A Σ^1_1 axiom of finite choice and Steel forcing

Jun Le Goh

University of Wisconsin-Madison

SEALS, Gainesville
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The answer is no:

Theorem (van Wesep '77)

For any theory T all of whose ω-models are hyp closed, there is some T' which is strictly weaker than T, all of whose ω-models are hyp closed.

Theories of hyp analysis

Definition (Montalbán '06, relativizing Steel '78)

T is a theory of hyp analysis if:

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"Clearly" Σ^1_1-AC and Δ^1_1-CA do not qualify.

Over $\mathsf{RCA}_0 + I\Sigma^1_1$:

Kleene '59, Kreisel '62,
Friedman '67, Harrison '68,
van Wesep '77, Steel '78,
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Σ^1_1-DC_0

$\Downarrow$

Σ^1_1-AC_0

$\Downarrow$

Π^1_1-SEP_0

$\Downarrow$

Δ^1_1-CA_0

$\Downarrow$

INDEC_0

$\Downarrow$

unique-Σ^1_1-AC_0

$\downarrow$

$L_{\omega_1\omega}$-CA_0

$\Downarrow$

JI

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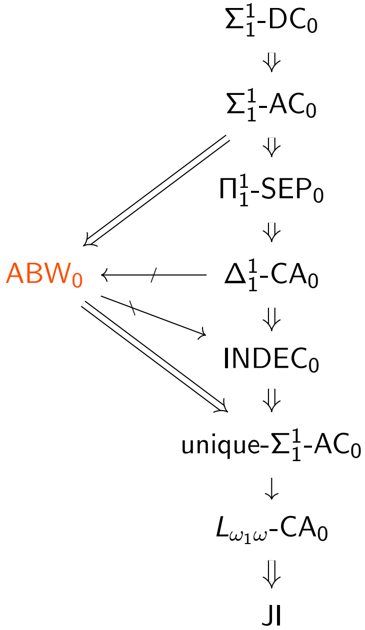

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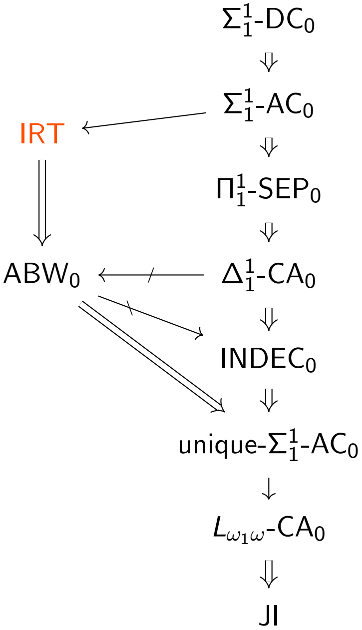

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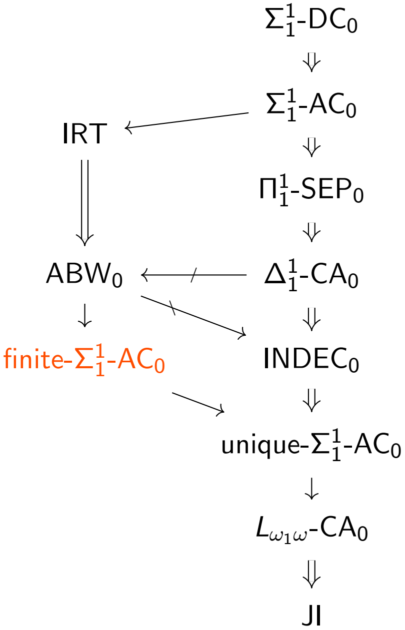

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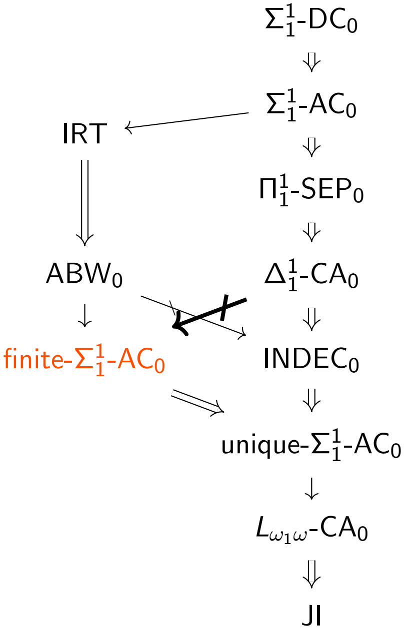

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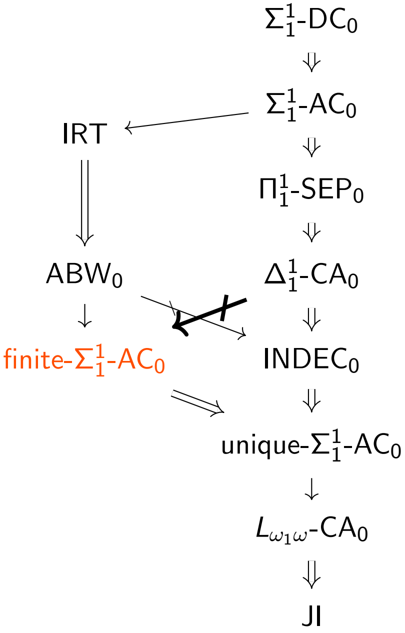
All of these separations (except Σ^1_1-$\mathsf{AC}_0 \nvdash \Sigma^1_1$-$\mathsf{DC}_0$) were proved using Steel forcing and variants thereof.



Kreisel '62: Σ^1_1-AC_0 consists of the sentences

$$(\forall n)(\exists X)\varphi(n,X) \to (\exists \langle X_n\rangle_n)(\forall n)\varphi(n,X_n)$$

for φ arithmetical.

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Finite-Σ^1_1-AC_0 consists of the sentences

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Finite-Σ^1_1-AC_0 is a theory of hyp analysis, since it is sandwiched between theories of hyp analysis.

Our results

Theorem (G.)

There is an ω-model which satisfies Δ^1_1-CA_0 but not finite-Σ^1_1-AC_0.

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We do not know if finite-Σ^1_1-AC_0 implies ABW_0.

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Steel constructs a generic tree $T^G \subseteq \omega^{<\omega}$ and generic paths $\langle \alpha_i^G \rangle_{i \in \omega}$ on T^G such that the α_i^G's are not easily definable from one another.

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For each finite $F \subset \omega$, define the model M_F to consist of all sets which are computable in the λ^{th} jump of $T^G \oplus \langle \alpha_i^G \rangle_{i\in F}$, for some $\lambda < \omega_1^{CK}$.

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Lemma

For each finite $F \subset \omega$, the set of paths on T^G in M_F is exactly $\{\alpha_i^G : i \in F\}$.

Finally, define $M_\infty = \bigcup_{F\subset\omega \text{ finite}} M_F$.

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Main ingredient of proof is to show that if two forcing conditions are sufficiently "alike", then they force the same Σ^1_1 formulas.

This helps to control the complexity of the forcing relation.

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