Exam 1.

No cooperation. The solutions are due Monday February 10 before class.

1. Prove without the use of the completeness theorem that in propositional
logic, $\vdash \phi \rightarrow \neg\neg\phi$ and $\vdash \neg\neg\phi \rightarrow \phi$ are both true.

2. Consider the theory of dense linear order without endpoints. Enrich its
language by two constant symbols $c, d$, and add to it the sentence $c < d$. Prove
that the resulting theory is complete.

In the following problems, a graph is a pair $\langle V, E \rangle$ where $V$ is a set (of
vertices) and $E \subset [V]^2$ is a set (of edges). Graphs are models for a first order
language $\mathcal{L}$ with a single binary relational symbol interpreted as $xRy$ if $x, y$ are
connected with an edge. A cycle of length $n$ is a sequence $x_0, x_1, \ldots, x_{n-1}$ of
pairwise distinct vertices such that $x_i, x_{i+1}$ are connected with an edge for all
$i < n - 1$ and also the vertices $x_0, x_{n-1}$ are connected with an edge. A graph
is then a model of the theory $\Gamma$ consisting of the sentences $\forall x, y \ xRy \leftrightarrow y Rx$
and $\forall x \neg xRx$.

3. For every natural number $n$, write a first order sentence in $\mathcal{L}$ equivalent to
"there are no cycles of length $n$".

4. Show that there is no first order sentence in $\mathcal{L}$ equivalent to "there are no
cycles".

5. Find some sentence $\phi$ such that the theory $\Gamma, \phi$ is complete (decides every
sentence).