Reduction games, provability, and compactness

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Joint work with Damir D. Dzhafarov and Sarah C. Reitzes

We work in second-order arithmetic, coding finite objects as numbers and countably infinite ones as sets of numbers.

We discuss some ways to compare statements of the form

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From now on, P and Q will denote such Π^1_2-problems.

Reverse Mathematics

Second-order arithmetic is a two-sorted first-order language.

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The usual base theory RCA_0 consists of the axioms for a discrete ordered commutative semiring and:

Δ^0_1-comprehension:

$$\forall n\,[\varphi(n) \leftrightarrow \psi(n)] \rightarrow \exists X\,\forall n\,[n \in X \leftrightarrow \varphi(n)]$$

for all φ, ψ s.t. φ is Σ^0_1 and ψ is Π^0_1, and X is not free in φ

Σ^0_1-induction:

$$(\varphi(0) \wedge \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for all Σ^0_1 formulas φ

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Jockusch gave a proof using the Compactness Theorem.

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If $\mathsf{RCA}_0 + Q \vdash P$ then $P \leqslant_\omega Q$, but not always vice-versa.

An Example: Versions of Ramsey’s Theorem

$[X]^n$ is the set of n-element subsets of X.

A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$. (We assume $k \geqslant 2$.)

$H \subseteq X$ is homogeneous for c if $|c([H]^n)| = 1$.

RT^n_k: Every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

RT^n: $\forall k\, \mathsf{RT}^n_k$.

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Theorem (Jockusch). Let $n \geqslant 2$.

Every computable instance of RT^n has a Π^0_n solution.

There is a computable instance of RT^n_2 with no Σ^0_n solution.

There is a computable instance of RT^n_2 s.t. every solution computes $\emptyset^{(n-2)}$.

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$\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$ for each k, so RT^1 is true in every ω-model of RCA_0, but:

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Corollary. $\mathsf{RT} \leqslant_\omega \mathsf{ACA}_0$ but $\mathsf{ACA}_0 \nvdash \mathsf{RT}$.

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Corollary. $\mathsf{RT} \leqslant_\omega \mathsf{ACA}_0$ but $\mathsf{ACA}_0 \nvdash \mathsf{RT}$.

Equivalently, $\mathsf{RT} \leqslant_\omega \mathsf{RT}^3_2$ but $\mathsf{RCA}_0 + \mathsf{RT}^3_2 \nvdash \mathsf{RT}$.

Computability-Theoretic Reductions

P is computably reducible to Q, written $P \leqslant_{\mathrm{c}} Q$, if

for every instance X of P,

there is an X-computable instance $\widehat{X}$ of Q s.t.,

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P is Weihrauch reducible to Q, written $P \leqslant_W Q$, if

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Theorem (Hirschfeldt and Jockusch / Brattka and Rakotoniaina / Patey). $\mathsf{RT}^n_2 <_{\mathrm{W}} \mathsf{RT}^n_3 <_{\mathrm{W}} \mathsf{RT}^n_4 <_{\mathrm{W}} \cdots$.

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Reduction Games

We will describe two-player reduction games for P and Q defined by Hirschfeldt and Jockusch.

Player 1 will play a P-instance X_0.

Player 2 will try to obtain a solution to X_0 by asking Player 1 to solve various Q-instances.

If Player 2 every plays such a solution, it wins, and the game ends.

If the game never ends then Player 1 wins.

If a player cannot make a move, the opponent wins.

Reduction Games over ω

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Third Move:

Player 1: A solution X_2 to Y_2.

Player 2: Either an $(X_0 \oplus X_1 \oplus X_2)$-computable solution to X_0, or an $(X_0 \oplus X_1 \oplus X_2)$-computable Q-instance Y_2.

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Write $P \leqslant^n_\omega Q$ if Player 2 has a winning strategy for $G(Q \to P)$ that wins in at most $n+1$ many moves, and similarly for gW.

Theorem (Hirschfeldt and Jockusch). Let $n \geqslant 3$ and $j \geqslant 1$, and let m be s.t.

$$n + (j-1)(n-2) < m \leqslant n + j(n-2).$$

Then

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Patey characterized the least m s.t. $\mathrm{RT}^n_k \leqslant^{m}_{\omega} \mathrm{RT}^n_j$ for $n \geqslant 2$ and $j < k$.

For $n \geqslant 3$, this m is always 2. For $n = 2$ it is more complicated and goes to infinity as k increases.

Reduction Games over Non-ω Models

Hirschfeldt and Jockusch also suggested considering games over models of RCA_0.

More generally, we work over models of a consistent extension Γ of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula.

Let $\mathcal{N}$ be a model of first-order arithmetic.

The notions of instance and solution of a Π^1_2-problem still make sense over $\mathcal{N}$.

For $X_0, \dots, X_n \in |\mathcal{N}|$, let $\mathcal{N}[X_0, \dots, X_n] = (\mathcal{N}, S)$ where S consists of all subsets of $|\mathcal{N}|$ that are Δ^0_1-definable from parameters in $|\mathcal{N}| \cup \{X_0, \dots, X_n\}$.

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Player 2: Either a solution to X_0 in $\mathcal{N}[X_0, X_1]$, or a Q-instance $Y_2 \in \mathcal{N}[X_0, X_1]$.

Third Move:

Player 1: A solution X_2 to Y_2 in S.

Player 2: Either a solution to X_1 in $\mathcal{N}[X_0, X_1, X_2]$, or a Q-instance $Y_3 \in \mathcal{N}[X_0, X_1, X_2]$.

$\vdots$

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Main Theorem. If $\Gamma + Q \vdash P$ then there is an $n \in \omega$ s.t. Player 2 has a winning strategy for $G^\Gamma(Q \to P)$ that wins in at most n many moves.

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Corollary. Let Γ be RCA_0 together with all Π^1_1 formulas true in the natural numbers. Then $\Gamma + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2$.

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Corollary. Let Γ be RCA_0 together with all Π^1_1 formulas true in the natural numbers. Then $\Gamma + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2$.

This theorem is a generalization of the aforementioned fact that if

$$\mathrm{ACA}_0 \vdash \forall X\, [\Theta(X) \ \to\ \exists Y\, \Delta(X, Y)]$$

where Θ and Δ are arithmetic, then there is an $n \in \omega$ s.t.

$$\mathrm{ACA}_0 \vdash \forall X\, [\Theta(X) \ \to\ \exists Y \in \Sigma^{0,X}_n\, \Delta(X, Y)].$$

Lemma. For $n \in \omega$, let $\Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n)$ state that

if X_0 is a P-instance then ($Y_0 = \Phi_{e_0}^{X_0} \wedge$(either Y_0 is a solution to X_0 or
(Y_0 is a Q-instance and if X_1 is a solution to Y_0 then ($Y_1 = \Phi_{e_1}^{X_0 \oplus X_1} \wedge$
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(Y_1 is a Q-instance and if X_2 is a solution to Y_1 then ($Y_2 = \Phi_{e_2}^{X_0 \oplus X_1 \oplus X_2} \wedge$
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$\ldots$($Y_n = \Phi_{e_n}^{X_0 \oplus \cdots \oplus X_n} \wedge Y_n$ is a solution to X_0))$\cdots$).

If $\Gamma + \mathrm{Q} \vdash \mathrm{P}$, then there is an $n \in \omega$ s.t.

$$\Gamma \vdash \forall X_0 \, \exists e_0, Y_0 \, \forall X_1 \, \exists e_1, Y_1 \cdots \forall X_n \, \exists e_n, Y_n \; \Theta_n(e_0, \ldots, e_n, X_0, \ldots, X_n, Y_0, \ldots, Y_n).$$

Proposition (essentially Hirschfeldt and Jockusch). If $\Gamma + P \vdash Q$ then Player 2 has a winning strategy for $G^\Gamma(Q \to P)$. Otherwise, Player 1 has a winning strategy for $G^\Gamma(Q \to P)$.

P is generalized Weihrauch reducible to Q over Γ, written $P \leqslant^\Gamma_{\mathrm{gW}} Q$, if Player 2 has a computable (i.e., Δ^0_1) winning strategy for $G^\Gamma(Q \to P)$.

We can similarly define computable reducibility and Weihrauch reducibility over Γ.

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There are connections here with intuitionistic provability, and in particular work of Kuyper and of Uftring, which we are still working out.

An Example: Limit-Homogeneous Sets

A 2-coloring c of $[\mathbb{N}]^2$ is stable if $\lim_y c(x, y)$ exists for all x.

$L \subseteq \mathbb{N}$ is limit-homogeneous for c if there is an i s.t. $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

SRT^2_2: Every stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

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Theorem (Cholak, Jockusch, and Slaman). $\mathsf{SRT}^2_2 \equiv_{\mathrm{c}} \mathsf{D}^2_2$.

Theorem (Chong, Lempp, and Yang). SRT^2_2 and D^2_2 are equivalent over RCA_0.

Theorem (Dzhafarov). $\mathsf{SRT}^2_2 \not\leqslant_{\mathrm{W}} \mathsf{D}^2_2$.

Theorem (Hirschfeldt and Jockusch). $\mathsf{SRT}^2_2 \leqslant^2_{\mathrm{gW}} \mathsf{D}^2_2$.

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LH: For every 2-coloring of $[\mathbb{N}]^2$, every infinite limit-homogeneous set has an infinite homogeneous subset.

An instance of LH includes the color i.

The Σ^0_2-bounding principle $B\Sigma^0_2$ plays a significant role in reverse mathematics.

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Proposition. LH is equivalent to Bound^* over RCA_0.

LH is Weihrauch-trivial, i.e. $\mathsf{LH} \leqslant_{\mathsf{W}} 1$, where 1 is the identity problem.

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Theorem. $\mathsf{LH} \nleqslant^{\mathsf{RCA}_0}_{\mathsf{gW}} \mathsf{Bound}^*$.

So the use of Bound^* in proving LH is “purely proof-theoretic”.

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Recall that $\mathsf{SRT}^2_2 \leqslant_{\mathrm{gW}} \mathsf{D}^2_2$.

Open Problem. Is $\mathsf{SRT}^2_2 \leqslant_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{D}^2_2$?