Pinned equivalence relations

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1 Introduction

In this note, I will provide a solution to a question of Kechris regarding unpinned Borel equivalence relations.

Definition 1.1. [1, Chapter 17] A Borel equivalence relation E on a Polish space X is *unpinned* if there is a forcing P and a P-name $\dot{x}$ for an element of X such that $P \times P$ forces the left and right evaluations of $\dot{x}$ to be E equivalent, and P forces $\dot{x}$ to be E-inequivalent to any ground model point of the space X. The equivalence is *pinned* if it is not unpinned.

Many equivalence relations are pinned, such as all K_σ equivalences, E_3, c_0 and others. There is a natural example of an unpinned equivalence relation:

Definition 1.2. F_2 is the Borel equivalence on $(2^\omega)^\omega$ defined by xF_2y if $\mathrm{rng}(x) = \mathrm{rng}(y)$.

The forcing witnessing the requisite property of F_2 is the collapse of 2^ω to countable size, with $\dot{x}$ a name for an enumeration of the set of ground model points of 2^ω.

It is not difficult to see that the class of pinned equivalence relations is closed downwards with respect to Borel reducibility, and so it offers a tool for proving irreducibility results: no unpinned relation can be reduced to pinned. As an example, F_2 is not reducible to E_3. Kechris asked whether this tool is really only checking whether F_2 is reducible to a given Borel equivalence relation:

Question 1.3. [1, Question 17.6.1] Is F_2 reducible to every unpinned Borel equivalence relation?

In this brief note, I will show that the answer is negative.

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Theorem 1.4. *There is an unpinned Borel equivalence relation strictly below F_2.*

2 Proof

The proof of the theorem begins with a fact essentially due to Shelah [3].

Fact 2.1. *There is an F_σ set $B \subset 2^\omega \times 2^\omega$ which contains an uncountable clique but no perfect clique.*

Here, a *clique* is a set $C \subset 2^\omega$ such that all pairs of elements of C are in the set B. Shelah proves something much stronger and does not stop to state this ZFC consequence of his results explicitly, so I will take a moment to prove it from statements explicitly appearing in his paper.

Proof. In [3, Theorem 1.13], Shelah shows that a set with these properties can be *forced*. I will be finished if I show that the two requisite properties of the F_σ set are absolute between any transitive models of a large fraction of ZFC that contain the (code for the) set B, including the countable models.

First of all, containing a perfect clique is equivalent to a Σ^1_1 statement. Suppose that $T \subset 2^{<\omega}$ is a perfect binary tree such that $[T]$ is a clique of B. Fix the closed sets $F_n : n \in \omega$ whose union gives B. Note that the collection of Borel subsets of $2^\omega \times 2^\omega$ which do not contain a perfect rectangle is a σ-ideal. Use this fact, thinning out the tree T if necessary, to find numbers $m(t), n(t)$ for every splitnode t of T such that $[T \restriction t^\frown 0] \times [T \restriction t^\frown 1] \subset F_{m(t)}$ and $[T \restriction t^\frown 1] \times [T \restriction t^\frown 0] \subset F_{n(t)}$. Existence of a tree with such numbers $m(t), n(t)$ associated to each splitnode t is clearly a Σ^1_1 statement.

Second, existence of an uncountable clique is equivalent to an existence of a model of a certain sentence in the language $L_{\omega_1\omega}(Q)$, where Q is the quantifier "there exist uncountably many". However, Keisler [2] showed that the relevant infinitary logic is complete, and so the existence of a model is equivalent to the consistency of the sentence. The consistency means nonexistence of a proof of contradiction, the proofs are hereditarily countable objects, and so "existence of a clique" is equivalent to a Π^1_1 sentence, and therefore absolute between transitive models of set theory. ☐

Let Y be the Borel set of those sequences $y \in (2^\omega)^\omega$ whose range is a clique in B, and let $E = F_2 \restriction Y$. It is immediate that $E \leq F_2$. I will show that E is unpinned and F_2 does not reduce to E.

First of all, the equivalence E is unpinned. Let $C \subset 2^\omega$ be an uncountable clique of B, let P be a forcing enumerating the set C in ordertype ω, and let $\dot{y}$ be the P-name for the generic enumeration. It is immediate that $\dot{y}$ witnesses the requisite property of the equivalence E.

To show that F_2 is not reducible to E, suppose that f is such a Borel reduction and work towards a contradiction. Note that f remains a reduction in every forcing extension by Shoenfield's absoluteness. I will produce a generic extension $V[G]$ such that in it, B contains no clique of size continuum. In the model $V[G]$, I will reach the contradiction in the following way. Consider the forcing P collapsing the size of the continuum to $\aleph_0$ and let $\dot{x}$ be a name for the generic enumeration of the ground model elements of 2^ω.

Claim 2.2. *$P \Vdash \operatorname{rng}(f(\dot{x})) \in V$.*

Proof. It must be the case that $P \Vdash \operatorname{rng}(f(\dot{x})) \subset V$. If some condition forced a new element into the set, one could pass to a forcing extension with mutually generic filters $H_0, H_1 \subset P$ containing that condition. Clearly, $(\dot{x}/H_0)F_2(\dot{x}/H_1)$, but the ranges of $f(\dot{x}/H_0)$ and $f(\dot{x}/H_1)$ are not equal by a mutual genericity argument. Thus f would not be a reduction in that extension.

It also must be the case that for every ground model element $y \in Y$, the largest condition in P must decide the statement $\check{y} \in \operatorname{rng}(f(\dot{x}))$. If $p, q \in P$ decided this statement in two different ways, then one could pass into a forcing extension with $V[G]$-generic filters with $p \in H_0, q \in H_1$. But then, $(\dot{x}/H_0)F_2(\dot{x}/H_1)$ while $y \in \operatorname{rng}(f(\dot{x}/H_0))\Delta\operatorname{rng}(f(\dot{x}/H_1))$ and f is not a reduction in this extension.

Consequently, $P \Vdash \operatorname{rng}(f(\dot{x})) = \{y : 1 \Vdash \check{y} \in \operatorname{rng}(\dot{x})\} \in V$. □

Let $C \subset 2^\omega$ be the set forced to be the ranges of $\dot{f}(\dot{x})$. Plainly, C is a clique in B, and therefore its size is less than the continuum. Thus there are two elementary submodels M_0, M_1 of a large enough structure which contain C as an element and a subset such that M_0, M_1 do not contain the same reals. Pass into a forcing extension in which there are filters $H_0 \subset M_0 \cap P$ and $H_1 \cap M_1 \cap P$ meeting all the dense sets in the respective models. By the forcing theorem applied in the models, $M_0[H_0] \models \operatorname{rng}(f(\dot{x}/H_0)) = C$ and $M_1[H_1] \models \operatorname{rng}(f(\dot{x}/H_1)) = C$, and by Borel absoluteness between the models $M_0[H_0], M_1[H_1]$ and the extension, it is the case that $\operatorname{rng}(f(\dot{x}/H_0)) = C = \operatorname{rng}(f(\dot{x}/H_1))$. However, the sequences $\dot{x}/H_0, \dot{x}/H_1$ are F_2 inequivalent, since the models M_0, M_1 did not contain the same reals. Thus f is not a reduction in the geenric extension, a contradiction.

Now I must describe how to obtain the generic extension $V[G]$ in which no clique of the set B has size continuum. The argument can be found in several places in the literature, including Shelah's [3]. Work in V and let κ be a regular cardinal larger than the continuum such that $\kappa^\omega = \kappa$. The model $V[G]$ is the extension of V with forcing Q adding κ many Cohen reals with finite support. To verify the requisite feature, suppose for contradiction that the poset Q forces that $\langle \dot{z}_\alpha : \alpha \in \kappa\rangle$ is a clique in the set B. For every ordinal $\alpha \in \kappa$, let M_α be a countable elementary submodel of a large structure containing α. Note that the c.c.c. of Q implies that $\dot{z}_\alpha \cap M_\alpha = \dot{z}_\alpha$ for every ordinal α. Use the cardinal arithmetic assumption to find a cofinal set $a \subset \kappa$ such that the models $M_\alpha : \alpha \in a$ form a Δ-system with root r. The simple form of the forcing Q implies that $Q \cap r$ is a regular subposet of $Q \cap \bar{M}$ which is in turn regular in Q and so there is a $Q \cap r$ name $\dot{u}_\alpha$ for the remainder of the name z_α. Thinning out the set a further if necessary I may assume that the structures $\langle M_\alpha, \dot{z}_\alpha, r, \dot{u}_\alpha\rangle : \alpha \in a$ are pairwise isomorphic, with the same transitive collapse $\bar{M}, \bar{z}, \bar{r}, \bar{u}$. Now, for every pair of ordinals $\alpha \neq \beta \in a$, Q forces that the filters $\dot{G} \cap M_\alpha \setminus r$ and

$\dot{G} \cap M_\beta \setminus r$ are mutually generic over $V[\dot{G} \cap r]$, and the evaluations of the names $\dot{u}_\alpha, \dot{u}_\beta$ according to these filters provide a pair of points in the set B. It follows that in the model $V[G \cap r]$, it is the case that the product of two copies of the poset $Q \cap \bar{M} \setminus \bar{r}$ force the two evaluations of the name $\bar{u}$ to form a point in the set B; moreover, the evaluations must be distinct by a mutual genericity argument. The last key point is that the forcing $Q \cap \bar{M} \setminus \bar{r}$ is countable and therefore in the forcing sense equivalent to Cohen forcing, and that adding a single Cohen real adds in fact a pairwise mutually generic perfect set of them. Thus, if $G \subset Q$ is a V-generic filter, in the model $V[G]$ there is a Cohen real over $V[G \cap r]$, so there is a perfect set P of pairwise mutually $V[G \cap r]$ generic filters over $Q \cap \bar{M} \setminus \bar{r}$, the set $\{\bar{u}/h : h \in P\}$ is an analytic uncountable clique of B, which then contains a perfect clique. But the set B contained no perfect clique in V, so it should contain no perfect clique in $V[G]$, contradiction!

As a final remark, Shelah's work in fact provides for a strictly increasing sequence of ω_1 many unpinned equivalence relations below F_2, with the proofs of irreducibility essentially repeating the above argument. The key point is that under $MA_{\aleph_{\omega_1}}$, for every countable ordinal α there is an F_σ set with cliques of size $aleph_\alpha$ but no larger.

References

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