Vershik's Conjecture, Ultraextensive Spaces, and the Herwig-Lascar Property of Groups

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Vershik's Conjecture

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Conjecture (Vershik, 2008)
The isometry group of the univeral Urysohn space (as well as the automorphism group of the random graph) contains a dense subgroup that is isomorphic to Hall's universal countable locally finite group.

Main Theorem

Theorem (EGLMM)

The following groups all contain dense locally finite groups isomorphic to Hall's group:

(i) the isometry group of the universal Urysohn space;
(ii) the isometry group of the universal rational Urysohn space;
(iii) the isometry group of any univeral Uryoshn Δ-metric space;
(iv) the isometry group of any universal ultrametric Urysohn space;
(v) the automorphism group of the random graph;
(vi) the automorphism group of any Henson graph.

Δ is a countable distance value set, i.e., for any $x, y \in \Delta$,

$$\min\{x + y, \sup(\Delta)\} \in \Delta.$$

Dense Locally Finite Subgroups

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Theorem (Vershik, 2005; Pestov, 2007)
The isometry group of the universal Urysohn space contains a countable dense locally finite subgroup.

Theorem (Solecki, 2009; alternative proof by Rosendal, 2011)
The isometry group of the universal rational Urysohn space contains a countable dense locally finite subgroup.

Hall's Group

Hall's Group $\mathbb{H}$

The class of all finite groups is a Fraïssé class, with Fraïssé limit $\mathbb{H}$.

Theorem (Hall, 1959)

1. Every finite group can be embedded in $\mathbb{H}$.
2. Any two isomorphic finite subgroups of $\mathbb{H}$ are conjugate in $\mathbb{H}$.
3. $\mathbb{H}$ is the unique countable group up to isomorphism with properties 1 and 2.
4. Every countable locally finite group can be embedded in $\mathbb{H}$, i.e., $\mathbb{H}$ is a universal countable locally finite group.

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For any ablian p-group A, B, $\mathbb{H} \oplus A \cong \mathbb{H} \oplus B$ iff $A \cong B$.

More Countable Dense Locally Finite Groups

We say that the group G satisfies a nontrivial mixed identity if there exists $w \in \mathbb{Z} * F_n \setminus F_n$ and $g_1, \ldots, g_n \in G$ such that

$$w(g; g_1, \ldots, g_n) = 1 \text{ for all } g \in G.$$

If G does not satisfy a nontrival mixed identity, we say that G is mixed identify free (MIF).

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A Proof of Vershik's Conjecture

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Theorem (EGLMM)

1. $\mathcal{K}_\Delta$ is a Fraïssé class.
2. Let (X_Δ, H_Δ) be the Fraïssë limit of $\mathcal{K}_\Delta$. Then $X_\Delta \cong \mathbb{U}_\Delta$ and $H_\Delta \cong \mathbb{H}$.
3. H_Δ is dense in $\mathrm{Iso}(X_\Delta)$.

Lemma $\mathbb{K}_\Delta$ has the amalgamation property.

I.e., let $X = X_1 \cup X_2$ be finite Δ-metric spaces,
$\Lambda \leq \Gamma_1, \Gamma_2$ be finite groups, and
Γ_1 acts isometrically on X_1 and Γ_2 acts isometrically on X_2
where the action of Δ on X are consistent with the actions of Γ_1 and Γ_2.

Then, there exists a finite Δ-metric space Y which is an extension of $X_1 \cup X_2$
on which $\Gamma_1 *_\Lambda \Gamma_2$ acts isometrically, with which
the actions of Γ_1 and Γ_2 are consistent.

Theorem (Rosendal, 2011)
G has the RZ-property iff any action of G by isometries on a metric space is finitely approximable, i.e. if G acts by isometries on a metric space (X, d_X), $A \subseteq X$ and $F \subseteq G$ are finite, then there is a finite metric space (Y, d_Y), an action of G on Y by isometries, and an embedding $\pi : A \to Y$ such that whenever $g \in F$ and $x, gx \in A$, then $\pi(gx) = g\pi(x)$ (F-embedding).

The Ribes–Zalesskiĭ Property

Groups with the RZ-property

Definition

A group G has the RZ-property if for any finitely generated subgroups $H_1, \ldots, H_n \leq G$, $H_1 \cdots H_n$ is closed in the profinite topology of G.

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$S \subseteq G$ is closed in the profinite topology iff for any $g \notin S$, there is $N \trianglelefteq G$ of finite index such that $gN \cap S = \emptyset$.

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Theorem (Ribes–Zalesskiĭ, 1993)

Finitely generated free groups have the RZ-property.

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If Γ_1 and Γ_2 are groups with the RZ-property, then so is $\Gamma_1 * \Gamma_2$.

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If $\Lambda \leq \Gamma$ is of finite index, then Γ has the RZ-property iff Λ has the RZ-property.

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Lemma
If $\Lambda \leq \Gamma_1, \Gamma_2$ are finite groups, then $\Gamma_1 *_\Lambda \Gamma_2$ has the RZ-property.

The Herwig–Lascar Property

Groups with the HL-Property

Definition (Herwig–Lascar, 1999)
Let G be a countable discrete group and $H_1, \ldots, H_n \leq G$. A left system of equations on $H_1, \ldots, H_n$ is a finite set of equations with variables $x_1, \ldots, x_m$ and constants $g_1, \ldots, g_l$ such that each equation is of the form

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Definition
We say that G has the HL-property if for every finitely generated $H_1, \ldots, H_n \leq G$ and left system of equations on $H_1, \ldots, H_n$ that does not have a solution in G, there exist normal subgroups of finite index $N_1, \ldots, N_n \trianglelefteq G$ such that the same left system of equations on $N_1 H_1, \ldots, N_n H_n$ does not have a solution.

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Lemma

If $\Lambda \leq \Gamma_1, \Gamma_2$ are finite groups, then $\Gamma_1 *_\Lambda \Gamma_2$ has the HL-property.

Groups with the HL-property

Theorem (Etedadialiabadi–G., 2019)

The following are equivalent:

(i) G has the HL-property;

(ii) Let $\mathcal{L}$ be a finite relational language with unary relation symbols $S_1, \ldots, S_n \in \mathcal{L}$. Let $\mathcal{T}$ be a finite set of finite $\mathcal{L}$-structures. Let D be a $\mathcal{T}$-free $\mathcal{L}$-structure such that $\{S_1^D, \ldots, S_n^D\}$ is a partition of the domain of D. Let C be a finite substructure of D. Let F be a finite subset of G. Suppose that G acts faithfully by isomorphisms on D and that G acts transitively on each S_i^D for $i = 1, \ldots, n$. Then there exists a finite $\mathcal{T}$-free $\mathcal{L}$-structure D' on which G acts by isomorphisms, and an F-embedding from C to D'.

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(iii) Clause (ii) with the additional assumption that every structure in T is a Gaifman clique.

An L-structure C is called a Gaifman clique if for every $a, b \in C$ there is a relation symbol $R \in L$ with arity $m \geq 2$, and $c_1, \ldots c_m \in C$ with $a, b \in \{c_1, \ldots, c_m\}$ and $R^C(c_1, \ldots, c_m)$.

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Theorem (Siniora–Solecki, 2019)
Let L be a finite relational language and T be a finite set of finite L-structures each of which is a Gaifman clique. Then the class of all finite T-free L-structures is a Fraïssé class with the free amalgamation property. Moreover, it has the coherent EPPA, and the automorphism group of its Fraïssé limit contains a dense locally finite subgroup.

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Theorem (EGLMM)

Let L be a finite relational language. Let T be a finite set of finite L-structures each of which is a Gaifman clique. Let $\mathcal{K}$ be the class of all pairs (M, G), where

- M is a finite T-free L-structure,
- G is a finite group, and
- G acts on M by isomorphism.

Then $\mathcal{K}$ is a Fraïssé class.

Let (M_∞, H_∞) be the Fraïssé limit of $\mathcal{K}$. Then $H_\infty \cong \mathbb{H}$ is a dense subgroup of $\mathrm{Aut}(M_\infty)$.