Exam 2.

Due Wednesday Nov. 1 before class.

1. Let \( f: \mathbb{R} \to \mathbb{R} \) be a strictly increasing function \((r < s \implies f(r) < f(s))\).
   Show that the set \( \{ r \in \mathbb{R}: f \text{ is not continuous at } r \} \) is countable. *Hint.*
   Produce an injection of the set into rationals.

2. Produce a function \( f: \mathbb{R} \to \mathbb{R} \) such that the set \( \{ r \in \mathbb{R}: f \text{ is not continuous at } r \} \) is uncountable.

3. Let \( \langle P, \leq \rangle \) be a partially ordered set. Define a transfinite recursive process
   by \( P_0 = P, P_{\alpha+1} = P_\alpha \setminus \{ p \in P_\alpha: p \text{ is minimal in } P_\alpha \} \), and
   \( P_\alpha = \bigcap_{\beta \in \alpha} P_\beta \)
   for a limit ordinal \( \alpha \). Show that there is an ordinal \( \alpha \) such that \( P_\alpha = P_{\alpha+1} \).

4. Show that the set \( P_\alpha = P_{\alpha+1} \) from (3) is the inclusion-largest subset of \( P \)
   which has no minimal elements.

5. Prove that every filter on \( \omega \) can be extended to an ultrafilter.