1 Categoricity

Definition 1.1. A theory Γ is *countably categorical* if all of its countable models are isomorphic to each other.

Example 1.2. Let Γ be the theory of linear dense ordering without endpoints. Then Γ is countably categorical.

Here, Γ is the theory in the language with single binary relational symbol, including the axioms that state $\leq$ is a linear ordering, plus $\forall x \forall y\ x < y \to \exists z\ x < z < y$ (density), $\forall x \exists y\ y < x$ (no lower endpoint) and $\forall x \exists y\ y > x$ (no upper endpoint). As usual, the statement $x < y$ is a shorthand for $x \leq y \wedge x \neq y$.

Proof. Suppose that $\langle L, \leq_L \rangle$ and $\langle K, \leq_K \rangle$ are dense linear orders without endpoints, both countable; we must show that they are isomorphic. List their respective underlyng sets as $L = \{l_n : n \in N\}$, $K = \{k_n : n \in N\}$. By induction on n build finite partial order-preserving injective functions $f_n : L \to K$ so that

- $f_0 = 0 \subset f_1 \subset f_2 \subset \dots$;
- $k_n \in \text{dom}(f_{n+1})$ and $l_n \in \text{rng}(f_{n+1})$.

If the induction succeeds, then $\bigcup_n f_n$ is going to be the desired isomorphism of L and K.

To perform the induction step, suppose that f_n is given. We will first extend f_n to a finite order preserving function g such that $l_n \in \text{dom}(g)$. There are several cases depending on the position of l_n. If $l_n \in \text{dom}(f_n)$ then let $g = f_n$. If l_n sits strictly between two successive elements of $\text{dom}(f_n)$, say $l < l_n < l'$, then use the density of the ordering K to find $k \in K$ strictly between $f_n(l)$ and $f_n(l')$, and define g to be the function f_n extended by a single value, $g(l_n) = k$. If l_n sits below the smallest element of $\text{dom}(f_n)$, say $l_n < l$, then use the fact that the ordering K has no endpoints to find a point $k \in K$ which is below $f_n(l)$, and define g to be the function f_n extended by a single value, $g(l_n) = k$.If l_n sits above the largest element of $\text{dom}(f_n)$, say $l < l_n$, then use the fact that the ordering K has no endpoints to find a point $k \in K$ which is above $f_n(l)$, and define g to be the function f_n extended by a single value, $g(l_n) = k$ again.

Now, by a symmetric process, with the roles of L, K reversed, extend the function g to a finite order preserving function h such that $k_n \in \text{rng}(h)$. The induction step is concluded by letting $f_{n+1} = h$. □

Example 1.3. Let Γ be the theory of linear order in which every element has an immediate successor and immediate predecessor. Then Γ is not countably categorical.

Here, Γ is the theory in the language with single binary relational symbol, including the axioms that state $\leq$ is a linear ordering, plus $\forall x \exists y\ x < y \wedge \forall z \neg (x < z < y)$ (immediate successor) and $\forall x \exists y\ y < x \wedge \forall z \neg (y < z < z)$ (immediate predecessor).

Proof. Consider the following two countable models of Γ: the model M which is isomorphic to a single copy of the integers with their usual ordering, and the model N which is isomorphic to two copies of the integers with their usual ordering, on top of each other. They are clearly not isomorphic, since in M, between any two elements m_0, m_1 there are only finitely many elements in between, while in N, if two elements n_0, n_1 are chosen from distinct copies of the integers, there are inifnitely many points in between. □

Example 1.4. The theory of abelian, divisible, torsion free groups is not countably categorical.

Here Γ is the theory in the language $\cdot, 1$ which includes the axioms of group theory, plus the axiom $\forall x \forall y\ x \cdot y = y \cdot x$ (abelian), for each natural number n the axiom $\forall x\ x \neq 1 \rightarrow x \cdot x \cdot \ldots n \text{ times} \ldots \cdot x \neq 1$ (torsion-free) and the axiom $\forall x \exists y\ y \cdot y \cdot \ldots n \text{ times} \ldots \cdot y = x$ (divisible).

Proof. Let Q be the rationals with their addition operation. Clearly, Q^n for distinct natural numbers n are models of the theory while they are pairwise nonisomorphic due to the dimension distinction. There is also one more model with infinite dimension. □

The most important reason for considering the notion of countable categoricity is the following theorem:

Theorem 1.5. *If Γ is a countably categorical theory in a countable language, then Γ is complete.*

Proof. Suppose that a theory Γ is not complete. Thus, there is a sentence ϕ such that Γ proves neither ϕ nor $\neg\phi$. Then both theories $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are consistent, and by the completeness theorem they both have countable models, say M, N respectively. Then M, N are both models of Γ, and they cannot be isomorphic. This means that the theory Γ is not countably categorical. □

Corollary 1.6. *The theory of an infinite set with equality, and the theory of a dense linear order without endpoints are both complete.*

The theorem above is an implication, not an equivalence. There are many complete theories which are not countably categorical. The proof of their completeness then has to be somewhat more involved. I provide only one example.

Theorem 1.7. *The theory of linear order with immediate successor and immediate predecessor is complete.*

Proof. Let $\langle L, \leq_L \rangle$ and $\langle K, \leq_K \rangle$ be models of the theory. They may not be isomorphic as we have seen above. Instead, consider the following feature. For every natural number n, consider the relation $\equiv_n$ between the tuples $\vec{l}$ and $\vec{k}$ of their underlying sets defined as follows: $\vec{l} \equiv_n \vec{k}$ if the tuples $\vec{k}$ and $\vec{l}$ have the same length, they are ordered in the same way, and if l_0, l_1 are elements of $\text{rng}(\vec{l})$??? and $k_0, k_1 \in \text{rng}(\vec{k})$ are the corresponding elements in K,

Claim 1.8. *If $\vec{l} \equiv_{n+1} \vec{k}$ and $i \in L$ is an arbitrary element, then there is $j \in K$ such that $\vec{l}^\frown i \equiv_n \vec{k}^\frown j$.*

Now, by induction on complexity of a formula ϕ prove the following statement. If $\phi(\vec{x})$ is a formula with of the language with $\leq n$ many quantifiers, $\vec{l}$ and $\vec{k}$ are tuples of elements of L, K respectively of the same length as $\vec{x}$, and $\vec{l} \equiv_n \vec{k}$, then $L \models \phi(\vec{l}/\vec{x})$ if and only if $K \models (\vec{k}/\vec{x})$. Once this is done, consider the case of a sentence ϕ. A sentence has no free variables, so the induction says that $L \models \phi$ just in case $K \models \phi$. Thus, all the countable models of the theory Γ satisfy the same sentences, and by the completeness theorem Γ is complete.

To perform the induction, note that for atomic formulas the statement is satisfied–an atomic formula only asserts that one of the entries on the variable list $\vec{x}$ is less or equal than another one, and the tuples $\vec{l}$ and $\vec{k}$ are assumed to be ordered in the same way. Passing the induction steps for formulas which are boolean combinations of simpler formulas is essentially trivial. The more difficult step appears when $\phi(\vec{x})$ is equal to $\exists y\ \psi(\vec{x}, y)$, and the statement has been verified for ψ. Let n be the number not smaller than the number of quantifiers in ϕ; note that $n \geq 1$. Suppose that $\vec{l}, \vec{k}$ are tuples of elements of L and K which are ordered in the same way and $\vec{l} \equiv_n \vec{k}$; we must show that $L \models \phi(\vec{l}/\vec{x})$ if and only if $K \models \phi(\vec{k}/\vec{x})$. For the left-to-right direction, if $L \models \phi(\vec{l}/\vec{k})$, then by the definition of the satisfaction relation, there must be $i \in L$ such that $L \models \psi(\vec{l}/x, i/y)$. Use the Claim to find $j \in K$ such that $\vec{l}^\frown i \equiv_{n-1} \vec{k}^\frown j$. Now note that the number of quantifiers in ψ is not greater than $n-1$, and so by the induction hypothesis $K \models \psi(\vec{k}/\vec{x}, j/y)$ and therefore $K \models \phi(\vec{k}/\vec{x})$. The argument for the right-toleft direction is symmetric. □

2 Definability and quantifier elimination

Definition 2.1. Let M be a model and n a natural number. A set $A \subset M^n$ is *definable* in the model M if there is a formula $\phi(\vec{x}, \vec{y})$ in the language of the model, all free variables of ϕ are listed, the string $\vec{x}$ has length n, the string $\vec{y}$ has length some k, and there is an k-tuple $\vec{p}$ of elements of M such that for every n-tuple $\vec{r}$ of elements of M, $\vec{r} \in M$ if and only if $M \models \phi(\vec{r}, \vec{p})$ holds. The tuple $\vec{p}$ is referred to as the *parameters* of the definition. If the string $\vec{y}$ is empty then the set A is said to be *definable without parameters*.

If M is a countably infinite model, then it has undefinable subsets, simply because there are only countably many definitions available, and there are uncountably many subsets of M. It is always interesting to know exactly which subsets of M or M^n are definable, and to be able to find the logically simplest definition of a given definable set.

Definition 2.2. A theory Γ has quantifier elimination if for every formula $\phi(\vec{x})$ there is a formula $\psi(\vec{x})$ with no quantifiers such that $\Gamma \vdash \phi \leftrightarrow \psi$. A model M has quantifier elimination if $Th(M)$ has it.

Example 2.3. Consider the model $\langle \mathbb{R}, 0, 1, \leq, +, \cdot \rangle$ and the formula $\phi(a,b,c) = \exists x\ ax^2 + bx + c = 0$. There is a quantifier-free formula $\psi(a,b,c)$ equivalent to it: $\psi(a,b,c) = b^2 - 4ac \geq 0$. The formula $\theta(a,b,c,d)$ saying that the two by two matrix with entries a,b,c,d has an inverse also has quantifier-free equivalent: $ac - bd \neq 0$.

Example 2.4. Consider the model $\langle \mathbb{Z}, \leq \rangle$ of integers with the usual ordering. It does not have the elimination of quantifiers: the formula $\phi(x,y) = \forall z \neg (x < z < y)$ has no quantifier-free equivalent.

Proof. Suppose that $\psi(x,y)$ is a formula which has no quantifiers. Let $f \mathbb{Z} \to \mathbb{Z}$ be the function $f(n) = 2n$. The map f preserves the ordering $\leq$ and therefore it preserves the validity of all quantifier-free formulas. In particular, $\mathbb{Z} \models \psi(0,1)$ if and only if $\mathbb{Z} \models \psi(0,2)$ since $0 = f(0)$ and $2 = f(1)$. At the same time, $\mathbb{Z} \models \phi(0,1)$ and $\neg\phi(0,2)$. This means that ϕ and ψ cannot be equivalent. □