0.4. *Let M be a transitive model of set theory and $\gamma \in \Gamma$ be a point P_Γ-generic over the model M. Let $\delta \in \Gamma$ be a point in M. Then $\gamma \cdot \delta$ is P_Γ-generic over M as well.*

Proof. Working in the model M, we see that the right multiplication by δ is a self-homeomorphism of Γ, and therefore it extends to a natural automorphism of the poset P_Γ, shifting each open set by right multiplication from the right. Finally, an automorphic image of a generic filter is a generic filter. □

The last auxiliary claim compares generic extensions of different models. It is the key tool.

Claim 0.5. *Let M_0, M_1 be transitive models of set theory containing F-related points $x_0, x_1 \in X$ respectively. Let $\gamma \in \Gamma$ be a point P_Γ-generic over some model containing both M_0, M_1. Then $M_0[\gamma] \cap M_1 \subseteq M_0$.*

Proof. For simplicity, we will argue that the inclusion holds for sets of ordinals. Suppose towards contradiction that it fails. By the forcing theorem, there would have to be a condition $p \in P_\Gamma$, a P_Γ-name $\tau \in M_0$ and a set $s \in M_1 \setminus M_0$ such that $p \Vdash \tau = \check{s}$. This means that $M_0 \models p \Vdash \tau \notin M_0$. Now, working in the model M_0, observe that there must be an ordinal α such that p does not decide the statement $p \Vdash \check{\alpha} \in \tau$: if the condition p decided all statements $\check{\alpha} \in \tau$, then one could form in M_0 the set $t = \{\alpha \colon p \Vdash \check{\alpha} \in \tau\}$, conclude that $p \Vdash \tau = \check{t}$ and so $s = t$, contradicting the assumption that $s \notin M_0$. □

Now, suppose towards contradiction that $h \colon Y \to X$ is a Borel reduction of $\mathbb{E}_1$ to F. Let M be a countable transitive model of a large fragment of ZFC containing the definition of h. Let $\langle z_n \colon n \in \omega \rangle$ be a sequence in the space Y which is generic over the model M for the modulo finite product of infinitely many copies of the Cohen forcing. For each number $m \in \omega$, let $y_m \in Y$ be the sequence which returns zero at the entries $n < m$, and for $n \geq m$ $y_m(n) = z_n$ holds. Thus, the points z_m for $m \in \omega$ are $\mathbb{E}_1$-equivalent. Let $M_m = M[y_m]$ and $x_m = h(y_m) \in X$. Note that the point x_m belongs to the model $M[y_m]$ for every $m \in \omega$. Let $\gamma \in \Gamma$ be a point generic over the model M_0, let $x_\omega = \gamma \cdot x_0$ and consider the model $M_\omega = M[x_\omega]$.

There must be a point $y_\omega \in M_\omega$ such that $h(y_\omega)\ F\ \gamma \cdot x_0$: the set $\{x \in X \colon \exists y\ h(y)\ F\ x\}$ is analytic, contains x_ω (as witnessed by any point y_m for $m \in \omega$) and so by Claim 0.3 it must be the case that $M \models \exists y\ h(y)\ F_x$. Since h is a Borel reduction of $\mathbb{E}_1$ to F, it must be the case that y_ω is $\mathbb{E}_1$-related to all the points y_m for $m \in \omega$. In particular, there must be a number $n \in \omega$ such that $z_n = y_\omega(n)$. We will reach a contradiction by showing that $z_n \notin M_\omega$.

Let $\delta \in \Gamma$ be a point in the model M_0 such that $\delta x_{n+1} = x_0$. Then the point $\gamma\delta$ is P_Γ-generic over M_0 by Claim 0.4. The point $\gamma\delta$ must be also P_Γ-generic over the model M_n since $M_n \subset M_0$ (M_n contains fewer open dense subsets of P_Γ than M_0). Look at the model $M_{n+1}[\gamma\delta]$ and observe that $x_\omega \in M_{n+1}[\gamma\delta]$ (since $x_\omega = \omega = \gamma\delta \cdot h(x_{n+1})$) and $y_\omega \in M_{n+1}[\gamma\delta]$ (since $y_\omega \in M[x_\omega]$) and $z_n \in M_{n+1}[\gamma\delta]$. At the same time, by Claim 0.5 applied to the models M_n and M_{n+1}, it must be the case that $M_{n+1}[\gamma\delta] \cap M_n \subset M_{n+1}$. This, however, is impossible since $z_n \in M_n$ and $z_n \notin M_{n+1}$. □

As a last remark, consider the group $\Gamma = (2^\omega)^\omega$: 2^ω is equipped by coordinatewise Boolean addition (the Cantor group structure) and $(2^\omega)^\omega$ is just the product group equipped with the product topology. Consider the subgroup $\Delta \subset \Gamma$ consisting of those elements $\gamma \in \Gamma$ such that for all but finitely many n, $\gamma(n)$ is the zero element of 2^ω; this is an F_σ-subgroup. Then Δ (as any subgroup) acts on the whole group Γ by left multiplication. The orbit equivalence relation of the action is exactly $\mathbb{E}_1$. Comparing with the theorem, we get

Corollary 0.6. Δ *is not a Polish group.*