Four and more*

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Abstract

We isolate several large classes of definable proper forcings and show how they include many partial orderings used in practice.

1 Introduction

The definable partial orderings used in practice share many common features. However, attempts to classify such partial orderings or their features in a combinatorial way can turn out to be very complex and, more importantly, can hide the real issues connecting the forcings to the motivating mathematical problems. This paper is a humble contribution to the classification problem. We isolate several classes of definable partial orderings. Each of them contains many forcings, directly defined from certain natural problems in abstract analysis. Moreover, many partial orders used in practice very naturally fall to one of those classes. However, these classes do not have any claim to completeness; in fact, we will show that there are natural partial orders which do not fall into any of them. Still, we believe that the results of the paper warrant further attention and investigation.

The notation of the paper follows the set theoretic standard of [13] and [15]. The symbol $\mathcal{P}(U)$ denotes the powerset of the set U, while $\mathcal{B}(X)$ denotes the collection of all Borel subsets of a Polish space X. If t is a finite binary sequence or a sequence of natural numbers then $[t]$ denotes the collection of all infinite binary sequences or all infinite sequences of natural numbers starting with t. There are many games in the paper, and we repeatedly use the following terminology. To say that one of the players dynamically produces a Borel set A on a fixed schedule is to say that this player really produces integer by integer a Borel code for the set A, in a predetermined way to be precisely specified later. By the Solovay model derived from an inaccessible cardinal κ we mean the

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intermediate model $V \subset V(\mathbb{R}) \subset V[G]$ where $G \subset \texttt{Coll}(\omega, <\kappa)$ is a V-generic filter, as opposed to the model $L(\mathbb{R}) \subset V[G]$.

2 The methodology of the paper

The methodology of the paper differs quite a bit from the standard approach to forcing. We believe the reader will quickly find out that it is quite efficient and flexible as far as definable forcing goes. Still we think it is necessary to introduce the basic concepts used. Note that the net effect of most lemmas and facts in this section is the elimination of the forcing relation from certain statements.

2.1 σ-ideals and forcing

Every suitably definable proper forcing adding a single real is of the form P_I for a suitable σ-ideal I on a Polish space, where

Definition 2.1. If X is a Polish space and I is a σ-ideal on it then P_I denotes the partial ordering of Borel I-positive sets ordered by inclusion.

It will be frequently clear from context exactly which Polish space X we are working on, and the dependence of the poset P_I on the space will be often neglected. The following lemmas record the most basic forcing properties of the posets of the form P_I. The reader should observe that they all are of the form which eliminates the forcing relation from certain statements. The proofs can be found in the first chapter of [35].

Fact 2.2. *Suppose that X is a Polish space and I is a σ-ideal on it. Then*

- *the forcing P_I adds a new element $\dot{r}_{gen} \in \dot{X}$ such that a Borel set $B \subset X$ is in the generic filter if and only if its realization in the generic extension contains $\dot{r}_{gen}$ as an element.*
- *the forcing P_I is proper if and only if for every countable elementary submodel M of a large enough structure and every condition $B \in P_I \cap M$ the set $\{r \in B : r \text{ is } M\text{-generic for the poset } P_I\}$ is I-positive.*

Fact 2.3. *Suppose that I is a σ-ideal such that P_I is proper. Then*

- *for every condition $B \in P_I$ and every name $\dot{g}$ for an element of a Polish space Y there is a stronger condition $C \subset B$ and a Borel function $G : C \to Y$ such that $C \Vdash \dot{g} = \dot{G}(\dot{r}_{gen})$. Moreover, if $A \subset Y$ is a universally Baire set and $B \Vdash \dot{g} \in \dot{A}$ then C, G can be chosen so that $G''C \subset A$.*
- *for every condition $B \in P_I$ and every name $\dot{D}$ for a Borel subset of a Polish space Y there is a stronger condition $C \subset B$ and a Borel set $E \subset C \times Y$ such that $C \Vdash \dot{D}$ is the $\dot{r}_{gen}$-th vertical section of the set $\dot{E}$.*

Note a simple uniformization corollary of the first item above: if $B \in P_I$ is a condition, Y a Polish space and $A \subset B \times Y$ is a universally Baire set with nonempty vertical sections then there is a condition $C \subset B$ and a Borel function $G : C \to Y$ such that for every point $r \in C$, $\langle r, G(r) \rangle \in A$. To see this, first use the universally Baire absoluteness to find a na me $\dot{g}$ for a real such that $B \Vdash \langle \dot{r}_{gen}, dotg \rangle \in \dot{A}$ and then apply the first item above.

Fact 2.4. *Suppose that I is a σ-ideal such that P_I is proper. The following are equivalent:*

- *P_I is bounding*
- *every Borel I-positive set has a compact I-positive subset, and every Borel function on a Borel I-positive set has a continuous restriction with an I-positive domain.*

Thus all bounding forcings of the form P_I are in fact forcings with finitely branching trees and have continuous reading of names. This means that they should be easy to understand from the combinatorial point of view, and we will pay special attention to the bounding cases.

Part of the point in this paper and the whole related theory is to find a correspondence between the topological and descriptive set theoretic properties of the ideal I and the forcing properties of the poset P_I. The reader should understand the basic difficulty in that there may be ideals I, J with wildly different topological properties while the posets P_I and P_J are essentially identical. This is recorded in the following definition.

Definition 2.5. Let I be a σ-ideal on a Polish space X. A *presentation* of the poset P_I is a Borel bijection $f : X \to Y$ for some Polish space Y together with the induced ideal $J = \{A \subset Y : f^{-1}A \in I\}$ and the poset P_J.

It is clear that if P_I is a presentation of P_J then P_J is a presentation of P_I, and moreover, they are isomorphic. The isomorphism map is $A \mapsto f''A$; note that Borel one-to-one images of Borel sets are Borel. However, the topological properties of the two presentations can be different–for example one of the ideals may have an F_σ basis while the other does not. Still, there are many topological properties invariant for different presentations–consider the previous lemma.

Note that the above problem includes the possibility of changing the topology on the underlying Polish space in such a way that the Borel structure and with it the poset P_I do not change. This possibility comes up several times in the paper, and it is really not clear how natural it can be in the various particular cases.

2.2 Determinacy arguments

Many arguments in the paper are stated in terms of infinite games. It seems to be the most efficient approach. The determinacy of the games in question is in general obtained by very complicated proofs from [23], requires large cardinals and an analysis of the definability of the games. A quick review of specific cases shows that the winning strategies are usually simple and readily at hand. Nevertheless, in order to state correct general theorems, we must use the following notion:

Definition 2.6. A set $A \subset \omega^\omega$ is *universally Baire* if there are trees $T, S \subset (\omega \times \mathrm{Ord})^{<\omega}$ such that $A = p[T]$ and T, S project to complements in every generic extension. An ideal I is universally Baire if for every universally Baire set $C \subset \omega^\omega \times X$ the set $\{r \in \omega^\omega$:the r-th vertical section of the set C is in $I\}$ is universally Baire again.

The following is a well-known unpublished extension of [19].

Fact 2.7. *(Martin) (LC) Suppose that T is a tree of height ω, $B \subset [T]$ a Borel set, $f : B \to \omega^\omega$ a continuous function and $A \subset \omega^\omega$ a universally Baire set. The following game is determined. In the game, players I and II play succesive nodes of the tree T, obtaining a branch $b \in [T]$. Player I wins if $b \in B$ and $f(b) \in A$.*

The large cardinal hypothesis used here is "two Woodin cardinals larger than the size of the tree T". It vanishes altogether if the set $A \subset \omega^\omega$ is Borel, a frequent case. A simple application:

Definition 2.8. Let P be a partial ordering. The *properness game $PG(P)$* is played by players Morphy and Andersen. Andersen produces an initial condition p_{ini} and then one-by-one open dense sets $D_n \subset P : n \in \omega$. Morphy produces one-by-one conditions $q_n^i \in D_n : n \in \omega$. Morphy wins if there is a condition $q \leq p_{ini}$ such that for every condition $r \leq q$ and every number n there is i such that q_n^i is compatible with r.

The following is well known (see e.g., [13] Theorem 31.9).

Fact 2.9. *Let P be a partial ordering. The following are equivalent:*

- *P is proper*
- *Morphy has a winning strategy in the properness game.*

For completeness we include a brief sketch of the prrof.

Proof. If P is proper then Morphy will win by producing on the side an $\in$-increasing sequence $\langle M_n : n \in \omega\rangle$ of countable elementary submodels of a large enough structures such that $p_{ini} \in M_0$ and $D_n \in M_n$ and playing so that the collection $\{q_n^i : i \in \omega\} \subset D_n$ enumerates the countable set $M \cap D_n$ for every number n, where $M = \bigcup_n M_n$. Any M-master condition $q \leq p_{ini}$ will then witness Morphy's win.

On the other hand, if Morphy has a winning strategy σ, for every countable elementary submodel M of a large enough structure containing σ and every condition $p_{ini} \in M \cap P$ it is possible to simulate a run of the properness game in which Andersen enumerates all the open dense subsets of the poset P in the model M. Since every initial segment of that play is in the model, necessarily $\{q_n^i : i \in \omega\} \subset D_n \cap M$ for every number n and every condition $q \leq p_{ini}$ witnessing Morphy's win will be a master condition for the model M. $\square$

The determinacy considerations then yield an interesting dichotomy.

Corollary 2.10. *(LC) Suppose that P is a universally Baire partial ordering on the reals. Exactly one of the following holds:*

- *P is proper*
- *some condition in P forces $([\mathbb{R}^V]^{\aleph_0})^V$ to be a nonstationary subset of $[\mathbb{R}^V]^{\aleph_0}$.*

Proof. Since the partial order P is definable, Fact 2.7 implies that the properness game is determined. There are two possible outcomes. Either Morphy has a winning strategy. In this case the partial ordering is proper.

Or, Andersen has a winning strategy σ indicating the initial condition p_{ini}. We claim that the condition p_{ini} works as desired in the second item above. To see this, consider a function $f : P^{<\omega} \to P$ in the extension given by $\dot{f}(\vec{p}) =$ some condition in $\dot{G} \cap \check{D}$, where $\dot{G}$ is the generic filter and $\check{D}$ is the open dense set the strategy σ produces after Morphy played the sequence $\vec{p}$. The claim is that no ground model countable subset of P is closed under the function $\dot{f}$. For suppose that $q \leq p_{ini}$ and $q \Vdash \check{x}$ is closed under $\dot{f}$ for some ground model countable set $x \subset P$. Then Morphy can win against the strategy σ by playing so that $D_n \cap x = \{p_n^i :\in \omega\}$, contradiction. $\square$

In particular, if the Continuum Hypothesis holds then a definable poset is proper if and only if it preserves $\aleph_1$. There is an example of definable partial ordering which, if $\delta_2^1 = \omega_2$, is not proper but still preserves $\aleph_1$. However, its definition is rather complex and we still think that for partial orders P of low complexity definition, P is proper if and only if it preserves $\aleph_1$, this outright in ZFC(+LC), without CH.

2.3 Preservation properties and Fubini type theorems

Another feature peculiar to the paper is the use of Fubini type statements in place of "forcing preservation" statements. This seems to be the most efficient approach.

Definition 2.11. Let I, J be σ-ideals on some Polish spaces. The symbol $I \perp J$ denotes the statement: there are an I-positive Borel set B_I and a Borel J-positive set B_J and a Borel set $C \subset B_I \times B_J$ such that the vertical sections of the set C are J-small and the horizontal sections of its complement are I-small.

Thus $I \perp J$ means that the Fubini theorem between the ideals I and J fails in a particularly violent manner. It turns out that most forcing preservation properties can be restated in terms of the relation $\perp$.

Lemma 2.12. *Let I be a σ-ideal on a Polish space such that P_I is proper. Let J be a universally Baire σ-ideal on some other Polish space, generated by Borel sets. The following are equivalent:*

- $P_I \Vdash \dot{B} \cap V \notin \dot{J}$ *for every Borel J-positive set B.*
- $\neg I \perp J$.

Proof. Suppose on one hand that there is a condition $B_I \in P_I$ and a Borel J-positive set B_J such that $B_I \Vdash \dot{B}_J \cap V \subset \dot{D}$ for some Borel $\dot{J}$-small set $\dot{D}$. By Lemma 2.3, thinning out the set B_I if necessary we may assume that there is a Borel set $C \subset B_I \times B_J$ with J-small vertical sections such that $B_I \Vdash \dot{D}$ is the $\dot{r}_{gen}$-th vertical section of the set $\dot{C}$. It is immediate to verify that the vertical sections of the complement of the set C are I-small.

On the other hand, if $C \subset B_I \times B_J$ is a Borel subset of some rectangle with Borel positive sides witnessing $I \perp J$ then it follows immediately from the definitions that $B_I \Vdash$ the $\dot{r}_{gen}$-th vertical section of the set $\dot{C}$ is J-small and covers the ground model reals in the set $\dot{B}_J$. □

Lemma 2.13. *Let I, J be respective σ-ideals on Polish spaces X, Y such that the posets P_I, P_J are proper. The following are equivalent:*

- *there are Borel sets $B_I \in P_I$ and $B_J \in P_J$, a P_I-name $\dot{r}$ for a real and a P_J-name $\dot{s}$ and a Borel relation C such that $B_I \Vdash \forall x \in V \cap \mathbb{R}\ \dot{r}\dot{C}x$ and $B_J \Vdash \forall x \in V \cap \mathbb{R}\ \neg x\dot{c}\dot{s}$.*
- $I \perp J$.

Proof. If $I \perp J$ and B_I, B_J, C witness it, then the first item is witnessed by B_I, B_J again, $\dot{r}$ =the P_I-generic, $\dot{s}$ =the P_J-generic and the relation $C' = C \cup \{\langle x, y\rangle : x \in B_I, y \notin B_J\}$.

On the other hand, if the first item holds as witnessed by $B_I, B_J, dotr, dots, C$, first reduce the names $\dot{r}$ and $\dot{s}$ to Borel functions f, g on some sets B'_I, B'_J as in Fact 2.3 and then use the relation $C' = \{\langle x, y\rangle : x \in B'_I, y \in B'_J, f(x)Cg(y)\}$ to show that $I \perp J$. □

There are two particular cases worth special attention.

Lemma 2.14. *Suppose that I is a σ-ideal on some Polish space such that P_I is proper. The following are equivalent:*

- *P_I is bounding*
- *$\neg I \perp J$ where J is the Laver ideal.*

Here the Laver ideal J on ω^ω is generated by sets $A_g = \{f \in \omega^\omega$: for infinitely many n, $f(n) \in g(f \restriction n)\}$ as g varies through all functions from $\omega^{<\omega}$ to ω. It is well-known that every analytic subset of ω^ω either is in the ideal J or contains all branches of some Laver tree.

Proof. Suppose that the poset P_I is bounding, $B_I \in P_I$ and $B_J \in P_J$ are Borel sets and $C \subset B_I \times B_J$ is a Borel set with J-small vertical sections. We must show that its complement contains an I-positive horizontal section. Thinning out the set B_I if necessary we may assume that there is a Borel function $G : B_I \to \omega^{\omega^{<\omega}}$ such that for every pair $\langle r, f \rangle \in C$ it is the case that for infinitely many numbers n, $f(n) \in G(r)(f \restriction n)$. Since the poset P_I is bounding, there is an I-positive Borel set $B \subset B_I$ and a function $h : \omega^{<\omega} \to \omega$ such that for every point $r \in B$ and every finite integer sequence s, $G(r)(s) \in h(s)$. Since the set B_J is J-positive, there must be a function $f \in B_J$ such that for all but finitely many numbers n, $h(f \restriction n) \in f(n)$. It is clear that $B \times \{f\}$ is the required I-positive horizontal section of the complement of the set C.

On the other hand, if the forcing P_I adds an unbounded real below some condition, Lemma 2.13 immediately shows that $I \perp J$: its first item will be witnessed with the relation of eventual dominance. $\square$

A set $a \subset \omega$ in some forcing extension of the ground model V is splitting if every infinite set $b \subset \omega$ in V has infinite intersection both with a and the complement of a. A subset A of a Boolean algebra B is *nowhere dense* if for every positive $b \in B$ there is a positive $c \leq b$ such that no positive $a \in A$ is below c.

Lemma 2.15. *Suppose that I is a σ-ideal on some Polish space such that P_I is proper in all σ-closed forcing extensions. The following are equivalent:*

- *P_I is bounding and does not add a splitting real.*
- *$\neg I \perp J$ where J is the ideal of subsets of $\mathcal{P}(\omega)$ which are nowhere dense in the algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.*

The assumption on the poset P_I may sound unnatural and in fact we do not have an example of a definable ideal I such that the poset P_I is proper and loses its properness after a σ-closed forcing. Certainly all examples described in this paper remain proper in all forcing extensions with the same reals as the ground model.

A third equivalence can be added to Lemma 2.15. If K is a coideal on ω such that $M(K)$ has the Mathias property (see §9.1) and $J(K)$ is a σ-ideal such that $M(K)$ is forcing equivalent to $P_{J(K)}$, then I satisfies conditions of Lemma 2.15 if and only if $\neg I \perp J(K)$. The proof is virtually identical to the proof below. The class of all K such that $M(K)$ has the Mathias property has several equivalent characterizations in terms of its forcing properties (see Theorem 9.10).

Proof. To understand the situation better, note that the Mathias forcing naturally densely embeds into P_J [35]. The Mathias forcing adds a dominating real and a real which is modulo finite included in or disjoint from every ground model set. So there are Borel functions $F : \mathcal{P}(\omega) \to \omega^\omega$ and $G : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that for every function $f \in \omega^\omega$ the set $\{r \in \mathcal{P}(\omega) : F(r)$ does not modulo finite dominate $f\}$ is J-small, and for every set $a \subset \omega$ the set $\{r \in \mathcal{P}(\omega) : G(r)$ is neither modulo finite included in nor modulo finite disjoint from $a\}$ is I-small.

The function G can be chosen to be the identity, but that is immaterial for our purposes.

Suppose that the first item fails. Then Lemma 2.13 immediately shows that $I \perp J$: its first item will be witnessed either with the eventual dominance relation or the splitting relation.

On the other hand suppose that the first item holds, and suppose that $C \subset B_I \times B_J$ is a Borel subset of a rectangle with positive Borel sides. We will show that either C or its complement contains a rectangle with positive Borel sides. By the homogeneity of the ideal J we may and will assume that $B_J = \mathcal{P}(\omega)$. Move to a forcing extension $V[u]$ where u is a Ramsey ultrafilter added by countable approximations. Analyze the poset P_I in this model. Note that the properties "P_I is bounding" and "P_I does not add splitting reals" are absolute between V and $V[u]$ by Lemma 2.3. Since it does not add splitting reals, a genericity argument for u shows that P_I preserves u as an ultrafilter, and since P_I is moreover bounding, it preserves u as a Ramsey ultrafilter (see e.g., [26] VI.5.1(2)). Still in the model $V[u]$, consider the standard c.c.c. poset Q diagonalizing the ultrafilter u; it adds a Mathias real $\dot{s}$ over V. Note that since the poset P_I preserves the Ramsey ultrafilter u, it commutes with Q in a very strong sense: if s is $V[u]$-generic for the poset Q and r is $V[u]$-generic for the poset P_I then $\langle r, s\rangle$ is $V[u]$-generic for $P_I \times Q$. This uses the Mathias property of the forcing Q in the P_I extension of the model $V[u]$.

Find a condition $\langle B_0, q\rangle \in P_I \restriction B_I \times Q$ deciding the statement $\langle \dot{r}_{gen}, \dot{s}\rangle \in \dot{C}$. For definiteness assume it decides it in the affirmative. Let M be a countable elementary submodel of a large enough structure containing all relevant objects. Let $B_1 \subset B_0$ be the set of all M-generic reals for the poset P_I. The set B_1 is Borel and I-positive by the assumption on the poset P_I. Let B_2 be the set of all reals diagonalizing the filter $u \cap M$ and meeting the condition q. Clearly B_2 consists only of Q-generic reals, and it is J-positive by the Mathias property of the forcing Q. By the strong commutativity mentioned in the previous paragraph, it must be the case that the rectangle $B_1 \times B_2$ with Borel positive sides consists of pairs M-generic for the poset $P_I \times Q$, and by an application of the forcing theorem and forcing absoluteness, it must be a subset of the Borel set C. $\square$

Note that the above argument shows that the statement $\neg I \perp J$ is equivalent to the apparently stronger rectangular Ramsey statement $MRR(I, J)$, "for every partition of a Borel rectangle with positive sides into countably many Borel pieces, one of the pieces contains a Borel rectangle with positive sides".

Lemma 2.16. *(LC) Suppose that I is a universally Baire σ-ideal such that the forcing P_I is proper. The following are equivalent:*

1. $\neg I \perp \texttt{meager}$
2. *for every Borel I-positive set B the Cohen forcing adds a real $\dot{r} \in \dot{B}$ which falls out of all ground model coded Borel I-small sets.*

Proof. The implication (2)→(1) is easy. Suppose (2) holds and (1) fails as witnessed by an I-positive Borel set B, a non-meager Borel set $C \subset 2^\omega$, and a Borel set $D \subset B \times C$ such that its horizontal sections are meager and the vertical sections of its complement are I-small. An easy homogeneity argument shows that we may assume $C = 2^\omega$. There is a Cohen name $\dot{r}$ for a real which is forced into the set $\dot{B}$ and out of all Borel ground model coded I-small set. It is immediate that the vertical section $\dot{D}_{\dot{r}}$ is forced to be a meager set containing all ground model reals. This contradicts the well-known fact that Cohen forcing preserves category, in our terminology $\neg\mathtt{meager} \perp \mathtt{meager}$.

The implication (1)→(2) needs large cardinal assumptions and the definability of the ideal I. It is proved in [33]. □

Corollary 2.17. *(LC) Suppose that I is a suitably definable σ-ideal containing all singletons such that every countable set is included in a G_δ set in the ideal, and such that P_I is proper. Then $I \perp \mathtt{meager}$.*

Proof. Let I be a suitably definable σ-ideal on some Polish space X satisfying the assumptions. By the previous Lemma, it is enough to show that the Cohen forcing does not add a real which falls out of all ground model coded Borel I-small sets.

And indeed, suppose that $f : 2^\omega \to X$ is any Borel function. We will find a G_δ set A in the ideal I such that $f^{-1}A$ is not meager. This will complete the proof. To find the set A, let M be a countable elementary submodel of a large enough structure and let $C \subset X$ be a countable set such that its f-preimage intersects every nonmeager set in the model M. Let $A = \bigcap_i O_i$ be a G_δ set in the ideal I containing the set C as a subset; the sets O_i are open. We claim that this set $A \subset X$ works.

Assume that $[s] \subset 2^\omega$ is a condition in the Cohen forcing for some sequence $s \in 2^{<\omega}$. By a standard density argument and the Baire theorem, it is enough to show that for every number $i \in \omega$ there is a basic open set $U \subset O_i$ such that the set $f^{-1}U \cap [s]$ is not meager. Suppose that for some number $i \in \omega$ this fails. Then the set $[s] \setminus \bigcup\{f^{-1}U : U \subset X$ is a basic open set with $f^{-1}U \cap [s]$ meager$\}$ is comeager in $[s]$, it is in the model M, and its f-image is disjoint from the set O_i and therefore from the set $C \subset O_i$. This contradicts the choice of the set C. □

2.4 Choiceless dichotomies

For all ideals studied in this paper, it turns out that in choiceless contexts such as the Solovay model or under AD+ they are closed under well-ordered unions and they satisfy a dichotomy–every positive set has a Borel positive subset. These two properties seem to be very closely related to the properness of the factor forcing, even though there seem to be no outright provable implications.

Example 2.18. The E_0 forcing. Let E_0 be the Vitali equivalence relation on the reals, and let I be the σ-ideal generated by Borel sets B which intersect every equivalence class in at most one point. The forcing P_I is proper, however the ideal I is not closed under well-ordered unions in the choiceless Solovay model or in $L(\mathbb{R})$–[35].

Example 2.19. Let I_n be the ideal of sets of σ-finite 2^{-n}-dimensional Hausdorff measure, for every number $n \in \omega$. These ideals form an inclusion decreasing sequence, and all of them are closed under well-ordered unions in the choiceless Solovay model by the results of Section 5. The ideal $I = \bigcap_n I_n$ is then also closed under well-ordered unions, however the poset P_I is not proper as the argument from [35], Subsection 2.3.15 shows.

On the other hand, the closure under well-ordered unions and the dichotomy are provably connected.

Definition 2.20. A *pullback* of an ideal I on a Polish space X is an ideal J on a space Y such that there is a Borel function $f : Y \to X$ such that for every set $B \subset Y$, $B \in J \leftrightarrow f''B \in I$.

Definition 2.21. A *strong dichotomy* for a σ-ideal I on a Polish space is the following statement: for every pullback J of I, every J-positive set has an analytic J-positive subset.

Lemma 2.22. *In the Solovay model, for every σ-ideal I, the ideal I satisfies the strong dichotomy if and only if it is closed under well-ordered unions.*

Proof. The right-to-left implication is easy. In the Solovay model, every subset of a Polish space is a well-ordered union of Borel sets. Now if I is a σ-ideal closed under well-ordered unions and J is its pullback, then J is closed under well-ordered unions as well and if B is a J-positive set, it can be written as a well-ordered union of some of its Borel subsets, and one of these must be J-positive.

For the left-to-right implication note that in the Solovay model, increasing unions of subsets of a Polish space X stabilize in $\aleph_1$ many steps, and so it is enough to prove that closure of a σ-ideal I on X under $\aleph_1$ unions follows from the strong dichotomy. For contradiction assume that $\langle B_\alpha : \alpha \in \omega_1 \rangle$ is an $\aleph_1$-collection of I-small sets with I-positive union. Let Z be the Polish space of all trees on ω with a natural topology, let $Y = X \times Z$ and let $A = \{\langle x, T \rangle \in Y : x \in B_\alpha$, the tree T is well-founded and has rank $\alpha\}$. Consider the pullback J of I on the space Y given by the projection function. The projection of the set A is exactly the union $\bigcup_\alpha B_\alpha$, and so $A \notin J$. Use the strong dichotomy to find an analytic J-positive subset $C \subset A$. By the boundedness theorem, there is a countable ordinal β such that whenever $\langle x, T \rangle \in C$ then the rank of the tree T is less than β. Then it must be the case that the projection of the set C to the space X is included in the set $\bigcup_{\alpha \in \beta} B_\alpha \in I$, contradicting the J-positivity of the set C. $\square$

In the context of determinacy, one implication of the above equivalence survives.

Lemma 2.23. *(ZF+AD+) If a σ-ideal on a Polish space satisfies the strong dichotomy then it is closed under well-ordered unions.*

Proof. Just the same as the previous argument with an additional ingredient, a theorem of Steel, [12] Theorem 1.1, asserting that under AD+, for every regular cardinal $\kappa \in \theta$ there is a set $Y \subset \mathbb{R}$ and a prewellorder $\leq$ on Y such that analytic subsets of Y meet just $< \kappa$ many $\leq$-equivalence classes. This result was pointed out to us by Stephen Jackson.

Assume AD+ and assume that I is a σ-ideal on a Polish space X satisfying the strong dichotomy. By transfinite induction on $\kappa \in \theta$ argue that whenever $\langle A_\alpha : \alpha \in \kappa \rangle$ is a collection of sets in I then its union is in the ideal I as well. The successor, countable and singular steps in the induction are trivial. So suppose $\kappa \in \theta$ is a regular cardinal and the statement has been verified up to κ. Fix a collection $\langle A_\alpha : \alpha \in \kappa \rangle$ of sets in the ideal I. If $\bigcup_\alpha A_\alpha \notin I$, use Steel's result to find a suitable prewellorder $\leq$ on a set $Y \subset \mathbb{R}$ of length κ and let $B \subset X \times \mathbb{R}$ be the set of all pairs $\langle x, r \rangle$ such that $x \in A_\alpha$ where r is in the α-th $\leq$-equivalence class. Consider the pullback J on the space $X \times R$ of the ideal I using the projection. The set B is J-positive and by the strong dichotomy it has an analytic positive subset $C \subset B$. The projection of C into the $\mathbb{R}$ coordinate is an analytic subset of Y, and therefore it meets only $< \kappa$ many $\leq$-classes, bounded by some ordinal $\beta \in \kappa$. The projection of C into the X coordinate is an I-positive set, and it is a subset of the set $\bigcup_{\alpha \in \beta} A_\alpha$. However, this contradicts the induction hypothesis, which implies that the set $\bigcup_{\alpha \in \beta} A_\alpha$ is in the ideal I. $\square$

Note that the argument shows that if strong dichotomy holds then we can find a *Borel* positive subset inside each positive set, not just an analytic one. There does not seem to be a bound on the complexity of the Borel set though.

2.5 The classes of ideals

The classes of ideals isolated in this paper are called ideals generated by closed sets, porosity ideals, σ-finite ideals, null ideals associated with dense submeasures and pavement submeasures, and P-cover ideals. These classes are mutually related. Every ideal σ-generated by closed sets is a porosity ideal, and every null ideal associated with pavement submeasure is a P-cover ideal. On the other hand, we do not know if every P-cover ideal is or can be presented as a null ideal. While there are porosity ideals which are not generated by closed sets, we do not know if there is a porosity ideal which cannot so presented.

Curiously enough, if $I_n : n \in \omega$ are ideals in the classes considered, the σ-ideal J generated by $\bigcup_n I_n$ also satisfies the two items above. While it seems that certain forcings used in practice are obtained by such a hybridization of ideals, we have no definite examples and for this reason we omit the proof. At any rate, the proof is just a spirited repetition of the determinacy arguments for the games $G(I, P, \dot{r})$. We could not find a general overarching theorem to the tune of "if $I_n : n \in \omega$ are ideals such that P_{I_n} are proper forcings then so is P_J" or " if ideals $I_n : n \in \omega$ are closed under well-ordered unions then so is J".

3 Ideals generated by closed sets

Ideals σ-generated by closed sets, that is, the σ-ideals with a basis consisting of F_σ sets occur very frequently indeed in the theory of definable forcing. The class of ideals σ-generated by closed sets is included in the class of porosity ideals from the next section, however it is an important enough subcase to deserve a special treatment.

Example 3.1. The Sacks forcing. Let I be the σ-ideal of countable subsets of 2^ω. The perfect set theorem shows that the Sacks forcing naturally densely embeds into the poset P_I.

Example 3.2. The Cohen forcing. The ideal I is the σ-ideal of meager sets, generated by closed nowhere dense sets.

Example 3.3. The Miller forcing. The σ-ideal I on ω^ω is σ-generated by compact subsets of ω^ω.

Example 3.4. The $c_{\min}$ forcing. Let $c_{\min}$ be the partition of pairs of infinite binary sequences into two classes, even and odd, depending on the parity of the smallest number where the two sequences differ. Let I be the σ-ideal generated by $c_{\min}$-homogeneous sets. Note that since $c_{\min}$ is a clopen partition, the closures of $c_{\min}$-homogeneous sets are still homogeneous.

Example 3.5. The Spinas forcing. Let I be the σ-ideal on 2^ω generated by the closed sets $C_a = \{f \in 2^\omega : f \restriction a \text{ is constant}\}$ for $a \subset \omega$ infinite. The poset P_I is bounding and it adds a splitting real.

Example 3.6. The $\mathtt{non}(\mathtt{null})$ forcing. Let $\{a_n : n \in \omega\}$ be a collection of nonempty finite pairwise disjoint subsets of ω, $|a_n| \geq n$. Let I be the ideal on 2^ω σ-generated by sets $B_f = \{g \in 2^\omega : \forall n\ f \restriction a_n \neq g \restriction a_n\}$ as f ranges over all infinite binary sequences. It is not difficult to see that in the P_I-extension the ground model reals have Lebesgue measure zero.

Example 3.7. Let I be the σ-ideal generated by closed measure zero sets. The partial order P_I has not been analysed.

3.1 Properness

Theorem 3.8. *If I is a σ-ideal generated by closed sets then the poset P_I is proper.*

Proof. This is proved in [35]. For the sake of completeness and to stress parallels with the later sections, we include the short proof. For definiteness assume that the underlying Polish space is just the Baire space ω^ω. Suppose that M is a countable elementary submodel of a large enough structure containing the ideal I, and let $B \in M \cap P_I$ be a condition. We must prove that the set $D = \{r \in B : r$ is M-generic$\}$ is I-positive. So suppose that $C_n : n \in \omega$ is a countable list of closed sets in the ideal I; we must find a real $r \in D \setminus \bigcup_n C_n$. Enumerate the open dense subsets of P_I in the model M by $\{E_n : n \in \omega\}$ and by induction on $n \in \omega$ construct a descending sequence $B = B_0 \supset B_1 \supset \dots$ of conditions in $M \cap P_I$ such that $B_{n+1} \in E_n$ and $B_{n+1} \cap C_n = 0$. Let $g \subset P_I \cap M$ be the resulting M-generic filter generated by the conditions B_n. By Lemma 2.3 applied in the model M, $M[g] \models \bigcap g$ is a singleton $\{r\}$ for some real r, and by Borel absoluteness indeed $\bigcap g = \{r\}$. The real r clearly has the desired properties.

To perform the induction, suppose the condition $B_n \in P_I \cap M$ is known. Thinning out the set B_n we may assume that for every basic open set O, $B_n \cap O \notin I$ or $B_n \cap O = 0$. There must be a basic open set O such that $B_n \cap O \neq 0$ and $B \cap O \cap C_n = 0$ since otherwise $B_n \subset C_n$ by the closedness of the set C_n; and this is impossible because $B_n \notin I$ while $C_n \in I$. Now note that the set $B_n \cap O$ is a condition in $P_I \cap M$. Any condition $B_{n+1} \in E_n \cap M$ with $B_{n+1} \subset B_n \cap O$ will work as desired. $\square$

The partial orders of this kind share several other properties.

Lemma 3.9. *Suppose that I is a σ-ideal generated by closed sets. Then:*

- *G_δ sets are dense in P_I–every I-positive Borel set has an I-positive G_δ subset C such that every set in the ideal is meager in C.*
- *continuous reading of names–for every positive Borel set B and every Borel function $f : B \to \omega^\omega$ there is a Borel I-positive set C such that $f \restriction C$ is continuous.*
- $\neg I \perp \texttt{meager}$
- *P_I can be embedded into a σ-closed*c.c.c. iteration.*

Proof. The first item follows from a result of Solecki [28]. First find any G_δ I-positive subset C' and then remove all basic open sets O such that $C' \cap O \in I$. A quick check shows that the resulting set C has the required properties. Note that the set B is a Polish space in the inherited topology, and $I \subset \texttt{meager}(C)$.

For the second, without loss of generality we may assume then that the set B is as in the previous paragraph. The set B with the inherited topology is then a Polish space, and there is a dense G_δ subset $C \subset B$ such that the function $f \restriction C$ is continuous. Now since all I-small sets are meager in the set B, it follows that $C \notin I$ is the required set.

The proof of the third item is similar. Suppose that $B \in P_I$ is an I-positive Borel set, and C is a Borel nonmeager set, and $D \subset B \times C$ is a Borel set. Thinning out the set B we may and will assume that it has the properties described in the first paragraph. The classical Kuratowski-Ulam theorem for the meager ideal says that either there is a vertical section of the set D which is somewhere comeager in C, or there is a horizontal section of the complement of D which is comeager in the set B. In both cases it follows that the triple B, C, D is not a witness to $\neg I \perp \texttt{meager}$.

The fourth item is proved in [35]. $\square$

3.2 Dichotomies

The dichotomies for σ-ideals generated by closed sets have been known for some time [28]. Nevertheless, we will restate them and reprove them since similar arguments will be applied in the following sections.

Theorem 3.10. *In the choiceless Solovay model, every σ-ideal generated by closed sets is closed under well-ordered unions.*

Proof. Let $V \subset W \subset V[G]$ be a ground model, its $\mathtt{Coll}(\omega, <\kappa)$ extension for some inaccessible cardinal κ, and the intermediate choiceless Solovay model. Let I be a σ-ideal on some Polish space X in the model W generated by closed sets. By the usual homogeneity argument we may and will assume that I is definable in W using ground model parameters.

We will show that for every set $B \subset X$ definable from the parameters in the ground model, either it is covered by I-small Borel sets coded in the ground model or else it is I-positive. Then consider a wellordered collection $\langle B_\alpha : \alpha \in \lambda \rangle$ of sets definable from ground model parameters. There are two cases: either $\bigcup_\alpha B_\alpha$ is covered by the countably many I-small Borel sets coded in the ground model, in which case it is I-small, or else there is a real $r \in \bigcup_\alpha B_\alpha$ which does not belong to any I-small Borel set coded in the ground model. In the latter case this real must belong to some set B_α which is definable from the ground model parameters and therefore then I-positive. The theorem follows by a standard homogeneity argument.

So fix a set B definable from ground model parameters such that B is not covered by the ground model coded I-small sets. There must then be a real $r \in B$ which falls out of all ground model coded I-small sets. It must be the case that there is in V a forcing P of size $< \kappa$ with a P-name $\dot r$ such that $P \Vdash \dot r$ falls out of all ground model coded I-small sets and $\mathtt{Coll}(\omega, <\kappa) \Vdash \dot r \in \dot B$. By a standard homogeneity argument it is enough to show that the set $\{r \in X : \exists g \subset P$ a V-generic filter such that $r = \dot r/g\}$ is I-positive, since it is a subset of the set B.

To this end, let $\{C_n : n \in \omega\}$ be a countable list of closed sets in the ideal I. We must produce a V-generic filter $g \subset P$ such that $\dot r/g \notin \bigcup_n C_n$. To this end, by induction build a descending chain of conditions $p_n \in P$ so that

- p_{n+1} belongs to the n-th open dense subset of the poset P in the model V, under some fixed enumeration in the model W
- the set $B_{n+1} = \{r \in X : \exists g \subset P$ V-generic with $p_{n+1} \in G$ and $\dot r/g = r\}$ is disjoint from the set C_n.

To perform the inductive step, suppose that the condition p_n has been obtained. The closure $\bar B_n$ of the set B_n has a code in the ground model, since $\bar B_n = X \setminus \bigcup\{O : O \subset X$ is a basic open set such that $p_n \Vdash \dot r \notin \dot O\}$. Note $p_n \Vdash \dot r \in \bar B_n$. By the assumptions on the name $\dot r$, this means that $\bar B_n \notin I$, so $\bar B_n \not\subset C_n$ and there must be a basic open set O such that $O \cap C_n = 0$ while $O \cap \bar B_n \neq 0$. By the definitions, there must be a condition $q \leq p_n$ forcing the real $\dot{r}$ into $\dot{O}$. Every condition $p_{n+1} \leq p_n$ in the n-th open dense subset of the forcing P will have the required properties. $\square$

Corollary 3.11. *In the Solovay model, every subset of X either has a Borel I-positive subset or a Borel I-small superset.*

Proof. Observe that in the Solovay model, every subset of the underlying Polish space X is a well-ordered union of Borel sets. Then apply the previous theorem. $\square$

Corollary 3.12. *(LC) Every universally Baire set has either a Borel I-positive subset or a Borel I-small superset.*

Proof. Note the previous corollary and refer to the universally Baire absoluteness. $\square$

A result of Solecki [28] provides the sharpest complexity estimate: whenever I is σ-ideal generated by closed sets, then every analytic I-positive subset of the underlying Polish space X has a G_δ I-positive subset.

3.3 Further preservation properties

A particular case of the ideals generated by closed sets is so frequent that it deserves further attention.

Definition 3.13. A σ-ideal I on a compact space X is generated by a σ-compact collection of compact sets if there is a σ-compact subset of the space $K(X)$ such that every set in the ideal is covered by a countable union of elements of the σ-compact set.

Among the examples mentioned in the beginning of this section, the σ-ideal of countable sets, the $c_{\min}$ ideal, and the $\mathtt{non}(\mathtt{null})$ ideal are generated by a σ-compact collection of compact sets. On the other hand, the ideals associated with Spinas forcing and Miller forcing are not, and even cannot, be represented that way, as Corollary 3.20 below shows.

Example 3.14. A typical not so well researched family of forcings representative of this section is connected with packing measures. Let $\langle X, d\rangle$ be a compact metric space and h be a positive real number. For a set $A \subset X$ and a positive number δ a *δ-packing* of A is a finite set p of mutually disjoint balls in X with centers in the set A and diameters $< \delta$. A *weight* $w(p)$ of that packing p is then $\Sigma\{\mathtt{diam}^h(b) : b \in p\}$. The h-dimensional packing premeasure $P^h_0(A)$ is defined as $\inf_\delta \sup\{w(p) : p \text{ a } \delta\text{-packing of } A\}$, and the h-dimensional packing measure $P^h(A)$ is $\inf\{\Sigma_n P^h_0(B_n) : \bigcup_n B_n = A\}$. Let I^h be the ideal on X σ-generated by sets of finite h-dimensional packing measure, which is identical to the ideal σ-generated by sets of finite h-dimensional packing premeasure. It is not difficult to see that packing premeasure is preserved by the closure operator and so the ideal I^h is generated by closed sets. In fact, the collections $K_{n,m} : n, m \in \omega$ of compact sets defined by $K_{n,m} = \{C \subset X \text{ compact: } \sup\{w(p) : p \text{ a } 2^{-m}\text{-packing of } C\} \leq n\}$ are clearly compact and together they generate the ideal I^h.

We will show that forcings associated with σ-ideals generated by a σ-compact collection of compact sets have many preservation properties. To quantify this precisely, the following definition will come handy.

Definition 3.15. A forcing P has the *weak Cohen property* if for every function $f \in \omega^\omega$ in the ground model and every function $g \in \omega^\omega$ in the extension dominated pointwise by f there exists a function $h \in \omega^\omega$ in the ground model such that $|h \cap g| = \aleph_0$.

In other words, the weak Cohen property is the statement that the poset does not add a bounded eventually different real. The reasoning behind the terminology is this: the property of not adding an eventually different real at all is certainly stronger, and since Cohen forcing has it, and it is the only definable c.c.c. forcing which has it [33], [2], 2.4.7, the Cohen property would be an apt name for it. Traditional examples include Mathias forcing and Silver forcing, as well as all σ-centered forcings, see below. There is a natural game theoretic counterpart to the weak Cohen property.

Definition 3.16. Supose that J is a σ-ideal on a Polish space. The *weak Cohen game* $WCG(J)$ is played between Steinitz and Andersen. Steinitz first indicates an I-positive Borel set $B_{ini} \in P_I$. After that, the game has infinitely rounds, in round n Steinitz plays a partition of the set B_{ini} into finitely many pieces and Andersen chooses one of them, call it B_n. Andersen wins if the *result* of the game, the Borel set $\bigcap_m \bigcup_{m \in n} B_n$, does not belong to the ideal J.

Example 3.17. If the poset P_J is σ-centered, then Andersen has a winning strategy. Suppose that Steinitz plays some initial move B_{ini}. The poset $Q = P_J$ below the condition B_{ini} can be decomposed into countably many centered pieces, $Q = \bigcup_i Q_i$. It is not difficult to see that for every number $i \in \omega$ and every partition P of the set B_{ini} into finitely many Borel pieces one of the pieces $C \in P$ has the following property $\phi_i(C)$: for every finite set $y \subset Q_i$ the set $C \cap \bigcap y$ is J-positive. Since if $\phi_i(C)$ failed for every element $C \in P$ as witnessed by some finite set $y_C \subset Q_i$, the set $y = \bigcup_{C \in P} y_C \subset Q_i$ is finite, by the centeredness $\bigcap y \notin J$, and for some element $C \in P$ of the partition, $C \cap \bigcap y \notin J$, contradicting the choice of the set $y_C \subset Q_i$. Now Andersen wins the game $WCG(J)$ in the following fashion. He splits $\omega = \bigcup_i a_i$ into countably many infinite pieces and at round $n \in \omega$ he finds the number $i \in \omega$ such that $n \in a_i$, and plays B_n to be some piece of Steinitz's partition with the property $\phi_i(B_n)$. We claim that this way Andersen wins. And indeed, looking at the result C of the play, it must even be the case that $B_{ini} \setminus C \in J$: if this set was positive, for some number m the set $C_m = B \setminus \bigcup_{m \in n} B_n$ must be J-positive as well and as such it would belong to the set Q_i for some $i \in \omega$. Now choose some number $n \in a_i$ bigger than m and observe that by the property $\phi_i(B_n)$ it must be the case that $C_m \cap B_n \notin J$ even though by the definition of the set C_m the intersection should be empty. Contradiction!

Lemma 3.18. *Suppose that J is a definable σ-ideal such that P_J is proper. The following are equivalent:*

- *The forcing P_J fails to have the weak Cohen property*
- *Steinitz has a winning strategy in the weak Cohen game $WCG(J)$.*

Proof. First suppose that Steinitz has a winning strategy σ in the game calling for some initial set B_{ini}. Then he actually has a winning positional strategy τ, a sequence $P_n = \{\{B_{j,n} : j \in j(n)\} : n \in \omega\}$ of partitions of the initial set B_{ini} such that he wins playing these partitions no matter what Andersen's answers are. To obtain the positional strategy note that at each round n there are only finitely many partitions the strategy σ can produce as n-th move for Steinitz, and choose P_n to be a finite partition refining them all. It is clear that τ must be a winning strategy if σ is. Now the partitions $\{P_n : n \in \omega\}$ generate a natural name $\dot{s}$ for a real, $\dot{s}(\check{n}) = j$ if and only if $\dot{r}_{gen} \in B_{j,n}$. It is clear that the function $\dot{s} \in \omega^\omega$ is dominated by the function $n \mapsto j(n)$, and it is forced to be eventually different from every ground model function. For if $C \subset B_i$ forced $|\dot{s} \cap \check{f}| = \aleph_0$ for some ground model function f, Andersen could play $B_n = B_{f(n),n}$ against the strategy τ, and writing B for the result of the game, the set $C \setminus B$ would be in the ideal I. Thus Andersen would win against the strategy τ, contradiction. In other words, the name $\dot{s}$ witnesses the failure of the weak Cohen property of the poset P_I.

On the other hand, a failure of the weak Cohen property provides a winning strategy for Steinitz. By Lemma 2.3, there must be a function $f \in \omega^\omega$, a Borel I-positive set $B \in P_I$ and a Borel function $G : B \to \omega^\omega$ such that for every element $r \in B$, $f(r)$ is pointwise dominated by f, and moreover for every function $h \in \omega^\omega$ the set $\{r \in B : h \cap G(r) \text{ is infinite}\}$ is in the ideal I. It is clear that Steinitz will win by playing the $B = B_{ini}$ and partitionining the set B according to the value $G(r)(n)$. $\square$

Lemma 3.19. *(LC) Suppose I is a σ-ideal generated by an F_σ collection of compact sets and J is a definable σ-ideal such that P_J is proper. The following are equivalent:*

1. *P_J has the weak Cohen property*
2. *$\neg I \perp J$.*

Of course it is the implication (1)$\to$(2) which is most interesting from the forcing preservation point of view.

Proof. The implication (2)$\to$(1) is easier. If P_J does not have the weak Cohen property then there is a σ-ideal I generated by an F_σ collection of compact sets such that $I \perp J$. Namely, suppose that $B \in P_J$ is a condition forcing that $\dot{g} \in \omega^\omega$ is a function pointwise dominated by some $\check{f} \in \omega^\omega$, yet eventually different from any ground model function. Thinning out the condition B if necessary we may assume that there is a Borel function $G : B \to \Pi_n f(n)$ such that $B \Vdash \dot{g} = \dot{G}(\dot{r}_{gen})$. Now let I be the σ-ideal on $\Pi_n f(n)$ which is σ-generated by sets $C_{g,n} = \{h \in \Pi_n f_n : h \text{ is different from } g \text{ at every input} \geq n\}$. This is an F_σ collection of compact sets. It turns out that $I \perp J$ as witnessed by the set $C \subset \Pi_n f(n) \times B$, where $\langle h, r\rangle \in C$ if $|h \cap G(r)| = \aleph_0$. Clearly, the vertical sections of the set C are J-small since B forces the real $\dot{g}$ to be eventually different from any given ground model function h. And the horizontal sections of the complement of the set C are I-small since P_I forces the generic real to have infinite intersection with any function $G(r)$ for r in the ground model.

Now the (1)→(2) implication. For definiteness assume that the σ-ideal I is on the Cantor space 2^ω, and fix compact sets $K_n : n \in \omega$ of compact subsets of 2^ω, the generators for the ideal I. We may and will assume that $K_0 \subset K_1 \subset \dots$, and for brevity we will identify the elements of the sets K_n with their respective trees on $2^{<\omega}$.

Suppose for contradiction that $I \perp J$, as witnessed by some sets $B_I \in P_I$ and $B_J \in P_J$ and a Borel set $C \subset B_I \times B_J$ such that its complement has I-small horizontal sections. We must find a J-positive vertical section of the set C. Thinning out the set B_J if necessary it is possible to find Borel functions $f_n : B_J \to K_n$ such that the horizontal section of the complement of the set C attached to a real $s \in B_J$ is covered by the I-small set $\bigcup_n f_n(s)$–Lemma 2.3.

Use the determinacy of the weak Cohen game $WCG(J)$ to find Andersen's winning strategy σ in it. Let M be a countable elementary submodel of a large enough substructure containing all the relevant objects. We will construct a play τ of the weak Cohen game respecting the strategy σ with Steinitz's moves in the model M as well as an M-generic filter $g \subset M \cap P_I$. To this end by induction build initial segments $\tau \restriction n \in M$ of the play τ as well as a descending chain $\{D_n : n \in \omega\}$ of conditions in $M \cap P_I$ so that

- $B_{ini} = B_J, D_0 = B_I$
- D_{n+1} is in the n-th open dense subset of P_I in M in some fixed enumeration
- for every real $s \in B_n$ the sets $\bigcup_{m \in n} f_m(s)$ and D_{n+1} are disjoint, where B_n is the set the strategy σ played at n-th round of the play τ.

In the end, let $B_\tau \subset B_{ini}$ be the result of the play against the strategy σ, and let $r \in B_I$ be the real such that $\{r\} = \bigcap_n D_n$. The choice of the functions f_n implies that for every real $s \in B_\tau$ it is the case that $\langle r, s\rangle \in C$, and the proof will be complete.

In order to perform the inductive construction, suppose that D_n as well as $\tau \restriction n \in M$ have been obtained. For every number $i \in \omega$ consider the equivalence relation E^i_n on the set B_{ini} given by $s \, E^i_n t$ if and only if $f_m(s) \restriction i = f_m(t) \restriction i$ for every number $m \leq n$. The equivalences induce partitions P^i_n of the set B_{ini} into finitely many Borel equivalence classes. The next move on the play against the strategy σ will be one of the partitions P^i_n, it is just necessary to decide which one:

Let B^i_n be the answer the strategy σ gives if the partition P^i_n is played. Let $T^i_m : m \leq n$ be the uniform values of $f_m(s) \restriction i : m \leq n$ for every real $s \in B^i_n$. So each T^i_m is a binary tree of height i. Let U be a nonprincipal ultrafilter on ω and define infinite binary trees $T_m : m \leq n$ by setting $t \in T_m$

iff $\{i \in \omega : t \in T^i_m\} \in U$. Since the sets $K_m : m \leq n$ are closed, it follows immediately that $T_m \in K_m$. Since the set D_n is I-positive and the closed sets $[T_m] : m \leq n$ are in the ideal I, there must be a finite binary sequence $t \notin \bigcup_{m \leq n} T_m$ such that $D_{n.5} = D_n \cap [t] \notin I$. Let $D_{n+1} \subset D_{n.5}$ be any condition in $M \cap P_I$ in the n-th open dense subset of P_I in the model M. By the definitions, there must be a number i larger than the length of the sequence t such that $t \notin T^i_m$, for all $m \leq n$. Then P^i_n will be the next Steinitz's move on the play against the strategy σ. It is immediate that the second item above continues to hold.

□

Corollary 3.20. *If I is a σ-ideal generated by an σ-compact collection of compact sets then P_I is bounding and does not add splitting reals.*

Proof. One way to prove this is to consider the ideal J on $\mathcal{P}(\omega)$ of sets nowhere dense in $\mathcal{P}(\omega)/\text{Fin}$. Mathias forcing naturally densely embeds into the factor poset P_J and it has the weak Laver property. The conclusion then follows from $\neg I \perp J$ and Lemma 2.15. Another way to argue is to use the Hechler forcing for the bounding part, and a σ-centered forcing diagonalizing an ultrafilter for the splitting part, using Example 3.17. Still another way is to use a fusion argument for the bounding part, and the dense-set version of Halpern-Laüchli theorem (see [17]) for the splitting part. The latter argument has the advantage of surviving into finite side-by-side products. □

Now suppose that I is an ideal on 2^ω, σ-generated by a σ-compact collection $\bigcup_n F_n$, where each F_n is a compact set of compact subsets of 2^ω, identified with their generating trees. The previous corollary together with Lemma 2.4 shows that compact sets are dense in the poset P_I but is there a simple description? The following corollary provides a combinatorially manageable subset of the poset P_I which facilitates fusion arguments. Call a nonempty tree $T \subset 2^{<\omega}$ *I-good* if for every node $t \in T$ and every number $n \in \omega$ there is m such that for no tree $S \in F_n$ the set $\{s \in 2^m \cap T : t \subset s\}$ is a subset of S.

Corollary 3.21. *A Borel set $B \subset 2^\omega$ is I-positive if and only if for some I-good tree $T \subset 2^{<\omega}$, $[T] \subset B$.*

Proof. The right to left direction is easy. If $T \subset 2^{<\omega}$ is an I-good tree and $S_m \in \bigcup_n F_n$ are trees for each $m \in \omega$, it is not difficult to build by induction nodes $0 = t_0 \subset t_1 \subset \dots$ of the tree T such that $t_{m+1} \notin S_m$. In the end the branch $\bigcup_m t_m$ is an element of the set $[T]$ which shows that $[T] \not\subset \bigcup_m [S_m]$.

For the other direction, first note that if $T \subset 2^{<\omega}$ is a tree and $t \in T$ is a node and $n \in \omega$ is a number such that for every number $m \in \omega$ the set $\{s \in 2^m \cap T : t \subset s\}$ is a subset of some tree $S_m \in F_n$, then there is a tree $S \in F_n$ containing all the nodes of the tree T compatible with t since the set F_n is compact. Now define a Cantor-Bedixson style operation on trees: given $T \subset 2^{<\omega}$, the tree $T' \subset T$ is the set of all nodes $t \in T$ such that for all n there is m such that no tree $S \in F_m$ contains all the nodes $s \in 2^m \cap T$ extending t. It is clear that $[T] \setminus [T'] \in I$. Repeating the operation transfinitely on any tree $T \subset 2^{<\omega}$, taking intersections at limit stages, we see that after countable number of stages the situation must stabilize either in the empty set, in which case $[T] \in I$, or in an I-good tree, in which case $[T] \notin I$. So if $B \notin I$ is a Borel I-positive set, by the previous corollary it contains an I-positive compact subset of the form $[T] \notin I$ for some tree T, which by the previous sentence contains an I-good subtree. □

The previous corollary can be also proved by a determinacy argument.

To conclude the subsection, a couple of borderline examples. The Spinas forcing is clearly not generated by a σ-compact collection of compact sets, because it adds a splitting real. Note that in this case the generating family of closed sets is G_δ in the space $K(2^\omega)$. Finally, the $\mathtt{non}(\mathtt{null})$ forcing makes the ground model reals null, and so $I \perp \mathtt{null}$ for the ideal associated to it. Note that the Solovay forcing is not σ-centered and does not have the weak Cohen property.

4 Porosity ideals

Definition 4.1. Let X be a Polish space and U a countable collection of its Borel subsets. An *abstract porosity* is an inclusion preserving map $\mathtt{por} : \mathcal{P}(U) \to \mathcal{B}(X)$, that is $a \subset b \to \mathtt{por}(a) \subset \mathtt{por}(b)$. The *porosity ideal* I associated with the porosity $\mathtt{por}$ is σ-generated by sets $\mathtt{por}(a) \setminus \bigcup a$, as a runs through all subsets of U. Such sets (and their subsets) are called *porous*.

Note that by extending the topology on the space X it is possible to assume that the sets in U are clopen, without changing the Borel structure or the poset P_I. However, this is not always a natural step to make–consider Example 4.5.

Example 4.2. The standard σ-ideal of σ-porous sets on the real line. The *metric porosity* of a set $A \subset \mathbb{R}$ at a point x is defined as

$$\limsup_{\delta \to 0} \frac{\lambda(A, x, \delta)}{\delta}$$

where $\lambda(A, x, \delta)$ is the length of the longest open subinterval of $(x - \delta, x + \delta)$ disjoint from the set A. Traditionally, a set A is called porous if it has metric porosity > 0 at all its points, and the σ-ideal of σ-porous sets is generated by the porous sets. It can be obtained in our setting as follows. Let U be the collection of all intervals with rational endpoints, and $r \in \mathtt{por}(a)$ if the metric porosity of the set $\mathbb{R} \setminus \bigcup a$ at r is greater than 0.

Example 4.3. The meager ideal. Let U be some basis for the topology of the space X, and let $\mathtt{por}(a)$ be the closure of $\bigcup a$. It is not difficult to see that porous sets are exactly those with a closed nowhere dense superset. Thus the resulting porosity ideal is the meager ideal.

Example 4.4. The monotonicity forcing. Consider the lexicographical ordering on the Cantor space 2^ω, and let $f : 2^\omega \to 2^\omega$ be a Borel function. The ideal I

σ-generated by the sets $B \subset 2^\omega$ such that $f \restriction B$ is monotonic, is a porosity ideal. In order to show this, let $U = \{u_{s,t} : s, t \in 2^{<\omega}\}$ where $u_{s,t} = \{r \in 2^\omega : s \subset r$ and $t \subset f(r)\}$. The abstract porosity $\mathtt{por}$ is then defined by $r \in \mathtt{por}(a)$ iff r is a point of monotonicity of the function $f \restriction (\{r\} \cup (X \setminus \bigcup a))$. The forcing P_I is either trivial or in the forcing sense equivalent to the c_{min} forcing from Example 3.4–[35], 2.3.7.

Example 4.5. Steprāns forcing. Let $f : 2^\omega \to 2^\omega$ be a Borel function. The ideal I σ-generated by the sets $B \subset 2^\omega$ such that $f \restriction B$ is continuous, is a porosity ideal. The argument is parallel to the previous example. The abstract porosity $\mathtt{por}$ is then defined by $r \in \mathtt{por}(a)$ iff r is a point of continuity of the function $f \restriction (\{r\} \cup X \setminus \bigcup a\}$. The forcing P_I is either trivial or in the forcing sense equivalent to the poset isolated in [31], see [35], 2.3.48.

Example 4.6. The differentiability forcing. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. The ideal I σ-generated by sets $B \subset 2^\omega$ such that $f \restriction B$ is differentiable, is a porosity ideal. The argument is parallel to the previous example.

In fact we have the following general

Lemma 4.7. *Every σ-ideal generated by closed sets is a porosity ideal.*

Proof. Fix the σ-ideal I on some Polish space X, and let U be some topology basis for X. For a set $a \subset U$ let $\mathtt{por}(a) = X$ if $X \setminus \bigcup a \in I$ and $\mathtt{por}(a) = 0$ otherwise. It is trivial to check that this is an abstract porosity which generates the ideal I. $\square$

We do not know if every porosity ideal can be presented as an ideal σ-generated by closed sets. For example it is possible to change the underlying topology of Example 4.5 to give the ideal a generating collection of closed sets.

A word about definability of porosities is in order. We do not know if for every porosity ideal $I \in L(\mathbb{R})$ there must be an abstract porosity in the model $L(\mathbb{R})$ which generates it. On the other hand, if the abstract porosity is suitably definable then there is a neat connection to the definability of the resulting porosity ideal.

Lemma 4.8. *Suppose that* $\mathtt{por}$ *is coanalytic. Then the associated porosity ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Here to say that $\mathtt{por}$ is coanalytic is to say that the set $\{\langle a, r\rangle \in \mathcal{P}(U) \times X : r \in \mathtt{por}(a)\} \subset \mathcal{P}(U) \times X$ is coanalytic. A brief survey of the preceding examples will show that in all of them the abstract porosity is coanalytic in this sense. To say that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ is to say that for every analytic set $A \subset \omega^\omega \times X$ the set $\{x \in \omega^\omega$:the vertical section A_x of the set A associated with the real x is in the ideal $I\} \subset \omega^\omega$ is coanalytic. This condition on the ideal I has traditionally been investigated in descriptive set theory–[15], 29.E. It significantly simplifies the theory of the countable support iteration of the poset P_I and the statements of the absoluteness theorems in [35].

Proof. To simplify the notation assume that the underlying space X is just the Baire space ω^ω and that the collection U is lightface Δ^1_1, and that the abstract porosity is lightface Π^1_1. For every real u we must prove that $\bigcup(I \cap \Sigma^1_1(u)) \in \Pi^1_1(u)$–see [35], Lemma C.0.8. For simplicity put $u = 0$. First, a small claim.

Claim 4.9. *If $A \subset \omega^\omega \times \omega^\omega$ is Σ^1_1 then the set $\{x \in \omega^\omega : A_x$ is porous$\}$ is Π^1_1.*

This is a direct computation. The vertical section A_x is porous if and only if $A_x \subset \mathtt{por}(a)$ where $a = \{u \in U : u \cap A_x = 0\}$ by the monotonicity of the abstract porosity. This can be restated as $\forall r \in A_x \forall b \subset U\ a \subset b \to r \in \mathtt{por}(b)$ by the monotonicity again, or $\forall r \in A_x \forall b \subset U\ r \in \mathtt{por}(b) \vee \exists v \in b\ A_x \cap v = 0$. This is a uniformly $\Pi^1_1(x)$ statement as desired.

The effective version of the First Reflection Theorem [15], 35.10 now shows that every Σ^1_1 porous set has a Δ^1_1 porous superset. A Π^1_1 coding of Δ^1_1 sets [18] then can be used to show that the set $C = \bigcup(\Sigma^1_1 \cap$porous sets$) = \bigcup(\Delta^1_1 \cap$porous sets) is Π^1_1. It will be enough to show that $C = \bigcup(I \cap \Sigma^1_1)$.

The right-to-left inclusion is clear. For the other, let $A \in \Sigma^1_1$ be a set such that $A \setminus C \neq 0$ and argue that $A \notin I$. Suppose that $\{a_n : n \in \omega\}$ is a countable collection of subsets of U; we must find an element $r \in A \setminus \bigcup_n(\mathtt{por}(a_n) \setminus \bigcup a_n)$. To this end, by induction construct recursive trees T_n as well as nodes $t^n_m \in T_m$ for $m \leq n$ so that

- T_0 is some recursive tree projecting into the Σ^1_1 set $A \setminus C$, $t^0_0 = 0$
- the nodes $t^i_n \in T_n$ are defined for all $i \geq n$ and form a strictly decreasing sequence in the tree
- the set $A_n = \bigcap_{m \leq n} \mathrm{proj}(T_m \restriction t^n_m)$ is nonempty
- $A_{n+1} \cap \mathtt{por}(a_n) \setminus \bigcup a_n = 0$.

It is clear that in the end the branches through the trees T_n obtained from the nodes t^i_n project into the same real $r \in A$, and the last item of the induction hypothesis will imply that $r \notin \bigcup_n(\mathtt{por}(a_n) \setminus \bigcup a_n)$ as desired. To find the tree T_{n+1} and the nodes t^{n+1}_m for $m \leq n+1$, consider the set $b = \{u \in U : A_n \cap u = 0\}$ and the set $\mathtt{por}(b)$. A similar complexity computation as in the proof of the claim shows that $\mathtt{por}(b)$ is a Π^1_1 set. The set $A_n \setminus \mathtt{por}(b)$ is then Σ^1_1 and nonempty, because if it were empty, the Σ^1_1 set A_n would be porous, covered by the set $\mathtt{por}(b) \setminus \bigcup b$ which contradicts the fact that $A_n \cap C = 0$ and the definition of the set C. There are now two cases. Either $\mathtt{por}(a_n) \subset \mathtt{por}(b)$. In this case let T_{n+1} be some recursive tree projecting into the nonempty Σ^1_1 set $A_n \setminus \mathtt{por}(b)$ and find suitable nodes $t^{n+1}_m : m \leq n+1$ in the trees. Or, $\mathtt{por}(a_n) \not\subset \mathtt{por}(b)$, and this means that $a_n \not\subset b$ by the monotonicity of the abstract porosity. Choose a set $u \in a_n \setminus b$, a recursive tree T_{n+1} projecting into the nonempty Σ^1_1 set $A_n \cap u$, and find suitable nodes $t^{n+1}_m : m \leq n+1$. This concludes the induction step and the proof.

□

4.1 Properness

Theorem 4.10. *The forcing P_I is proper for every porosity ideal I.*

Proof. Let $\mathtt{por}, U$ be the abstract porosity used to define the ideal I. First, a small abstract claim. Call a set $B \subset X$ *supporting* if, writing $a = \{u \in U : u \cap B = 0\}$, it is the case that $B \cap u \notin I$ for all $u \in U \setminus a$, and moreover $B \cap \mathtt{por}(a) = 0$.

Claim 4.11. *Every I-positive Borel set has a supporting Borel I-positive subset.*

Proof. Let $B \in P_I$ and write $a = \{u \in U : u \cap B \in I\}$. Let $C = (B \setminus \bigcup a) \setminus \mathtt{por}(a)$. We claim the set $C \subset B$ is positive and supporting. For the positivity, note that the first set difference removed only the set $B \setminus \bigcup a$ which is in the ideal I by σ-additivity. The second set difference then removed only a porous set, namely the set $\mathtt{por}(a) \setminus \bigcup a$. For the supporting property, note that $a = \{u \in U : u \cap C = 0\}$ and observe that $C \cap \mathtt{por}(a) = 0$ since the set $\mathtt{por}(a)$ was explicitly removed from C. □

Note that the claim proves much more really: every set can be turned into a supporting one by removing an I-small set which is moreover suitably definable from the original set.

Now let M be a countable elementary submodel of a large enough structure and $B \in P_I \cap M$ is a condition. We must prove that the set $\{r \in B : r \text{ is } M\text{-generic for } P_I\}$ is I-positive. That means, given countably many subsets $\{a_n : n \in \omega\}$ of U, we must produce an M-generic real $r \in B \setminus \bigcup_n (\mathtt{por}(a_n) \setminus \bigcup a_n)$.

Let $\{D_n : n \in \omega\}$ enumerate all open dense subsets of P_I in the model M, and by induction build a decreasing sequence $B = B_0 \supset B_1 \supset \dots$ of conditions in $P_I \cap M$ such that each of them is supporting, $B_{n+1} \in D_n$ and $B_{n+1} \cap \mathtt{por}(a_n) \setminus \bigcup a_n = 0$. This will certainly conclude the proof since then the unique real in the intersection $\bigcap_n B_n$ has the desired properties.

The inductive step is divided into two cases. Assume first the set B_n is disjoint from $\mathtt{por}(a_n)$. In such a case any supporting condition $B_{n+1} \subset B_n$ in the set $D \cap B_n$ will do the job. Otherwise, there is some real $s \in B_n \cap \mathtt{por}(a_n)$. Here, writing $b = \{u \in U : u \cap B_n = 0\}$, the supporting property of the condition B_n implies that $s \notin \mathtt{por}(b)$, and by the monotonicity of the porosity, $a_n \not\subset b$. Find a set $u \in a_n \setminus b$ and note that the set $C = B_n \cap u \in M$ is I-positive by the supporting property of the condition B_n, and moreover $C \cap \mathtt{por} a_n \setminus \bigcup a_n = 0$ since $u \in B$. Then any supporting condition $B_{n+1} \subset C$ in the set $D_n \cap M$ will do the job. □

There are very few forcing preservation properties we can prove for porosity ideals in general.

Lemma 4.12. *$\neg I \perp \mathtt{meager}$ holds for every porosity ideal I.*

Note that the previous proof actually gives strong properness in the sense of [35] 4.1.4. Strongly proper forcings do not make the ground model reals meager by the argument from [33] 6.3.

4.2 Dichotomies

Theorem 4.13. *In the choiceless Solovay model, every porosity ideal I is closed under well-ordered unions.*

Proof. Let $V \subset W \subset V[G]$ be a ground model, its $\mathtt{Coll}(\omega, <\kappa)$ extension for some inaccessible cardinal κ, and the intermediate choiceless Solovay model. Let $I \in W$ be a porosity ideal on some Polish space X, generated by some abstract porosity $\mathtt{por}, U \in W$. By a standard homogeneity argument we may and will assume that all of I, X, U, and $\mathtt{por}$ are definable from parameters in the ground model.

Using the argument from the proof of Theorem 3.10 it is clear that it is sufficient to prove that if $P \in V$ is a poset of size $< \kappa$ adding an element $\dot{r} \in \dot{X}$ such that $\mathtt{Coll}(\omega, <\kappa) \Vdash \dot{r} \notin \bigcup_{a \in V} \mathtt{por}(a) \setminus \bigcup a$, then the set $\{r \in X : \exists g \subset P$ a V-generic filter such that $r = \dot{r}/g\}$ is I-positive.

To this end, let $\{a_n : n \in \omega\}$ be a countable collection of subsets of the set U. We must produce a V-generic filter $g \subset P$ such that $\dot{r}/g \notin \bigcup_n (\mathtt{por}(a_n) \setminus \bigcup a_n)$. Well, enumerate the open dense subsets of the poset P in V by $\{D_n : n \in \omega\}$ and build a sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ of conditions in P such that

- $p_{n+1} \in D_n$
- the set $B_{n+1} = \{r \in X : \exists g \subset P$ V-generic with $p_{n+1} \in G$ and $\dot{r}/g = r\}$ is disjoint from the set $\mathtt{por}(a_n) \setminus \bigcup a_n$.

The inductive step is divided into two cases. Either the set B_n is disjoint from $\mathtt{por}(a_n)$. Here any condition $p_{n+1} \leq p_n$ in the set D_n will do because certainly $B_{n+1} \subset B_n$. Or the set $B_n \cap \mathtt{por}(a_n)$ is nonempty, containing some real r. In this case, look at the set $b \subset U, b = \{u \in U : p_n \Vdash \dot{r} \notin \dot{u}\} \in V$. By the assumption on the name $\dot{r}$, $r \notin \mathtt{por}(b) \setminus \bigcup b$, and by the definition of the set b, $r \notin \bigcup b$. Therefore $r \notin \mathtt{por}(b)$. However, $r \in \mathtt{por}(a)$ and so, by the monotonicity of porosity, $a \not\subset b$. Let $u \in a \setminus b$. By the definition of the set b, there is a condition $q \leq p_n$ forcing the real $\dot{r}$ into $\dot{u}$. Any condition $p_{n+1} \leq q$ in the open dense set D_n will work as required. $\square$

Corollary 4.14. *In the Solovay model, every subset of X has either a Borel I-positive subset or a Borel I-small superset.*

Proof. In the Solovay model, every subset of X is a well-ordered union of Borel sets. Apply the previous theorem. $\square$

Corollary 4.15. *(LC) Suppose that* $\mathtt{por}$ *is universally Baire. Then every universally Baire set has either a Borel I-positive subset or a Borel I-small superset.*

A similar game-theoretic argument as in [28] can be used to show that under a suitable large cardinal assumption, the above corollary holds even for porosities which are not universally Baire. As usual, the part concerning analytic sets can be proved in ZFC alone.

Theorem 4.16. *If I is a porosity ideal, every I-positive analytic set has a Borel I-positive subset.*

Proof. For the simplicity of notation assume that the underlying space is the Baire space ω^ω. Let $A \subset \omega^\omega$ be an I-positive analytic set. Consider the partial ordering Q of I-positive analytic sets ordered by inclusion. Similarly to the treatment of the poset P_I let $\dot{r}_{gen}$ be the Q-name for the real given by $\dot{r}_{gen}(n) = m$ if and only if $\{r \in \omega^\omega : r(n) = m\} \in \dot{G}$ where $\dot{G}$ is the generic filter. The following claim is a variation of the basic Lemma 2.1.1 in [35]. Note that in the end the poset Q will turn out to contain P_I as a dense subset, however as long as this conclusion is not available, care must be exercised to perform the proof correctly.

Claim 4.17. *Every condition $B \in Q$ forces $\dot{r}_{gen} \in \dot{B}$, and the generic real is outright forced to fall out of every I-small set.*

Proof. For the first part, let T be a tree projecting into the set B. Suppose that $G \subset Q$ is a generic filter containing the condition B and work in the generic extension. Let $S \subset T$ be the set defined by $t \in S$ if and only if the projection of the tree $T \restriction t$ is in the generic filter. The σ-additivity of the ideal I implies that $S \subset T$ is a tree without terminal nodes, and each of its branches projects into the real r_{gen}. Therefore $r_{gen} \in B$ as desired.

For the second part, note that the ideal I is generated by Borel sets. If $B \in Q$ is a condition and $C \in I$ is a set, increase the set C if necessary into a Borel I-small set and consider the condition $B \setminus C \in I$. The first part of the claim and an absoluteness argument implies $B \setminus C \Vdash \dot{r}_{gen} \notin \dot{C}$ as required. $\square$

Now let M be a countable elementary submodel of a large enough structure containing all the vital information, in particular the set A. The set $B = \{r \in \omega^\omega :$ there is an M-generic filter $g \subset M \cap Q$ such that $A \in g$ and $r = \dot{r}_{gen}/g\}$ is an I-positive Borel subset of the set A, completing the proof of the theorem. To see why it is Borel, let R be the complete subalgebra of the algebra $RO(Q)$ generated by the name $\dot{r}_{gen}$ and let $p \in R$ be the projection of the set $A \in Q$. Then $B = \{r \in \omega^\omega :$ there is a *unique* M-generic filter $g \subset M \cap R$ such that $p \in g$ and $r = \dot{r}_{gen}/g\}$ and this is a Borel set by [15], 15.A. To see why the set B is I-positive repeat the argument from the previous Theorem, and use the Claim. To see that $B \subset A$, use the forcing theorem and the Claim to see that for every real $r \in B$, $M[r] \models r \in A$, and by an analytic absoluteness argument $r \in A$ as desired. $\square$

A similar argument can be repeated in the following sections.

4.3 Other preservation properties

There is an interesting class of porosities for which the related partial ordering is bounding.

Fact 4.18. *[24] If X is a compact separable metric space and I is the ideal of σ-porous subsets of X as in Example 4.2, then every I-positive analytic set has a compact I-positive subset. Moreover [32], every Borel function $f : B \to \mathbb{R}$ with an I-positive Borel domain has a continuous restriction with an I-positive Borel domain.*

Corollary 4.19. *If I is as in Example 4.2, then P_I is a bounding forcing.*

Proof. By Fact 4.18 and Fact 2.4. $\square$

It is not clear how large a class of forcings is described in the previous corollary and how and if it depends on the original metric space X. It does not include all porosity ideals, since the meager ideal is a porosity ideal. The proof in [24] is quite complicated and very much unlike anything in this paper. There is a limited number of cases in which a parallel result can be obtained by an application of an integer game.

Example 4.20. Consider the Cantor space 2^ω, the set $U = \{[t] : t \in 2^{<\omega}\}$, and the porosity $\mathtt{por}$ given by $x \in \mathtt{por}(a)$ if there is a number m such that for infinitely many $n \in \omega$ there is $t \in 2^{<\omega}$ such that $x \restriction n \subset t$, $|t| \leq nm$, and $[t] \in a$. Then compact sets are dense in P_I, where I is the associated porosity ideal.

To see this, for a set $B \subset 2^\omega$ consider a game $G(B)$ played by Lasker and Steinitz. Lasker produces dynamically on a fixed schedule a σ-porous set A and Steinitz produces a binary sequence x. Steinitz wins if $x \in B \setminus A$. To make this precise, Lasker produces subsets $a_k : k \in \omega$ of U and the set A is then extracted as $\bigcup_k(\mathtt{por}(a_k) \setminus \bigcup a_k)$. At round n Lasker indicates all the pairs $\langle t, k\rangle$ such that $t \in 2^{n^2}$, $k \in n$ and $[t] \in a_k$, and Steinitz answers with the n-th bit of the sequence x. It is clear that given any set $C \in I$, Lasker can play in such a way that $C \subset A$ for his resulting set $A \in I$.

Claim 4.21. *$B \in I$ if and only if Lasker has a winning strategy in the game $G(B)$.*

Clearly, if $B \in I$ then Lasker can win by indicating the subsets $\{a_k : k \in \omega\}$ of the set U such that $B \subset \bigcup_k(\mathtt{por}(a_k) \setminus \bigcup a_k)$, disregarding Steinitz's moves entirely. On the other hand, if Lasker has a winning strategy σ then for every number $k \in \omega$ let $B_k = \{x \in B :$ if Steinitz produces x then the winning strategy produces a set $a_k = a_k(x)$ such that $x \in \mathtt{por}(a_k) \setminus \bigcup a_k\}$. Since the strategy σ is winning, it is the case that $\bigcup_k B_k = B$, and the proof of the claim will be complete if we show that each set B_k is porous. Let $a = \{[t] \in U : B_k \cap [t] = 0\}$ and argue that $B_k \subset \mathtt{por}(a)$. Well, suppose $x \in B_k$. If $m \in \omega$ is the number witnessing that $x \in \mathtt{por}(a_k(x))$ then m actually witnesses that $x \in \mathtt{por}(a)$: for infinitely many $n > k, m$ there is a binary sequence $t \in 2^{<\omega}$ such that $x \restriction n \subset t$, $|t| \leq nm$ and $[t] \in a_k(x)$. But for every such number n and such a sequence t it must be the case that $[t] \in a$, since $[t] \in a_k(y)$ for every infinite binary sequence y extending $x \restriction n$ by the rules of the game, and so no infinite binary sequence y extending t and $x \restriction n$ can be an element of the set B_k by the definition of the set B_k!

To conclude the argument for the Example, let $B \in P_I$ be a Borel I-positive set. The game $G(B)$ is determined, and by the above Claim it must be Steinitz who has a winning strategy σ. Now consider the space X of all Lasker's counterplays and the set $C = \sigma''X$. Then

- $C \subset B$ since σ is a winning strategy
- C is compact since it is a continuous image of the compact space X
- C is I-positive since σ is still a winning strategy for Steinitz in the game $G(C)$.

This is exactly what we set out to prove.

Diego Rojas generalized this argument to give a short determinacy proof of the following. Whenever $\langle X, d\rangle$ is a zero dimensional compact metric space and I is the σ-ideal on X generated by the metric porosity, then every I-positive analytic set has an I-positive compact subset.

Example 4.22. Let $f : 2^\omega \to 2^\omega$ be a Borel function and let I be the σ-ideal generated by the sets X such that $f \restriction X$ is continuous. Then compact sets are dense in P_I. So this is the ideal from Example 4.5 and this result has been proved in [35], 2.3.46. There is an extremely simple integer game which gives this result in a manner similar to the previous Example. Let $B \subset 2^\omega$ be a set, and consider another game $H(B)$ between Steinitz and Lasker. Lasker produces binary sequences $y_n : n \in \omega$ and Steinitz produces a binary sequence x. Steinitz wins if $x \in B$ and $\forall n\ f(x) \neq y_n$. To specify the schedule for both players, at round n Steinitz must indicate n-th bit on his sequence, while Lasker is allowed to hesitate before placing more bits on his respective sequences. Lasker may even fail to finish the production of some of his sequences y_n, and then in the end the value $f(x)$ is compared only with those sequences y_n he finished.

Claim 4.23. *$B \in I$ if and only if Lasker has a winning strategy in the game $H(B)$.*

The argument then follows closely that for the previous Example. Note though that in this case the forcing is not bounding since it fails the continuous reading of names criterion of Lemma 2.4 by its very definition.

5 σ-finite ideals

Definition 5.1. Suppose X is a Polish space, U is a countable collection of its Borel subsets, and $\mathtt{diam} : U \to \mathbb{R}^+$ is a *diameter* function such that the diameters converge to zero. Suppose moreover that $w : \mathcal{P}(U) \to \mathbb{R}^+ \cup \{\infty\}$ is a Borel *weight* function such that

- w is monotone: $a \subset b \subset U$ implies $w(a) \leq w(b)$

- w is weakly subadditive: there is a function $f \in \omega^\omega$ such that $w(a), w(b) < n$ implies $w(a \cup b) < f(n)$.

Then let $\mu : \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be defined by $\mu(B) = \sup\{\inf\{w(a) : a \subset U$ consists of sets of diameter $\leq \delta$ such that $B \subset \bigcup a\} : \delta > 0\}$. The function μ will be called the *Hausdorff submeasure* defined from $\mathtt{diam}, w$. The *σ-finite ideal* associated with μ is the σ-ideal generated by sets B with $\mu(B) < \infty$.

Note that by extending the topology on the space X if necessary it is possible to enter the situation in which the sets in the collection U are clopen, without changing the partial order P_I. In such a case the ideal is clearly generated by $G_{\delta\sigma}$ sets. The terminology chosen is a little misleading in that the function μ may not be subadditive but just weakly subadditive if the original weight function was.

Example 5.2. The usual h-dimensional Hausdorff measure μ_h on the unit interval. It may not be clear how to obtain it in this context due to the demand that the diameters of sets in U converge to zero. Consider the set U of all intervals $[\frac{n}{2^{-m}}, \frac{n+1}{2^{-m}}]$ for natural numbers $m \in \omega$ and $n \in 2^m$, their diameters equal to their length. For $a \subset U$ put $w(a) = \Sigma_{u \in a} \mathtt{diam}(u)^h$. Since every interval of length $< 2^{-m-1}$ but $\geq 2^{-m}$ can be covered by two intervals from the set U of length 2^{-m}, it follows that for the derived Hausdorff submeasure μ, $\mu_h \leq \mu \leq 2\mu_h$ and the σ-finite ideal derived from μ is the same as the ideal of σ-finite sets for the h-dimensional submeasure μ_h.

Example 5.3. Laver forcing. Look at the Baire space ω^ω, the collection $U = \{[t] : t \in \omega^{<\omega}\}$ with the diameter function $\mathtt{diam}([t]) = 2^{-m}$ if t is the m-th element of $\omega^{<\omega}$ under some fixed enumeration, the weight function $w(a) = 1$ if for every sequence $t \in \omega^{<\omega}$ the set $\{n \in \omega : [t^\frown n] \in a\}$ is finite, and $w(a) = \infty$ otherwise. It is not difficult to see that the null ideal arising form the resulting Hausdorff submeasure is exactly the Laver ideal.

Example 5.4. Fat tree forcing (also called profusely branching tree forcing). Let $\{X_n : n \in \omega\}$ be a collection of finite sets, and $\{\theta_n : n \in \omega\}$ respective finite submeasures on each such that $\theta_n(X_n) \to \infty$ as $n \to \infty$. Call a finite sequence t suitable if for every number $n \in \mathrm{dom}(t)$ it is the case that $t(n) \in X_n$. A *fat tree* is a tree T of height ω consisting of suitable sequences such that for every number $m \in \omega$ there is $k \in \omega$ such that for every $n > k$ and every sequence $t \in T$ of length n the set $\{x \in X_n : t^\frown x \in T\}$ has θ_n submeasure at least m.

Consider the forcing notion consisting of all fat trees ordered by inclusion. The case when each θ_n is a counting measure is well-known to be proper and bounding ([16]). In fact, this forcing naturally densely embeds into a poset P_I for a suitable σ-finite ideal I. Namely, let $X = \Pi_n X_n$, let U be the collection of all sets of the form $[t]$, where t is a suitable sequence and $[t] = \{x \in \Pi_n X_n : t \subset x\}$. Let $\mathtt{diam}([t]) = 2^{-|t|}$ and let $w(a) = \sup_t\{\theta_{|t|}(\{x \in X_{|t|} : [t^\frown x] \in a\})\}$. Let I be the associated σ-finite ideal. We will show in Subsection 5.4 that a Borel set $B \subset X$ is I-positive if and only if it contains a subset of the form $[T]$ for some fat tree T.

Example 5.5. Using the notation of Example 5.4, let $\hat{g}_n\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing unbounded function and consider trees such that for every number $m \in \omega$ there is $k \in \omega$ such that for every $n > k$ and every sequence $t \in T$ of length n we have $g_n \circ \theta_n(\{x \in X_n : t^\frown x \in T\}) \geq m$.

Such generalizations of Example 5.4 include $\mathbf{PT}_{f,g}$ defined in [2], 7.3.B.

Example 5.6. Let J be an F_σ ideal on ω. By Mazur's theorem [21] there is a lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega : \phi(a) < \infty\}$. Let $X = \mathcal{P}(\omega)$, $U = \{A_n : n \in \omega\}$ where $A_n = \{x \subset \omega : n \in x\}$, $\mathtt{diam}(A_n) = 2^{-n}$, and $w(a) = \phi(\bar{a})$ whenever $a \subset U$ and $\bar{a} = \{n \in \omega : A_n \in a\}$. Let μ be the derived Hausdorff measure on $\mathcal{P}(\omega)$ and I the derived σ-finite ideal.

What is the forcing P_I? It follows from the work in Section 5.3 that it is proper and bounding. The generic subset of ω has finite intersection with all ground model elements of the ideal J. If $x \in J$ and $\phi(x) < \delta$ then it is immediate to verify that writing $B_x = \{y \subset \omega : x \cap y \text{ is infinite}\}$, we have $\mu(B_x) < \delta$ and therefore $B_x \in I$. The ideal I is not generated by the sets B_x though. The forcings P_I are close if not identical to the partial orders isolated by Claude Laflamme [16] in a combinatorial form.

5.1 Properness

Theorem 5.7. *If I is a σ-finite ideal for some Hausdorff submeasure, then the forcing P_I is proper.*

The proof is really a game theoretic argument.

Definition 5.8. Suppose that I is a σ-finite ideal, P is a partial order and $\dot{r}$ is a P-name for a real. The σ-finite game $SFG(I, P, \dot{r})$ is a game of length ω between Alechin and Capablanca played in the following fashion. In the beginning Alechin indicates an initial condition p_{ini} and then he produces one-by-one open dense subsets $\{D_n : n \in \omega\}$ of the poset P, and dynamically on a fixed schedule a Borel set A in the ideal I. Capablanca plays one by one decreasing conditions $p_{ini} \geq p_0 \geq p_1 \geq \dots$ so that $p_n \in D_n$ and p_n decides the n-th digit of the real $\dot{r}$. He is allowed to hesitate for any number of rounds before placing his next move. Capablanca wins if either the set A Alechin played failed to be σ-finite or else, writing g for the filter Capablanca obtained, it is the case that $\dot{r}/g \notin A$.

To make this precise, Alechin produces subsets $\{a^l_k : k, l \in \omega\}$ of the set U so that all elements of a^l_k have diameter $\leq 2^{-l}$, and $w(a^l_k) \leq k$. The Borel set A above is then extracted as $\bigcup_k \bigcap_l \bigcup a^l_k$. Note that this is indeed a set in the ideal I as each set $\bigcap_l \bigcup a^l_k$ has Hausdorff submeasure at most k. Enumerating the set U as $\{u_i : i \in \omega\}$, we demand Alechin to indicate at round n which among the sets $u_i : i \in n$ fall into which set $a^l_k : k, l \in n$. Note that in this way Alechin's moves related to the set A can be coded as natural numbers. Note also that given any set $B \in I$, Alechin can play so that his resulting set $A \in I$ is a superset of B.

Lemma 5.9. *The following are equivalent:*

- *$P \Vdash \dot{r}$ is not contained in any ground model Borel I-small set*
- *Capablanca has a winning strategy in the game $SFG(I, P, \dot{r})$.*

Theorem 5.7 immediately follows. Suppose that I is a σ-finite ideal. An application of the lemma to the poset P_I and its generic real shows that Capablanca has a winning strategy σ in the game $SFG(I, P_I, \dot{r}_{gen})$. Now suppose M is a countable elementary submodel of a large enough structure containing the ideal I and the strategy σ, and let $B \in M \cap P_I$ be a condition. We must show that the set $\{r \in B : r \text{ is } M\text{-generic}\}$ is I-positive. Well, if A is a Borel I-small set then consider the play of the game in which Capablanca follows the strategy σ and Alechin indicates $B = p_{ini}$, enumerates the open dense sets in the model M, and dynamically produces the set A. Clearly, all moves of the play will be in the ground model, therefore the filter $g \subset M \cap P_I$ Capablanca creates will be M-generic, and the resulting real $\dot{r}_{gen}/g \in B$ will be M-generic and outside the I-small set A as desired.

One direction of the lemma is easy. If some condition $p \in P$ forces the real $\dot{r}$ to belong to some ground model coded I-small Borel set B, then Alechin has a simple winning strategy. He will indicate $p_{ini} = p$, dynamically produce a suitable superset $A \supset B, A \in I$, and on the side he will find an inclusion increasing sequence $\{M_n : n \in \omega\}$ of countable elementary submodels of some large enough structure containing $A, we, P, \dot{r}$ such that the n-th Capablanca's move p_n belongs to the model M_n, and he will make sure to enumerate all open dense subsets of the poset P that occur in the model $N = \bigcup_n M_n$. This is certainly easily possible. In the end Capablanca's filter g will be N-generic, by the forcing theorem $N[g] \models \dot{r}/g \in A$, by Borel absoluteness $\dot{r}/g \in A$, and Alechin won.

For the other direction of the lemma note that the game is Borel and therefore determined–Fact 2.7. Thus it will be enough to obtain a contradiction from the assumption that $P \Vdash \dot{r}$ is not contained in any ground model coded Borel I-small set and yet Alechin has a winning strategy σ. A small claim will be used repeatedly:

Claim 5.10. *For every condition $p \in P$ and every number $k \in \omega$ there is a number $l(p, k) > 0$ such that for every set $a \subset U$ of weight $\leq k$ consisting of sets of diameter $\leq 2^{-l(p,k)}$ there is a condition $q \leq p$ forcing $\dot{r} \notin \bigcup a$.*

Proof. Suppose this fails for some p, k, and for every natural number $l \in \omega$ find a set $a_l \subset U$ of weight $\leq k$ consisting of sets of diameter $\leq 2^{-l}$ such that $p \Vdash \dot{r} \in \bigcup a_l$. But then, $p \Vdash \dot{r} \in \bigcap_l \bigcup a_l$, and the latter set is certainly in the ideal I, being of Hausdorff submeasure $\leq k$. Contradiction! □

First we must fix some objects instrumental in the construction of the counterplay. Fix an enumeration $U = \{u_i : i \in \omega\}$ from which Alechin's schedule is derived. Fix a function $g \in \omega^\omega$ such that for every number n, for every collection of $\leq n$ many subsets of U of weight $\leq n$, their union has weight $\leq g(n)$. Such a function exists by the weak subadditivity of the weight function w. Fix also a countable elementary submodel M of a large enough structure containing the weight function and the strategy σ.

Capablanca will obtain a winning counterplay against the strategy σ by induction. His moves will be denoted by p_n, played at rounds i_n, and on the side he will produce numbers l_n. The intention is that the resulting filter g will be M-generic, and the resulting real $\dot{r}/g$ will fall out of all sets $a_k^{l_k}$. For the convenience of notation let τ_n be the initial segment of the counterplay ending after the round i_n. The induction hypothesis is the following:

- $p_n \in M$ and in fact $p_n \in D_n$ where D_n is the n-th open dense subset of the poset P in the model M in some fixed enumeration.
- $l_n > l(p_n, g(n))$, diameters of all sets $u_i : i \leq i_n$ are greater than 2^{-l_n} and the diameters of all sets $u_i : i > i_n$ are less than $2^{-l(p_n, g(n))}$. Also $i_n > \max\{n, l_k : k \in n\}$.
- $p_n \in E_n$ where E_n is the n-th open dense subset of P Alechin produced in the play τ_n
- for every number $i \leq i_n$ and every number $k \in n$ if Alechin decided during the play τ_n that $u_i \in a_k^{l_k}$, then $p_n \Vdash \dot{r} \notin \dot{u}$.

This will certainly conclude the proof. Let $\tau = \bigcup_n \tau_n$ and argue that Capablanca won this run of the game $SFG(I, P, \dot{r})$. To see this note that whenever $u = u_i \in a_k^{l_k}$ is a Borel set, then every condition Alechin played after round i forces $\dot{r} \notin u$ by the third item of the induction hypothesis. Now since the resulting filter $g \subset P$ is M-generic, by the forcing theorem $M[g] \models \dot{r}/g \notin u$, and by Borel absoluteness $\dot{r}/g \notin u$. This means that $\dot{r}/g \notin \bigcup_k a_k^{l_k} \supset A$, and Capablanca won.

To get p_0, l_0, i_0 just find a condition $p_0 \in D_0 \cap E_0 \cap M$ below p_{ini}, let i_0 be some number such that all sets $u_i : i > i_n$ have diameter less than $l(p_0, g(0))$, and let l_0 be a large enough number. The induction hypotheses are satisfied. Now suppose that the play τ_n and the numbers $l_k : k \leq n$ have been constructed. Let $\bar{a}_k^{l_k} : k \leq n$ be the sets the strategy σ produces if Capablanca forever hesitates to place his next move after the play τ_n. Now note that the set $b = \bigcup_{k \in n} \bar{a}_k^{l_k}$ has weight $\leq g(n)$ by the definition of the function g, and the only sets of diameter $\leq 2^{-l(p_n, g(n))}$ in the set b are in the collection $\{u_i : i \leq i_n\}$. Note that the last item of the induction hypothesis shows that $p_n \Vdash \dot{r} \notin \bigcup(b \cap \{u_i : i \leq i_n\})$. By the second item of the induction hypothesis then, there is a condition $q \leq p_n$ int he model M such that $q \Vdash \dot{r} \notin \bigcup b$. Let $p_{n+1} \leq q$ be some condition in the model M which belongs to the sets D_n and E_n and decides the n-th bit of the real $\dot{r}$. The condition p_{n+1} will then be played at some round i_{n+1} such that the sets of diameter $\geq 2^{-l(p_{n+1}, g(n+1))}$ are indexed by numbers $i \leq i_n$, and let l_{n+1} be a sufficiently large number. This concludes the inductive step and the proof of Theorem 5.7.

The heavy use of the determinacy of the σ-finite game in the above proof has the unpleasant side effect that it is impossible to extend the argument to cover the case of undefinable weight function w. We do not know if in such a case the conclusion of Theorem 5.7 can in fact fail.

5.2 Dichotomies

Theorem 5.11. *In the Solovay model, every σ-finite ideal is closed under well-ordered unions.*

Proof. As was the case before, it is just enough to prove that if κ is an inaccessible cardinal, I is a σ-finite ideal on some Polish space X derived from $U, \mathtt{diam}$, and a weight function w, and P is a forcing of size $< \kappa$ adding a real $\dot{r}$ which falls out of all Borel ground model coded I-small sets, then in the choiceless Solovay model derived from the cardinal κ the set $C = \{r \in X : \exists g \subset P\ g$ is V-generic and $r = \dot{r}/g\}$ is I-positive. In order to prove this, fix Capablanca's winning strategy σ in the game $SFG(I, P, \dot{r})$ in the ground model and move to the Solovay extension. There, the strategy σ is still winning in the ground model version of the game $SFG(I, P, \dot{r})$ since the nonexistence of a defeating counterplay is a wellfoundedness statement. Now for every I-small set $B \subset X$ in the Solovay extension consider the play of the game $SFG(I, P, \dot{r})$ against σ in which Alechin enumerates all the open dense subsets of the poset P in the ground model and dynamically produces some Borel set $A \supset B$ in the ideal I. Since the strategy σ is still winning, the resulting real must belong to the set $C \setminus A$, showing that the set C cannot be I-small. $\square$

Corollary 5.12. *Let I be a σ-finite ideal on some Polish space X. In the Solovay model, every subset of X has either a Borel I-positive subset or a Borel I-small superset.*

Corollary 5.13. *(LC) Let I be a σ-finite ideal on some Polish space X. Every universally Baire subset of X has either a Borel I-positive subset or a Borel I-small superset. For analytic sets this is true in ZFC.*

This can be viewed as a consequence of universally Baire absoluteness and the previous corollary, however there is an alternative integer game argument. For the simplicity of notation suppose that the underlying Polish space is just the Cantor space 2^ω. Let $B \subset 2^\omega$ be a set and consider the following integer counterpart $iSFG(I, B)$ to the σ-finite game $SFG(I, P, \dot{r})$ above. Let Alechin and Capablanca play for ω many steps. Alechin dynamically and on a fixed schedule produces a σ-finite Borel set A, and Capablanca produces a binary sequence $r \in 2^\omega$. Alechin's schedule for producing the set A is the same as in the game $SFG(I, P, \dot{r})$, and Capablanca is allowed to hesitate before announcing the next bit on the sequence r. Capablanca wins if $r \in B \setminus A$.

Lemma 5.14. *Let $B \subset 2^\omega$. The following are equivalent.*

- *$B \in I$*
- *Alechin has a winning strategy in the game $iSFG(I, B)$.*

It now follows that under AD, every subset of 2^ω either has an analytic I-positive subset, or a Borel I-small superset. To see this, suppose that $B \subset 2^\omega$ is I-positive. Then Alechin has no winning strategy in the game $G(B)$, and by the determinacy of the game, Capablanca has a winning strategy σ. Let C be the set of all binary sequences the strategy σ can come up with. Clearly, $C \subset B$ since the strategy σ is winning, C is analytic since it is the image of a Borel set under the continuous function σ, and C is I-positive because σ remains a winning strategy for Capablanca in the game $G(C)$. An additional argument is then needed to show that every I-positive analytic set has an I-positive Borel subset.

One direction of the lemma is trivial. If $B \in I$ then Alechin can win by producing a suitable superset $A \supset B$ in the ideal I, ignoring Capablanca's moves entirely. In the other direction, suppose for contradiction that $B \notin I$ and still Alechin has a winning strategy σ. First a bit of notation. If τ is a partial play respecting the strategy and if $k, l \in \omega$ are natural numbers then $c^l_k(\tau)$ is the set of all u_j for $j >$the length of τ which the strategy σ puts into the collection a^l_k if Capablanca hesitates forever to place his next move after τ. Similarly, for finite sequences $\vec{k}, \vec{l}$ of natural numbers of the same length, $c^{\vec{l}}_{\vec{k}}(\tau)$ is the set of all u_j for $j >$the length of τ such that the strategy σ puts them into the collection $a^{\vec{l}(m)}_{\vec{k}(m)}$ for some number m if Capablanca hesitates forever to place his next move after τ. Finally, for $i \in \omega$ and $b \in 2$ let $\tau i b$ denote the extension of the play τ in which Capablanca makes just one more nontrivial move–at round i he places the bit b on his sequence.

Note that all the sets $\bigcup_k \bigcup_j \bigcap_{l>j} \bigcup c^l_k(\tau)$ and $\bigcap_i \bigcup c^{\vec{l}}_{\vec{k}}(\tau i b)$ are of σ-finite Hausdorff submeasure, for all finite plays τ observing the strategy σ, all bits $b \in 2$ and all finite sequences $\vec{k}, \vec{l}$. This follows in the second case from the weak subadditivity of the weight function and the fact that the diameters converge to 0. Since $B \notin I$, there must be a real $r \in B$ falling out of all the abovementioned sets. We will construct Capablanca's winning counterplay against the strategy σ in which he produces the real r. This will complete the proof of the Lemma.

By induction on $n \in \omega$ build natural numbers i_n and l_n and look at the partial play τ_n in which Alechin follows the strategy σ while Capablanca places the m-th bit of the real r at the round i_m for $m \leq n$ and which ends right after the round i_n. The induction hypotheses are

- $r \notin u_i$ for any $i \in \omega$ less than length of τ_n which the strategy σ put into the set $a^{l_k}_k$ for some $k \in n$
- $r \notin \bigcup_{k \leq n} \bigcup c^{l_k}_k(\tau_n)$

Of course the first item immediately implies that Capablanca won this run of the game $iSFG$ against the strategy σ, since the resulting real r is in $B \setminus \bigcup_k \bigcup a^{l_k}_k$ by the first item of the induction hypothesis. The second item is here just to keep the induction going.

To construct the play τ_0 and the numbers l_0, i_0, proceed as in the induction step from $\tau_{-1} = 0$. To perform the induction step, suppose the play τ_n as well as the numbers $i_k, l_k : k \in n$ have been found. To find the number l_{n+1} note that $r \notin \bigcup_{k\in\omega} \bigcup_j \bigcap_{l>j} \bigcup c^l_k(\tau_n)$ and therefore there must be a number l_{n+1} large enough so that no set $u_i : i <$the length of τ_n has diameter $< 2^{-l_{n+1}}$ and moreover, $r \notin c^{l_{n+1}}_{n+1}(\tau_n)$. To find the number i_{n+1}, let b be the $n+1$-th bit on the sequence r, let $\vec{k} = \langle 0, 1, \dots n+1\rangle$, $\vec{l} = \langle l_0, l_1 \dots l_{n+1}\rangle$ and note that $r \notin \bigcap_i \bigcup c^{\vec{l}}_{\vec{k}}(\tau_n ib)$. It follows that there is a number i such that $r \notin \bigcup_{k\leq n+1} c^{l_k}_k(\tau ib)$. Such number $i = i_{n+1}$ and the play $\tau_{n+1} = \tau_n i_{n+1} b$ work as desired.

5.3 Other preservation properties

The most important special case of σ-finite ideals is the one associated with lower semicontinuous Hausdorff submeasures:

Definition 5.15. A weight function w is *lower semicontinuous* if $w(a) = \sup\{w(b) : b \subset a \text{ finite}\}$ for every set $a \subset U$. A Hausdorff submeasure is *lower semicontinuous* if it is derived from a lower semicontinuous weight function.

The posets P_I, where I is a σ-finite ideal derived from a lower semicontinuous Hausdorff submeasure, share some forcing properties, in particular they are bounding. This is exactly quantified and proved below.

Definition 5.16. A forcing P has the *Laver property* if for every ground model function $f \in \omega^\omega$ and every ground model nondecreasing function $g \in \omega^\omega$ converging to infinity, for every function $h \in \omega^\omega$ dominated pointwise by f in the extension, there is a ground model function $e : \omega \to [\omega]^{<\aleph_0}$ such that the set $e(n)$ has size $\leq g(n) + 1$ and contains the value $h(n)$.

A basic definable example of a partial ordering with Laver property is the Mathias forcing. It seems to be difficult to come up with substantially more complex examples (see Theorem 9.10). There is a natural game theoretic counterpart to the Laver property.

Definition 5.17. Fix a σ-ideal J on ω^ω. The *Laver game* $LG(J)$ between Botvinnik and Tal is played in the following fashion. First, Botvinnik indicates an initial condition $B_{ini} \in P_J$. After that, in each round n he chooses a number $g(n) \in \omega$ and a finite partition the initial set B_{ini} into Borel pieces. Tal then responds by a Borel set B_n which is the union of at most $g(n) + 1$ many sets in the partition. Tal wins if either the function $g \in \omega^\omega$ is not nondecreasing and diverging to infinity, or else the *result* of the play, the set $\bigcap_n B_n$, is J-positive.

Lemma 5.18. *Suppose that J is a σ-ideal such that P_J is proper. The following are equivalent:*

- *P_J fails the Laver property*
- *Botvinnik has a winning strategy in the game $LG(J)$.*

Proof. Suppose first that Botvinnik has a winning strategy σ in the game. It is then not difficult to see that Botvinnik has a *positional* winning strategy τ, that is, a nondecreasing function $g \in \omega^\omega$ and partitions P_n of the initial condition such that he wins playing these objects regardless of Tal's moves. To see this, note that at every move there are only finitely many options for Tal and so there are only finitely many possible answers the strategy σ can supply. Let P_n be a partition refining all the finitely many partitions the strategy σ can supply at round n. Finally, use a compactness argument to find an increasing sequence $\{n_m : m \in \omega\}$ of natural numbers such that the strategy σ asks for at least m many pieces of the partition at each round after round n_m, no matter what Tal plays. Then define the function g by $g(n) = m$ if $n_m \leq n < n_{m+1}$. It is not difficult to check that the positional winning strategy τ given by g and $\{P_n : n \in \omega\}$ is winning since it is a better strategy than σ. But then, if $B = B_{ini}$ is the initial condition dictated by the strategy, $f \in \omega^\omega$ is a function defined by $f(n) = |P_n|$, and $\dot{h}$ is a name for a function in ω^ω defined by $\dot{h}(n) = m$ if $\dot{r}_{gen}$ belongs to the m-th piece of the partition P_n in some fixed enumeration $P_n = \{C^k_n : k \in f(n)\}$, it is immediate that $B, f, g, \dot{h}$ witness the failure of the Laver property of the poset P_J. For if $e : \omega \to [\omega]^{<\aleph_0}$ is a function such that $|e(n)| \leq g(n)$, the Borel set $C = \bigcap_n \bigcup_{k \in e(n)} C^k_n$ must be I-small, since it is a result of the play respecting the strategy τ; and so no condition below the set B can force $\forall n \in \omega\ \dot{h}(n) \in \check{e}(n)$.

On the other hand, if Botvinnik has no winning strategy in the game $LG(J)$ then the Laver property is rather easy to check. Suppose $B \in P_J$ is a J-positive Borel set, $f \in \omega^\omega$ a function, $g \in \omega^\omega$ a nondecreasing function diverging to infinity, and $\dot{h}$ a name for a function in ω^ω dominated by f. Strengthening the condition B if necessary we may and will assume that there is a Borel function $c : B \to \Pi_n f(n)$ such that $B \Vdash \dot{h} = \dot{k}(\dot{r}_{gen})$. Let $P_n = \{C^k_n : k \in f(n)\}$ be a partition of the set B defined by $C^k_n = \{r \in B : c(r)(n) = k\}$. Now the set $B = B_{ini}$ together with the partitions $\{P_n : n \in \omega\}$ and the function g does not constitute a positional winning strategy for Botvinnik, and there must be a winning counterplay for Tal, with moves $B_n = \bigcup_{k \in e(n)} C^k_n$ for some function $e : \omega \to [\omega]^{<\aleph_0}$ such that $|e(n)| \leq g(n) + 1$. The result of the game, some I-positive Borel set $C \subset B$, then clearly forces $\forall n \in \omega\ \dot{h}(n) \in \check{e}(n)$ as desired. $\square$

Compare the following with Lemma 2.14.

Theorem 5.19. *(LC) Suppose that I is a σ-finite ideal derived from some lower semicontinuous Hausdorff submeasure and J is a universally Baire σ-ideal such that P_J is proper. The following are equivalent:*

1. *P_J has the Laver property*
2. *$\neg I \perp J$.*

Of course from the point of view of forcing preservation it is the implication (1)→(2) that is most interesting.

Proof. The (2)→(1) implication is easier. Suppose that J is a σ-ideal such that P_J is proper and fails to have the Laver property, as witnessed by some ground model functions $f, g \in \omega^\omega$ and a name $\dot{h}$ for a function in the extension dominated by f. Then $I \perp J$ for some σ-finite ideal I derived from a lower semicontinuous Hausdorff submeasure. Namely, let $X_n = [f(n)]^{g(n)}$ and let θ_n be the submeasure on X_n defined by $\theta_n(Y)$ =the smallest possible size of a set $z \subset X_n$ such that Y contains no superset of z, and use Example 5.4 to obtain a σ-finite ideal I on the space $\Pi_n X_n$. It is not difficult to see that the forcing P_I adds a function $\dot{e} \in \Pi_n X_n$ such that every ground model function h dominated by f satisfies $h(n) \in \dot{e}(n)$ for all but finitely many n. To see that $I \perp J$, find a J-positive Borel set B_J and a Borel function $k : B_J \to \omega^\omega$ such that $B_J \Vdash \dot{h} = \dot{k}(\dot{r}_{gen})$ and let $C \subset \Pi_n X_n \times B_J$ be the Borel set $C = \{\langle e, r\rangle :$ for all but finitely many n, $k(r)(n) \in e(n)\}$. Now the vertical sections of the set C are J-small since for every given $e \in \Pi_n X_n$ the condition B_J forces $\dot{h}$ to avoid the prediction by e infinitely many times. And the horizontal sections of the complement of the set C are I-small since for every given real r the poset P_I forces the function $\dot{e}$ to predict $k(r)$ at all but finitely many values.

For the other implication, suppose $B_J \in P_J$ and $B_I \in P_I$ are positive Borel sets and $C \subset B_J \times B_I$ is a Borel set with I-small vertical sections. We must produce a J-positive horizontal section of the complement of the set C.

Use Fact 2.3 to see that thinning out the set B_J if necessary we may assume that there are Borel maps $a^l_k : B_J \to \mathcal{P}(U) : k, l \in \omega$ such that for every element $r \in B_J$, the set $a^l_k(r)$ has weight $\leq k$ and consists of sets of diameter $\leq 2^{-l}$, and the vertical section C_r of the set C above r is covered by the σ-finite set $\bigcup_k \bigcap_l \bigcup a^l_k(r)$. Fix also an enumeration $U = \{u_i : i \in \omega\}$, a function $g \in \omega^\omega$ such that unions of $\leq n$ many subsets of U of weight $\leq n$ have weight $\leq g(n)$, fix Tal's winning strategy σ in the Laver game $LG(J)$, and let M be a countable elementary submodel of a large enough structure containing the strategy σ as well as other relevant objects.

By induction on $n \in \omega$ build plays $\tau_0 \subset \tau_1 \subset \tau_2 \subset \dots$ of the game $LG(I)$ of the respective length $i_0, i_1, i_2, \dots$, conditions $B_I = A_0 \supset A_1 \supset A_2 \supset \dots$ and numbers $l_0, l_1, l_2, \dots$. The intention is that the resulting set B_τ of the play $\tau = \bigcup_n \tau_n$ is J-positive, the intersection $\bigcap_n A_n$ is a singleton containing some unique $x \in X$ and the set $B_\tau \times \{x\}$ is a subset of the complement of C, secured by the fact that for every $r \in B_\tau$, $x \notin \bigcup_k \bigcup a^{l_k}_k$. The induction hypotheses are:

- $l_n > l(A_n, g(n))$, and for every number $i \leq i_n$ it is the case that $\mathtt{diam}(u_i) \geq 2^{-l_n}$ and for every number $i > i_n$ it is the case that $\mathtt{diam}(u_i) \leq l(A_n, g(n))$.
- For every number j, $i_n \leq j < i_{n+1}$, Botvinnik places the following move in τ_{n+1} at round j. Consider the equivalence relation E^j_n given by $r \, E^j_n \, s$ if and only if for every number i, $i_n \leq i \leq j$, and for every number $k \leq n$, $u_i \in a^{l_k}_k(r) \leftrightarrow u_i \in a^{l_k}_k(s)$. Botvinnik plays the partition of B_J into the finitely many Borel E^j_n equivalence classes, asking Tal to choose $n+1$ many of them. Tal answers according to the strategy σ by a set B_j, a union of at most $n+1$ many equivalence classes. Let

$b_n^j = \{u_i \in U : i_n \leq i \leq j, \exists k \leq n\ \exists r \in B_j\ u_i \in a_k^{l_k}\}$. Note that the collection b_n^j consists of sets of diameter at most $l(A_n, g(n))$ and has weight at most $g(n)$ since it is a union of $n+1$ many sets of weight $\leq n$.

- Whenever $i \in i_n$ and $\exists k \leq n\ \exists r \in \bigcap_{j \in i_n} B_j\ u_i \in a_k^{l_k}$ then $A_n \cap u_i = 0$.

This will certainly be enough. The first item implies that the intersection $\bigcap_n A_n$ will be a singleton by Fact 2.2. The resulting set B_τ of the play τ is J-positive. The third item then implies that $x \in \bigcap_{n\in\omega} A_n$ and $r \in \bigcap_{j\in\omega} B_j$ then $\langle x, r\rangle \notin C$ as required, since $x \notin \bigcup_k \bigcup a_k^{l_k}$. The second item is there only to keep the induction going.

Now suppose the play τ_n, the set A_n and the numbers l_n, i_n have been found. Consider the infinite run of the Laver game extending τ_n according to the third inductive item, and the collection $b_n = \bigcup_{j\in\omega} b_n^j$. This collection consists of sets of diameter $\geq l(A_n, g(n))$ and by the lower semicontinuity of the Hausdorff submeasure in question, it has weight at most $g(n)$. Therefore the Borel set $A_{n.5} = A_n \setminus \bigcup b_n$ is I-positive. Note that this set already satisfies the third item of the induction hypothesis. Now find an arbitrary set $A_{n+1} \subset A_{n.5}$ in the n-th open dense set in the model M and consider the number $l(A_{n+1}, g(n+1))$. Let i_{n+1} be some number such that $\texttt{diam}(u_i) \leq 2^{-l(A_{n+1}, g(n+1))}$ for every $i > i_n$, and let $l_{n+1} > l$ be some number such that $\texttt{diam}(u_i) \geq l_{n+1}$ for every $i \leq i_n$. This completes the inductive step and the proof of the Theorem. □

Corollary 5.20. *If I is the σ-finite ideal derived from some lower semicontinuous Hausdorff submeasure, then P_I is bounding and does not add splitting reals.*

Proof. Use Theorem 5.19 with the ideal J of sets nowhere dense in the algebra $\mathcal{P}(\omega)/\text{Fin}$ and then Lemma 2.14. □

Corollary 5.21. *[27] Let $\langle X, d\rangle$ be a compact metric space and h a positive real number. Every analytic subset of X of non-σ-finite h-dimensional Hausdorff measure has a compact subset with the same property.*

Proof. First we will find a Hausdorff submeasure μ on the space X in the sense of this section such that $\mu_h \leq \mu \leq 4^h \cdot \mu_h$ where μ_h is the usual h-dimensional Hausdorff measure derived from the metric d. Use the compactness of the space X to find a finite set $Y_n \subset X$ such that every point of X lies within distance 2^{-n} from some element of Y_n. The submeasure μ is derived from the set $U = \{B(x, 2^{-n+1}) : x \in Y_n, n \in \omega\}$, the usual metric diameter function, and the weight function $w(a) = \Sigma\{\texttt{diam}^h(u) : u \in a\}$. Observe that for every positive δ there are finitely many sets in U with diameter greater than δ, and the weight function is lower semicontinuous, so that the results of this subsection are applicable.

It is clear that $\mu_h \leq \mu$. On the other hand, if $A \subset X$ is a set and b is some collection of open balls of diameter $< \delta < 1$ covering the set A, replace each ball $u \in b$ by a ball $v_u \in U$ such that $u \subset v$ and $\texttt{diam}(u) \leq \texttt{diam}(v_u) \leq 4\texttt{diam}(u)$.

This is easily possible by the choice of the set U, and the resulting set $a = \{v_u : u \in b\} \subset U$ will still cover the set A, it will consist of sets of diameter $< 4\delta$ and it will have weight $w(a) \leq 4^h \cdot w(b)$. It immediately follows that $\mu(A) \leq \mu_h(A)$.

It now follows that the ideals I, J of sets of σ-finite measure coincide for μ and μ_h. If $A \subset X$ is an analytic set, $A \notin I$, then there is a Borel set $B \subset A$ with $B \notin I$ by Corollary 5.13. Since the submeasure μ is lower continuous, the poset P_I is bounding and by the basic Lemma 2.4 there is a compact set $C \subset B$, $C \notin I$. This is the sought compact subset of A of non-σ-finite h-dimensional Hausdorff measure. $\square$

There is an alternative integer game theoretic argument for the bounding part of the lemma, which then can be used in a very concise determinacy proof of the above classical result. For definiteness assume that the underlying space is $X = \omega^\omega$. Given a set $B \subset \omega^\omega$ and a function $f : B \to \omega^\omega$, consider the following game $G(B, f)$ between Alechin and Euwe. Alechin produces subsets $a^l_k : l \in \omega$ of U, each a^l_k of weight $\leq k$ consisting of sets of diameter $\leq 2^{-l}$; Euwe produces sequences $x, y \in \omega^\omega$. Alechin's schedule is identical to that of games SFG and $iSFG$, Euwe is allowed to hesitate before placing next number on his sequences x, y. Euwe wins if $x \in B \setminus \bigcup_k \bigcap_l \bigcup a^l_k$ and $y = f(x)$.

Lemma 5.22. *Alechin has a winning strategy if and only if $B \in I$.*

The proof follows the line of argument for Lemma 5.14. Now suppose that B is an I-positive Borel set, and $f : B \to \omega^\omega$ is a Borel function. We will find a compact I-positive set $C \subset B$ such that $f \restriction C$ is continuous, which by Lemma 2.4 is equivalent to the bounding property of the poset P_I. The game $G(B, f)$ is Borel, therefore determined, and by the Claim it must be Euwe who has a winning strategy σ. Look at the space Y of legal Alechin counterplays and note that the lower continuity of the weight function implies that this space is compact. Consider the set D of all pairs $\langle x, y\rangle \in \omega^\omega \times \omega^\omega$ which the strategy σ can come up with against some Alechin's counterplay. The set D is a continuous image of the compact space Y, and therefore it must be compact. It follows that $C \subset \omega^\omega$, the projection of the set C into the first coordinate, must be compact as well and $f \restriction C$ is continuous. Finally, the set $C \subset B$ must be I-positive since the strategy σ remains winning in the game $G(C, f)$.

There is an important anti-preservation theorem regarding this class of forcings as well as many others.

5.4 The fat tree forcings

It now remains to complete the argument for Example 5.4. Let $\{X_n : n \in \omega\}$ be a sequence of finite sets, θ_n submeasures on them, and derive the diameters, weights as well as the σ-finite ideal I as in that Example. We must prove

Lemma 5.23. *A Borel subset of $\Pi_n X_n$ is I-positive if and only if it has a subset of the form $[T]$ for some fat tree T.*

Proof. The right-to-left implication is easy and the reader who made it up to here can certainly prove this on his own. For the opposite direction, suppose that B is an I-positive set. Using Corollary 5.20 and Lemma 2.4, thinning out the set B if necessary we may assume it is closed, $B = [S]$ for some tree S.

Consider the game G between Petrosian and Spassky. At round $n \in \omega$, a node $t_n \in S$ of length n will be known; $t_0 = 0$. Petrosian plays a number $m_n \in \omega$ and a nonempty set $Y_n \subset X_n$ consisting of immediate successors of the node $t \in S$ which has θ_n submeasure at least m_n. Spassky chooses an element $x_n \in Y_n$ and puts $t_{n+1} = t_n^\frown \langle x_n \rangle$. Petrosian wins if the seqeunce of his numbers $m_n : n \in \omega$ is nondecreasing and converges to infinity.

The game G is Borel and therefore determined. If Petrosian has a winning strategy then it is easy to use it to construct a fat subtree $T \subset S$. Thus the proof will be complete once we derive a contradiction from the assumption that it is Spassky who has a winning strategy σ.

By induction build partial plays $0 = \tau_0 \subset \tau_1 \subset \dots$ against the strategy σ and a decreasing sequence $[S] = [S_0] \supset [S_1] \supset \dots$ of I-positive closed sets such that

- the numbers m_n played in the play τ_{k+1} after τ_k are all equal to k
- every branch in the closed set $[S_k]$ can result from some infinite extension of the play τ_k in which Spassky follows the strategy σ and Petrosian plays numbers $m_n = k$ after the play τ_k.

Of course then in the end the play $\tau = \bigcup_k \tau_k$ follows the strategy σ and Petrosian won in it, reaching a contradiction. Now the play $0 = \tau_0$ and the tree $S = S_0$ satisfy the induction hypotheses. Assume that the play τ_k and the tree S_k are known. Let $l = l(S_k, k+1)$. An inspection of the definitions reveals that there must be a node $t \in S_n$ of length l such that $l = l(S_k \restriction t, k+1)$. Let τ_{k+1} be any extension of the play τ_k which results in the node t. There must be such an extension by the second induction hypothesis. To construct the tree $S_{k+1} \subset S_k \restriction t$, analyze the strategy σ to find a collection $a \subset U$ of weight $\leq k+1$, consisting of sets of diameter $\leq 2^{-l}$, such that every element in the closed set $[S_{k+1}] = [S_k \restriction t] \setminus \bigcup a$ can result from an infinite extension of the play τ_{k+1} in which Spassky follows the strategy σ Petrosian plays $m_n = k+1$ after the play τ_{k+1}. To conclude the induction step, note that the closed set $[S_{k+1}]$ is I-positive by the choice of the number l. $\square$

There is an important anti-preservation result concerning forcing with σ-finite ideals. Extending the topology while preserving the Borel structure it is possible to make sure that all sets in the generating collection U are clopen, and therefore every set of finite submeasure will be included in a G_δ set of the smae finite submeasure. In the natural examples it invariably so happens that countable sets have finite submeasure. Corollary 2.17 then shows that unlike in the previous sections, the quotient forcing makes ground model sets meager. It is not necessary to invoke large cardinal hypotheses for this conclusion.

6 Submeasurable forcings

Definition 6.1. Let X be a set. A *submeasure* on X is a function $\mu : \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ such that

1. it is monotone: $\mu(0) = 0$, $A \subset B \subset X$ implies $\mu(A) \leq \mu(B)$
2. it is countably subadditive: $\mu(\bigcup_{n\in\omega} A_n) \leq \Sigma_n \mu(A_n)$.

The submeasure μ is *normalized* if $\mu(X) = 1$.

Clearly, if μ is a submeasure on a Polish space X then the collection $I_\mu = \{A \subset X : \mu(A) = 0\}$ is a σ-ideal and we would like to investigate the quotient forcing P_{I_μ}. There is not much to say in general, since every σ-ideal I is equal to I_μ for the submeasure μ defined by $\mu(A) = 0$ if $A \in I$ and $\mu(A) = 1$ otherwise. Rather, the idea is to start with some natural submeasures μ and show how their measure-theoretic properties influence the forcing properties of the quotient P_{I_μ}.

Definition 6.2. A submeasure μ on 2^ω is *dense* if

1. it is outer regular: for every Borel set $A \subset 2^\omega$, $\mu(A) = \inf\{\mu(B) : A \subset B$ open $\}$
2. it satisfies a version of Lebesgue density theorem. Given a set $A \subset 2^\omega$ and a point $x \in 2^\omega$, define $\bar{d}(A, x) = \limsup_n \frac{\mu(A)\cap\mu([x\restriction n])}{\mu([x\restriction n])}$ and $\bar{d}A = \{x \in 2^\omega : \bar{d}(A, x) = 1\}$. We require that for every Borel set $A \subset 2^\omega$, $\mu(A) = \mu(\bar{d}A)$.
3. it is suitably definable: the set $Y \subset \mathbb{Q} \times \mathcal{P}(2^{<\omega})$ is Borel, where $\langle q, a\rangle \in Y$ if $\mu(\bigcup_{t\in a}[t]) \leq q$.

Similar definition can be used on spaces of the form $\Pi_n X_n$ where X_n are finite sets.

It is clear that a null ideal associated with a dense submeasure is generated by G_δ sets.

Example 6.3. The Solovay forcing carries Lebesgue measure on it which is dense.

Example 6.4. A quite general way of constructing dense submeasures is implicit in Steprāns's work [30]. A *good norm* on a set X is a norm $n : \mathbb{R}^X \to \mathbb{R}^+$ such that $|f| \leq |g|$ implies $n(f) \leq n(g)$, and $n(1) = 1$. If n, m are good norms on sets X, Y respectively, their *iteration* is the good norm $n * m$ on $X \times Y$ defined by $(n * m)(f) = n(x \mapsto m(y \mapsto f(x, y)))$. Now if $X_i : i \in \omega$ are finite sets with respective good norms $n_i : i \in \omega$ on them, consider the sequence of the good norms $m_j = n_0 * n_1 * \cdots * n_{j-1}$ on the sets $\Pi_{i\in j} X_i$. This sequence of norms has a natural limit, a good norm p on $X = \Pi_i X_i$ described in the following way:

- Suppose $f \in \mathbb{R}^X$ is a step function, i.e. there is a partition $X = \bigcup_{k\in l} Y_k$ of the space X into finitely many clopen sets and real numbers $r_k : k \in l$ such that $f(x) = r_k \leftrightarrow x \in Y_k$. Then find a number i large enough such that $f(x)$ depends only on $x \restriction i$, write $f(x) = f^*(x \restriction i)$, and let $p(f) = m_i(f^*)$.

- If $g \in \mathbb{R}^X$ is a nonnegative lower semicontinuous function, let $p(g) = \sup\{p(f) : |f| \leq g$ and f is a step function$\}$.
- For all other functions $h \in \mathbb{R}^X$, let $p(h) = \inf\{p(g) : g \geq |h|$ is a lower semicontinuous function$\}$.

Let μ be the submeasure on $\mathcal{P}(X)$ defined by $\mu(B) = p(\chi_B)$. The work of Steprāns in [30] can be used to show that μ is a dense submeasure. Fremlin in unpublished work [9] showed that many of these submeasures are in fact capacities, and therefore the tree posets obtained by Steprāns are dense in P_{I_μ}.

Definition 6.5. A submeasure μ on a Polish space X is a *pavement submeasure* if it is obtained by an application of the following process. There is a countable collection U of Borel subsets of X, with a weight function $w : U \to \mathbb{R}^+$. The weight function is naturally extended to $w : \mathcal{P}(U) \to \mathbb{R}^+ \cup \{\infty\}$ by $w(a) = \Sigma_{u \in a} w(u)$. The submeasure μ is then obtained as $\mu(A) = \inf\{w(a) : A \subset \bigcup a\}$.

Extending the topology of the space X and preserving its Borel structure it is possible to bring about a situation in which the sets in the collection U are clopen. In such a case the associated null ideal is again generated by G_δ sets. It is quite clear that there are very many null ideals for pavement submeasures. The problem is that we do not understand their factor forcings except for a couple of select cases:

Example 6.6. Solovay forcing. This obtains when $X = [0,1]$, U is the collection of rational intervals, and a weight of an interval is just its length.

Example 6.7. Laver forcing. Let $\{\delta_t : t \in \omega^{<\omega}\}$ be an arbitrary sequence of positive real numbers with finite sum. For every sequence $t \in \omega^{<\omega}$ and every number n assign weight δ_t to the set $A_{t,n} = \{g \in \omega^\omega : t \subset f \wedge f(|t|) \in n\} \subset \omega^\omega$. It is not difficult to see that the null ideal I for the resulting submeasure is generated by the sets $A_h = \{f \in \omega^\omega :$ for infinitely many $i \in \omega$, $f(i) \in h(f \restriction i)\}$ as h varies through all functions from $\omega^{<\omega}$ to ω. In other words, the null ideal is precisely the Laver ideal.

Example 6.8. Popov forcing ([25]). Let $X = \prod_{n \in \omega} X_n$ for finite sets X_n whose sizes approach ∞. Consider X with the product topology. For $f \in X$ and finite $s \subseteq \omega$ let $a_{f,s} = \{x \in X | x(i) \neq f(i)$ for all $i \in s\}$, and let $w(a_{f,s}) = 1/(|s|+1)$. Then ϕ is a what is sometimes called a *pathological submeasure*: it does not dominate a finitely additive positive functional.

Laver and Popov forcings are different. By Lemma 2.14, Laver ideal satisfies $\neg J \perp$ `null`. On the other hand, if I is the null ideal of a pathological submeasure, then $I \perp$ `null`. By [4, Theorem 6, second part], there is a Borel set $C \subseteq [0,1] \times X$ such that every vertical section of C has complement in I while every horizontal section of C is Lebesgue null. The result of [4] was proved under the additional assumption that the submeasure is Maharam, but it is not difficult to see that the proof works without this assumption.

7 Properness

Theorem 7.1. *Suppose I is a null ideal derived from some dense submeasure μ on 2^ω. The forcing P_I is proper.*

Proof. This is again a game theoretic argument. Let P be a forcing and $P \Vdash \dot{r} \in 2^\omega$. Consider a game $DG(P, I, \dot{r})$ between Kramnik and Kasparov. In it Kasparov first indicates an initial condition p_{ini}, then in each round n an open dense set $D_n \subset P$, and on a fixed schedule to be specified later he creates a G_δ set $B \in I$. Kramnik plays a descending sequence $p_{ini} \geq p_0 \geq p_1 \geq \dots$ so that $p_n \in D_n$ and the condition p_n decides the value of $\dot{r}(\check{n})$. Kramnik is allowed to tread water, that is, wait for any finite number of steps before placing another nontrivial move. Let g be the filter on the poset P generated by the conditions $\{p_n : n \in \omega\}$. Kramnik wins if the real $\dot{r}/g$ falls out of the set B.

It only remains to specify Kasparov's schedule used to construct the set B. In fact, for every number i he constructs an open set B_i of submeasure $< 2^{-i}$ and the set B is recovered as $B = \bigcap_i B_i$. Moreover, for all numbers i, j he must identify all those sequences $s \in 2^{<\omega}$ such that $\mu(B_i \cap [s]) \geq (1-2^{-j})\mu([s])$. To do this, in the beginning of the game fix some bijection $f : \omega \to \omega \times \omega \times 2^{<\omega} \times 2^{<\omega}$ and demand that in any round n Kasparov announce whether $[t] \in B_i$ and $\mu(B_i \cap [s]) \geq (1 - 2^{-j})\mu([s])$, where $f(n) = \langle i, j, s, t \rangle$. Note that in this way Kasparov's moves used to construct the set B are integers, and he can produce a G_δ superset of any set in the ideal I.

Lemma 7.2. *The following are equivalent:*

1. *P forces the real $\dot{r}$ to fall out of all Borel ground model coded I-small sets*
2. *Kramnik has a winning strategy in the game $DG(P, I, \dot{r})$.*

Once this lemma is proved, the theorem follows by the same argument as in Theorem 5.7. It is clear that if (1) fails then Kasparov has a winning strategy and therefore (2) must fail: Kasparov will just choose a condition $p_{ini} \in P$ which forces the real $\dot{r}$ into some I-small Borel set C. By the outer regularity of the submeasure μ the set C is a subset of a G_δ set B which Kasparov can produce and win no matter what Kramnik does.

The implication (1)$\to$(2) is the heart of the matter. Since the game $DG(P, I, \dot{r})$ is Borel, it is determined. So it is enough to assume (1) and derive a contradiction from the assumption that Kasparov has a winning strategy σ. A small claim will be used repeatedly:

Claim 7.3. *For every condition p and every number $j \in \omega$ there is a sequence $s \in 2^{<\omega}$ such that for every Borel set $A \subset [s]$ of submeasure $\mu(A) < (1 - 2^{-j})\mu([s])$ there is a condition $q \leq p$ forcing $\dot{r} \in [s] \setminus \dot{A}$.*

Proof. Suppose this fails for some p, j, and for every sequence $s \in 2^{<\omega}$ find a Borel set $A_s \subset [s]$ such that $\mu(A_s) < (1 - 2^{-j})\mu([s])$ and $p \Vdash \dot{r} \notin [s] \setminus \dot{A}_s$. Let $B = \{r \in 2^\omega : \forall n \in \omega \; r \in A_{r \restriction n}\}$. It is immediate that B is a Borel set, $p \Vdash \dot{r} \in \dot{B}$, and since for every sequence $s \in 2^{<\omega}$ the set $B \cap [s] \subset A_s$ has measure $< (1-2^-j)\mu([s])$, by the density condition $\mu(B) = \mu(\bar{d}B) = 0$. This contradicts the assumption that $P \Vdash \dot{r}$ falls out of all Borel ground model coded I-small sets. $\square$

Let p_{ini} be the condition the strategy σ plays as the initial move. Observe that there must be a number $j \in \omega$ such that for every Borel set A of submeasure $< 2^{-j}$ there is a condition $q \leq p_{ini}$ forcing $\dot{r} \notin \dot{A}$. If this failed and for every number j there was a Borel set $A_j \subset 2^\omega$ of submeasure $< 2^{-j}$ such that $p_{ini} \Vdash \dot{r} \in \dot{A}_j$, then $p_{ini} \Vdash \dot{r} \in \bigcap_j A_j$ and the latter set has zero submeasure, contradicting the assumptions on the name $\dot{r}$. Let j be such a number, and fix some number $i > j$. Kramnik will construct a counterplay against the strategy σ such that his resulting number $\dot{r}/g$ will fall out of the set B_i the strategy σ will produce. This will provide the desired contradiction.

By induction on $n \in \omega$ construct conditions p_n, numbers $m_n \in \omega$, $j_n \in \omega$, and sequences $s_n \in 2^{<\omega}$ so that

- $p_{ini} \geq p_0 \geq p_1 \geq \dots$
- writing τ_n for the partial play of the game ending at round m_n in which Kasparov follows his strategy σ and Kramnik plays p_k at round m_k for all $k \leq n$, it is the case that the play τ_n follows the rules of the game, in particular $p_n \in D_n$
- $s_0 \subset s_1 \subset \dots$, s_n is a sequence which witnesses the Claim for p_n and j_n, p_{n+1} forces $\check{s}_n \subset \dot{r}$
- the strategy σ indicated during the play τ_n that $\mu(B_i \cap [s_n]) < (1 - 2^{-j_n})\mu([s_n])$.

Of course, in the end it is the case that $\dot{r}/g = \bigcup_n s_n$, and by the last item for no number n it is the case that $[s_n] \subset B_i$. Therefore $\dot{r}/g \notin B_i$ and Kramnik has won.

To obtain p_0, m_0, j_0 and s_0 consider the open set $C \subset 2^\omega$ which the strategy σ produces as B_i if Kramnik makes no nontrivial move. There is a condition $q \leq p_{ini}$ for cing the real $\dot{r}$ out of the set $\bar{d}C$, and strengthening the condition q we can find a number j_0 such that $q \Vdash \forall k \in \omega\ \mu(C \cap [\dot{r} \restriction k]) < (1 - 2^{-j_0})\mu([\dot{r} \restriction k])$. Find a condition $p_0 \leq p$ which decides the value $\dot{r}(0)$ and belongs to the open dense set D_0 the strategy σ produced, let $s_0 \in 2^{<\omega}$ be a sequence which witnesses the statement of the Claim for p_0 and j_0, and let m_0 be an integer large enough so that the strategy σ announced before round m_0 that $\mu(C \cap [s_0]) < (1 - 2^{-j_0})\mu([s_0])$. The induction hypothesis is satisfied.

Suppose now that p_n, m_n, j_n and s_n have been constructed. Let $C \subset 2^\omega$ be the open set which the strategy σ produces as B_i in the infinite play which starts with τ_n and proceeds without further nontrivial Kramnik's move. By the induction hypothesis, $\mu(\bar{d}C \cap [s_n]) = \mu(C \cap [s_n]) < (1 - 2^{-j_n})\mu([s_n])$. By the induction hypothesis there is a condition $q \leq p_n$ forcing the real $\dot{r}$ into the set $[s_n] \setminus \bar{d}C$, and strengthening this condition if necessary we can find a number

$j_{n+1} \in \omega$ so that $q \Vdash \forall k \in \omega\ \mu(C \cap [\dot{r} \restriction k]) < (1 - 2^{-j_n})\mu([\dot{r} \restriction k])$. Find a condition $p_{n+1} \leq q$ in the open dense set $D_n \subset P$. Find a sequence $s_{n+1} \in 2^{<\omega}$ witnessing the Claim for the condition p_{n+1} and the number j_{n+1}; note that $s_n \subset s_{n+1}$. Finally, find a number m_n such that the strategy σ indicated $\mu(C \cap [s_{n+1}]) < (1 - 2^{-j_0})\mu([s_{n+1}])$ before the round m_n. The induction hypothesis is clearly satisfied. □

Theorem 7.4. *Let I be a null ideal derived from some pavement submeasure. Then P_I is a proper forcing.*

Proof. The proof again uses a determined infinite game. For notational simplicity assume that the underlying space is 2^ω, and fix the weight function $w : U \to \mathbb{R}^+$ generating the submeasure. Suppose that I is a null ideal, P is a partial order and $\dot{r}$ is a P-name for a real. The null game $NG(I, P, \dot{r})$ is a game of length ω between Fischer and Spassky played in the following fashion. In the beginning Fischer indicates an initial condition p_{ini} and then he produces one-by-one open dense subsets $\{D_n : n \in \omega\}$ of the poset P, and dynamically on a fixed schedule a Borel set A in the ideal I. Spassky plays one by one decreasing conditions $p_{ini} \geq p_0 \geq p_1 \geq \dots$ so that $p_n \in D_n$ and p_n decides the n-th digit of the real $\dot{r}$. He is allowed to hesitate for any number of rounds before placing his next move. Spassky wins if, writing g for the filter he obtained, it is the case that $\dot{r}/g \notin A$.

To make this precise, Fischer plays subsets $\{a_k : k \in \omega\}$ of the collection U with $w(a_k) \leq 2^{-k}$, and the Borel set A in the previous paragraph is extracted as $\bigcap_k \bigcup a_k$. To obtain the sets a_k, at round n Fischer indicates finite sets $a_k^n \subset U$ for all $k \in n$ in such a way that $\Sigma_n w(a_k^n) \leq 2^{-k}$ and $\Sigma_{n>m} a_k^n \leq 2^{-m}$.

The sets a_k are then obtained as $\bigcup_n a_k^n$. It is clear that $\Sigma_n w(a_k) \leq 2^{-k}$ and that given any set $B \in I$ Fischer can play so that $B \subset A$ for his resulting set A.

Lemma 7.5. *The following are equivalent:*

1. *$P \Vdash \dot{r}$ is not contained in any ground model Borel I-small set*
2. *Spassky has a winning strategy in the game $NG(I, P, \dot{r})$.*

Granted this lemma, the whole treatment transfers from the previous section, including the dichotomy results. One direction of the lemma is easy. If there is a condition $p \in P$ such that $p \Vdash \dot{r} \in \dot{B}$ for some ground model coded Borel I-small set B, then Fischer can easily win by indicating $p_{ini} = p$, dynamically producing a suitable I-small superset A of the set B, and mentioning all the open dense sets necessary to make sure that the result of the game falls into the set $B \subset A$.

For the other direction of the lemma note that the game is Borel and therefore determined. Thus it will be enough to obtain a contradiction from the assumption that $P \Vdash \dot{r}$ is not contained in any ground model coded Borel I-small set and yet Fischer has a winning strategy σ. A small claim will be used repeatedly:

Claim 7.6. *For every condition $p \in P$ there is a number $l(p) > 0$ such that for every set $a \subset U$ with $w(a) < 2^{-l(p)}$ there is a condition $q \leq p$ forcing $\dot{r} \notin \bigcup a$.*

Proof. Suppose this fails and for every number $l \in \omega$ find a set $a_l \subset U$ with $w(a_l) < 2^{-l}$ such that $p \Vdash \dot{r} \in \bigcup a_l$. But then $p \Vdash \dot{r} \in \bigcap_l \bigcup a_l$ and the latter set is in the null ideal, contradicting the properties of the name $\dot{r}$. □

Spassky will find a winning counterplay against the strategy σ in the following fashion. Fix $k = l(p_{ini})$ and a countable elementary submodel M of a large enough structure containing the strategy σ and the ideal I. The intention is to build a counterplay with moves in the model M such that the resulting filter $g \subset P$ is M-generic and the resulting real $\dot{r}/g$ will not belong to the set $\bigcup a_k$. This will prove the theorem.

The counterplay will be built by induction, Spassky's moves denoted by p_n, played at rounds i_n. The initial segment of the play ending after the round i_{n-1} will be denoted by τ_n, and for notational convenience let $p_{-1} = p_{ini}$ and $\tau_0 = \langle p_{ini} \rangle$. The following induction hypotheses will be satisfied:

- $l(p_n) \leq i_n$
- the condition $p_n \in M$ is in the sets D_n and E_n, it decides the n-th bit of the real $\dot{r}$ and for every number m, $k < m \leq i_n$ it forces $\dot{r} \notin \bigcup a_k^m$. Here the symbols E_n and a_k^m refer to Fischer's moves in the play τ_{n+1}, and D_n is the n-th open dense subset of the poset P in the model M in some fixed enumeration.

This will certainly be sufficient. Let $\tau = \bigcup_n \tau_n$ and argue that Spassky has won. And indeed, look at the set $a_k = \bigcup_n a_k^n$. For every Borel set $u \in a_k^n$, every condition Spassky played at or after round n forces $\dot{r} \notin \dot{u}$, by the forcing theorem and the fact that the resulting filter g is M-generic it follows that $M[g] \models \dot{r}/g \notin u$ and by Borel absoluteness $\dot{r}/g \notin u$. Note that this argument uses just the second item of the induction hypothesis, the first item just helps keep the induction going.

To perform the induction, suppose the play τ_n has been constructed. Let $\bar{a}_k$ be the set the strategy σ produces if Spassky forever hesitates to place another move after the play τ_n. The rules of Fischer's schedule for the construction of the set $\bar{a}_k$ imply that $w(\bar{a}_k \setminus \bigcup_{m \leq i_n} a_k^m) \leq 2^{-i_n}$. The first item of the induction hypothesis implies that there is a condition $q \leq p_{n-1}$ such that $q \Vdash \dot{r} \notin \bigcup(\bar{a}_k \setminus \bigcup_{m \leq i_n} a_k^m)$. Note that since $p_{n-1} \Vdash \dot{r} \notin \bigcup \bigcup_{m \leq i_n} a_k^m$ by the second item of the induction hypothesis, it is in fact the case that $q \Vdash \dot{r} \notin \bigcup \bar{a}_k$. Find a condition $p_n \leq q$ in the open dense sets D_n and E_n which decides the n-th bit of the real $\dot{r}$, and play it at round i_n such that $l(p_n) \leq i_n$. This concludes the construction of the play τ_{n+1} and the proof of the theorem. □

7.1 Dichotomies

Theorem 7.7. *In the Solovay model, every null ideal associated with a pavement submeasure or a dense submeasure is closed under well-ordered unions.*

Proof. Like in the proof of Theorem 3.10, it suffices to show that if P is a small forcing notion and $\dot{r}$ is a P-name for a real that falls out of all ground model sets in I, then the set $\{r \in X : \exists g \subset P \text{ a } V\text{-generic filter such that } r = \dot{r}/g\}$ is I-positive. The argument follows literally the proof of Theorem 5.11. $\square$

Corollary 7.8. *In the Solovay model, every subset of X has either a Borel I-positive subset or a Borel I-small superset.*

Corollary 7.9. *(LC) Every universally Baire subset of X has either a Borel I-positive subset or a Borel I-small superset. For analytic sets this is true in ZFC.*

This can be viewed as a mere application of $L(\mathbb{R})$ absoluteness and the previous corollary. However, there is a direct integer game argument both in the case of a pavement submeasure or a dense submeasure. We will describe the case of a pavement submeasure. Suppose for definiteness that the underlying space X is just the Cantor space 2^ω, and for every set $B \subset 2^\omega$ consider the integer variation $iNG(I, B)$ of the game NG. Here, Fischer produces dynamically on a fixed schedule an I-small set A and Spassky produces a binary sequence x. Fischer's schedule is the same as in the case of the game NG while Spassky is allowed to hesitate for an arbitrary finite number of rounds before placing the next bit on the sequence x. Spassky wins if $x \in B \setminus A$.

Claim 7.10. *Fischer has a winning strategy in the game $iNG(I, B)$ if and only if $B \in I$.*

The claim shows that under AD, every set $B \notin I$ has an analytic subset $C \notin I$. To see this, note that by the determinacy of the game $iNG(I, B)$ it must be the case that Spassky has a winning strategy σ in it. Let Y be the space of all possible Adam's counterplays with the natural topology, and let $C = \sigma''Y$. It is clear that $C \subset B$ since the strategy σ is winning, and $C \notin I$ because the strategy σ remains winning in the game $iNG(I, C)$. Moreover, the set C is analytic since it is an image of the Polish space Y under the continuous function σ. A separate argument is then necessary to show that every I-positive analytic set has an I-positive Borel set.

To prove the claim, note that the right-to-left direction is easy. If the set B is I-small then there are subsets $\{a_k : k \in \omega\}$ of U such that $w(a_k) \leq 2^{-k}$ and $B \subset \bigcap_k \bigcup a_k$. Fischer can then produce these sets under his schedule, disregarding Spassky's moves entirely. The other direction is more difficult. Suppose that σ is Fischer's winning strategy.

First a bit of notation. For every finite partial play τ observing the strategy σ and every number $k \in \omega$ let $\bar{a}_k(\tau)$ be the set of those $u \in U$ which the strategy σ throws into the set a_k after the last move of τ if Spassky forever hesitates to make another nontrivial move after τ. Also, for a number $i \in \omega$ greater than the length of τ, and a bit $b \in 2$ let $\tau i b$ be the play of length i extending τ in which Fischer follows his strategy and Spassky makes only one nontrivial move after τ, namely places the bit b on the sequence x on the last, $i-1$-th round of the play.

Now consider the sets $C = \bigcap_k \bigcup \bar{a}_k(0)$ and $D_{k\tau b} = \bigcap_i \bigcup \bar{a}_k(\tau ib)$, for every finite play τ observing the strategy σ, number $k \in \omega$ and a bit $b \in 2$. It is immediate that all of these sets are null, and it will be enough to show that $B \subset C \cup \bigcup_{k\tau b} D_{k\tau b}$. Well, suppose that $x \in B \setminus C \cup \bigcup_{k\tau b} D_{k\tau b}$ is some sequence. We will complete the proof of the Claim by showing that Spassky can defeat the strategy σ by producing the sequence x in a suitable manner. Find a number k such that $x \notin \bigcup \bar{a}_k(0)$. The counterplay will be constructed so that $x \notin \bigcup a_k$, where a_k is the k-th set the strategy σ will produce. By induction build plays $0 = \tau_0 \subset \tau_1 \subset \dots$ so that

- Spassky produced the first n bits of the sequence x during the play τ_n with some hesitation, while Fischer observed his strategy σ
- x does not belong to any set u such that the strategy σ put u into the set a_k during the play τ_n
- $x \notin \bar{a}_k(\tau_n)$

This is easily possible. The trivial play $0 = \tau_0$ satisfies the induction hypotheses. If the play τ_n has been constructed, note that $x \notin D_{k\tau b}$ where b is the n-th bit on the sequence x, and therefore there exists an i such that the play $\tau_{n+1} = \tau ib$ satisfies the induction hypotheses again. Clearly, the play $\tau = \bigcup_n \tau_n$ defeats the winning strategy σ since Spassky produced the sequence x while by the second item of the induction hypothesis the set $\bigcup a_k$ produced by the strategy σ does not contain x.

7.2 Other preservation properties

The submeasurable forcings are bounding in one very natural case. Recall:

Definition 7.11. Suppose that X is a Polish space. A function $\mu : \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ is a *Choquet capacity* [15, 30.B], [10, 432J] if

1. it is monotonic: $A \subset B \subset X$ implies $\mu(A) \leq \mu(B)$
2. it is outer regular:$\mu(K) = \inf\{\mu(A) : A \supseteq K,\ A \text{ open}\}$ for every compact K
3. it is continuous on increasing unions: whenever $A_n : n \in \omega$ is a nondecreasing sequence of subsets of X with union A, we have $\sup_n \mu(A_n) = \mu(A)$
4. capacities of compact sets are finite. We will in fact always deal with situations in which $\mu(X)$ is finite.

The most important feature of capacities is the Choquet theorem.

Fact 7.12. *(Inner regularity) If μ is a Choquet capacity on a Polish space X and $A \subset X$ is an analytic set then $\mu(A) = \sup\{\mu(C) : C \subset A$ is compact$\}$.*

Theorem 7.13. *If the submeasure μ is a Choquet capacity and the forcing P_{I_μ} is proper, then it is bounding.*

Proof. It is exactly enough to show that compact sets are dense in P_I and the continuous reading of names holds. Well, if $B \in P_I$ is a positive Borel set then the Choquet capacitability theorem shows that it has a compact positive subset. For the continuous reading of names let $B \in P_I$ be a positive Borel set and $f : B \to \omega^\omega$ be a Borel function. Let $\mu^* : \mathcal{P}(X \times \omega^\omega) \to \mathbb{R}$ be the function defined by $\mu^*(A) = \mu(\text{proj}(A))$. It is easy to verify that μ^* is a capacity–[15], 30.B.2. The graph of the function f is μ^*-positive, and by Choquet capacitability theorem again it has a compact μ^*-positive subset C. Let D be the projection of the set C. Clearly, $D \subset B$ is a compact μ-positive set and the function $f \restriction D$ is continuous, since it has a compact graph! $\square$

It can be verified that none of the bounding forcings in the previous sections can be presented as P_I for some capacitable σ-ideal I. The reason is that in all of the examples the collection of closed sets in the ideal is $\mathbf{\Pi}^1_1$-hard, and this feature even persists to all possible presentations. However, it is not difficult to show that the collection of closed sets of 0 capacity is G_δ, and this for every capacity whatsoever– [15], Exercise 30.15.

As in the previous section it is the case that the forcings associated with dense submeasures or pavement submeasures make the set of the ground model reals meager because they can be presented as quotient forcings of σ-ideals generated by G_δ sets.

8 P-cover ideals

Definition 8.1. Suppose that K is an analytic P-ideal on ω. Recall that an ideal K on ω is a P-ideal if for every sequence A_n ($n \in \omega$) of sets in K there is $A \in K$ such that $A_n \setminus A$ is finite for all n. The associated *P-cover ideal* I on $\mathcal{P}(\omega)$ is generated by sets $A_x = \{y \subset \omega : x \setminus y \text{ is infinite}\}$ as x varies through all elements of K.

I-positive sets are sometimes called *approximations* to K. The family of compact hereditary sets in I plays an important role in the proof of the structure theorem for analytic P-ideals ([29]). Note that since K is a P-ideal, the sets A_x with all their subsets form a σ-ideal and so they form a basis for the ideal I consisting of G_δ sets. It is quite obvious that the ideal I does not contain all singletons, for example $\{\omega\} \notin I$. However, the ideal I does contain all singletons when restricted to some interesting Borel sets B, such as $B = K$.

Example 8.2. Laver forcing. Let K be the collection of sets $x \subset \omega \times \omega$ with finite vertical sections. It is not difficult to see that $P_I \restriction K$ is isomorphic to the poset P_J where J is a σ-ideal of nondominating subsets of ω^ω. It has been known for some time that P_J is in the forcing sense equivalent to the Laver forcing (see [3]).

Example 8.3. The optimal amoeba forcing for measure. Let K be the collection of sets $x \subset 2^{<\omega}$ such that the set $B_x = \{r \in 2^\omega :$ for infinitely many numbers $n \in \omega$, $r \restriction n \in x\} \subset 2^\omega$ is Lebesgue null. It is well-known and not difficult to verify that K is an analytic P-ideal (see [29]). The poset $P_I \restriction K$ adds a Lebesgue null set containing all ground model coded Lebesgue null sets. It is not the same as the standard amoeba forcing for measure, in particular it is not c.c.c. Note that the same procedure would work for the hypothetical Maharam submeasures in place of the Lebesgue measure.

Example 8.4. Every quotient forcing associated with a pavement submeasure is isomorphic to $P_I \restriction B$ for a suitable P-cover ideal I and Borel I-positive set B. To see this, let U, w be the weight generating the null ideal J, and let X be the underlying space. Let $K = \{a \subset U : w(a) \text{ is finite}\}$; so this is a typical F_σ P-ideal on the set $\mathcal{P}(U)$. Let I be the associated P-cover ideal on $\mathcal{PP}(U)$. Consider the function $\pi : X \to \mathcal{P}(U)$ defined by $\pi(r) = \{u \in U : r \notin u\}$ and the set $B = \text{rng}(\pi) \subset \mathcal{P}(U)$. We claim that B is an I-positive Borel set and the bijection $\pi : X \to B$ moves the ideal J to the ideal I below B. If $A \subset X$ is a set in the ideal J, for every $n \in \omega$ find a set $a_n \subset U$ such that $w(a_n) \leq 2^{-n}$ and $A \subset \bigcup a_n$, and set $b = \bigcup_n a_n \subset U$. Clearly, $b \in K$ and the image $\pi'' A$ is included in the I-small set $\{c \subset U : b \setminus c \text{ is infinite}\}$. On the other hand, if $A \subset \mathcal{PP}U$ is a set in the ideal I, find a set $b \subset U$ of finite weight such that $A \subset \{c \subset U : b \setminus c \text{ is infinite}\}$ and note that the preimage $\pi^{-1}A$ is J-small since it is covered by the union of every cofinite subset of b.

8.1 Properness

Theorem 8.5. *If I is a P-cover ideal then the forcing P_I is proper.*

Proof. Fix the analytic P-ideal K on ω which generates the ideal I. Use the classical result of Solecki [29] to find a finite lower semicontinuous submeasure $\mu : \mathcal{P}(\omega) \to \mathbb{R}^+$ such that $K = Exh(\mu)$. That is to say, $\mu(y) = \sup\{\mu(x) : x \subset y$ finite$\}$ for every set $y \subset \omega$, and $K = \{y \subset \omega : \lim_n \mu(y \setminus n) = 0\}$. Note that in fact K is Borel.

Suppose that P is a forcing and $\dot{x}$ is a P-name for a subset of ω. Consider the P-cover game $PCG(P, \dot{x}, I)$ between Karpov and Korchnoi. In it, Karpov produces an initial condition p_{ini}, one by one open dense sets $D_n \subset P$ and dynamically on a fixed schedule a set $y \subset \omega, y \in K$. Korchnoi produces one by one a descending chain $p_{ini} \geq p_0 \geq p_1 \geq \dots$ of conditions such that $p_n \in D_n$ and p_n decides the statement $\check{n} \in \dot{x}$. He can hesitate for an arbitrary finite number of steps before placing his next move. In the end, let $g \subset P$ be the filter Korchnoi created. Korchnoi wins if $y \subset \dot{x}/g$ modulo a finite set.

To make this precise, we need to specify Karpov's schedule for the set y. At round n Karpov decides whether $n \in y$ or not and specifies a number $m_n \in \omega$ such that $\mu(y \setminus m_n) \leq 2^{-m_n}$. The latter demand is equivalent to the condition that for every number $k \in \omega$, $\mu(y \cap k \setminus m_n) \leq 2^{-m_n}$. It is quite clear that Karpov can produce any give set in the ideal K under this schedule.

As in the previous sections, it will be enough to prove the following lemma.

Lemma 8.6. *The following are equivalent.*

- *$P \Vdash \dot{x}$ falls out of all ground model coded Borel I-small sets.*
- *Korchnoi has a winning strategy in the game $PCG(P, \dot{x}, I)$.*

One direction of the lemma is again trivial. If there is a condition $p \in P$ forcing the set $y \setminus \dot{x}$ to be infinite, then Karpov can win by playing on the side an increasing sequence $\langle M_n : n \in \omega \rangle$ of countable elementary submodels of some large structure, enumerating all the open dense subsets of P in $M = \bigcup_n M_n$, producing $p = p_{ini}$ and the set y, and playing so that Korchnoi's filter g is M-generic. In the end, $M[g] \models y \setminus \dot{x}/g$ is infinite by the forcing theorem, and so $y \setminus \dot{x}/g$ is infinite and Karpov won.

The opposite direction is harder. Suppose that the first item of the lemma is satisfied. A small claim will be used repeatedly.

Claim 8.7. *For every condition $p \in P$ there are numbers $m(p)$ and $k(p)$ such that for every set $y \in K$ of submeasure $\leq 2^{-m(p)}$ there is a condition $q \leq p$ forcing $\check{y} \setminus \dot{x} \subset \check{k}(p)$.*

Proof. Suppose that this fails for some p. By induction on $n \in \omega$ find sets $y_n \in K$ and increasing numbers k_n such that

- $\mu(y_n) \leq 2^{-n}$ and $\mu(\bigcup_{m \in n} y_m) \setminus k_n \leq 2^{-n}$.
- $y_n \cap k_n = 0$.
- $p \Vdash \check{y}_n \setminus \dot{x} \neq 0$.

To start, let $k_0 = 0$. To find the set y_n and the number k_{n+1} once the sets $y_m : m \in n$ and the number k_n are known, use the failure of the claim at $p, -n$ and k_n to find a set $y_n \in K$ such that $y_n \cap k_n = 0$, $\mu(y_n) \leq 2^{-n}$ and $p \Vdash \check{y}_n \setminus \dot{x} \neq 0$. Then $z = \bigcup_{m \in n+1} y_n \in K$ and therefore there is a number $k_{n+1} \in \omega$ such that $\mu(z \setminus k_{n+1}) \leq 2^{-n-1}$. This concludes the inductive step.

In the end, let $y = \bigcup_n y_n$. It is not difficult to verify from the first induction hypothesis that $\mu(y \setminus k_n) \leq 2^{-n} + \Sigma_{m \geq n} 2^{-m}$ and therefore $y \in K$. The last two induction hypotheses then show that $p \Vdash \check{y} \setminus \dot{x}$ is infinite, contradiction. $\square$

The game $PCG(P, \dot{x}, I)$ is Borel and therefore determined. To conclude the proof of the lemma, it will be enough to derive a contradiction from the assumption that Karpov has a winning strategy σ. To find Korchnoi's counterplay, let M be a countable elementary submodel of a large enough structure and let $p = p_{ini} \in P$ be Karpov's initial condition. Let $m(p), k(p)$ be the numbers from the claim. The idea now is to construct a counterplay such that the resulting filter $g \subset P \cap M$ is M-generic and $y \setminus \dot{x}/g \subset \max\{k(p), m_{m(p)}\}$. In order to do that, find Korchnoi's moves $p_n \in P \cap M$ played at rounds i_n in such a way that

- the condition $p_n \in M$ belongs to the n-th open dense set Karpov played, to the n-th open dense subset of P in the model M under some fixed enumeration, and it decides the statement $\check{n} \in \dot{x}$

- $p_n \Vdash \dot{x} \cap \check{y} \cap i_n \subset \max\{k(p), m_{m_p}\}$; note that the set $y \cap i_n$ is known at round i_n.
- the number $m_{m(p_n)}$ is known at round i_n, and $i_n > k(p_n), m_{m(p_n)}$.

The second induction hypothesis then immediately implies that Korchnoi won the resulting play of the game, obtaining the desired contradiction. To construct p_0, i_0, let $y \in K$ be the set the strategy σ produces if Korchnoi forever hesitates to place a nontrivial move in the play. By the claim, there is a condition $q \leq p$ forcing $\check{y} \setminus \dot{x} \subset \max\{m_{m(p)}, k(p)\}$. Let $p_0 \leq q$, $p_0 \in M$ be a condition in the first open dense subset of the poset P in the model M and in the first open dense set Karpov played, deciding the statement $0 \in \dot{x}$. Let i_0 be a sufficiently large number so that the last induction hypothesis is satisfied. The induction step is similar. Going through the same motions as in the previous sections will then conclude the proof of the theorem. $\square$

The dependence on Solecki's result and the determinacy of the PCG game make it difficult to extend the result to the case of P-cover ideals generated by undefinable P-ideals. It is not difficult to observe that if K is the complement of a Ramsey ultrafilter F, I' is the P-cover ideal derived from K and I is the ideal generated by I' and $\{F\}$ then P_I is in the forcing sense equivalent to the standard c.c.c. poset Q diagonalizing the Ramsey ultrafilter F, since it adds a diagonalizing real and such a real is Q-generic by the Mathias criterion for Q-genericity.

The posets P_I associated with an analytic P-ideal K are strongly inhomogeneous, and some singletons such as $\{\omega\}$ are positive in the ideal I. The P-ideal K itself is a condition in the forcing P_I and below this condition the poset has much more reasonable properties. Note that it adds an element of the analytic P-ideal K which modulo finite includes all ground model elements of K.

Lemma 8.8. *The ideal I is homogeneous below K.*

Proof. Recall that an ideal $I \restriction K$ is homogeneous if and only if for every Borel I-positive set $B \subset K$ there is a function $f : K \to B$ such that f-preimages of I-small sets are I-small [35], Definition 2.3.1. In this case, every function f mapping a set $y \in K$ to a set $x \in B$ which covers y modulo a finite set will clearly work. $\square$

The homogeneity of ideals considerably simplifies the statements of absoluteness theorems in [35]. It is not immediately clear if the poset $P_I \restriction K$ is homogeneous per se.

8.2 Dichotomies

Again a routine modification of the proofs in §3.2 gives the following.

Theorem 8.9. *In the choiceless Solovay model, every P-cover ideal is closed under well-ordered unions.*

Corollary 8.10. *In the choiceless Solovay model, every set has either a Borel I-small superset or a Borel I-positive subset.*

Corollary 8.11. *(LC) Every universally Baire set has either a Borel I-small superset or a Borel I-positive subset.*

There is an alternative integer game proof for the previous corollary. The argument should be more or less obvious to the interested reader at this point.

9 Other forcings

It is not difficult to find definable proper forcings which do not fall into any of the classes described above. This follows from the following general theorem.

Theorem 9.1. *(LC) If I is a universally Baire ideal such that P_I is $< \omega_1$-proper in every σ-closed extension, and every universally Baire set has either a Borel I-positive subset or a Borel I-small superset, then every intermediate extension $V \subset W \subset V^{P_I}$ is either a c.c.c. extension of V or it is equal to V^{P_I}.*

An inspection of the proofs of properness in the previous sections will show that they in fact yield $< \omega_1$-properness, and this in every σ-closed extension.

Proof. The argument follows the lines of [34]. Consider the game G between Fischer and Karpov ... errr, this is not a game proof. Instead, consider a P_I-name $\dot{A}$ for a set of ordinals and suppose that there is a condition $B \in P_I$ forcing that $V[\dot{A}]$ is not a c.c.c. extension of V. We must show that the model $V[A]$ contains the P_I-generic real.

First fix some natural objects. Let $\{M_n : n \in \omega\}$ be an $\in$-increasing sequence of countable elementary submodels of a large structure containing all the relevant objects, with union $M = \bigcup_n M_n$. Let $C = \{r \in B : r \text{ is } M\text{-generic}\}$; this is an I-positive Borel set. Let $G : C \to \mathcal{P}(M)$ be the function defined by $G(r) = \dot{A}/r$; this is a function which is in a suitable sense Borel, and $C \Vdash \dot{A} \cap \check{M} = G(\dot{r}_{gen})$.

For every set $a \subset \omega$ let C_a be the set of those reals $r \in C$ which are M_n-generic for every number $n \in a$, and not M_n-generic for every $n \notin a$, and in the latter case even $G(r) \cap M_n$ is not M_n-generic. The abstract argument of [34] shows that the sets $C_a \subset C$ are I-positive–this uses the forcing assumptions on the poset P_I and the name $\dot{A}$. It is clear that these sets are Borel, mutually disjoint, and even their images under the function G are mutually disjoint.

Finally we are in the position to make use of the descriptive set theoretic assumptions on the ideal I. Let $Y \subset \mathcal{P}(\omega) \times \mathbb{R}$ be a universal analytic set. Use the universally Baire uniformization to find a universally Baire function $F : \mathcal{P}(\omega) \to \mathbb{R}$ such that $F(a) \in C_a \setminus Y_a$ in the case that the vertical section Y_a belongs to the ideal I, and $F(a) \in C_a$ otherwise. The range $\mathrm{rng}(F) \subset C$ of this function is a universally Baire set which has no analytic superset in the ideal I. This is so because every such putative superset would have to be indexed as a vertical section $Y_a \supset \mathrm{rng}(F)$ of the set Y, but then the definition of the function F shows that $F(a) \notin a$, contradiction. By the descriptive set theoretic assumptions on the ideal I, there is a Borel I-positive set $D \subset \operatorname{rng}(F)$. Note that the function $G \restriction D$ is one-to-one. By Borel absoluteness then, $D \Vdash \dot{r}_{gen} = \dot{G}^{-1}(\dot{A} \cap \check{M})$ and $\dot{r}_{gen} \in V[\dot{A}]$ as desired. $\square$

It is well-known that Mathias forcing can be decomposed into a σ-closed*c.c.c. iteration, and Silver forcing can be decomposed as a σ-closed*Grigorieff iteration. Similarly, the E_0 forcing can be decomposed into a σ-closed*c.c.c. iteration. Thus Theorem 9.1 shows that these posets do not fit into the classes described previously for purely descriptive set theoretic reasons. A direct descriptive proof is also possible. Theorem 9.1 does not give any information as to which c.c.c. forcings fit into the classes described above. The only ones we can see to fit are the Cohen forcing (the porosity class or the closed set class) and the Solovay forcing (the null class). We do not have any negative results in this direction.

9.1 Mathias forcings, $M(K)$

Let us describe a class of forcings associated with Borel ideals on ω that has recently attracted a considerable attention (see [6]). If K is a Borel (analytic, projective) ideal on ω, consider the quotient Boolean algebra $\mathcal{P}(\omega)/K$. As a forcing notion, this quotient is frequently proper. For convenience, we sometimes consider K as an ideal on some other countable set.

Example 9.2. Let $\mathrm{NWD}(\mathbb{Q})$ be the ideal of all nowhere dense subsets of the rationals. Balcar, Hernández Hernández and Hrušák ([1]) proved that $\mathcal{P}(\mathbb{Q})/\mathrm{NWD}(\mathbb{Q})$ is proper.

Example 9.3. Steprāns ([30]) has defined a family of $2^{\aleph_0}$ coanalytic ideals whose quotients are pairwise nonequivalent proper forcing notions. Each one of these forcings is an iteration of a forcing that adds a real of minimal degree followed by $\mathcal{P}(\omega)/\mathrm{Fin}$.

Example 9.4. Let $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ be the ideals of sets of density zero and logarithmic density zero, respectively. Steprāns and the first author ([8, Theorem 1.3]) proved that a quotient over either of these two ideals (and over many other density ideals) is forcing equivalent to the iteration of $\mathcal{P}(\omega)/\mathrm{Fin}$ and a measure algebra of Maharam character equal to continuum, and therefore proper. Since by a result of W. Just the quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ can be nonisomorphic, this example also shows that nonisomorphic quotients $\mathcal{P}(\omega)/K$ can be forcing equivalent.

All three proofs of properness of quotients of the form $\mathcal{P}(\omega)/K$ in the above examples are different. By [11] there is an analytic P-ideal such that the forcing $\mathcal{P}(\omega)/K$ is improper. None of these forcings are of the form P_I, since they are not completely countably generated.

Definition 9.5. If K is an ideal on ω, define the *Mathias forcing* $M(K)$ associated to K as follows. Conditions are pairs (s, A) where $s \subseteq \omega$ is finite, $A \subseteq \omega$ is K-positive, and $\max(s) < \min(A)$. Let $(s, A) \leq (t, B)$ if $s \supseteq t$, $A \subseteq B$, and $s \setminus t \subseteq B$.

The case $K = \mathrm{Fin}$ is the Mathias forcing [20]. Every forcing of the form $M(K)$ is equivalent to one of the form P_I and it is definable whenever K is definable. As mentioned before, in the case of $M(\mathrm{Fin})$ the ideal I is the ideal of all nowhere dense subsets of $\mathcal{P}(\omega)/\mathrm{Fin}$. If K is definable then the quotient algebra $\mathcal{P}(\omega)/K$ is not ccc: it even contains a (not necessarily regular) subalgebra isomorphic of the nowhere c.c.c. quotient $\mathcal{P}(\omega)/\mathrm{Fin}$ by a result of Mathias. Therefore Theorem 9.1 implies that no $M(K)$ belongs to any of the classes of forcing notions considered earlier in the present paper.

By changing Definition 9.5 to require $\omega \setminus A \in K$, we obtain *Prikry forcing* $P(K^*)$ corresponding to the dual filter K^* of K. The following lemma is immediate. For the second part note that $P(K)$ is, being σ-centered, always proper. The case of $K = \mathrm{Fin}$ is due to Mathias.

Lemma 9.6. *Let K be any ideal on ω.*

1. *$M(K)$ is forcing equivalent to the iteration of $\mathcal{P}(\omega)/K$ and $P(\dot{F})$, where $\dot{F}$ is the $\mathcal{P}(\omega)$-generic filter.*
2. *$M(K)$ is proper if and only if $\mathcal{P}(\omega)/K$ is proper.*

By Theorem 9.1 and the fact that $\mathcal{P}(\omega)/K$ is not ccc if K is definable, neither of the forcings $M(K)$ fits any of the classes described previously. Note that the generic filter $\dot{F}$ is forced to be an ultrafilter on ω only when $\mathcal{P}(\omega)$ does not add reals.

While many forcings of the form $M(K)$ have the Laver property defined in 5.16, this is not true in general since the forcing in Example 9.2 adds Cohen reals. As a matter of fact, we have a dichotomy.

Theorem 9.7. *Assume K is any ideal on ω. Then $M(K)$ either has the Laver property or it adds Cohen reals.*

This will follow from a more general Theorem 9.10 below. The proof assumes reader's familiarity with the theory of semiselective coideals [5]. Semiselectivity is a property of coideals introduced in [5] as a weakening of selectivity or 'happiness,' of [20], where it was shown to be equivalent to several natural properties of the forcing $M(K)$, like Prikry property of Mathias property. Here, a forcing $M(K)$ has the *Prikry property* if for every sentence ϕ of the forcing language and every condition (s, A) there is $(s, B) \leq (s, A)$ deciding ϕ. It has the *Mathias property* if every subset of the generic subset of ω is generic.

Lemma 9.8 adds two more equivalent reformulations of semiselectivity.

Lemma 9.8. *For an ideal K the following are equivalent.*

1. *Forcing $M(K)$ has the Prikry property.*

2. Forcing $M(K)$ does not add Cohen reals.

Proof. Assume $M(K)$ has the Prikry property. By [5], $M(K)$ has the Prikry property if and only if the coideal $\mathcal{H} = K^+$ is semiselective. Let $\dot{r}$ be a name for a new real. Since $\mathcal{H}$ is semiselective, we can find $A \in \mathcal{H}$ such that $|A \cap (n+1)| < \sqrt{n}$ for all $n \in A$. By using the Prikry property of $M(K)$ ([5, Theorem 4.1]), find $(\emptyset, B) \leq (\emptyset, A)$ such that for every $n \in B$ and every $s \subseteq B \cap (n+1)$ there is $t_s \in 2^n$ such that $(s, B/n) \Vdash \dot{r} \restriction n = t_s$. Then $(\emptyset, B)$ forces that $\dot{r} \in [T]$, where T is the tree determined by t_s ($s \subseteq B$, s finite). Note that the n-th level of T has size at most $2^{|B \cap (n+1)|} < 2^{\sqrt{n}}$. Therefore $[T]$ is a closed null set, and $\dot{r}$ is not Cohen or random over V.

Now assume $M(K)$ does not have the Prikry property, equivalently $\mathcal{H}$ is not semiselective. Then there is $A \in \mathcal{H}$ and maximal antichains $\{\mathcal{A}_n : n \in \omega\}$ in $(\mathcal{H}, \subseteq^*)$ such that no $B \subseteq A$ in $\mathcal{H}$ diagonalizes all $\mathcal{A}_n$. We may assume that $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$. In $V[x]$, where x is a $M(K)$-generic subset of ω, let A_n be the unique element of $\mathcal{A}_n$ such that $x \subseteq^* A_n$. Recursively define sequences $\langle n_i, m_i : i \in \omega \rangle$ as follows. Let $n_0 = \min x$. If n_i has been defined and $x/n_i \subseteq A_{n_i}$, then let $n_{i+1} = m_{i+1} = \min(x/n_i)$. Otherwise, let $m_{i+1} = \min(x/n_i \setminus A_{n_i})$ and $n_{i+1} = \min(x/m_{i+1})$. Finally let

$$g(i) = \begin{cases} 0, & \text{if } x/n_i \subseteq A_{n_i} \\ 1, & \text{if } x/n_i \not\subseteq A_{n_i}. \end{cases}$$

We will denote $M(K)$ names for these objects by $\dot{x}$, $\dot{n}_i$, $\dot{g}$.

Claim 9.9. *Assume $(s, B) \leq (\emptyset, A)$ decides $\dot{g} \restriction i$ and n_k, m_k $(k < i)$ but not n_i for some $i \geq 1$, then for every $j \in \{0, 1\}$, there is $(t, C) \leq (s, B)$ and forcing $g(i) = j$ and deciding n_i, m_i but not n_{i+1}.*

Proof. Since (s, B) does not decide n_i, we must have $A_{n_{i-1}}/n_{n-i} \not\subseteq B$ and also $m_{i-1} = \max(s)$. If $j = 0$, let $n = \min(B/m_{i-1})$, find $D \in \mathcal{A}_n$ such that $C = B \cap D$ is in $\mathcal{H}$ and let $t = s \cup \{n\}$. Then (t, C) forces $n_i = m_i = n$ and $g(i) = 0$.

Now assume $j = 1$. Since B/m_{i-1} is not a diagonalization of $\mathcal{A}_i$ for any $i \in \omega$, there is $n \in B/m_{i-1}$ such that $B/n \not\subseteq A_n$. Pick $m \in B/n \setminus A_n$ and let $t = s \cup \{n, m\}$, $C = B/m$. Then (t, C) forces $n_i = n$, $m_i = m$ and $g(i) = 1$.

Note that in either case (t, C) forces that n_{i+1} is equal to $\min x/m_i$, and the value of this expression is not yet decided by (t, C). □

To see that $(\emptyset, A)$ forces g is Cohen over V, fix a dense open subset U of 2^ω and a condition $(s, B) \leq (\emptyset, A)$. Since B is not a diagonalization of the family $\{A_n\}$, (s, B) decides only a finite initial segment u of g. Find v extending u such that $[v] \subseteq U$, and use Claim to find an extension of (s, B), digit by digit, that forces $g \in [v]$. □

In many cases the forcing $M(K)$ is σ-closed. By [14] this is equivalent to $\mathcal{P}(\omega)/K$ being countably saturated (in the model-theoretic sense) and this holds for a large class of ideals K (see [7, §6]). For example, the ideal $\mathcal{I}_{\omega^2}$ of all subsets of ω^2 whose order type is less than ω^2 has countably saturated quotient. Therefore under CH its quotient is isomorphic to $\mathcal{P}(\omega)/\operatorname{Fin}$, these two being saturated elementary equivalent models of the same cardinality. It is natural to ask whether $M(\mathrm{Fin})$ and $M(\mathcal{I}_{\omega^2})$ are forcing equivalent. The negative answer follows from the equivalence of (6) and (7) in the following theorem.

An ideal K on ω is *dense* if every infinite set has an infinite subset in K. It is *nowhere dense* if every positive set has an infinite subset B such that the restriction of K to B coincides with Fin.

Theorem 9.10. *The following are equivalent for any ideal K on ω.*

1. *$M(K)$ does not add Cohen reals.*
2. *$M(K)$ has the Laver property.*
3. *$M(K)$ has the Prikry property.*
4. *$M(K)$ has the Mathias property.*
5. *Coideal K^+ is semiselective.*

If K is moreover definable then under a large cardinal assumption the following are equivalent to the above.

6. *K is nowhere dense.*
7. *$M(K)$ is forcing equivalent to $M(\mathrm{Fin})$.*

Proof. The equivalence of (3), (4) and (5) was proved in [5]. Lemma 9.8 gives the equivalence of (1) and (3). The implication from (3) to (2) is standard [2, the proof of Corollary 7.4.7]. Finally, (2) implies (1) since Cohen forcing does not have the Laver property.

Since (6) implies (7) implies (1) is clear, we only need to check that (5) implies (6). Assume K is definable and there is $A \in K^+$ such that every infinite subset of A has an infinite subset in K. By the semiselective version of Mathias' theorem [5, Theorem 4.3], there is $B \subseteq A$ in K^+ such that either all infinite subsets of B are in K or all infinite subsets of B lie outside K. Since both alternatives are false, K is not semiselective. $\square$

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