Dimension theory and forcing

Jindřich Zapletal[a,b]

[a]Department of Mathematics, University of Florida, Gainesville FL32611-0281, U.S.A.
[b]Institute of Mathematics, Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic

Abstract

There is a proper Baire category preserving forcing which adds infinitely equal real but no Cohen real. This resolves a long-standing open problem of David Fremlin. The forcing has a natural description in terms of infinite-dimensional topology.

Keywords: Cohen real, infinite dimension, calibrated ideal

1. Introduction

The purpose of this paper is to give a positive answer to a long-standing open question of David Fremlin in forcing theory.

Definition 1.1. Let M be a transitive model of set theory. A function $x \in \omega^\omega$ is an *infinitely equal real* over M if for every $y \in \omega^\omega \cap M$, we have $y \cap x \neq 0$. A function $x \in \omega^\omega$ is a *Cohen real* over M if it belongs to every dense G_δ set with a code in the model M.

It is not difficult to see that a Cohen real is an infinitely equal real. The converse implication fails badly. However, it is not clear whether one can obtain a Cohen real from an infinitely equal real via some more or less elementary manipulations. This leads to the following:

Question 1.2. [3, Problem DQ] Are there transitive models $M \subset N$ of set theory, N containing an infinitely equal real over M but no Cohen real over M?

I resolve this question completely by providing an example:

Theorem 1.3. *There is a proper, Baire category preserving forcing which adds an infinitely equal real over the ground model, but no Cohen real.*

The solution is somewhat unusual in that the forcing is concisely defined and analysed in terms of infinite-dimensional topology; however, its combinatorial description is not readily available.

Email address: `zapletal@math.ufl.edu` (Jindřich Zapletal)
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Theorem 1.3 has certain repercussions in forcing iteration theory. By a result of Bartoszynski, it implies that there is an iteration of two proper, category preserving forcings such that neither of the iterands adds a Cohen real while the iteration does. This phenomenon does not appear at limit stages of iterations: if a countable support iteration of limit length of proper, category preserving forcings is such that no intermediate stage of the iteration adds Cohen reals, then even the whole iteration adds no Cohen reals. This is a result of Shelah [10, Conclusion VI.2.13D(1)]. In fact, the successor case of [10, Conclusion VI.2.13D(1)] is in error, as the present paper shows.

A remark describing the history of the result is in place here. Michal Morayne first asked how many Cantor sets are needed to cover the Hilbert cube. Towards the solution of this problem, Márton Elekes proposed the forcing of the form Borel sets modulo the σ-ideal σ-generated by the compact finite-dimensional subsets of the Hilbert cube, and asked whether this poset adds Cohen reals (2009). This was answered by Pol and Zakrzewski in [8] in the negative via the one-to-one or constant property of the ideal [9]. Later Banakh, Morayne, Rałowski and Żeberski [1] resolved the original question in a way that suggested that the forcing must add an infinitely equal real. The present paper essentially only connects and cleans up these pieces.

The notation of the paper follows the set theoretic standard of [5]. Every Polish space X and every analytic subset $A \subset X$ have canonical interpretations in any forcing extension, and the forcing names for these interpretations are denoted by adding a dot superscript to the notation for them: $\dot{A}$, $\dot{X}$. In the context of spaces considered in this paper, all the basic notions of topological dimension (small inductive, large inductive, covering) coincide, and the word "dimension" denotes interchangeably one of them.

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2. Proof of Theorem 1.3

It is not difficult to identify the proper forcing P that has the required properties. Let X be a compact metrizable space which is infinite-dimensional, and all of its compact subsets are either infinite-dimensional or zero-dimensional. Such spaces have been constructed in [4, 11] and elsewhere. Let I be the σ-ideal σ-generated by the compact zero-dimensional subsets of X. The poset P of Borel I-positive subsets of X ordered by inclusion (denoted by P_I) has the properties advertised in Theorem 1.3 as the rest of this section shows.

The argument hinges on two lemmas that forgo the use of the forcing relation entirely.

Lemma 2.1. *For every Borel I-positive set $B \subset X$ and every Borel function $f : B \to \omega^\omega$ there is a Borel I-positive set $C \subset B$ such that the set $f''C \subset \omega^\omega$ is meager.*

Lemma 2.2. *There is a Borel function $f : X \to \omega^\omega$ such that for every point $z \in \omega^\omega$, the set $\{x \in X : f(x) \cap z = 0\}$ belongs to I.*

Theorem 1.3 follows from the two lemmas immediately using basic definable forcing machinery developed in [12]. The P_I-extension contains a point $\dot{x}_{gen} \in X$ which is in the intersection of all Borel sets in the generic ultrafilter [12, Proposition 2.1.2]. The poset P_I is proper and preserves Baire category as the σ-ideal I is σ-generated by closed sets [12, Theorem 4.1.2]. Every element of ω^ω in the P_I-extension is of the form $f(\dot{x}_{gen})$ where f is the canonical interpretation of some Borel function from X to ω^ω coded in the ground model [12, Proposition 2.3.1].

To prove that P_I adds no Cohen reals, suppose that $B \in P_I$ is a condition and τ is a P_I-name for an element of ω^ω; I have to find a condition $C \subset B$ and a meager set A in the ground model such that $C \Vdash \tau \in \dot{A}$. Thinning out the set B if necessary, find a Borel function $f : X \to \omega^\omega$ such that $B \Vdash \tau = \dot{f}(\dot{x}_{gen})$. Use Lemma 2.1 to find a Borel I-positive set $C \subset B$ such that $A = f''C \subset \omega^\omega$ is meager. By a Shoenfield absoluteness argument, $C \Vdash \tau = \dot{f}(\dot{x}_{gen}) \in \dot{A}$ as desired.

To prove that P_I adds an infinitely equal real, find a Borel function $f : X \to \omega^\omega$ as in Lemma 2.2, and argue that $P_I \Vdash \dot{f}(\dot{x}_{gen})$ is an infinitely equal real over the ground model. In other words, if $B \in P_I$ is a condition, $z \in \omega^\omega$ is a function, and $n \in \omega$, I must find a condition $C \subset B$ and a number $m > n$ such that $C \Vdash \dot{f}(\dot{x}_{gen})(m) = \check{z}(m)$. To see this, use the σ-completeness of the ideal I to conclude that the set $\{x \in X : \forall m > n\ z(m) \neq f(x)(m)\}$ is in the ideal I. The σ-completeness applied again provides a definite number $m > n$ such that the set $C = \{x \in B : f(x)(m) = z(m)\}$ is I-positive. A Shoenfield absoluteness argument shows that $C \Vdash \dot{f}(\dot{x}_{gen})(m) = \check{z}(m)$ as desired.

Thus, it is only necessary to prove Lemmas 2.1 and 2.2. This is a task that is already forcing-free, and uses only tools from basic dimension theory and descriptive set theory. The following fact contains all the dimension theory I will need.

Fact 2.3. [2] *Let X be a compact metric space.*

1. *The union of a countable collection of closed zero-dimensional subsets of X is zero-dimensional;*
2. *likewise for a countable product of closed zero-dimensional subsets of X;*
3. *the union of two zero-dimensional subsets of X has dimension at most 1;*
4. *every zero-dimensional subset of X is covered by zero-dimensional G_δ-subset of X;*
5. *if X has finite dimension then it is homeomorphic to a compact subset of $[0,1]^n$ for suitable natural $n \in \omega$.*

Proof of Lemma 2.1. This is the heart of the matter. The argument is best packaged using a hard canonization theorem of Pol and Zakrzewski which originally motivated the paper. I will first verify two key descriptive properties of the ideal I. Recall [7, Section 29.E] that a σ-ideal J on a Polish space Y is

$\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$ if for every Polish space Z and an analytic set $A \subset Z \times Y$ the set $\{z \in Z : A_z \in J\}$ is coanalytic.

Claim 2.4. *The σ-ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. I will first prove that $I \cap K(X)$, which by Fact 2.3(1) is precisely the collection of zero-dimensional compact subsets of X, is a G_δ subset of $K(X)$ in the Vietoris topology. The simplest argument uses the Lebesgue covering dimension. A compact set $K \subset X$ is zero-dimensional if and only if every open cover of K has a refinement such that every point of K belongs to exactly one element of the refinement. Let $\mathcal{O}$ be a countable basis of open subsets of X, closed under finite unions and intersections. By a straightforward compactness argument, K is zero-dimensional if and only if every finite open cover of K consisting of elements of $\mathcal{O}$ has a refinement which consists of pairwise disjoint elements of $\mathcal{O}$ and still covers K. It is easy to check that this formula defines a G_δ subset of $K(X)$ in the Vietoris topology.

The claim now follows from a general fact [7, Theorem 35.38]: hereditary coanalytic families of closed sets σ-generate $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals. □

Now, recall [6] that a σ-ideal J on a Polish space Y is *calibrated* if for every closed set $C \notin J$ and every countable collection $\{D_n : n \in \omega\}$ of closed sets in J, the set $C \setminus \bigcup_n D_n$ contains a closed J-positive set.

Claim 2.5. *The σ-ideal I is calibrated.*

Proof. Suppose that $C \subset X$ is I-positive and closed and $\{D_n : n \in \omega\}$ are closed sets in I. Thus, the set C is infinite-dimensional while the sets $\{D_n : n \in \omega\}$ are zero-dimensional. The set $\bigcup_n D_n$ is zero-dimensional by (1), and it is covered by a G_δ zero-dimensional set E by Fact 2.3(4). Find compact sets $F_n \subset X$ such that $\bigcup_n F_n = X \setminus E$. One of these compact sets must fail to be zero-dimensional, since otherwise $\bigcup_n F_n$ is zero-dimensional by Fact 2.3(1), the set C would break into two zero-dimensional pieces E and $\bigcup_n F_n$, and this contradicts its infinite dimension by Fact 2.3(3). Thus, find $n \in \omega$ such that the set F_n is not zero-dimensional. It cannot be covered by countably many compact zero-dimensional sets by Fact 2.3(1), and therefore $F_n \notin I$. The set F_n witnesses the calibration of the σ-ideal I. □

Now, I am ready to state the key tool in this proof:

Fact 2.6. [8] *Let J be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ calibrated σ-ideal on a Polish space Y, σ-generated by closed sets . Let $B \subset Y$ be a Borel J-positive set and $g : B \to \omega^\omega$ be a Borel function. Then there is a Borel J-positive set $C \subset B$ such that $g \restriction B$ is constant or one-to-one.*

To prove Lemma 2.1, suppose that $B \subset X$ is a Borel I-positive set and $f : B \to \omega^\omega$ is a Borel function. Write $\pi : \omega \to \omega$ for the function defined by $\pi(n) = 2n$, and let $g : B \to \omega^\omega$ be the function defined by $g(x) = f(x) \circ \pi$. Use Fact 2.6 to find a Borel I-positive set $C \subset B$ such that the function g is either one-to-one or constant. In both cases, $f''C \subset \omega^\omega$ is a meager Borel set:

- if $g \restriction C$ is constant with the constant value z then $f''C$ is a subset of the meager set $\{y \in \omega^\omega : y \circ \pi = z\}$.;
- if $g \restriction C$ is one-to-one then the Borel set $f''C$ is meager again. If it were not, there would be points $z_0 \neq z_1 \in f''C$ such that $z_0 \circ \pi = z_1 \circ \pi$. Since the function $g \restriction C$ is one-to-one, there is only one point $x \in C$ such that $g(x) = z_0 \circ \pi$ and then $z_0 = f(x) = z_1$, a contradiction.

This completes the proof. $\square$

Proof of Lemma 2.2. This is much easier than the argument for Lemma 2.1. The original motivation was a construction of [1].

For every $z \in \omega^\omega$ let $A_z = \{y \in \omega^\omega : z \cap y = 0\}$. I will first find a Borel bijection $h : \omega^\omega \to [0, 1)$ such that for every point $z \in \omega^\omega$, the set $h''A_z$ is nowhere dense in the real interval $[0, 1)$. For every sequence $x \in \omega^{\leq\omega}$ let $g(x)$ be the binary sequence ($x(0)$ many 1's)0($x(1)$ many 1's)0($x(2)$ many 1's)0($\dots$; in case that x is finite of length n $g(x)$ ends with $x(n-1)$-many 1's. Define $h : \omega^\omega \to [0, 1)$ by setting $h(x)$ to be the point with binary expansion $0.g(x)$. It is not difficult to see that the function h is a Borel bijection.

Suppose that $z \in \omega^\omega$ is a point and $J \subset [0, 1)$ is a nonempty open interval. I must find a nonempty open interval $J' \subset J$ containing no elements of the set $h''A_z$. Find a finite sequence $s \in \omega^n$ for some n such that the interval $[0.g(s)^\frown 0, 0.g(s)^\frown 1]$ is a subset of J. Let $J' = [0.g(s)^\frown 0^\frown(z(n)$ many 1's$)^\frown 00, 0.g(s)^\frown 0^\frown(z(n)$ many 1's$)^\frown 01]$ and note that $J' \subset J$ works.

Now, use the universality properties of the Hilbert cube to view the compact space X as a compact subset of $[0, 1/2]^\omega$. Find injections $\pi_n : \omega \to \omega$ for all $n \in \omega$ such that the ranges of π_n form a partition of ω. Let $f : X \to \omega^\omega$ be the Borel function defined by $f(x) \circ \pi_n = h^{-1}(x(n))$ for every $n \in \omega$. The function f works as desired in Lemma 2.2. To see this, suppose that $z \in \omega^\omega$ is an arbitrary point, and for every $n \in \omega$ let $z_n = z \circ \pi_n$. For every $n \in \omega$, the closure C_n of the set $h''A_{z_n}$ is nowhere dense in $[0, 1]$, so it is zero-dimensional, and by Fact 2.3(2), even the product $\prod_n C_n$ is zero-dimensional. A review of the definitions shows that $\{x \in X : f(x) \cap z = 0\} \subset \prod_n C_n \in I$ as desired. $\square$

3. Concluding remarks

It is instructive to see why an infinite-dimensional example must be used to prove Theorem 1.3. Suppose that X is a compact metric space of finite dimension say $n \in \omega$, and let I be the σ-ideal σ-generated by compact subsets of X of dimension zero. I will show that the poset P_I adds a Cohen real.

The space X can be viewed as a compact subset of $[0, 1]^m$ for some $m \in \omega$ by Fact 2.3(5). I claim that P_I forces that one of the coordinates of the generic m-tuple $\dot{x}_{gen}$ is a Cohen element of the real interval $[0, 1]$. Suppose this fails; then there is a condition $B \in P_I$ and ground model coded closed nowhere dense sets C_i for $i \in m$ such that $B \Vdash \forall i \in m\ \dot{x}_{gen}(i) \in \dot{C}_i$. The product $\prod_i C_i$ is zero-dimensional, therefore belongs to the ideal I, and thinning the set B if necessary we may assume that $B \cap \prod_i C_i = 0$. Now, the condition B simultaneously forces $\dot{x}_{gen} \in \dot{B}$ and $\dot{x}_{gen} \in \prod_i C_i$, which is impossible as the two sets are disjoint.

On the other hand, it is not essential for the initial space X to have the property that every compact subset is either zero-dimensional or infinitely-dimensional. If the space X cannot be covered by countably many closed sets of finite dimension (as is the case for the Hilbert cube, for example), then the σ-ideal I on X σ-generated by compact sets of finite dimension has all the properties required for the proof of Theorem 1.3, and the proof changes only in minor, if notationaly somewhat awkward, respects.

The topological presentation of the poset P_I seems to depend on certain initial choices. It would be interesting to know whether this dependence is real or just formal.

Question 3.1. How does the forcing P_I (or its forcing properties) depend on the initial choice of the infinite-dimensional space X?

Finally, as the usual approach towards forcing problems includes a direct combinatorial construction of a suitable poset, the following question is natural.

Question 3.2. Is there a combinatorial description of a forcing satisfying Theorem 1.3 which does not mention topological dimension?

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