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On the structure of stationary sets

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Abstract We isolate several classes of stationary sets of $[\kappa]^\omega$ and investigate implications among them. Under a large cardinal assumption, we prove a structure theorem for stationary sets.

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1 Introduction

We investigate stationary sets in the space $[\kappa]^\omega$ of countable subsets of an uncountable cardinal. We concentrate on the following particular classes of stationary sets:

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              full
 ↗                            ↘
    club → local club → reflective → projective stationary
 ↘                            ↗
            spanning
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Figure 1.1

In Figure 1.1, the → represents the implication. We show among others that under a suitable large cardinal assumption (e.g., under Martin's Maximum), the diagram collapses to just two classes:

$$\left\{\begin{array}{r}\text{club}\\ \text{local club}\\ \text{full}\end{array}\right\} \longrightarrow \left\{\begin{array}{r}\text{projective stationary}\\ \text{spanning}\\ \text{reflective}\end{array}\right\}$$

Figure 1.2

Under the same large cardinal assumption, we prove a structure theorem for stationary sets: for every stationary set S there exists a stationary set $A \subset \omega_1$ such that S is spanning above A and nonstationary above $\omega_1 - A$.

We also investigate the relation between some of the above properties of stationary sets on the one hand, and the properties of forcing on the other, in particular the forcing that

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shoots a continuous ω_1-chain through a stationary set. We show that the equality of the classes of projective stationary sets and spanning sets is equivalent to the equality of the class of stationary-set-preserving forcings and the class of semiproper forcings.

The work is in a sense a continuation of [1] and [2] of the first two authors, and the ultimate of the groundbreaking work of Foreman, Magidor and Shelah[3].

2 Definitions

We work in the spaces $[\kappa]^\omega$ and $[H_\lambda]^\omega$, where κ and λ are the uncountable cardinals. The concept of a closed unbounded set and a stationary set has been generalized to the context of these spaces (cf. [4]) and the generalization gains a considerable prominence following the work [5] of Shelah on proper forcing.

The space $[\kappa]^\omega$ is the set of all countable subsets of κ, ordered by inclusion; similarly for $[H_\lambda]^\omega$, where H_λ denotes the set of all sets hereditarily of cardinality less than λ. A set C in this space is closed unbounded (club) if it is closed under unions of increasing countable chains, and cofinal in the ordering by inclusion. A set S is stationary if it meets every club set. We shall (with some exceptions) only consider κ and λ that are greater than ω_1; note that the set ω_1 is a club in the space $[\omega_1]^\omega$ (which motivates the generalization). In order to simplify some statements and some arguments, we shall only consider those $x \in [\kappa]^\omega$ (those $M \in [H_\lambda]^\omega$) whose intersection with ω_1 is a countable ordinal (these objects form a club set); we denote this countable ordinal by δ_x or δ_M respectively:

$$\delta_x = x \cap \omega_1, \quad \delta_M = M \cap \omega_1. \tag{2.1}$$

The filter generated by the club sets in $[\kappa]^\omega$ is generated by the club sets of the form

$$C_F = \{x \mid x \text{ is closed under } F\}, \tag{2.2}$$

where F is an operation, $F : \kappa^{<\omega} \to \kappa$; similarly for H_λ. In the case of H_λ, we consider only those $M \in [H_\lambda]^\omega$ that are submodels of the model $(H_\lambda, \in, <)$, where $<$ is some fixed well ordering; in particular, the M's are closed under the canonical Skolem functions obtained from the well ordering.

For technical reasons, when considering continuous chains in $[\kappa]^\omega$ or $[H_\lambda]^\omega$, we always assume that when $\langle x_\alpha \mid \alpha < \gamma \rangle$ is such a chain then for every $\alpha, \beta < \gamma$,

$$\text{if } \alpha < \beta \text{ then } \delta_{x_\alpha} < \delta_{x_\beta}. \tag{2.3}$$

The term ω_1-chain or $(\gamma+1)$-chain, where $\gamma < \omega_1$, is an abbreviation for "a continuous ω_1-chain that satisfies (2.3)."

We also note that in one instance we consider club (stationary) sets in the spaces $[\kappa]^{\omega_1}$ (where $\kappa \geqslant \omega_2$) which are defined appropriately.

Throughout the paper we employ the operations of projection and lifting, which move the sets between the spaces $[\kappa]^\omega$ for different κ:

If $\kappa_1 < \kappa_2$ and if S is a set in $[\kappa_2]^\omega$, then the projection of S to κ_1 is the set

$$\pi(S) = \{x \cap \kappa_1 \mid x \in S\}. \tag{2.4}$$

If S is a set in $[\kappa_1]^\omega$ then the lifting of S to κ_2 is the set

$$\hat{S} = \{x \in [\kappa_2]^\omega \mid x \cap \kappa_1 \in S\}. \tag{2.5}$$

We recall that the lifting of a club set is a club set and the projection of a club set contains a club set. Hence, the stationarity is preserved under lifting and projection.

The special case of projection and lifting is when $\kappa = \omega_1$:

$$\pi(S) = \{\delta_x \mid x \in S\}, \quad \hat{A} = \{x \mid \delta_x \in A\},\ A \subset \omega_1.$$

Definition 2.1 *A set $S \subset [\kappa]^\omega$ is a local club if the set*

$$\{X \in [\kappa]^{\aleph_1} \mid S \cap [X]^\omega \ \ \textit{contains a club in } [X]^\omega\}$$

contains a club in $[\kappa]^{\aleph_1}$.

Definition 2.2 *A set $S \subset [\kappa]^\omega$ is full if for every stationary $A \subset \omega_1$ there exist a stationary $B \subset A$ and a club C in $[\kappa]^\omega$ such that*

$$\{x \in C \mid \delta_x \in B\} \subset S.$$

(S contains a club above densely many stationary $B \subset \omega_1$.)

Definition 2.3 *A set $S \subset [\kappa]^\omega$ is projective stationary if for every stationary set $A \subset \omega_1$, the set $\{x \in S \mid \delta_x \in A\}$ is stationary. ("S is stationary above every stationary $A \subset \omega_1$.")*

Definition 2.4 *A set $S \subset [\kappa]^\omega$ is reflective if for every club C in $[\kappa]^\omega$, $S \cap C$ contains an ω_1–chain.*

Definition 2.5 *If x and y are in $[\kappa]^\omega$, then y is an ω_1-extension of x if $x \subset y$ and $\delta_x = \delta_y$.*

Definition 2.6 *A set $S \subset [\kappa]^\omega$ is spanning if for every $\lambda \geqslant \kappa$, for every club set C in $[\lambda]^\omega$ there exists a club D in $[\lambda]^\omega$ such that every $x \in D$ has an ω_1-extension $y \in C$ such that $y \cap \kappa \in S$.*

Local clubs were defined in [1]. Projective stationary sets were defined in [2]; so were full sets (without the name). Note that all five properties defined are invariant under the equivalence mod club filter. All five properties are also preserved under lifting and projection. For instance, let $S \subset [\kappa_1]^\omega$ be reflective and let us show that the lifting $\hat{S}$ to $[\kappa_2]^\omega$ is reflective. Let C be a club set in $[\kappa_2]^\omega$ and let $F : \kappa_2^{<\omega} \to \kappa_2$ such that $C_F \subset C$. If we let for every $e \in [\kappa_1]^{<\omega}$,

$$f(e) = \kappa_1 \cap \mathrm{cl}_F(e),$$

where $\mathrm{cl}_F(e)$ is the closure of e under F, then C_f is a club in $[\kappa_1]^\omega$. Also for every $x \in C_f$, if y is the closure of x under F then $y \cap \kappa_1 = x$. Let $\langle x_\alpha \mid \alpha < \omega_1\rangle$ be an ω_1-chain in $S \cap C_f$, we then let y_α be the closure of x_α under F, then $\langle y_\alpha \mid \alpha < \omega_1\rangle$ is an ω_1-chain in $\hat{S} \cap C_F$. The arguments are simpler for the other four properties as well as for projection.

It is not difficult to see that all the implications in Figure 1.1 hold. For instance, to see that every spanning set is projective stationary, note that the definition of the projective stationary set can be reformulated as follows: for every club C in $[\kappa]^\omega$, the projection of $S \cap C$ to ω_1 contains a club in ω_1. So let C be a club in $[\kappa]^\omega$. If S is spanning, then there is a club D in $[\kappa]^\omega$ such that all $x \in D$ have an ω_1-extension in $S \cap C$. Hence $\pi(D) \subset \pi(S \cap C)$, where π denotes the projection to ω_1.

3 Local clubs and full sets

Local clubs form a σ-complete normal filter that extends the club filter. Local clubs need not contain a club, but they do under the large cardinal assumption "Weak Reflection Principle" (WRP).

Definition 3.1[3]**.** *Weak Reflection Principle at κ: for every stationary set $S \subset [\kappa]^\omega$ there exists a set X of size $\aleph_1$ such that $\omega_1 \subset X$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$ (S reflects at X).*

It is not hard to show[1] that WRP at κ implies a stronger version, namely that for every stationary set $S \subset [\kappa]^\omega$, the set of all $X \in [\kappa]^{\omega_1}$ at which S reflects is stationary in $[\kappa]^{\omega_1}$. In other words, every local club in $[\kappa]^\omega$ contains a club.

Thus WRP is equivalent to the statement that every local club contains a club. And clearly, WRP at $\lambda > \kappa$ implies WRP at κ. The consistency strength of WRP at ω_2 is exactly that of the existence of a weakly compact cardinal; the consistency of full WRP is considerably stronger but not known exactly at this time.

Example 3.2. For every ordinal η such that $\omega_1 \leqslant \eta < \omega_2$, let C_η be a club set of $[\eta]^\omega$ of order-type ω_1 (therefore $|C_\eta| = \aleph_1$). Let $S = \bigcup\{C_\eta \mid \omega_1 \leqslant \eta < \omega_2\}$. Then S is a local club in $[\omega_2]^\omega$ and has cardinality $\aleph_2$. By a theorem of Baumgartner and Taylor[6], every club set in $[\omega_2]^\omega$ has size $\aleph_2^{\aleph_0}$. Therefore, WRP at ω_2 implies $2^{\aleph_0} \leqslant \aleph_2$, a result of Todorčević[7].

Let P be a notion of forcing and assume that $|P| \geqslant \aleph_1$. Let $\lambda \geqslant |P|^+$ and consider the model H_λ whose language has the predicates for forcing P as well as the forcing relation. Note that every countable ordinal has a P-name in H_λ.

If $M \in [H_\lambda]^\omega$, a condition q is semi-generic for M if for every name $\dot\alpha$ for a countable ordinal such that $\dot\alpha \in M$, $q \Vdash \dot\alpha \in M$.

The forcing P is semiproper (see [5]) if the set

$$\{M \in [H_\lambda]^\omega \mid \forall p \in M,\ \exists q < p,\ q \text{ is semigeneric for } M\} \tag{3.1}$$

contains a club in $[H_\lambda]^\omega$.

In [1], it is proved that P preserves the stationary sets (in ω_1) if and only if the set (3.1) is a local club. Since $|H_{|P|^+}| = 2^{|P|}$, we conclude that if P is stationary-set-preserving, then WRP at $2^{|P|}$ implies that P is semiproper. Consequently, we have

Theorem 3.3[3]**.** WRP *implies that the class of stationary-set-preserving forcing notions equals the class of semiproper forcing notions.*

Example 3.4 (Namba forcing)[8]**.** This is a forcing (of cardinality $2^{\aleph_2}$) that adds a countable cofinal subset of ω_2 without adding new reals (cf. [9]). It preserves the stationary subsets of ω_1 and by [5], it is not semiproper unless $0^\#$ exists.

We use the Namba forcing to get a partial converse of Theorem 3.3: if the stationary-set-preserving equals semiproper, then WRP holds at ω_2.

Theorem 3.5. *If there exists a stationary set $S \subset [\omega_2]^\omega$ that does not reflect, then the Namba forcing is not semiproper. Hence if every stationary-set-preserving forcing of size $2^{\aleph_2}$ is semiproper, then* WRP *holds at ω_2, and every local club in $[\omega_2]^\omega$ contains a club.*

As Stevo Todorcevic points out, this theorem follows from his result in [10] (CC^* implies WRP) combined with Shelah's result in [5, p. 398], that if the Namba forcing is semiproper then CC^* holds.

Proof. Let $S \subset [\omega_2]^\omega$ be a nonreflecting stationary set and assume that the Namba forcing P is semiproper.

Since S does not reflect, there exists for each α, $\omega_1 \leqslant \alpha < \omega_2$, an operation $F_\alpha : \alpha^{<\omega} \to \alpha$ such that no $x \in S$ is closed under F_α.

Let $\lambda = (2^{\aleph_2})^+$. As the set (3.1) contains a club, there exists some $M \in [H_\lambda]^\omega$ such that $M \cap \omega_2 \in S$, $\langle F_\alpha \mid \omega_1 \leqslant \alpha < \omega_2\rangle \in M$ and there exists some $q \in P$ semigeneric for M.

Let G be a P-generic filter (over V) such that $q \in G$. In $V[G]$, look at $M[G]$, where $M[G] = \{\dot{x}/G \mid \dot{x} \in M\}$. Since G produces a countable cofinal subset of ω_2^V, $M[G] \cap \omega_2^V$ is cofinal in ω_2^V. Let $\alpha < \omega_2^V$ be the least ordinal in $M[G]$ that is not in M. Since G contains a semigeneric condition for M, we have $M[G] \cap \omega_1 = M \cap \omega_1$ and so $\omega_1 \leqslant \alpha < \omega_2^V$ and $M[G] \cap \alpha = M \cap \omega_2$.

Since $\alpha \in M[G]$, $F_\alpha \in M[G]$. Hence $M[G] \cap \alpha$ is closed under F_α. It follows that $x = M \cap \alpha = M[G] \cap \alpha$ belongs to S and is closed under F_α. This is a contradiction.

Now we turn our attention to full sets. First we reformulate the definition: $S \subset [\kappa]^\omega$ is full if and only if there exists a maximal antichain W of stationary subsets of ω_1 such that for every $A \in W$, there exists a club C_A in $[\kappa]^\omega$ with $\hat{A} \cap C_A \subset S$, where $\hat{A}$ is the lifting of A from ω_1 to $[\kappa]^\omega$.

We remark that the full sets form a filter, not necessarily σ-complete. It is proved in [2] that σ-completeness of the filter of full sets is equivalent to the presaturation of the nonstationary ideal on ω_1. It is also known that presaturation follows from WRP which shows that WRP is a large cardinal assumption.

Example 3.6. Let W be a maximal antichain of stationary subsets of ω_1 and consider the model $\langle H_\lambda, \in, <, \ldots \rangle$, whose language has a predicate for W. Let

$$S_W = \{M \in [H_\lambda]^\omega \mid (\exists\, A \in W \cap M)\ \delta_M \in A\}.$$

The clubs $C_A = \{M \in [H_\lambda]^\omega \mid A \in M\}$ for $A \in W$ witness that S_W is full.

We will now show that the sets S_W from Example 3.5 generate the filter of full sets

Lemma 3.7. *Let S be a full set in $[\kappa]^\omega$. There exists a model $\langle H_\lambda, \in, <, \ldots \rangle$, where $\lambda = \kappa^+$, and a maximal antichain W of stationary subsets of ω_1 such that $S_W \subset \hat{S}$.*

Proof. Let S be full in $[\kappa]^\omega$. By the reformulation of full sets, let W be a maximal antichain and for each $A \in W$, let $F_A : \kappa^{<\omega} \to \kappa$ be an operation such that

$$\{x \in C_{F_A} \mid \delta_x \in A\} \subset S.$$

Consider a model $\langle H_\lambda, \in, <, \ldots \rangle$, $\lambda = \kappa^+$, whose language has a predicate for W as well as for the function assigning the operation F_A to each $A \in W$. We claim that for every $M \in S_W$, $M \cap \kappa \in S$. To see this, let $M \in [H_\lambda]^\omega$ and let $A \in W \cap M$ be such that $\delta_M \in A$. Then M is closed under F_A and so $M \cap \kappa \in C_{F_A}$ and $\delta_{M\cap\kappa} = \delta_M \in A$. Hence, $M \cap \kappa \in S$.

Consequently, the filter of full sets on $[\kappa]^\omega$ is generated by the projections of the sets S_W on $[H_\lambda]^\omega$ with $\lambda = \kappa^+$.

In [2], it is proved that the statement that every full set contains a club is equivalent to the saturation of the nonstationary ideal on ω_1 (and so is the statement that every full set contains an ω_1–chain). More precisely,

Theorem 3.8[2]**.** (a) *If the nonstationary ideal on ω_1 is saturated then for every $\kappa \geqslant \omega_2$, every full set in $[\kappa]^\omega$ contains a club.*

(b) *If every full set in $[H_{\omega_2}]^\omega$ contains an ω_1-chain, then the ideal of nonstationary subsets of ω_1 is saturated.*

Consequently, "every full set is reflective" is equivalent to "every full set contains a club" and follows from large cardinal assumptions (such as MM). The consistency of "full = club", being that of the saturation of NS_{ω_1}, is quite strong. Neither "local club = club" nor "full = club" implies the other: WRP has a model in which NS_{ω_1} is not saturated, while the saturation of

NS_{ω_1} is consistent with $2^{\aleph_0} > \aleph_2$ which contradicts WRP. Both are the consequences of MM which therefore implies that "club = local club = full".

4 Projective stationary and spanning sets

In this section, we investigate the projective stationary and spanning sets and particularly a forcing notion associated with such sets. Among others we show that WRP implies that every projective stationary set is spanning (and then spanning = projective stationary).

First we prove a theorem (that generalizes Baumgartner and Taylor's result[6] on clubs) that shows that the equality does not hold in ZFC. Every spanning subset of $[\omega_2]^\omega$ has size $\aleph_2^{\aleph_0}$ while Example 3.2 gives a projective stationary (even a local club) set of $[\omega_2]^\omega$ of size $\aleph_2$. Thus the equality "spanning = projective stationary" implies $2^{\aleph_0} \leqslant \aleph_2$.

Theorem 4.1. *Every spanning set in $[\omega_2]^\omega$ has size $\aleph_2^{\aleph_0}$.*

Proof. Let $S \subset [\omega_2]^\omega$ be spanning. We shall find $2^{\aleph_0}$ distinct elements of S. Let $F : [\omega_2]^2 \to \omega_1$ be such that for each $\eta < \omega_2$, the function F_η, defined by $F_\eta(\xi) = F(\{\xi, \eta\})$, is a one-to-one mapping of η to ω_1. As S is spanning, there exists an operation G on ω_2 such that every $M \in [\omega_2]^\omega$ closed under G has an ω_1-extension N that is closed under F and $N \in S$.

We shall find models M_f, $f \in 2^\omega$, closed under G, and $\delta < \omega_1$ such that

(a) $\delta_{M_f} \leqslant \delta$ for each f, and (4.1)

(b) if $f \neq g$ then there exist $\xi \in M_f$ and $\eta \in M_g$ such that $F(\xi, \eta) \geqslant \delta$.

Now assume that we have models M_f that satisfy (4.1). If $f \neq g$ and if $x \in [\omega_2]^\omega$ is such that $M_f \cup M_g \subset x$ and x is closed under F, then $\delta_x > \delta$. Hence if N_f and N_g are ω_1-extensions of M_f and M_g, respectively, and are closed under F, then $N_f \neq N_g$. Thus we get $\{N_f, \mid f \in 2^\omega\}$ such that the N_f's are $2^{\aleph_0}$ elements of S.

Toward the construction of the models M_f, let $c_\alpha \subset \alpha$, for each $\alpha < \omega_2$ of cofinality ω, be a set of order type ω with $\sup c_\alpha = \alpha$ and let M_α be the closure of c_α under G. Let $Z \subset \omega_2$ and $\delta < \omega_1$ be such that Z is stationary and for each $\alpha \in Z$, $M_\alpha \subset \alpha$ and $\delta_{M_\alpha} = \delta$.

We shall find, for each $s \in 2^{<\omega}$ (the set of all finite 0-1-sequences), a stationary set Z_s and an ordinal $\xi_s < \omega_2$ such that

(i) if $s \subset t$, then $Z_t \subset Z_s$,

(ii) $(\forall\, \alpha \in Z_s)\ \xi_s \in c_\alpha$,

(iii) $\xi_{\langle s0\rangle} < \xi_{\langle s1\rangle}$ and $F(\xi_{\langle s0\rangle}, \xi_{\langle s1\rangle}) \geqslant \delta$. (4.2)

Once we have the ordinals ξ_s, we let, for each $f \in 2^\omega$, M_f be the closure under G of the set $\{\xi_{f\restriction n} \mid n < \omega\}$. Clearly,

$$M_f = \bigcup_{n=0}^{\infty} M_{f\restriction n},$$

where for each $s \in 2^{<\omega}$, M_s is the closure under G of $\{\xi_{s\restriction 0}, \dots, \xi_s\}$. Since $M_s \subset M_\alpha$ for $\alpha \in Z_s$, we have $\delta_{M_f} \leqslant \delta$ for every $f \in 2^\omega$. The condition (4.2)(iii) guarantees that the models M_f satisfy (4.1).

The Z_s and ξ_s are constructed by induction on $|s|$. Given Z_s, there are $\aleph_2$ ordinals ξ such that $S_\xi = \{\alpha \in Z_s \mid \xi \in c_\alpha\}$ is stationary. Consider the first $\omega_1 + 1$ of these ξ's and let $\eta = \xi_{\langle s1\rangle}$ be the $\omega_1 + 1$st element, and $Z_{\langle s1\rangle} = S_\eta$. Then find some $\xi < \eta$ among the first ω_1 elements such that $F_\eta(\xi) \geqslant \delta$ and let $\xi_{\langle s0\rangle}$ be such ordinal ξ and let $Z_{\langle s0\rangle} = S_\xi$.

In Definition 2.6, we have defined spanning sets in $[\kappa]^\omega$ as satisfying a certain condition at every $\lambda \geqslant \kappa$. The following lemma shows that it is enough to consider the condition at H_{κ^+}.

Lemma 4.2. *A set $S \subset [\kappa]^\omega$ is spanning if and only if for every club C in $[H_{\kappa^+}]^\omega$ there exists a club D in $[H_{\kappa^+}]^\omega$ such that every $M \in D$ has an ω_1-extension $N \in C$ such that $N \cap \kappa \in S$.*

Proof. It is easy to verify that if the condition $\forall C,\ \exists D$ holds at some $\mu > \lambda$ then it holds at λ. Thus assume that $\lambda \geqslant \kappa^+$ and the condition of the lemma holds, and let us prove that for every club C in $[H_\lambda]^\omega$ there exists a club D in $[H_\lambda]^\omega$ such that every $M \in D$ has an ω_1-extension $N \in C$ such that $N \cap \kappa \in S$.

Let C be a club in $[H_\lambda]^\omega$ and let F be an operation on H_λ such that $C_F \subset C$. Let C_0 be a club in $[H_{\kappa^+}]^\omega$ be such that $\hat{C}_0 \subset C_F$. Let D_0 be a club in $[H_{\kappa^+}]^\omega$ such that every $M_0 \in D_0$ has an ω_1-extension $N_0 \in C_0$ with $N_0 \cap \kappa \in S$. Let $D = \hat{D}_0$ be the set of all $M \in [H_\lambda]^\omega$ such that $M \cap H_{\kappa^+} \in D_0$. Let $M \in D$ and let $M_0 = M \cap H_{\kappa^+}$. Then $M_0 \in D_0$. Let $N_0 \in C_0$ be an ω_1-extension of M_0 such that $N_0 \cap \kappa \in S$. We let N be the F-closure of $M \cup (N_0 \cap \kappa)$ in H_λ. The model N is in C_F. We claim that $N \cap \kappa = N_0 \cap \kappa$. This shall give us that $N \cap \kappa \in S$ and N is an ω_1-extension of M.

Let $\alpha \in N \cap \kappa$. Let τ be a skolem term in $(H_\lambda, \in, <, F)$ and let $a \in M$ and $\alpha_0, \dots, \alpha_n \in N_0 \cap \kappa$ be such that $\alpha = \tau(a, \alpha_0, \dots, \alpha_n)$. Define $h : [\kappa]^{n+1} \to \kappa$ by

$$h(\beta_0, \dots, \beta_n) = \begin{cases} \tau(a, \beta_0, \dots, \beta_n), & \text{if } \tau(a, \beta_0, \dots, \beta_n) < \kappa, \\ 0, & \text{otherwise.} \end{cases}$$

Then $h \in M$ and hence $h \in M \cap H_{\kappa^+} = M_0 \subset N_0$. Therefore,

$$\alpha = h(\alpha_0, \dots, \alpha_n) \in N_0 \cap \kappa.$$

Definition 4.3. *Let $S \subset [\kappa]^\omega$ be a stationary set. P_S is the forcing notion that shoots an ω_1-chain through S: forcing conditions are continuous $(\gamma+1)$-chains, $\langle x_\alpha \mid \alpha \leqslant \gamma \rangle$, $\gamma < \omega_1$, such that $x_\alpha \in S$ for each α, and $\delta_{x_\alpha} < \delta_{x_\beta}$ when $\alpha < \beta \leqslant \gamma$. The ordering is by extension.*

The forcing P_S does not add new countable sets and so ω_1 is preserved. The generic ω_1-chain is cofinal in $[\kappa]^\omega$ and so κ is collapsed to ω_1.

The following theorem gives a characterization of projective stationary sets and spanning sets in terms of the forcing P_S:

Theorem 4.4. (a) *A set $S \subset [\kappa]^\omega$ is projective stationary if and only if the forcing P_S preserves stationary subsets of ω_1.*

(b) *A set $S \subset [\kappa]^\omega$ is spanning if and only if the forcing P_S is semiproper.*

Proof. (a) This equivalence was proved in [2]; we include the proof for the sake of completeness.

Let A be a stationary subset of ω_1. We will show that P_S preserves A if and only if $\hat{A} \cap S$ is stationary.

First assume that $\hat{A} \cap S$ is nonstationary and let $C \subset [\kappa]^\omega$ be a club such that for every $x \in S \cap C$, $\delta_x \notin A$.

Let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be a generic ω_1-chain; there exists a club $D \subset \omega_1$ such that for each $\alpha \in D$, $\alpha = \delta_{x_\alpha}$ and $x_\alpha \in C$. Then D is a club in $V[G]$ disjoint from A.

Conversely, assume that $\hat{A} \cap S$ is stationary. We will show that A remains stationary in $V[G]$. Let $\dot{C}$ be a name for a club in ω_1 and let p be a condition. Let λ be sufficiently large. Since $\hat{A} \cap \hat{S}$ is stationary in $[H_\lambda]^\omega$, there exists a countable model M containing $\dot{C}$ and p such that $\delta_M \in A$ and $M \cap \kappa \in S$. Let $\langle x_\alpha \mid \alpha < \delta_M \rangle$ be an M-generic δ_M-chain extending p. By genericity, $M \cap \kappa = \bigcup \{x_\alpha \mid \alpha < \delta_M\}$. Since $M \cap \kappa \in S$, it can be added on the top of the chain $\langle x_\alpha \mid \alpha < \delta_M \rangle$ to form a condition q. This condition extends p and forces that δ_M is a limit point of $\dot{C}$, and hence q forces that $\delta_M \in \dot{C} \cap A$. Therefore, A is stationary in $V[G]$.

(b) First let S be a spanning set in $[\kappa]^\omega$. Let $\lambda \geqslant (2^\kappa)^+$ (note that $|P_S| \leqslant 2^\kappa$) and let us prove that the set (3.1) contains a club in $[H_\lambda]^\omega$.

Let C be the club of all models $N \in [H_\lambda]^\omega$ that contain S, the forcing P_S and the forcing relation. By Definition 2.6, there exists a club D in $[H_\lambda]^\omega$ such that every $M \in D$ has an ω_1-extension $N \in C$ such that $N \cap \kappa \in S$. We claim that the set (3.1) contains D.

Let $M \in D$ and $p \in M$. Let $N \in C$ be an ω_1-extension of M such that $N \cap \kappa \in S$. We enumerate all ordinals in $N \cap \kappa$ and all names $\dot\alpha \in N$ for ordinals. Starting with $p_0 = p$, construct a sequence of conditions $p_0 > p_1 > \cdots > p_n > \cdots$ such that $p_n \in N$ for each n, and for every $\dot\alpha \in N$ there are some p_n and $\beta \in N$ such that $p_n \Vdash \dot\alpha = \beta$, and that for every $\gamma \in N \cap \kappa$ there is some $p_n = \langle x_\xi \mid \xi \leqslant \alpha\rangle$ such that $\gamma \in x_\alpha$. The sequence produces a continuous chain whose limit is the set $N \cap \kappa$. Since $N \cap \kappa \in S$, it can be put on the top of this chain to form a condition $q < p$ that decides every ordinal name in N as an ordinal in N. Now since N is an ω_1-extension of M, they have the same set of countable ordinals and it follows that q is semigeneric for M.

Conversely, assume that P_S is semiproper. Let $\lambda \geqslant (2^\kappa)^+$ and let C be a club in $[H_\lambda]^\omega$. Let F be an operation on H_λ such that $C_F \subset C$. Let $\mu > \lambda$ be such that $F \in H_\mu$. Since P_S is semiproper, there is a club $D \subset [H_\lambda]^\omega$ such that every model in D has the form $M \cap H_\lambda$, where $F \in M \in [H_\mu]^\omega$, and there is a semigeneric condition for M. We shall prove that every $M \cap H_\lambda \in D$ has an ω_1-extension N in C_F such that $N \cap \kappa \in S$.

Let $M \cap H_\lambda \in D$ and let q be a semigeneric condition for $M \in [H_\mu]^\omega$. Let G be a generic filter on P_S over V such that $q \in G$. Working in $V[G]$, let $M[G]$ be the set of all $\dot a/G$ for $\dot a \in M$, and let $N = M[G] \cap (H_\lambda)^V$. Since P_S does not add new countable sets, $N \in V$. Since $F \in M[G]$, $M[G]$ is closed under F, and so is N. Hence $N \in C_F$. Since q is semigeneric for M, $M[G] \cap \omega_1 = M \cap \omega_1$, and so N is an ω_1-extension of $M \cap H_\lambda$. Since the union of the generic ω_1-chain $\langle x_\alpha \mid \alpha < \omega_1\rangle$ is κ, we claim that the union of $\langle x_\alpha \mid \alpha < \delta_M\rangle$ is $M[G] \cap \kappa = N \cap \kappa$. Granting this claim, this union is x_{δ_M} and $\langle x_\alpha \mid \alpha \leqslant \delta_M\rangle$ is a condition in P_S. Therefore, $x_{\delta_M} \in S$, and hence $N \cap \kappa \in S$.

We now proceed to prove the claim. We just need to check that $x_{\delta_M} = M[G] \cap \kappa$. We have $M \subset M[G]$ and $G \in M[G]$. In $V[G]$, G defines a bijection $f : \omega_1 \to \kappa$. Let $\dot f \in M$ be a canonical name for this f. We then have

$$\Vdash \forall p \in \dot G\, \exists\, \alpha < \omega_1\, \forall\, \gamma < \mathrm{dom}(p)\; p(\gamma) \subset \dot f''\alpha$$

and

$$\Vdash \forall\, \alpha < \omega_1\, \exists\, p \in \dot G\, \forall \gamma < \alpha\, \exists\, \beta < \mathrm{dom}(p)\; \dot f(\gamma) \in p(\beta).$$

Also, $\dot f/G \in M[G]$ and $\dot f/G \cap M[G] : \delta_M \to M[G] \cap \kappa$ is a bijection.

First we check that $M[G] \cap \kappa \subset x_{\delta_M}$.

Let $\alpha \in M[G] \cap \kappa$. Let $\dot\alpha \in M$ be a name such that $\Vdash \dot\alpha < \kappa$ and $\alpha = \dot\alpha/G$. Then

$$\Vdash \exists\, p \in \dot G\big(\dot\alpha \in \textstyle\bigcup p\big).$$

Hence $M \models (\Vdash \exists\, \xi < \omega_1\; \dot\alpha \in \dot x_\xi)$. Let $\dot\xi \in M$ be a name for a countable ordinal such that

$$\Vdash \dot\alpha \in \dot x_{\dot\xi}.$$

Since the semigeneric condition q is in G, let $\xi < \delta_M$ and $r \in G$ be such that $r \Vdash \dot\alpha \in \dot x_\xi$. It follows that

$$\alpha = \dot\alpha/G \in (\dot x/G)_\xi \subset x_{\delta_M}.$$

Secondly, we check that $x_{\delta_M} \subset M[G] \cap \kappa$.

Let $\alpha < \delta_M$. Let $\beta \in x_\alpha$. We show that $\beta \in M[G]$.

Let $p \in M[G] \cap G$ be such that $x_\alpha = p(\alpha)$. Let $\dot{p} \in M$ be such that $\dot{p}/G = p$. Let $\dot{\alpha} \in M$ be such that $\dot{\alpha}/G = \alpha$. Let $\dot{\xi} \in M$ be such that

$$q \Vdash \dot{p}(\dot{\alpha}) \subset \dot{f}''\dot{\xi}.$$

It follows that $\beta \in M[G] \cap \kappa$.

As a corollary, if stationary-set-preserving = semiproper, then projective stationary = spanning. We shall prove the converse later in this section.

It follows that WRP implies that projective stationary = spanning. More precisely,

Corollary 4.5. *If every local club in* $[H_{(2^\kappa)^+}]^\omega$ *contains a club, then every projective stationary set in* $[\kappa]^\omega$ *is spanning.*

Looking at the proof of (b), we observe that the club D in the definition of spanning is the club that witnesses the semiproperness of P_S. If we replace "club" by "local club", the proof goes through as before and we get the following characterization of projective stationary sets.

Lemma 4.6. *A set* $S \subset [\kappa]^\omega$ *is projective stationary if and only if for every* $\lambda \geqslant \kappa$*, for every club* $C \subset [\lambda]^\omega$*, there exists a local club* D *in* $[\lambda]^\omega$ *such that every* $x \in D$ *has an* ω_1*-extension* y *in* C *such that* $y \cap \kappa \in S$.

The quantifier $\forall\, C$ in Definition 2.6 and Lemma 4.6 can be removed by the following trick. Let S be a stationary set in $[\kappa]^\omega$ and let $\lambda \geqslant \kappa^+$ and $\mu = \lambda^+$. Let

$$S^*_\lambda = \{M \cap H_\lambda \mid M \in [H_\mu]^\omega,\ S \in M \text{ and } M \cap \kappa \in S\}, \tag{4.3}$$

and

$$Sub(S^*_\lambda) = \{M \in [H_\lambda]^\omega \mid M \text{ has an } \omega_1\text{-extension} N \in S^*_\lambda\}. \tag{4.4}$$

Here we assume that H_μ has the Skolem functions and $M \in [H_\mu]^\omega$ is an elementary submodel. The set S^*_λ is a stationary subset of $[H_\lambda]^\omega$ and is equivalent to the lifting of S.

Lemma 4.7. (a) *S is spanning if and only if* $\mathrm{Sub}(S^*_\lambda)$ *contains a club.*

(b) *S is projective stationary if and only if* $\mathrm{Sub}(S^*_\lambda)$ *is a local club.*

Proof. We prove (a) as (b) is proved similarly.

Let $\lambda \geqslant \kappa^+$ and $\mu = \lambda^+$.

First assume that S is spanning. Let

$$C = \{M \cap H_\lambda \mid M \in [H_\mu]^\omega \text{ and } S \in M\}.$$

Let D be a club in $[H_\lambda]^\omega$ such that every $M \in D$ has an ω_1-extension $N \in C$ with $N \cap \kappa \in S$. Then $D \subset \mathrm{Sub}(S^*_\lambda)$.

Conversely, assume that S is not spanning. Let $C = C_F$ be the least counterexample. As F is definable in H_μ from S, it belongs to every elementary countable submodel M of H_μ such that $S \in M$. Hence every $N \in S^*_\lambda$ is closed under F and it follows that $S^*_\lambda \subset C$. Therefore, every $M \in \mathrm{Sub}(S^*_\lambda)$ has an ω_1-extension $N \in C$ such that $N \cap \kappa \in S$. Since C is a counterexample, $\mathrm{Sub}(S^*_\lambda)$ does not contain a club.

Now we prove that projective stationary = spanning implies that stationary-set-preserving = semiproper. This is a consequence of the following lemma.

Lemma 4.8. *Let P be a forcing* ($|P| \geqslant \aleph_1$) *and let* $\lambda \geqslant |P|^+$.

(a) *P is semiproper if and only if the set* (3.1) *is spanning.*

(b) *P preserves stationary sets in* ω_1 *if and only if the set* (3.1) *is projective stationary.*

Proof. Both (a) and (b) have the same proof, using Definition 2.6 and Lemma 4.6. The left-to- right implications are obvious, as the club is spanning and the local club is projective stationary. Thus assume (for (a)) that the set (3.1) is spanning. If follows from Definition 2.6 that there exists a club D in $[H_\lambda]^\omega$ such that every $M \in D$ has an ω_1-extension in the set (3.1). But since every condition that is semigeneric for an ω_1-extension of M is semigeneric for M, it follows that every $M \in D$ belongs to the set (3.1). Thus the set (3.1) contains a club and P is semiproper.

Corollary 4.9. *If every projective stationary set is spanning, then every forcing that preserves stationary sets of ω_1 is semiproper.*

We conclude sec. 4 with Figure 4.1 describing the implications under the assumption of WRP.

full

↗ ↘

club = local club projective stationary = spanning

↘ ↗

reflective

Figure 4.1

5 Strong reflection principle

The Strong Reflection Principle (SRP) is the statement that every projective stationary set contains an ω_1-chain. Thus SRP implies that every projective stationary set is reflective and that every full set contains a club. As SRP implies WRP (cf. [1]) we also have local club = club and projective stationary = spanning, obtaining Figure 1.2 from the introduction.

We shall now look more closely at spanning sets and prove, among others, that if all spanning sets contain an ω_1-chain then SRP holds.

Definition 5.1. *For $X \subset [\kappa]^\omega$, let*

$$X^\perp = \{M \in [H_{\kappa^+}]^\omega \mid M \textit{ has no } \omega_1\textit{-extension } N \textit{ such that } N \cap \kappa \in X\}.$$

The set $X^\perp$ is a subset of $[H_{\kappa^+}]^\omega$ and is disjoint from $\hat{X}$. If X is nonstationary, then $X^\perp$ contains a club. Let us therefore restrict ourselves to stationary sets $X \subset [\kappa]^\omega$.

Lemma 5.2. (i) *If $S_1 \subset S_2 \subset [\kappa]^\omega$, then $S_2^\perp \subset S_1^\perp$.*

(ii) *If $S_1 \equiv S_2$ mod club filter, then $S_1^\perp \equiv S_2^\perp$ mod club filter.*

(iii) *$\hat{S} \cup S^\perp$ is spanning (where $\hat{S}$ is the lifting of S to H_{κ^+}).*

(iv) *S is spanning if and only if $S^\perp$ is nonstationary.*

Proof. (ii) Let $F : \kappa^{<\omega} \to \kappa$ be such that $S_1 \cap C_F = S_2 \cap C_F$. Let $D = \{M \in [H_{\kappa^+}]^\omega \mid F \in M\}$. D is a club in $[H_{\kappa^+}]^\omega$. We claim that $S_1^\perp \cap D = S_2^\perp \cap D$.

If $M \in D$ and $M \notin S_1^\perp$, then M has an ω_1-extension N such that $N \cap \kappa \in S_1$. Since $F \in M \subset N$, $N \cap \kappa$ is closed under F. So $N \cap \kappa \in S_2$. Hence $M \notin S_2^\perp$. Similarly for the other direction, and so we have $S_1^\perp \cap D = S_2^\perp \cap D$.

(iii) Let $\lambda \geqslant \kappa^+$ be arbitrary and let C be a club in $[H_\lambda]^\omega$. Let $F : H_\lambda^{<\omega} \to H_\lambda$ be such that $C_F \subset C$. We claim that every $M \in C_F$ has an ω_1-extension $N \in C$ such that $N \cap H_{\kappa^+} \in \hat{S} \cup S^\perp$, i.e., either $N \cap \kappa \in S$ or $N \cap H_{\kappa^+} \in S^\perp$.

Let $M \in C_F$. If $M \cap H_{\kappa^+} \in S^\perp$, then we are done. Otherwise, let $M_0 = M \cap H_{\kappa^+}$. M_0 has an ω_1-extension $N_0 \in [H_{\kappa^+}]^\omega$ such that $N_0 \cap \kappa \in S$. Let N be the closure of $M \cup (N_0 \cap \kappa)$ under F. We have that $N \in C$ and $M \subset N$. By an argument exactly as in the proof of Lemma 4.2, we conclude that $N \cap \kappa = N_0 \cap \kappa$. Hence N is an ω_1-extension of M and $N \cap \kappa \in S$.

(iv) If S is spanning then by definition the set of all $M \in [H_{\kappa^+}]^\omega$ that do have an ω_1-extension N with $N \cap \kappa \in S$ contains a club, and hence $S^\perp$ is nonstationary. If $S^\perp$ is nonstationary, then, since $\hat{S} \cup S^\perp$ is spanning, $\hat{S}$ must be spanning. Hence S is spanning.

Theorem 5.3. *If every spanning set in $[H_{\kappa^+}]^\omega$ contains an ω_1-chain, then every projective stationary set in $[\kappa]^\omega$ contains an ω_1-chain.*

Proof. Let S be a projective stationary set in $[\kappa]^\omega$. By Lemma 5.2(iii), $\hat{S} \cup S^\perp$ is spanning in $[H_{\kappa^+}]^\omega$ and therefore contains an ω_1-chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$. We claim that $\{\alpha < \omega_1 \mid M_\alpha \cap \kappa \in S\}$ contains a club and therefore S contains an ω_1-chain.

Suppose not. The set $A = \{\alpha < \omega_1 \mid M_\alpha \in S^\perp \text{ and } \alpha = \delta_{M_\alpha}\}$ is stationary. Let

$$C = \{N \in [H_{\kappa^+}]^\omega \mid \kappa \in N \text{ and } (\forall\, \beta \in N \cap \omega_1)\ M_\beta \in N\}.$$

C is a club in $[H_{\kappa^+}]^\omega$. Since S is projective stationary, there exists an $N \in C$ such that $\delta_N \in A$ and $N \cap \kappa \in S$. For every $\alpha < \delta_N$ we have $M_\alpha \subset N$. Hence $M_{\delta_N} \subset N$ and $M_{\delta_N} \cap \omega_1 = N \cap \omega_1 = \delta_N$. Therefore, $M_{\delta_N} \notin S^\perp$. This is a contradiction.

Corollary 5.4. *If every spanning set contains an ω_1-chain, then* SRP *holds.*

6 A structure theorem

The following definition relativizes projective stationary and spanning.

Definition 6.1. *Let A be a stationary set of countable ordinals and let $S \subset [\kappa]^\omega$.*

(a) *S is projective stationary above A if for every stationary $B \subset A$, the set $\{x \in S \mid \delta_x \in B\}$ is stationary.*

(b) *S is spanning above A if for every club $C \subset [H_{\kappa^+}]^\omega$ there exists a club D in $[H_{\kappa^+}]^\omega$ such that every $M \in D$ with $\delta_M \in A$ has an ω_1-extension $N \in C$ such that $N \cap \kappa \in S$.*

The following result is proved in [2].

Lemma 6.2. *If the nonstationary ideal on ω_1 is saturated, then for every stationary set $S \subset [\kappa]^\omega$ there exists a stationary $A \subset \omega_1$ such that S is projective above A.*

Notice that the conclusion of the lemma can be stated as: the complement of S is not full. Thus Lemma 6.2 is a reformulation of Theorem 3.8(a).

Corollary 6.3. *If the nonstationary ideal on ω_1 is saturated then for every stationary $S \subset [\kappa]^\omega$ there exists a stationary $A \subset \omega_1$ such that*

(i) *S is projective stationary above A, and*

(ii) *$\{x \in S \mid \delta_x \notin A\}$ is nonstationary.*

Proof. Let W be a maximal antichain of stationary sets $A \subset \omega_1$ such that S is projective stationary above A. Since $|W| \leqslant \aleph_1$, there exists a stationary A_S such that

$$A_S = \Sigma\{A \mid A \in W\}$$

in the Boolean algebra $P(\omega_1)/NS$. It is easy to verify that A_S has the two properties.

Corollary 6.4. SRP *implies* WRP. *In fact, assuming* SRP, *for every stationary $S \subset [\kappa]^\omega$ there exists a set X of size $\aleph_1$ such that $\omega_1 \subset X$ and an ω_1-chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ with $\alpha = \delta_{N_\alpha}$ for all $\alpha < \omega_1$ such that $X = \bigcup_{\alpha<\omega_1} N_\alpha$ and $N_\alpha \in S$ for every $\alpha \in A_S$.*

Proof. The set $S \cup \{x \mid \delta_x \notin A_S\}$ is projective stationary and by SRP it contains an ω_1-chain.

The proof that WRP implies that projective stationary = spanning applies to the relativized notions, i.e., projective stationary above A = spanning above A. Thus we obtain the following theorem.

Theorem 6.5. *Assume* SRP. *Let $\kappa \geqslant \omega_2$ and let $S \subset [\kappa]^\omega$ be stationary. There exists a stationary A_S such that*

(i) *for almost all $x \in S$, $\delta_x \in A_S$,*

(ii) *almost all x with $\delta_x \in A_S$ have an ω_1-extension $y \in S$.*

Moreover, the set A_S is unique mod club filter and if $S_1 \equiv S_2$ then $A_{S_1} \equiv A_{S_2}$.

Also, a stronger version of (ii) holds: for every $\lambda \geqslant \kappa$ and every model $(\lambda, \ldots)$, almost all countable $M \prec (\lambda, \ldots)$ with $\delta_M \in A_S$ have an ω_1-extension $N \prec (\lambda, \ldots)$ such that $N \cap \kappa \in S$.

7 Order types and canonical functions

Two functions $f, g : \omega_1 \to \omega_1$ are equivalent (mod club filter) if the set $\{\alpha < \omega_1 \mid f(\alpha) = g(\alpha)\}$ contains a club. $f < g$ if and only if $\{\alpha < \omega_1 \mid f(\alpha) < g(\alpha)\}$ contains a club. Then $<$ is a well-founded partial order of the equivalence classes and every function can be assigned a rank in this partial order. For all $\eta < \omega_2$, there exist the canonical function f_η such that each f_η has rank η and when η is a limit ordinal then f_η is the least upper bound of $\{f_\xi \mid \xi < \eta\}$. The canonical functions are unique and for $\omega_1 \leqslant \eta < \omega_2$, if g_η is any one-to-one mapping of ω_1 onto η, then for almost all $\alpha < \omega_1$,

$$f_\eta(\alpha) = \text{order type of } \{g_\eta(\beta) \mid \beta < \alpha\}. \tag{7.1}$$

The Boundedness Principle is the statement

$$(\forall\, g : \omega_1 \to \omega_1)(\exists\, \eta < \omega_2)\; g < f_\eta. \tag{7.2}$$

This follows from the saturation of the nonstationary ideal on ω_1 (but the consistency strength is considerably less).

Theorem 7.1. *The boundedness principle is equivalent to the following statement: for every club $C \subset \omega_1$, the set*

$$\{x \in [\omega_2]^\omega \mid \textit{order-type}(x) \in C\} \textit{ is a local club.} \tag{7.3}$$

Proof. First assume that for every club C the set (7.3) is a local club. Let $g : \omega_1 \to \omega_1$ be an arbitrary function.

Let $C = \{\gamma < \omega_1 \mid (\forall\, \alpha < \gamma)\; g(\alpha) < \gamma\}$.

Let η and $\langle x_\alpha \mid \alpha < \omega_1\rangle$ be such that $\omega_1 < \eta < \omega_2$ and $\langle x_\alpha \mid \alpha < \omega_1\rangle$ is an ω_1-chain which is a club in $[\eta]^\omega$ and for all $\alpha < \omega_1$ order-type $(x_\alpha) \in C$. By our assumption, such η exists.

We claim that $g < f_\eta$. By (7.1), $f_\eta(\alpha) = \text{order-type}(x_\alpha)$ for almost all $\alpha < \omega_1$. Let

$$D = \{\alpha \in C \mid \alpha < f_\eta(\alpha) = \text{order-type}(x_\alpha)\}.$$

For each $\alpha \in D$ we have $f_\eta(\alpha) \in C$ and $f_\eta(\alpha) > \alpha$, while $g(\alpha) < \alpha'$, where α' is the least element of C greater than α. Thus $g < f_\eta$, witnessed by D.

Conversely, assume that for every $g : \omega_1 \to \omega_1$, there exists an $\eta < \omega_2$ such that $g < f_\eta$. Let $C \subset \omega_1$. Consider the set

$$D = \{\eta < \omega_2 \mid \{\alpha < \omega_1 \mid f_\eta(\alpha) \in C\} \text{ contains a club}\}.$$

Using canonicity, it is easy to verify that D is closed. We claim that D is unbounded.

Let $\eta_0 < \omega_2$. We construct a sequence of functions $\langle g_k \mid k < \omega\rangle$ and a sequence of ordinals $\langle \eta_k \mid k < \omega\rangle$ so that

$$f_{\eta_0} < g_0 < f_{\eta_1} < g_1 < \cdots$$

and that $g_k(\alpha) \in C$ for every k and every α. This can be done since C is unbounded and by our assumption. Let

$$\eta = \lim_k \eta_k.$$

Then for almost α,

$$f_\eta(\alpha) = \lim_k f_{\eta_k}(\alpha) = \lim_k g_k(\alpha).$$

Since C is closed, we have $f_\eta(\alpha) \in C$ for almost α, and so $\eta \in D$.

Now if $\eta \in D$ and $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ is a club in $[\eta]^\omega$, then by (7.1) the order type of x_α is $f_\eta(\alpha)$ for almost all $\alpha < \omega_1$, and therefore

$$\{x \in [\eta]^\omega \mid \text{order-type}(x) \in C\}$$

contains a club in $[\eta]^\omega$. Thus (7.3) is a local club.

Corollary 7.2. *If SRP holds then for every stationary set $S \subset [\kappa]^\omega$, the set* $\{\text{order-type}(x \cap \omega_2) \mid x \in S\}$ *is stationary.*

Proof. SRP implies both the boundedness principle and local club = club, and so the set

$$\{x \in [\kappa]^\omega \mid \text{order-type}(x \cap \omega_2) \in C\}$$

contains a club for every club $C \subset \omega_1$.

References

[1] Feng Q, Jech T. Local clubs, reflection, and preserving stationary sets. *Proc London Math Soc* (3), **58**(2): 237–257 (1989)

[2] Feng Q, Jech T. Projective stationary sets and a strong reflection principle. *J London Math Soc* (2), **58**(2): 271–283 (1998)

[3] Foreman M, Magidor M, Shelah S. Martin's maximum, saturated ideals, and nonregular ultrafilters, I. *Ann of Math* (2), **127**(1): 1–47 (1988)

[4] Jech T. Some combinatorial problems concerning uncountable cardinals. *Ann Math Logic*, **5**: 165–198 (1972/1973)

[5] Shelah S. Proper Forcing. Lecture Notes in Mathematics, vol 1476. Berlin: Springer-Verlag, 1982

[6] Baumgartner J E, Taylor A D. Saturation properties of ideals in generic extensions, I. *Trans Amer Math Soc*, **270**(2): 557–574 (1982)

[7] Bekkali M. Topics in Set Theory. Lecture Notes in Mathematics, vol 1376. Berlin: Springer-Verlag, 1991.

[8] Namba K. Independence proof of (ω, ω_α)-distributive law in complete Boolean algebras. *Comment Math Univ St Paul*, **19**: 1–12 (1971)

[9] Jech T. Set Theory. Springer Monographs in Mathematics. Berlin: Springer-Verlag, 2003

[10] Todorčević S. Conjectures of Rado and Chang and cardinal arithmetic. In: Finite and infinite combinatorics in Sets and Logic (Banff, AB, 1991), *NATO Adv Sci Inst Ser C Math Phys Sci*, vol 411, pp. 385–398. Dordrecht: Kluwer Acad Publ, 1993