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Orderable Magmas

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- Magma is a nonempty set with a binary operation: $(M, \cdot)$

- A *left order* on a magma $(M, \cdot)$ is
  a (strict) linear ordering $\prec$ of the domain M,
  which is left invariant with respect to the magma operation:
  $(\forall x, y, z)[x \prec y \;\Rightarrow\; z \cdot x \prec z \cdot y]$

- $\prec$ is a *bi-order* on $(M, \cdot)$ if
  $(\forall x, y, z)[x \prec y \Rightarrow (z \cdot x \prec z \cdot y \;\&\; x \cdot z \prec y \cdot z)]$

- $LO(M)$ the set of all left orders on M
  $RO(M)$ the set of all right orders on M
  $BiO(M)$ the set of all bi-orders on M

- Given a left order $\prec_l$ on a group G, we have a right order $\prec_r$:

$$x \prec_r y \Leftrightarrow y^{-1} \prec_l x^{-1}$$

- G is a left orderable group $\Rightarrow G$ is *torsion-free*

$$e \prec x \;\Rightarrow\; x \prec x^2 \prec x^3 \prec \cdots \prec x^n$$

- G is an abelian and torsion-free group $\Rightarrow G$ is orderable

- Every torsion-free nilpotent group is bi-orderable.

- Torsion-free, but not left orderable group:

$$G = \langle x, y \mid xy^2x^{-1}y^2 = e,\;\; yx^2y^{-1}x^2 = e \rangle$$

- A magma $(M, \cdot)$ is *computable* if its domain M is a computable set and its operation $\cdot$ is computable.

- $(\mathbb{Z}, +)$ has two orders, both computable.

  $(\mathbb{Z}^2, +)$ has $2^{\aleph_0}$ orders.

  $\mathbb{Z}^\omega = \bigoplus_{i \in \omega} \mathbb{Z}$, the direct sum of ω copies of $\mathbb{Z}$

  $(\mathbb{Z}^\omega, +)$ has $2^{\aleph_0}$ orders.

- (Downey and Kurtz)

  There is a computable torsion-free abelian group G (hence orderable) such that G has no computable order.

- Klein bottle group Kl=$\langle a, b \mid a^{-1}ba = b^{-1} \rangle$

  $ba = ab^{-1}$

- Kl is left orderable, but not bi-orderable.

  $b \succ e \Longleftrightarrow a^{-1}ba = b^{-1} \succ a^{-1}a = e$

- For a bi-order $\prec$: $(a \prec b \;\&\; c \prec d \Rightarrow a \cdot c \prec b \cdot d)$

  $a \prec b \Rightarrow a \cdot c \prec b \cdot c$

  $c \prec d \Rightarrow b \cdot c \prec b \cdot d$

- Not necessarily true for a left order. In Kl:

  $ba \neq a$ but $(ba)^2 = baba = ab^{-1}ba = a^2$

- Let R and S be binary relations on M.

  $R^{-1} = \{(y, x) : \ (x, y) \in R\}$

  $S \circ R = \{(x, z) : \ (\exists y)[(x, y) \in R \ \& \ (y, z) \in S\}$

  $\Delta_M = \{(a, a) : a \in M\}$

- R is a left order on $(M, \cdot)$ iff the following conditions are satisfied:

  $R \circ R \subseteq R$

  $R \cap R^{-1} = \varnothing$

  $R \cup R^{-1} = (M \times M) - \Delta_M$

  $\Delta_M R \subseteq R$

- A magma $(Q, *)$ is called a *quandle* if:

  (1) $a{*}a = a$ (idempotence)

  (2) For every $b \in Q$, the mapping $*_b : Q \to Q$ defined by

  $$*_b(a) = a * b$$

  is bijective.

  (3) $(a{*}b) * c = (a * c) * (b * c)$ (right self-distributivity)

- A quandle Q_0 is called *trivial* if the operation $*$ is defined by:

  $$a * b = a$$

  Every linear ordering of the elements of Q_0 is right invariant.

- For a group G, the *conjugate* quandle Conj(G) has domain G and the operation $*$ given by $a * b = b^{-1}ab$.

- (1) $a * a = a^{-1}aa = a$

  (2) $\forall b \ \forall c \ \exists! a \ [a * b = c]$

  $b^{-1}ab = c \Rightarrow a = bcb^{-1}$

  (3) $(a * b) * c = c^{-1}(a * b)c = c^{-1}b^{-1}abc$

  $(a * c) * (b * c) = (c^{-1}ac) * (c^{-1}bc) = (c^{-1}bc)^{-1}(c^{-1}ac)(c^{-1}bc) = c^{-1}b^{-1}cc^{-1}abc = c^{-1}b^{-1}abc$

- Every bi-order on G induces a right order on Conj(G).

- Given a bi-order B on G, define R on Conj(G) by:

$$(a,b) \in R \Leftrightarrow (a,b) \in B$$

- The order R is right invariant because for $(a,b) \in R$ and $c \in$ Conj(G):

$$(a,b) \in R \Rightarrow (a,b) \in B \Rightarrow (c^{-1}ac, c^{-1}bc) \in B \Rightarrow (a * c, b * c) \in R$$

- Not all right orders on Conj(G) are always induced by bi-orders on G.

- It is possible to have $BiO(G) = \varnothing$, while $RO(\text{Conj}(G)) \neq \varnothing$.

  Let G be an abelian group with torsion. Then $BiO(G) = \varnothing$, but Conj(G) is a trivial quandle, so it admits right orders.

- Topology defined on $LO(M)$ by *subbasis* $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\Delta_M}$:

$$S_{(a,b)} = \{R \in LO(M) \ : \ (a,b) \in R\}$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki and Veve)

  Let M be a left orderable magma with cardinality $|M| = \mathfrak{m} \geq \aleph_0$.

  Then $LO(M)$ is a compact space. $BiO(M)$ is also a compact space.

- By Vedenissoff's theorem, $LO(M)$ can be

  homeomorphically embedded into the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

  Moreover, $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use Conrad's theorem to establish that $LO(G)$ is compact.

- (Conrad)

  A partial left order on G with a non-negative cone P can be extended to a total left order iff for every finite set $\{x_1, \ldots, x_n\} \subseteq G \backslash \{e\}$, there are $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$ such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}).$$

  Here, $sgr(X)$ stands for the sub-semigroup of G generated by X.

- For a countable left-orderable group G, $LO(G)$ is homeomorphic to the Cantor set iff for any finitely many pairs of elements, $(a_1, b_1), \ldots, (a_k, b_k)$, $S_{(a_1,b_1)} \cap \cdots \cap S_{(a_k,b_k)}$ is either empty or infinite.

- (Sikora) The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska) The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- Let $F_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \ \rangle$ be a free group of rank n.

- *Conjecture* (Sikora, 2004) For $n > 1$, the spaces $LO(F_n)$ and $BiO(F_n)$ are homeomorphic to the Cantor set.

- (Linnell) The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

  (There are countable groups with infinitely countably many bi-orders.)

- (Navas-Flores) The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Chubb, Dabkowski and Harizanov) For a computable group G isomorphic to a free group F_n of rank $n > 1$, we have a bi-order in every truth-table degree.

- (Ha and Harizanov) For every infinite right-orderable computable magma M, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $RO(M)$ to the set of all infinite paths of $\mathcal{T}$.

  Chubb had a similar result for semigroups.

- Corollaries

  (i) By the *Low Basis Theorem* of Jockusch and Soare, M has an order of *low* Turing degree.

  (ii) If M has only finitely many orders, they must be all computable.

  (iii) If M has countably infinitely many orders, then it has infinitely many computable orders.

  (iv) If M does not have a computable order, then the space of orders on M is homeomorphic to the Cantor set.

- Let $F_\infty = \langle x_0, x_1, \ldots \mid \ \rangle$ be a free group of rank $\aleph_0$.

- (Ha and Harizanov)

  There is an isomorphic computable copy G of F_∞ such that its conjugate quandle $Conj(G)$ has no computable right order (hence no computable bi-order).

- *Corollary*
  The spaces $RO(Conj(G))$ and $BiO(Conj(G))$ are homeomorphic to the Cantor set.

- THANK YOU!