Separation problems and forcing*

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Abstract

Certain separation problems in descriptive set theory correspond to a forcing preservation property, with a fusion type infinite game associated to it. As an application, it is consistent with the axioms of set theory that the circle $\mathbb{T}$ can be covered by $\aleph_1$ many closed sets of uniqueness while a much larger number of H-sets is necessary to cover it.

1 Introduction

Let X be a Polish space and $K(X)$ its hyperspace of compact subsets of X. If $J \subseteq I \subseteq K(X)$ are two collections of compact sets, one may ask whether it is possible to separate J from $K(X) \setminus I$ by an analytic set, i.e. if there is an analytic set $A \subseteq K(X)$ such that $J \subseteq A \subseteq I$. A typical context arises when I, J are both coanalytic σ-ideals of compact sets or even $J = K_{\aleph_0}(X)$, the collection of countable compact subsets of X. For example, Pelant and Zelený [9, Theorem 4.8] showed that $K_{\aleph_0}(X)$ *cannot* be separated by an analytic set from the collection of non-σ-porous sets, where X is an arbitrary compact separable metric space. On the other hand, Loomis [7] [5, page 129 and 185] showed that $K_{\aleph_0}(\mathbb{T})$ *can* be separated by an analytic set from the collection of compact sets of multiplicity, where $\mathbb{T}$ is the unit circle.

In this paper, I explore the connection of such separation problems with the theory of definable proper forcing [13]. I first isolate a natural infinite game which equivalently characterizes the separation problem (Theorem 2.2). The winning strategies for the good player in this game are immediately reminiscent of fusion arguments in forcing. I then define the related forcing property (the *overspill property* of Definition 3.1) and show that it is a true forcing preservation property (Theorem 3.2). Within the realm of suitably definable proper forcings, it is preserved under the operation of countable support product (Theorem 3.5),

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countable union (Theorem 3.8) as well as the countable support iteration (not proved in this paper). I also provide several subtle examples of forcings that do or do not have the overspill property (Section 4) and explain the relationship with the preexisting Sacks-type property (Section 5).

A good application of these ideas can be found in harmonic analysis. I compare the cardinal invariants of the σ-ideal H_σ σ-generated by H-sets, introduced by Rajchman [10], and the σ-ideal U_σ σ-generated by closed sets of uniqueness [5] on the unit circle $\mathbb{T}$. It is well-known that $H_\sigma \subset U_\sigma$.

Theorem 1.1. (Theorem 6.3) *Suppose that the Generalized Continuum Hypothesis holds and $\kappa \geq \aleph_1$ is a regular cardinal. Then there is a cardinal preserving forcing extension in which $\mathbb{T}$ is covered by $\aleph_1$ many closed sets of uniqueness but not fewer than κ many H-sets.*

The main point of this result is in the methodology through which it is obtained. The combinatorics of both H-sets and sets of uniqueness is very complex, seemingly prohibiting any straightforward approach via combinatorial proper forcing. The present proof attacks directly the main descriptive set theoretic difference between the σ-ideals H_σ and U_σ: the collection $K_{\aleph_0}(\mathbb{T})$ cannot be separated from the former by an analytic set, but it can be so separated from the latter. This difference survives the various forcing manipulations necessary and yields the independence result in a conceptual, compartmentalized way. The arguments seem to be impossible to replicate meaningfully through the combinatorial approach to proper forcing.

The notation in the paper follows the set theoretic standard of [4]. I use [5] as a canonical reference for harmonic analysis, [6] for descriptive set theory, and [13] for definable forcing. If I is a σ-ideal on a Polish space X, then P_I denotes the partial order of Borel I-positive sets ordered by inclusion. For a Polish space X, $K(X)$ denotes its hyperspace, i.e. the space of compact subsets of X with the Vietoris topology. A subset $I \subset K(X)$ is hereditary if it is closed under taking subsets: $K \subset L \in I$ implies $K \in I$. The closure of a set A in a topological space is denoted by a bar: $\overline{A}$. If $t : \omega \to 2$ is a partial finite function then $[t] = \{x \in 2^\omega : t \subset x\}$; a similar notation prevails for the Baire space ω^ω. If $A \subset X \times Y$ is a set then $p(A)$ denotes its projection into the first (i.e. X) coordinate. A tree is a set of finite sequences closed under initial segment. For a tree T, the expression $[T]$ denotes the set of infinite branches through T. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every Polish space Y and every analytic set $D \subset Y \times X$ the set $\{y \in Y :$ the vertical section D_y of D above y is in the ideal $I\}$ is coanalytic. A forcing is bounding if every function in ω^ω in its generic extension is pointwise dominated by a ground model function.

2 The separation game

Definition 2.1. Let X be a compact metric space. Let $I \subset K(X)$ be a coanalytic collection of compact subsets of X, closed under subsets. Let J be a σ-ideal of compact subsets of X, containing all singletons. The game $G(J, I, X)$

is played between Player I and II for infinitely many rounds. In the n-th round of the game $G(J, I, X)$, Player I produces a compact set $K_n \in J$ and Player II responds with an open set $O_n \supset K_n$. In the end, Player II wins if the *result of the play*, the intersection $\bigcap_n \overline{O}_n$ is a compact element of I.

Note that Player II has a simple strategy for making the result of the play compact. He can just make sure that the n-th open set is a subset of 2^{-n}-neighborhood of the set K_n in some fixed complete metric on X, and then the result of the play must be totally bounded and therefore compact.

Theorem 2.2. *Suppose that X is a Polish space, $J \subset K(X)$ is a σ-ideal, and $I \subset K(X)$ is hereditary. Then*

1. *Player II has no winning strategy in the game $G(J, I, X)$ if and only if there is no analytic set $A \subset K(X)$ with $J \subseteq A \subseteq I$;*
2. *if I is coanalytic then the game $G(J, I, X)$ is determined.*

Proof. To prove the left-to-right direction of (1), suppose first that Player II has a winning strategy σ in the game $G(J, I, X)$. To produce the desired analytic set $A \subset K(X)$, fix a countable basis $\mathcal{O}$ for the space X closed under finite unions and intersections, and by tree induction build a countable tree T of partial finite plays according to the strategy σ ending with a move of Player II, such that if $t \in T$ is a node and $O \in \mathcal{O}$ is a basic open set such that Player I can make a move after t which forces the strategy σ to respond with a superset of O, then there is an immediate successor $s \in T$ of the node t that indeed ends with the strategy σ playing a superset O. Now, if $b \in [T]$ is a branch through the tree T, it is an infinite play against the strategy σ, so Player II won and the end result of it is in the collection I. Consider the set $A = \{C \in K(X) :$ for some branch $b \in [T]$, K is covered by the end result of the play $b\}$. This is an analytic collection of compact sets, and since I is closed under subsets, $A \subset I$. Moreover, $J \subset A$: if $C \in J$ is a compact set, then by induction on n build nodes $t_n \in T$ so that C is a subset of all moves of Player II in the play t_n. The induction step is possible to perform, since the set C is a possible move of Player I in the next round past t_n and the strategy σ must respond to it with some open set $P \supset C$. By the compactness of the set C, there must ba a basic open set $O \subset P$ still covering C. The construction of the tree T guarantees that there is an immediate successor t_{n+1} of t_n whose last move is still a superset of both O and C as desired. In the end, the end result of the play $\bigcup_n t_n$ is a superset of C and shows that $C \in A$.

For the right-to-left direction of (1), suppose that $A \subset K(X)$ is an analytic collection containing J, and $A \subset I$. To produce a winning strategy σ for Player II, fix a continuous function $g : \omega^\omega \to K(X)$ such that $A = \operatorname{rng}(g)$. Player II will win by producing, along with the moves of the game, sequences $t_n \in \omega^n$ so that $0 = t_0 \subset t_1 \subset \dots$ and for every number $n \in \omega$, the following statement $(*n)$ holds: for every compact set $K \subset \bigcap_{m \in n} \overline{O}_m$ in J there is a point $y \in \omega^\omega$ extending t_n such that $K \subset g(y)$. That way, the end result $L \subset X$ of the play will be a subset of $g(y)$ where $y = \bigcup_n t_n$, and as $g(y) \in I$ and I is closed under subsets, $L \in I$ and Player II won. To see that $L \subset g(y)$, observe that if some point $x \in L$ did not belong to the compact set $g(y)$, by the continuity of the function g there would have to be a number n such that for every point $y' \in \omega^\omega$ with $t_n \subset y'$ it is the case that $x \notin L$, contradicting $(*n)$ as $x \in \bigcap_{m\in n} \overline{O}_m$ and $\{x\} \in J$.

It is necessary to prove that Player II can maintain $(*n)$ at every stage n of the play. $(*0)$ by the assumptions on the set A no matter what the open set O_0 is. Now suppose that $(*n)$ holds and Player I produces a set $K_n \in J$. I must show that there is an open set O containing K_n and a number $i \in \omega$ such that $(*n+1)$ holds with $O_{n+1} = O$ and $t_{n+1} = t_n^\frown i$. Suppose for contradiction that this is not the case. Choose inclusion decreasing basic open sets $\langle P_i : i \in \omega\rangle$ such that each of them is a legal move for Player II at this stage and $K_n = \bigcap_i P_i$. Since $(*n+1)$ must fail for each of them, there are countable compact sets $L_i \subset \bigcap_{m\in n} \overline{O}_m \cap \overline{P}_i$ such that for every point $y \in \omega^\omega$ with $t_n^\frown i \subset y$ the inclusion $L_i \subset g(y)$ fails. Consider the closure $L \bigcap_{m\in n} \overline{O}_m$ of the union $\bigcup_i L_i$. The set L contains only the points in $\bigcup_i L_i$ and points in K_n, and as J is a σ-ideal, it follows that $L \in J$. By the induction hypothesis, there must be a point $y \in \omega^\omega$ extending t_n such that $L \subset g(y)$. This, however, contradicts the choice of the set L_i where i is the first entry of the sequence y past t_n!

(2) of the theorem is proved by a standard unraveling argument. Since the collection I is coanalytic, there is a continuous function $g : \omega^\omega \to K(X)$ whose image is the complement of I. Consider the game $G'(J, I, X)$ which is slightly more difficult than $G(J, I, X)$ for Player I. The game $G'(I, J, X)$ proceeds in the same way as the previous one, except in some rounds, Player I also indicates a number i_n. The number i_n does not have to be played at round n but perhaps at a later round at Player I's will, and it must be smaller than the index of that round. Player I wins if he played all numbers i_n for $n \in \omega$, thereby creating a sequence $y \in \omega^\omega$, and $g(y) \subset \bigcap_n \overline{O}_n$. Thus, if Player I wins in a play of the game $G'(J, I, X)$, then he also won the associated play of the game $G(J, I, X)$: the set $g(y)$ is not in I, and as the collection I is closed under subsets, the set $\bigcap_n O_n$ cannot belong to I either. In the wide tree of all possible plays of the game $G'(J, I, X)$, the plays in which Player I wins forms a G_δ set, and the game $G'(I, J, X)$ is therefore determined [8]. I will show that winning strategies for both players in the new game translate to winning strategies in the old game.

It is clear that if Player I has a winning strategy in the game $G'(J, I, X)$, then the same strategy, merely omitting the additional moves, will be his winning strategy in the game $G(I, J, X)$. Now suppose that σ' is a winning strategy for Player II in the game $G'(J, I, X)$. To get a winning strategy for this player in the original game, note that σ' can be easily improved not to depend on the choices of the numbers i_n. Simply at each round consider the finitely many possibilities for such choices of these numbers in the previous round and play the intersection of all sets that σ' advises to play against each. I claim that this improved strategy σ is in fact winning for Player II in the original game $G(J, I, X)$. Indeed, if there is a play p in the game $G(J, I, X)$ against this strategy in which Player II loses, then the result L of that play cannot be in I and there is a point $y \in \omega^\omega$ such that $g(y) = L$. Consider the play p' against the strategy σ' in which Player I plays the same compact sets as in p and produces the point y in such a way that each number on it is added at a round with index larger than that number. The definition of the strategy σ implies that the moves of the strategy σ' in p' will be supersets of the corresponding moves of the strategy σ in p, therefore the moves of Player I in p' are legal and the result L' of the Play p' will be a superset of $L = g(y)$, resulting in Player I's victory. This of course contradicts the choice of the strategy σ'. $\square$

Corollary 2.3. *If $J \subset K(X)$ is a σ-ideal of compact sets containing all singletons and $I_0, I_1 \subset X$ are coanalytic families of compact sets such that $J \subset I_0 \cap I_1$, and there is no analytic set A such that $J \subset A \subset I_0$ or $J \subset A \subset I_1$, then there is no analytic set $A \subset K(X)$ such that $J \subset A \subset I_0 \cup I_1$.*

Proof. Finitely many strategies for Player I can be combined together, with Player I playing unions of the compact sets indicated by all of them in any given round. $\square$

As the most trivial example for Player I, he has a winning strategy if $I = J$ is the collection of countable compact subsets of the Cantor space $X = 2^\omega$. He will win by playing finite sets C_n such that $C_0 \subset C_1 \subset \dots$ such that for every number n and every point $x \in C_n$ there is another point $x \neq y \in C_{n+1}$ such that x, y agree on the first n positions. In the end, the result of the play must contain the closure of the set $\bigcup_n C_n$, which is perfect, therefore uncountable and winning for Player I. Note the similarity between this winning strategy and the fusion arguments for the Sacks forcing (which is isomorphic to a dense subset of P_I).

As the most trivial example for Player II, he has a winning strategy if I equals to the Lebesgue null sets. He will simply make use of the fact that every countable set is null and at the n-th move, he will cover the move K_n with an open set of mass $\leq 2^{-n}$. In this way, the result of the play will be Lebesgue null and therefore winning for Player II.

3 The forcing connection

The winning strategies for Player I in the game $G(J, I, X)$ certainly remind the alert reader of various forcing fusion arguments. To exploit this parallel, we will consider only the case of J equal to the σ-ideal of compact countable sets, where we will make use of the closure of the class of countable sets under arbitrary images as well as finite products. We will also look at the case of σ-ideal I only and make a connection with the forcing properties of the quotient poset P_I of Borel I-positive sets ordered by inclusion.

Definition 3.1. Let I be a σ-ideal on a Polish space X. If Y is a Polish space and $C \subset X \times Y$ is closed, write $K_{\aleph_0}(C)$ and $K_I(C)$ (or $K_{\aleph_0}$ and K_I if no confusion is possible) of the σ-ideals of countable compact subsets of C and the compact subsets of C with projection in I. Say that I has the *overspill property* if for every Polish space Y and every closed set $C \subset X \times Y$ with I-positive projection, there is no analytic set $A \subset K(C)$ with $K_{\aleph_0}(C) \subseteq A \subseteq K_I(C)$.

As every Polish space is a continuous injective image of a closed subset of ω^ω, the overspill property needs to be verified only for the case $Y = \omega^\omega$. I will now proceed to derive several forcing consequences of the overspill property.

Theorem 3.2. *Suppose that I is a σ-ideal on a Polish space X such that the quotient poset P_I is proper. The following are equivalent:*

1. *I has the overspill property;*
2. *every I-positive analytic set has an I-positive Borel subset, and for every Polish space Y and every analytic family $A \subset K(Y)$ containing all countable compact subsets of Y, P_I forces the space Y to be covered by the ground model coded elements of the set A.*

Proof. For the (1)→(2) implication, assume the overspill property. Suppose that Y is a Polish space, $A \subset K(Y)$ an analytic set containing all countable compact subsets, and suppose that $B \Vdash \dot{y} \in Y$ is a point. I must find a strengthening $B' \subset B$ and a compact set $K \in A$ such that $B' \Vdash \dot{y} \in \dot{K}$. Use the Borel reading of names to thin out the set B if necessary to find a Borel function $f : B \to Y$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Find a closed set $C \subset X \times Y \times \omega^\omega$ projecting into the graph of f. Let $A' \subset K(C)$ be the collection of those compact subsets $L \subset C$ whose projection $p_Y(L)$ into Y belongs to A. This is an analytic subset of $K(C)$ containing all countable compact sets, and by the overspill property it contains an element L whose projection $p_X(L)$ into X is I-positive. The Mostowski absoluteness implies that $p_X(L) \Vdash \dot{y} \in p_Y(L)$ and (2) has been verified.

For the (2)→(1) implication, assume (2) holds, and towards the verification of the overspill property choose a closed subset $C \subset X \times Y$ for some Polish space Y with I-positive projection, and an analytic set $A \subset K(C)$ containing all countable compact sets. I must find a set $K \in A$ with an I-positive projection. To do this, first produce an I-positive Borel set $B \subset p(C)$. The properness of P_I implies that it is possible to thin out B if necessary and find a Borel function $f : B \to Y$ whose graph is a subset of C–[13, Proposition 2.3.4]. (2) shows that it is possible to thin out B further to find a set $K \in A$ such that $B \Vdash \langle \dot{x}_{gen}, \dot{f}(\dot{x}_{gen}) \rangle \in \dot{K}$. Now, the set K must have an I-positive projection since there is a condition (namely B) which forces the generic point to belong to that projection. The overspill property follows. □

The two most prominent consequences:

Corollary 3.3. *Suppose that I is a σ-ideal on a Polish space X such that the poset P_I is proper, and I has the overspill property. Then*

1. *the poset P_I is bounding;*
2. *the poset P_I forces the set of ground model reals to be nonmeager.*

Proof. For the bounding property, [13, Theorem 3.3.2] shows that it is enough to verify that every closed subset of $X \times \omega^\omega$ with I-positive projection has a compact subset with I-positive projection. This follows immediately from the overspill property applied to the collection $A = K(C)$.

For (2), assume for contradiction that it fails. Then, there must be an I-positive Borel set $B \subset X$ and a name $\dot{K}$ for a compact nowhere dense subset of 2^ω such that $B \Vdash 2^\omega \cap V$ is a subset of the union of all rational translates of $\dot{K}$ in the Cantor group. Use the bounding property to thin out B if necessary to make sure that B is compact and there is a continuous function $f : B \to K(2^\omega)$ such that $B \Vdash \dot{K} = \dot{f}(\dot{x}_{gen})$. Let $Y = \operatorname{rng}(f)$; this is a compact subset of $K(2^\omega)$. Let $A \subset K(Y)$ be the collection of all compact subsets $D \subset Y$ such that there is $z \in 2^\omega$ such that for every $L \in D$, no rational translate of z belongs to L. This is an analytic collection of compact sets, and it contains all countable compact sets. By Theorem 3.2, there is $z \in 2^\omega$ and an I-positive compact set $C \subset B$ such that for all $x \in C$, z belongs to no rational translate of $f(x)$. Then, $C \Vdash \check{x}$ belongs to no rational translate of $\dot{K}$, contradicting the choice of the name $\dot{K}$. $\square$

With a reformulation of overspill as a forcing preservation property, a question immediately arises whether it persists under the usual forcing operations. The game characterization of overspill leads to preservation theorems for the countable support product of definable forcings. First, recall the basic definitions and results about the countable support product. Let $\{I_n : n \in \omega\}$ be σ-ideals on respective Polish spaces $\{X_n : n \in \omega\}$. The *box product* $\prod_n I_n$ is the collection of all analytic sets $A \subset \prod_n X_n$ which do not contain a subset of the form $\prod_n A_n$, where each $A_n \subset X_n$ is an I_n-positive Borel set. It is not clear why this collection should be a σ-ideal; however, in a good number of cases it is, and its quotient corresponds to the countable support product $\prod_n P_{I_n}$.

Fact 3.4. [13, Theorem 5.2.6] *Let $\{I_n : n \in \omega\}$ be σ-ideals on respective Polish spaces $\{X_n : n \in \omega\}$, such that for every number n,*

1. *every I_n-positive analytic set has a Borel I_n-positive subset;*
2. *the product poset P_{I_n} is proper, bounding, and forces the set of the ground model reals to be non-meager;*
3. *I_n is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Then the box product $J = \prod_n I_n$ is a σ-ideal and the σ-ideal J shares the properties (1,2,3).

Theorem 3.5. *Let $\{I_n : n \in \omega\}$ be σ-ideals on respective Polish spaces $\{X_n : n \in \omega\}$, such that for every number n,*

1. *the product poset P_{I_n} is proper;*
2. *I_n is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;*

3. I_n *has the overspill property.*

Then the box product $J = \prod_n I_n$ *is a* σ*-ideal and the* σ*-ideal* J *shares the properties (1,2,3).*

Proof. The collection J is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal and the quotient P_J is proper and bounding by Fact 3.4. Note that the overspill property of each I_n implies that every I_n-positive analytic set has an I_n-positive Borel subset and the quotient P_{I_n} is bounding and preserves Baire category by Corollary 3.3.

To prove (3), suppose that $C \subset \prod_n X_n \times \omega^\omega$ is a closed set with a J-positive projection. I must produce a winning strategy for Player I in the game $G(K_{\aleph_0}, K_J, C)$. For every number $n \in \omega$ find an I_n-positive compact set $B_n \subset X_n$ such that $\prod_n B_n \subset p(C)$. Use the properness and the bounding property of the poset P_J to thin out the sets B_n if necessary and find a continuous function $f : \prod_n B_n \to C$ such that for every $x \in \prod_n B_n$, $f(x) = \langle x, y\rangle$ for some $y \in \omega^\omega$. For each $n \in \omega$ pick an arbitrary point $x_n \in B_n$. The winning strategy σ in the game $G(K_{\aleph_0}, K_J, C)$ for Player I is described as follows. As the play $\langle K_i, O_i : i \in \omega\rangle$ of the game proceeds, at round i Player I will on the side create countable compact sets $L^i_n \subset X_i$ and open sets $P^i_n \subset X_i$ for all $n \in \omega$ and find a strategy σ_i so that for each i,

- $L^i_n = \{x_n\}$ for all $i \in n$ and $K_i = f'' \prod_n L^i_n$;
- $L^i_n \subset P^i_n$ and all but finitely many sets P^i_n are equal to X_n. Moreover, $\prod_n P^i_n \subset f^{-1}O_i$;
- σ_i is a winning strategy for Player I in the game $G(K_{\aleph_0}, K_{I_n}, B_n \cap \bigcap_{j\in i} \overline{P}^j_i)$,
- for each $j \leq i$ the sequence $\langle L^k_j, P^k_j : j \leq k \leq i\rangle$ is a play against the strategy σ_j.

The construction is straightforward, using an elementary claim at each stage:

Claim 3.6. *Whenever* $\langle Y_i : i \in j\rangle$ *are topological spaces,* $\langle L_i : i \in j\rangle$ *compact subsets of each, and* $O \subset \prod_i Y_i$ *is an open set covering* $\prod_i L_i$*, then there are open sets* $\langle O_i : i \in j\rangle$ *of the respective spaces such that* $K_i \subset O_i$ *and* $\prod_i O_i \subset O$.

Proof. It is enough to treat the case $j = 2$. First, argue that for every point $y \in L_0$, there are open sets $O_{0x} \subset Y_0$ and $O_{1x} \subset Y_1$ such that $x \in O_{0x}$, $K_1 \subset O_{1x}$, and $O_{0x} \times O_{1x} \subset O$. To see this, note that the set $\{x\} \times K_1$ is compact, and therefore it is covered by finitely many open rectangles included in O; let $\{P_k \times Q_k : k \in l\}$ be such a finite list, with the demand that $x \in P_k$ for each $k \in l$. Let $O_{0x} = \bigcap_k P_k$ and $O_{1x} = \bigcup_k Q_k$; these two open sets work.

Now, by compactness of the set K_0, there is a finite list $\{x_k : k \in l\}$ of points in K_0 such that $K \subset \bigcup_{k\in l} O_{0x_k}$. Let $O_0 = \bigcup_{k\in l} O_{0x_k}$ and $O_1 = \bigcap_{k\in l} O_{1x_k}$. The sets O_0, O_1 work as requested in the statement of the claim. $\square$

At the end of the play, consider the results $C_n \subset B_n$ of the plays against the strategies σ_n constructed on the side; so $C_n \subset B_n$ is an I-positive compact set. It is not difficult to see that $f'' \prod_n C_n \subset C$ will be a subset of the result of the main play, and therefore Player I won. The proof of the theorem is complete.

$\square$

Corollary 3.7. *Let κ be a cardinal and $\{I_\alpha : \alpha \in \kappa\}$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces with the overspill property such that the quotient P_{I_α} is proper for every $\alpha \in \kappa$. Let P be the countable support product of the posets $\{P_{I_\alpha} : \alpha \in \kappa\}$. Then for every Polish space Y and every analytic set $A \subset K(Y)$ containing all countable compact sets, P forces Y to be covered by the ground model elements of the set A.*

Proof. Suppose for Y is a Polish space and $A \subset K(Y)$ is an analytic set containing all countable compact sets. Let $\dot{y}$ be a P-name for an element of Y and $p \in P$ be a condition; I must find a set $C \in A$ and a condition $q \leq p$ which forces $\dot{y} \in \dot{C}$. Let M be a countable elementary submodel of a large enough structure containing $p, \dot{x}$. A standard argument shows that there are I_α-positive compact sets $\{K_\alpha : \alpha \in \kappa \cap M\}$ such that the product $L = \Pi_{\alpha \in \kappa \cap M} K_\alpha$ consists of M-generic sequences only for the product forcing meeting the condition p, and the function $g : L \to Y$ given by $g(\vec{x}) = \dot{y}/\vec{x}$ is continuous. Let $A' \subset K(L)$ be the collection of all compact subsets of L whose images are covered by sets in A; this is an analytic collection of sets containing all countable sets. By the overspill property of the product ideal $\Pi_{\alpha \in \kappa \cap M} I_\alpha$, there are compact sets $\{K'_\alpha : \alpha \in \kappa \cap M\}$ such that $K'_\alpha \subset K_\alpha$ and $L' = \Pi_{\alpha \in \kappa \cap M} K'_\alpha \in A'$. The g-image of L' is then covered by some set $C \in A$, and a review of the definitions shows that L' is a condition below p that forces $\dot{y} \in \dot{C}$ as desired. $\square$

Finally, I prove that the overspill property is preserved under countable union of σ-ideals generated by countable sets.

Theorem 3.8. *Let $\{I_n : n \in \omega\}$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals generated by closed sets, each with the overspill property, on some fixed Polish space X. The σ-ideal σ-generated by $\bigcup_n I_n$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and it has the overspill property again.*

Proof. The family $A \subset F(X)$ given by $C \in A$ if and only if there is $n \in \omega$ such that $C \in I_n$ is hereditary and coanalytic by the definability assumptions on the σ-ideals I_n for $n \in \omega$. It σ-generates the σ-ideal I which then is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [6, Theorem 35.38].

For the verification of the overspill property of the σ-ideal I, fix a closed set $C \subset X \times \omega^\omega$ whose projection is I-positive. To construct the winning strategy for Player I in the game $G(K_{\aleph_0}, K_I, C)$, first thin out C if necessary so that its intesections with basic open sets are either empty or else have I-positive projections. Let $\{\langle P_i, m_i\rangle : i \in \omega\}$ be an enumeration of pairs consisting of a basic open subset of C and a natural number, with infinite repetitions. The strategy for Player I is described in the following way. As the play $\langle K_i, O_i : i \in \omega\rangle$ proceeds, at round $i \in \omega$ Player I on the side constructs countable compact sets $\langle L^j_i : j \leq i\rangle$ as well as a strategy σ_i so that

- if $\bigcap_{j\in i}\overline{O}_j\cap P_i$ is nonempty then σ_i is a winning strategy for Player I in the game $G(K_{\aleph_0}, K_{I_{m_i}}, C\cap\bigcap_{j\in i}\overline{O}_j\cap\overline{P}_i)$;
- for every $j\in\omega$, the sequence $p_j=\langle L^j_i, O_i : i\geq j\rangle$ is a play of the game according to the strategy σ_j;
- $K_i=\bigcup_{j\leq i}L_j$.

This is easy to arrange from the assumption that each ideal I_n for $n\in\omega$ has the overspill property. Let D be a (compact) outcome of the play; we must show that $p(D)$ is an I-positive set. Since the σ-ideals I_n for $n\in\omega$ are σ-generated by closed sets, it will be enough to show that whenever O is an open set with nonempty intersection with $p(D)$ and $n\in\omega$ then $p(D)\cap O\notin I_n$–for then, the closed sets in the generating σ-ideals must be relatively meager in the set $p(D)$. Let P be an open subset of C such that $p(\overline{P})\subset O$, and let $i\in\omega$ be an index such that $\langle P_i, m_i\rangle=\langle P, m\rangle$. The outcome E of the play p_i is a compact subset of $\overline{P}\cap D$ with I_n-positive projection. Since the play p_i was played according to the winning strategy σ_i, it must be the case that $p(E)\notin I_n$. Now $p(E)\subset p(D)\cap O\notin I_n$ as required. This completes the proof.

□

The attentive reader should not fail to notice how overspill fits into the doctrine of [13, Section 3.10]. Many forcing properties can be restated as the bad player not having a winning strategy in a certain game. If the forcing in question is suitably definable, then the game in question is determined, and the winning strategies for the good player can serve as a tool for proving preservation theorems for product or iteration.

4 Examples

In order to efficiently use the ideas developed in the previous section for independence results, one must develop a dictionary of σ-ideals with and without the overspill property. In this section, I will provide a number of examples. They are typically associated with forcings adding a certain highly independent real.

One large class of examples is best motivated by the following consideration.

Definition 4.1. Let I be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X with the overspill property. The *overspill ordinal* is the smallest ordinal α such that in all separation games of Definition 3.1 Player I has a winning strategy using only countable compact sets of Cantor–Bendixson rank $<\alpha$.

The overspill ordinal is a finer invariant that can discern between the various σ-ideals with the overspill property. It will turn out that the σ-ideal H_σ σ-generated by H-sets has overspill ordinal 3. The countable union operation does not increase the overspill ordinal. The countable support product may increase the overspill ordinal by adding ω to it, as products of n-many sets of rank α

may have rank $\alpha + n$. Here I show that there are quite natural σ-ideals with arbitrarily large prescribed countable overspill ordinal.

Let $\alpha \in \omega_1$ be a nonzero countable ordinal and let $f : \omega \to \alpha$ be any surjection with infinite fibers. Call a set $a \subset \omega$ *small* if $f_\alpha \restriction a$ is a decreasing function; in particular, such sets must be finite. Let I_α be the σ-ideal on 2^ω σ-generated by all sets $K \subset 2^\omega$ such that for every $n \in \omega$ there is a small set $a \subset \omega$ such that $n < \min(a)$ and a function $u \in 2^a$ such that every point $x \in K$ has nonempty intersection with u. The notation neglects the possibly nontrivial dependence of the σ-ideal I_α on the choice of the surjection f.

Theorem 4.2. *Let $\alpha \in \omega_1$ be a countable ordinal. The σ-ideal I_α*

1. *is σ-generated by closed sets and is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$;*
2. *has the overspill property, with the overspill ordinal $\leq \alpha + 2$;*
3. *if $\omega \cdot \alpha = \alpha$ then the overspill ordinal of I_α is exactly $\alpha + 2$.*

The case $\alpha = 1$ corresponds to a poset introduced by Shelah [11, Proposition 1.10] and studied also by Spinas [12] in a combinatorial form.

Proof. For (1), it is clear that a closure of a generating set of I_α is again a generating set. Moreover, the collection of closed generating sets is hereditary and G_δ, and therefore the σ-ideal I_α is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [6, Theorem 35.38].

The argument for (2) begins with an auxiliary claim.

Claim 4.3. *Suppose that $C \subset 2^\omega \times \omega^\omega$ is a closed set such that its intersections with clopen sets are either empty or have I_α-positive intersection. There is a countable compact set $K \subset C$ of rank $\leq \alpha + 1$ such that for every small set $a \subset \omega$ and every function $u \in 2^a$, if there is $x \in p(C)$ such that $x \cap u = 0$ then there is $x \in p(K)$ such that $x \cap u = 0$.*

Proof. By induction on $\beta \leq \alpha$ prove the following stronger sentence $\Theta(\beta)$: for every clopen set O with nonempty intersection with C, there is a countable compact set $K \subset C \cap O$ of rank $\leq \beta + 1$ such that for every small set $a \subset \omega$ with $f(\min(a)) < \beta$ and every function $u \in 2^a$, if there is $x \in p(C \cap O)$ such that $x \cap u = 0$ then there is $x \in p(K)$ such that $x \cap u = 0$.

The sentence $\Theta(0)$ is trivially satisfied. Suppose $\Theta(\gamma)$ holds for all $\gamma < \beta$, and verify $\Theta(\beta)$. For simplicity assume that $O = C$. Pick a point $\langle x, y \rangle \in C \cap [s, t]$ and by induction find numbers n_i so that

- $0 = n_0 < n_1 < \dots;$
- for every small set a with $\min(a) > n_{i+1}$ and every $u \in 2^a$ there is a point $\langle x', y' \rangle \in C \cap O \cap [x \restriction n_i, y \restriction n_i]$ such that $x' \cap u = 0$.

This is possible as the nonempty intersections of C with clopen sets are I_α-positive. For every sequence $p \in 2^{n_{i+2}}$ extending $x \restriction n_i$ and every number $m \in n_{i+2}$ such that $f(m) < \beta$ use the induction hypothesis to find a set $K_{p,m} \subset$

$C\cap[p,y\restriction n_i]$ of rank $f(m)$ such that for every small set $a\subset\omega$ with $\min(a))=m$ and every function $u\in 2^a$, if there is $\langle x',y'\rangle\in C\cap[p,y\restriction n_i]$ such that $x\cap u=0$ then there is such a point in $K_{p,m}$. Let K be the union of all sets thus obtained together with the point $\langle x,y\rangle$. This is a countable compact set of rank at most β. I claim that this set confirms $\Theta(\beta)$.

Indeed, if $a\subset\omega$ is a finite small set and $u\in 2^a$ is a set such that there is $x\in C$ with $x\cap u=0$, look at $m=\min(a)$ and find the number $i\in\omega$ such that $n_i\leq m<n_{i+1}$. For definiteness assume that $i>0$. Then there is a point $\langle x',y'\rangle\in C\restriction x\restriction n_i, y\restriction n_i$ such that $x'\cap u=0$. Let $p=x'\restriction n_{i+1}$, and observe that the choice of the set $K_{p,m}$ ensures that there is such a point $\langle x',y'\rangle\in K_{p,m}\subset K$ as required. □

Let $C\subset 2^\omega\times\omega^\omega$ be a closed set with an I-positive projection; I must describe a winning strategy for Player I in the game $G(K_{\aleph_0},K_{I_\alpha},C)$ that uses only sets of rank at most α. First, thin out the set C so that its nonempty intersections with basic open sets have I-positive projections. Let $\langle P_n:n\in\omega\rangle$ enumerate all basic open subsets of C, and let Player I proceed in the following way. As the play $\langle K_n,O_n:n\in\omega\rangle$ proceeds, he makes sure that $K_0\subset K_1\subset\dots$, and if the set P_n has nonempty intersection with $\bigcap_{m\in n}O_n$, then he uses the claim to provide a compact set K_n such that for every small set $a\subset\omega$ and every function $u\in 2^a$, if there is $x\in p(C\cap\bigcap_{m\in n}O_n\cap P_n)$ such that $x\cap u=0$ then there is such an x in $p(K_n)$.

To prove that this is a winning strategy for Player I, let $L\subset C$ be the result of the game and show that for every generating set $D\subset 2^\omega$ the set $\{x\in L:p(x)\in D\}$ is meager in L. Suppose that small sets a_n and functions $u_n\in 2^{a_n}$ with $n<\min(a_n)$ witness that D is a generating set of I. Let $P\subset C$ be an open set with nonempty intersection with L, say $P=P_n$ for some n. There must be a point $x\in p(C\cap\bigcap_{m\in n}O_m\cap P_m)$ such that $x\cap u_k=0$ for some $k\in\omega$, since the set $p(C\cap\bigcap_{m\in n}O_m\cap P_m)$ is I-positive by the choice of the set C. By the choice of the set K_n, there must be a point $\langle x,y\rangle\in K_n\cap\bigcap_{m\in n}O_m\cap P_m$, such that $x\cap u_k=0$, and this point will appear in all later sets K_m for $m>n$ and also in the result L. This point has a neighborhood given by $x\restriction\max(a_k)+1$ whose projection is disjoint with the set D. This means that the set $\{x\in L:p(x)\in D\}$ is meager in L as desired.

The proof of (3) is best packaged with the help of a certain forcing preservation property.

Claim 4.4. *Suppose that J is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space Y such that the quotient forcing P_J is proper. Suppose that $\alpha\in\omega_1$ is a countable ordinal such that $\alpha=\omega\cdot\alpha$. Suppose that J has the overspill property, with the overspill ordinal $\leq\alpha$. Then P_J forces that 2^ω is covered by the ground model coded sets in the σ-ideal I_α.*

Proof. First argue that for every $\beta\in\alpha$ and every countable compact set $K\subset 2^\omega$ of Cantor–Bendixson rank $<\beta$ and every $n\in\omega$ there is a small set $a\subset\omega$ such that $n<\min(a)$ and $f(\min(a))<\omega\cdot\beta$, and a function $u\in 2^a$ such that every point $x\in K$ has nonempty intersection with K. The proof proceeds by induction on β. The case $\beta = 0$ is trivial. Now suppose that $K \in K(2^\omega)$ is a countable compact set of rank $< \beta$ and for ordinals below β the statement has been verified. Let $\{x_i : i \in k\}$ be the finite list of points of K of the largest rank, say of rank $\gamma \in \beta$. Find a small set $b \subset \omega$ of size k such that $f''b \subset [\omega\cdot\gamma, \omega\cdot(\gamma+1))$ and a function $v \in 2^b$ such that for every $i \in k$, the point x_i has nonempty intersection with v. Now, consider the countable compact set $L = \{x \in K : x \cap v = 0\}$. This is a set of rank $< \gamma$ and so by the induction hypothesis there is a small set $c \subset \omega$ with $\max(b) < \min(c)$, $f(\min(c)) < \beta \cdot \gamma$, and a function $w \in 2^c$ such that every point $x \in L$ has nonempty intersection with w. The set $a = b \cup c$ and the function $u = v \cup w$ work as desired for the set K.

Now, look at the poset P_J, and suppose that $B \in P_J$ is a condition and $\dot{y}$ is a P_J-name for an element of 2^ω. Use the bounding assumption to find an I-positive compact set $C \subset B$ and a continuous function $h : C \to 2^\omega$ such that $C \Vdash \dot{y} = h(\dot{x}_{gen})$. Find a winning strategy σ for Player I in the game $G(K_{\aleph_0}, K_J, C)$ which uses only sets of rank $< \alpha$. Let Player II construct a counterplay against the strategy σ as follows: whenever Player I indicates a compact set K_n, the image $h''K_n \subset 2^\omega$ is a countable compact set of rank $< \alpha$ and so by the previous paragraph there is a small set $a_n \subset \omega$ with $n < \min(a_n)$ and a function $u_n \in 2^{a_n}$ such that every point of $h''K_n$ has nonempty intersection with u_n. Let Player II answer with the set $O = \{x \in C : h(x) \cap u \neq 0\}$; this is a clopen neighborhood of the set K_n. Let $D \subset C$ be the result of the play thus constructed. Then for every point $x \in D$ and every $n \in \omega$, $h(x) \cap u_n = 0$, and so $D \Vdash \forall n \in \omega\ \dot{y} \cap \check{u}_n \neq 0$. The claim follows. $\square$

Since the quotient forcing of the σ-ideal I_α forces its generic point not to belong to any I_α-small set, its overspill ordinal must be at least $\alpha + 2$ by the previous claim and (3) follows. $\square$

Theorem 4.2 yields an interesting corollary about the complexity of the bounding condition on definable proper partial orderings.

Theorem 4.5. *There is a Polish space Y and a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I on $Y \times 2^\omega$ σ-generated by closed sets, such that*

1. *for every $y \in Y$ the vertical section $B_y = \{\langle y, x\rangle : x \in 2^\omega\}$ is I-positive;*
2. *the sets $\{y \in Y : P_I \restriction B_y$ is bounding$\}$, $\{y \in Y : P_I \restriction B_y$ adds no Cohen real$\}$, $\{y \in Y : I \restriction B_y$ has the overspill property$\}$ are all equal and they are complete coanalytic sets.*

In other words, checking whether a given quotient forcing is bounding (or has overspill, or adds no Cohen reals) is at least a complete coanalytic job, even if it is known that the σ-ideal is σ-generated by closed sets.

Proof. Let Y be the collection of all linear orders on ω topologized as a closed subset of $\mathcal{P}(\omega \times \omega)$. Fix a surjection $\pi : \omega \to \omega$ with infinite fibers. Let I be the σ-ideal σ-generated by sets $K \subset Y \times 2^\omega$ such that K is a subset of single vertical section B_y of $Y \times 2^\omega$ and for every $n \in \omega$ there is a finite set $a \subset \omega$ such that $\min(a) > n$ and for $k, m \in a$, $k < m$ implies $\pi(k) <_y \pi(l)$, and a function $u \in 2^a$ such that for every $x \in K$, $x \cap u \neq 0$. I claim that this σ-ideal works as required.

It is not difficult to see that a closure of a generating set is a generating set again, so the σ-ideal I is generated by closed sets. It is also clear that the collection of generating compact sets is hereditary and G_δ, and so the σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [6, Theorem 35.38]. Now let $C \subset Y$ be the set of all wellorders. This is a complete coanalytic set. Theorem 4.2 shows that if $y \in C$ then $I \restriction B_y$ has overspill. The proof of the theorem will be complete if I show that whenever $y \notin C$ then $P_I \restriction B_y$ adds a Cohen real.

Thus, assume that $y \in Y$ is not a wellorder and let $a \subset \omega$ be an infinite set such that for $k, l \in a$, $l < k$ implies $\pi(l) <_y \pi(k)$. I will show that $P_I \restriction B_y$ forces $\dot{x}_{gen} \restriction a$ to be a Cohen real where $\dot{x}_{gen} \in 2^\omega$ is name for the 2^ω-coordinate of the P_I-generic pair. Indeed, if $O \subset 2^a$ is an open dense set, then for every $n \in \omega$ there is a finite partial function $u : a \to 2$ with $\min(\mathrm{dom}(u_n)) > n$ such that for every $z \in 2^a$, $z \cap u_n = 0$ implies $z \in O$. A brief review of the definition of generating sets for the σ-ideal I shows that $B_y \Vdash \exists n\ \dot{x}_{gen} \cap u_n = 0$ and so $\dot{x}_{gen} \restriction a \in \dot{O}$ as required. $\square$

I will now produce a family of σ-ideals generated by closed sets generated in a similar way to the σ-ideals I_α of Theorem 4.2, with different properties. The important representatives of this class fail to have the overspill property, while the quotient forcings are bounding.

Let ϕ be a lower semicontinuous submeasure on ω which assigns finite nonzero masses to finite nonempty sets. If $x \in 2^\omega$, $a \subset \omega$ is a finite set, $u \in 2^a$ and $\varepsilon > 0$ is a real number, say that x *ε-coheres* with u if $\phi(\{m \in a : x(m) = u(m)\}) > (1-\varepsilon)\phi(a)$. Let I_ϕ be the σ-ideal σ-generated by sets $K \subset 2^\omega$ such that there is $\varepsilon > 0$ such that for every $n \in \omega$ there is a finite set $a \subset \omega$ with $n < \min(a)$, a function $u \in 2^a$, and no point $x \in K$ ε-coheres with u.

Theorem 4.6. *Let ϕ be a lower semicontinuous submeasure on ω assigning finite nonzero masses to nonempty finite sets.*

1. *I_ϕ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal σ-generated by closed sets.*
2. *If ϕ is strongly subadditive then the poset P_{I_ϕ} is bounding.*
3. *If ϕ is the measure with $\phi(\{n\}) = \frac{1}{n+1}$ then I_ϕ fails to have the overspill property: there is an analytic set $A \subset K(2^\omega) \cap I_\phi$ containing all countable compact sets.*
4. *if ϕ is the chromatic number of the random graph, then P_{I_ϕ} adds a Cohen real.*

Here, the submeasure ϕ is *strongly subadditive* if $\phi(a \cup b) \leq \phi(a) + \phi(b) - \phi(a \cap b)$ for all sets $a, b \in \mathrm{dom}(\phi)$. Good examples of strongly subadditive submeasures are measures or suprema of measures. If G is a graph on ω and ϕ

assigns to each set $b \subset \omega$ the chromatic number of $G \restriction b$, then ϕ will typically fail to be strongly subadditive.

Proof. I will start with the proof of (1). A closure of a generating set is again a generating set, so the σ-ideal is σ-generated by closed sets. The collection of compact generating sets is hereditary and $G_{\delta\sigma}$, therefore the σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [6, Theorem 35.38].

The argument for (2) starts with a definition. For every set $B \subset 2^\omega$ and every $\varepsilon > 0$, write $n(B, \varepsilon)$ for the smallest number n such that for every finite set $a \subset \omega$ with $n < \min(a)$ and every $u \in 2^a$ there is $x \in B$ which ε-coheres with u. Note that if $B \notin I_\phi$ then the number $n(B, \varepsilon)$ must exist. The proof of (2) hinges on an auxiliary claim.

Claim 4.7. *Whenever $\{B_k : k \in \omega\}$ are I_ϕ-positive sets whose intersections with open sets are I_ϕ-positive or empty, and $\varepsilon, \delta > 0$ are positive real numbers, then there is $l \in \omega$ such that $n(\bigcup_{k\in l} B_k, \varepsilon + \delta) \leq n(\bigcup_{k\in\omega} B_k, \varepsilon)$.*

Proof. Suppose that this fails, and for each $l \in \omega$ find finite sets $a_l \subset \omega$ and functions $u_l \in 2^{a_l}$ witnessing the failure of the inequality. There are three cases.

Case 1. The numbers $\phi(a_l)$ are unbounded. Find $\gamma > 0$ such that $(1 - \varepsilon)(1 - \gamma) > (1 - \varepsilon - \delta)$. Consider the number $m = n(B_0, \gamma)$. Find a number $l \in \omega$ such that $\varepsilon\phi(a_l) > \phi(a_l \cap m)$, so $\phi(a_l \setminus m) > (1 - \varepsilon)\phi(a_l)$. Find a point $x \in B_0$ such that x γ-coheres with $u_l \setminus m$. Then x $\varepsilon + \delta$-coheres with u_l, and this contradicts the choice of the function u_l.

Case 2. The numbers $\phi(a_l)$ are not bounded away from zero. Consider the number $m = n(B_0, \varepsilon)$ and find $l \in \omega$ large enough so that $\min(a_l) > m$. Find $x \in B_0$ which ε-coheres with u_l. This point x contradicts the choice of the function u_l.

Case 3. Both of the previous cases fail. Passing to a subsequence if necessary, we may assume that the sets a_l converge to some $a \subset \omega$, the functions u_l converge to some $u \in 2^a$, and there are positive real numbers $0 < r < s$ such that for every $l \in \omega$, $r < \phi(a_l) < s$. Note that $n < \min(a)$. Let $b \subset a$ be a finite set such that $\phi(a) - \phi(b) < \delta r/2$. Let $x \in \bigcup_{n\in\omega} B_n$ be a point that ε-coheres with $u \restriction b$. Let $k \in \omega$ be such that $x \in B_k$. Consider the nonempty, and therefore I_ϕ-positive, set $C = B_k \cap [x \restriction \max(b) + 1]$. Let $\gamma > 0$ be a real such that $\gamma s < \delta r$, and consider the number $n(C, \gamma)$. Find a number $l > n$ such that $n(C, \gamma) < \min(a_l \setminus a)$ and find a point $y \in C$ which γ-coheres with $u_l \restriction a_l \setminus n(C, \gamma)$. The strong subadditivity of ϕ now shows that y $\varepsilon + \delta$-coheres with u_l, contradicting the choice of the function u_l.

A contradiction results in all cases, completing the proof of the claim. $\square$

Since the σ-ideal I is σ-generated by compact sets, to verify the bounding property it is sufficient to show that every I-positive analytic set contains an I-positive compact subset. Let $A \subset 2^\omega$ be an I-positive analytic set, say $A = \mathrm{rng}(f)$ for some continuous function $f : \omega^\omega \to 2^\omega$. Thinning out the set A and manipulating the function f if necessary, we may assume that f-images of nonempty open sets are I-positive. For a (finite or infinite) function $h \in \omega^{\leq\omega}$

write $D_h = \{y \in \omega^\omega : \forall m \in \mathrm{dom}(h)\ y(m) \leq h(m)\}$. By induction on $i \in \omega$ build a function $h \in \omega^\omega$ such that for every $i \in \omega$, every $j \leq i$ and every $t \in \omega^j$, $n(f''D_{h\restriction i} \cap [t], 2^{-j+1} - 2^{-i}) \leq n(f''D_{h\restriction i+1} \cap [t], 2^{-j+1} - 2^{-i-1})$. This is easily possible by a repetitive application of the previous claim. It will be enough to show that the compact set $f''D_h \subset A$ is I_ϕ-positive.

To this end, choose positive reals $\varepsilon_i > 0$ and finite partial functions $u_i^j : \omega \to 2$ such that $j < \min(\mathrm{dom}(u_i^j))$; I must produce a point $y \in D_h$ such that for every $i \in \omega$, the point $f(y)$ ε_i-coheres with some u_i^j. To do this, by induction on $i \in \omega$ build sequences $t_i \in \omega^{<\omega}$ so that

- $0 = t_0 \subset t_1 \subset \dots$ and the sequences remain below h;
- for every $z \in [t_{i+1}]$, the point $f(z)$ ε_i-coheres with some function u_i^j.

Once the induction is performed, setting $y = \bigcup_i t_i$ will complete the proof. Suppose t_i has been found. Find $k > |t_i|$ such that $2^{-k+1} < \varepsilon_i$ and find a sequence $s \in \omega^k$ extending t_i below the function h. Consider the number $j = n(f''[t], 2^{-j})$ and the function u_i^j. By the construction of the function h, for every m there must be a point $y_m \in D_{h\restriction m} \cap [s]$ such that $f(y)$ ε_i-coheres with u_i^j. Some subsequence of the points $\langle y_m : m \in \omega\rangle$ converges to some $z \in D_h$, and then $f(z)$ also has to ε_i-cohere with u_i^j. Choose an initial segment $t_{i+1} \subset z$ longer than t_i such that all points in $f''[t_{i+1}]$ ε_i-cohere with u_i^j. This completes the induction step and the proof of (2).

For (3), let $A \subset K(2^\omega)$ be the collection of generating compact subsets of I_ϕ. This is a $G_{\delta\sigma}$ collection of compact sets, and it is clearly a subset of I_ϕ. I must show that A contains all countable compact sets. This will be accomplished by proving a stronger statement that has the virtue of surviving the induction on the Cantor-Bendixson rank. A provisional definition: if $K \subset 2^\omega$ is a set, $\varepsilon, \delta > 0$ are positive real numbers and $a \subset \omega$ is a set, say that a is K, ε, δ-*good* if for every set $c \subset a$ of ϕ-mass $> \delta$, no point $x \in K$ $\frac{1}{2} + \varepsilon$-coheres with any of the two *flipping functions* on the set c. By the flipping functions I mean the two functions on the set c whose values constantly oscillate between 0 and 1.

Claim 4.8. *For every countable compact set $K \subset 2^\omega$, all positive numbers $\varepsilon, \delta > 0$ and every set $a \subset \omega$ with $\phi(a) = \infty$ there is $b \subset a$ such that $\phi(b) = \infty$ and b is K, ε, δ-good.*

Proof. By induction on the Cantor–Bendixson rank of K. For the basis of induction, if the rank is 1 (and so K is finite), then there is a set $b \subset a$ such that $\phi(b) = \infty$ on which all elements of K are constant and $\frac{1}{\min(b)+1} < \varepsilon\delta$. Then, the set b is K, ε, δ-good: for every set $c \subset b$ with $\phi(c) > \delta$, every flipping function u on c and every $x \in K$, since $x \restriction c$ is constant and the ϕ-masses of singletons decrease, it must be the case that $\phi(\{m \in c : x(m) = u(m)\}) < \frac{1}{2}\phi(c) + \phi(\{\min(c)\}) < \frac{1}{2}\phi(c) + \varepsilon\delta < (\frac{1}{2} + \varepsilon)\phi(c)$ as required.

For the induction step, suppose that $K \subset 2^\omega$ is a countable compact set and the statement of the claim has been verified for all countable compact sets of smaller rank. It is enough to check the case in which K has a single point (call it y) of the highest rank–if there are finitely many such points then decompose K into finitely many compact pieces, each with a single point of the highest rank and perform the thinning of the set a repeatedly for each of these pieces. First, thin out the set a if necessary, we may assume that $y \restriction a$ is constant. By induction on $i \in \omega$ build sets $a_i, b_i \subset \omega$ so that

I1 $a \supset a_0 \supset a_1 \supset \dots$ are all sets with infinite ϕ-mass;

I2 b_i are finite sets with $\frac{1}{4}\varepsilon\delta < \phi(b_i) < \frac{1}{3}\varepsilon\delta$, $\frac{1}{\min(b_0)} < \frac{1}{3}\varepsilon\delta$, $\max(b_i) < \min(b_{i+1})$, and $b_i \subset a_i$;

I3 the set a_i is $K \cap [y \restriction i^\frown(1 - y(i))], \frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon\delta$-good.

This is easy to do, using the fact that the statement of the claim has been verified for all sets $K \cap [y \restriction i^\frown(1 - y(i))]$, which have rank smaller than K. In the end, let $b = \bigcup_i b_i$; I claim that this is the required K, ε, δ-good set.

It is clear that $\phi(b) = \infty$ by the demand I2. Suppose that $c \subset b$ is a finite set with $\phi(c) > \delta$, u is a flipping function on c and $x \in c$. If $x = y$ then certainly x does not $\frac{1}{2} + \varepsilon$-cohere with u as $x \restriction c$ is constant, just as in the first paragraph of this proof. If $x \neq y$ then find the least $i \in \omega$ such that $x(i) \neq y(i)$ and consider the sets $c_0 = c \cap \bigcup_{j \in i} b_j$, $c_1 = c \cap b_i$, and $c_2 = c \setminus \bigcup_{j \leq i} b_j$. Now, $x \restriction c_0$ is constant and so $\phi(\{m \in c_0 : x(m) = u(m)\}) < \frac{1}{2}\phi(c_0) + \frac{1}{3}\varepsilon\delta$. Also, $\phi(c_1) < \frac{1}{3}\varepsilon\delta$. For the set c_2, there are two cases: either $\phi(c_2) < \frac{1}{3}\varepsilon\delta$, or else $\phi(\{m \in c_2 : x(m) = u(m)\}) < (\frac{1}{2} + \frac{\varepsilon}{3})\phi(c_2)$ by the demand I3 since $c_2 \subset a_{i+1}$. In both cases, we conclude that $\phi(\{m \in c : x(m) = u(m)\}) < (\frac{1}{2} + \varepsilon)\phi(c)$ as desired. $\square$

Now suppose that K is a countable compact set, and use the claim to find a set $a \subset \omega$ such that $\phi(a) = \infty$ and a is $K, \frac{1}{4}, 1$-good. Thus, for every $n \in \omega$ there is a finite set $a_n \subset a$ with $n < \min(a_n)$ and a function $u \in 2^a$ (namely, a flipping function on a_n) such that no point of K 3/4-coheres with u_n. We have just verified that $K \in A$.

For (4), let G be a random graph on ω, let ϕ be the submeasure on ω which assigns the chromatic number of $G \restriction b$ to every set $b \subset \omega$, and let $a \subset \omega$ be an infinite G-independent set. I will show that $P_{I_\phi} \Vdash \dot{x}_{gen} \restriction a$ is a Cohen real. To this end, suppose that $O \subset 2^a$ is a dense open set; it will be enough to prove that the set $\{x \in 2^\omega : x \restriction a \notin O\}$ belongs to I_ϕ.

For every number $n \in \omega$ find a finite set $a_n \subset a$ with $n < \min(a_n)$ and a function $u_n \in 2^{a_n}$ such that $[u_n] \subset O$ in 2^a. This is possible as the set O is open dense. Now, for every number $n \in \omega$ find a finite set $b_n \subset \omega$ such that $n < \min(b_n)$, $a_n \subset b_n$, and $G \restriction b$ is a cycle of odd length. Thus, $\phi(b_n) = 3$ while the ϕ-mass of any proper subset of b_n is at most 2. Let $v_n \in 2^{b_n}$ be any function such that $v_n \restriction a_n = u_n$. The number $\varepsilon = 1/3$ and functions v_n for $n \in \omega$ now witness that $\{x \in 2^\omega : x \restriction a \notin O\} \in I_\phi$ as desired.

$\square$

5 The weak Sacks property

The simplest case of the overspill property arises when the overspill ordinal is equal to 2; in other words, Player I has winning strategies that use finite sets only. In a remarkable turn of events, this feature can be immediately translated into a preexisting forcing property.

Definition 5.1. A forcing P is said to have *the weak Sacks property* if for every function $f \in \omega^\omega$ in the P-extension there is a ground model infinite set $a \subset \omega$ and a ground model function g with domain a such that for every $n \in a$, $|g(n)| \leq 2^n$ and $f(n) \in g(n)$.

The weak Sacks property is an obvious weakening of Sacks property which requires $a = \omega$ [1, Definition 6.3.37]. It clearly implies the bounding property, and in a suitably definable case, its conjunction with adding no independent reals is in fact equivalent to the conjunction of the bounding property and P-point preservation [14]. The main point here is

Theorem 5.2. *Let I be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X such that the poset P_I is proper and every I-positive analytic set contains an I-positive Borel subset. Then the following are equivalent:*

1. *P_I has the weak Sacks property;*
2. *I has the overspill property with the overspill ordinal equal to* 2.

Proof. (2) immediately implies (1). Suppose that $B \in P_I$ is a condition and $\dot{y}$ a name for a point in the Baire space ω^ω. Since the forcing P_I is bounding, there is a compact I-positive set $C \subset B$ and a continuous function $f : C \to \omega^\omega$ such that $C \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ and Player I has a winning strategy σ in the game $G(K_{\aleph_0}, I, C)$ that uses only finite sets as moves. Now consider the counterplay against the strategy σ in which Player II at round n finds a number $m = m_n$ such that $2^m > |K_n|$ and plays an open set O_n covering K_n on which the continuous function $x \mapsto f(x)(m)$ takes fewer than 2^m many values, collected in some set $g(m)$ of size $< 2^m$. In the end, the result of the play is an I-positive compact set $D \subset C$, and, writing $a = \{m_n : n \in \omega\}$, it forces $\forall n \in a\ \dot{y}(n) \in \check{g}(n)$ as desired.

The other direction is more difficult. Suppose that (1) holds and (2) fails in some closed set $C \subset X \times \omega^\omega$ with I-positive projection. Use the bounding property to find a compact I-positive set B and a continuous function $h : B \to C$ such that every $x \in B$ is the first coordinate of $h(x)$. Use the bounding property to thin out B further if necessary so that all open sets from the countable basis of the space X are relatively clopen in B. Note that (2) fails in the set B as well, since any winning strategy for Player I in $G(K_{\aleph_0}, I, B)$ is transported by h to a winning strategy in the game $G(K_{\aleph_0}, K_I, C)$ without increasing the rank of the sets used. Thus, it must be the case that Player II has a winning strategy σ in the game G similar to $G(K_{\aleph_0}, I, B)$ except Player I is allowed to play finite sets only in the game G. Now, by induction on $n \in \omega$ build increasing finite sets e_n of finite plays of the game G in which Player II follows the strategy σ and, whenever $t \in e_n$ is a play with the last move the strategy σ made in it a certain open set O, whenever $K \subset O \cap B$ is a set of size 2^n then there is a one round extension s of t in the set e_{n+1} such that the last move of strategy σ in s contains K as a subset.

In order to see how to make the induction step, choose $t \in e_n$ and note that the set $[O \cap B]^{2^n}$ is compact, and the set $U = \{P^{2^n}$:there is a move $K \in [O \cap B]^{2^n}$ of Player I that provokes the strategy σ to play $P\}$ covers it, since every set of size 2^n will provoke σ's answer that covers it. A compactness argument will yield a finite subcover of U, which will lead immediately to the construction of the finite set e_{n+1} on the next stage of induction.

Once the induction is complete, consider the function f defined on the set B so that $f(x)(n) =$ the intersection of the collection of those open sets used as last moves of plays in the set e_n to which x belongs. I claim that the name $\dot{f}(\dot{x}_{gen})$ violates the weak Sacks property: there is no condition $B' \subset B$, with an infinite set $a \subset \omega$ and a function g on a such that for every $n \in a$, $|g(n)| < 2^n$ and $B' \Vdash \dot{f}(\dot{x}_{gen})(n) \in \check{g}(n)$. Suppose for contradiction that such C, a, g exist and thin out B' so that for every $x \in B'$ and every $n \in a$, $f(x)(n) \in g(n)$. Let $\{n_i : i \in \omega\}$ enumerate the set a in an increasing order and by induction on i build plays $t_i \in e_{n_i}$ so that $t_0 \subset t_1 \subset \dots$, t_{i+1} is a one move extension of t_i, and its last move still contains C as a subset. If this succeeds, then in the end the result of the play $\bigcup_i t_i$ contains B' as a subset and Player I won, contradicting the assumption that σ was a winning strategy for Player I. The induction step is simple: given t_i, find a set $K \subset B'$ of size $2^{n_{i+1}}$ such that the values $f(x)(n_{i+1})$ for $x \in K$ exhaust all possibilities in C. Note that there are fewer than $2^{n_{i+1}}$ possibilities for this value at the set B' since they are controlled by the function g. By the construction of the set $e_{n_{i+1}}$, there must be a one round extension t_{i+1} of t_i such that the last move O on it contains K as a subset. But then, O also contains C as a subset: for every point $x \in B'$, there is $x' \in K$ such that $f(x)(n) = f(x')(n)$, and by the definition of the function f, $x \in f(x)(n) = f(x')(n) \subset O$ as desired! ☐

This theorem yields many examples of σ-ideals with the overspill property, since Sacks or weak Sacks property are fairly common in the realm of definable forcing. Thus, the σ-ideal σ-generated by Borel subsets of 2^ω consisting of pairwise non-modulo-finite-equal sequences has the overspill property, since the quotient forcing is proper and has the Sacks property [13, Section 4.7.1].

6 The σ-ideal generated by H-sets

In this section, I will apply the work of previous sections to obtain an independence result for two σ-ideals from harmonic analysis. Let $\mathbb{T}$ be the unit circle, understood as the group $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 6.1. A set $A \subset \mathbb{T}$ is a *set of uniqueness* if every trigonometric series converging to zero pointwise off A is trivial. U_σ is the σ-ideal σ-generated by closed sets of uniqueness.

Fourier showed that the empty set is a set of uniqueness, and Cantor proved that every countable closed set is a set of uniqueness. While it is not true that the union of two sets of uniqueness is a set of uniqueness, and it is not known whether this holds for Borel sets, Bary [2] showed that the union of countably many *closed* sets of uniqueness is a set of uniqueness.

Definition 6.2. A set $A \subset \mathbb{T}$ is an *H-set* if there is an infinite set $b \subset \omega$ and a nontrivial open set $O \subset \mathbb{T}$ such that for every $n \in b$, $nA \cap O = 0$. H_σ is the σ-ideal σ-generated by H-sets.

Rajchman [10] defined H-sets in a search for perfect sets of uniqueness. He proved that H-sets are sets of uniqueness, and since the closure of an H-set is again an H-set, it follows that $H_\sigma \subset U_\sigma$. He also showed that the Cantor middle third set is an H-set, producing a perfect set of uniqueness. The combinatorics of both H-sets and sets of uniqueness is quite complicated [5]. I will show

Theorem 6.3. *Suppose that the Generalized Continuum Hypothesis holds and $\kappa \geq \aleph_1$ is a regular cardinal. Then there is a cardinal preserving forcing extension in which $\mathtt{cov}(U_\sigma) = \aleph_1$ and $\mathtt{cov}(H_\sigma) \geq \kappa$.*

Proof. The plan of attack is straightforward. Consider the quotient forcing P_{H_σ} of Borel H_σ-positive sets ordered by inclusion. H_σ turns out to be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal σ-generated by closed sets, with the overspill property; the poset P_{H_σ} is bounding and preserves Baire category. Consider the countable support product P of κ many copies of P_{H_σ}. This is a proper bounding forcing preserving Baire category; a standard argument shows that it is $\aleph_2$-c.c. and therefore preserves all cardinals. It is not difficult to show that $P \Vdash \mathtt{cov}(H_\sigma) \geq \kappa$, since any H-set in the extension can cover at most countably many among the κ many P_{H_σ}-generic points added by the product. Most importantly, a result of Loomis [7] shows that there is an analytic (in fact $G_{\delta\sigma}$) set $A \subset K(\mathbb{T})$ that contains all countable closed sets and consists only of sets of uniqueness; in other words, the σ-ideal U_σ does not have the overspill property. By Theorem 3.7, in the P-extension, $\mathbb{T} = \bigcup(A \cap V)$ and therefore $\mathtt{cov}(U_\sigma) = \aleph_1$.

In order to fill in the details of this plan, I must first prove the requisite properties of the poset P_{H_σ}. To do that, I will consider a different poset that seems to have nothing to do with harmonic analysis. Let $\omega = \bigcup_k a_k$ be a partition into infinite sets, and let I be the σ-ideal on 2^ω σ-generated by sets $K \subset 2^\omega$ for which there is $k \in \omega$ such that for every $n \in \omega$ there is $m > n$ in the set a_k and $b \in 2$ such that for every $x \in K$, $x(m) = b$.

Claim 6.4. *I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal with the overspill property, and the quotient forcing P_I is proper.*

Proof. Observe that I is σ-generated by $\bigcup_k I_k$ where each I_k is a σ-ideal on 2^ω σ-generated by sets $K \subset 2^\omega$ such that for every $n \in \omega$ there is $m > n$ in a_k and $b \in 2$ such that for every $x \in K$, $x(m) = b$. The σ-ideals I_k are pairwise isomorphic, and they have been treated in Theorem 4.2, case $\alpha = 1$. Thus, they are σ-generated by closed sets, $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and have the overspill property. The overspill property of I then follows from Theorem 3.8. $\square$

The quotient poset P_I is quite complicated, in particular, it seems to be highly inhomogeneous. It can be combinatorially presented as the poset of those binary trees $T \subset 2^{<\omega}$ such that for every node $t \in T$ and every number $k \in \omega$ there is a number $m \in \omega$ such that for every $n \in a_k$ greater than m there are extensions s_0, s_1 of t in the tree T such that $s_0(n) = 0$ and $s_1(1) = 1$.

To see the connection between the poset P_I and P_{H_σ}, enumerate the nontrivial rational open intervals of $\mathbb{T}$ by $\{O_k : k \in \omega\}$ and consider the map $h : \mathbb{T} \to 2^\omega$ defined by $h(x)(m) = 0$ if $nx \in O_k$, where m is the n-th element of a_k. It is immediate that h is a one-to-one Borel function, thus its range $\mathrm{rng}(h) \subset 2^\omega$ is a Borel set, and the function h also carries the σ-ideal H_σ to $I \restriction \mathrm{rng}(h)$. It immediately follows that the σ-ideal H_σ has all the properties of I claimed in the above claim.

The rest of the plan outlined in the first paragraph follows from the references there, with perhaps one exception–that any H-set in the P-extension contains only countably many P_{H_σ}-generic points. To see this, suppose that $p \Vdash \dot{A}$ is an H-set, for some condition $p \in P$. By the standard analysis of the countable support product of definable forcing, there is a countable set $b \subset \kappa$, H_σ-positive sets $\{K_\alpha : \alpha \in b\}$ and a Borel set $D \subset \Pi_{\alpha\in b} K_\alpha \times \mathbb{T}$ such that the vertical sections of D are all H-sets and the condition $q \leq p$, $q = \Pi_{\alpha\in b} K_\alpha$ forces $\dot{A} \subset D_{\vec{x}_{gen}\restriction b}$. I claim that the only generic points in the product that can belong to the set $\dot{A}$ are indexed by the ordinals in the set b. Indeed, choose a condition $r \leq q$ and an ordinal $\gamma \in \kappa \setminus b$; thinning out if necessary I may find a countable set $c \supset b \cup \{\gamma\}$ and H_σ-positive sets $\{L_\alpha : \alpha \in c\}$ such that $r = \Pi_{\alpha\in c} L_\alpha$. The set $E \subset \Pi L_\alpha$, $E = \{\vec{x} : \vec{x}(\gamma) \in D_{\vec{x}\restriction b}\}$ is Borel and has H_σ-small sections in γ-th coordinate. Therefore, it cannot contain a Borel rectangular box with I_α-positive sides, and by Fact 3.4, it must be the complement of E that contains such a box $s = \Pi_{\alpha\in c} M_\alpha$. The condition $s \leq r$ must force the γ'th generic point not to belong to $\dot{A}$, as an immediate absoluteness argument shows.

$\square$

The properties of the poset P_{H_σ} beyond those that follow from its overspill property are not easy to identify. I will just observe that the poset adds an independent real: if $x \in \mathbb{T}$ is the generic point and $O \subset \mathbb{T}$ is a nontrivial rational interval, then neither the set $\{n \in \omega : nx \in O\}$ nor its complement in ω can contain an infinite set by the definition of the σ-ideal H_σ.

As the last point in the paper, I will prove two independence results complementary to Theorem 6.3. They show that there is a great degree of freedom in moving the covering numbers of the σ-ideals H_σ, U_σ around by forcing.

Theorem 6.5. *It is consistent with ZFC that $\mathbb{T}$ is covered with $\aleph_1$ many H-sets while the continuum is very large.*

Proof. It is enough to reach for a model of ZFC in which the continuum is large while there is a P-point basis of size $\aleph_1$, such as in the product Sacks extension. For every point $x \in \mathbb{T}$ there is a set $a \subset \omega$ in the P-point ultrafilter such that the points $\{nx : n \in a\}$ converge, and therefore they avoid a certain nonempty open interval in the circle $\mathbb{T}$. This shows that the $\aleph_1$ many sets $B_{a,O} = \{x \in \mathbb{T} : \forall n \in a \; nx \notin O\}$, as a ranges over the P-point basis of size $\aleph_1$ and O ranges over all possible open intervals with rational endpoints, cover the circle $\mathbb{T}$, and they are all H-sets. $\square$

Theorem 6.6. *It is consistent with ZFC that $\mathbb{T}$ cannot be covered by fewer than $\aleph_2$ many closed sets of uniqueness while there are dominating, nonmeager and nonnull sets of size $\aleph_1$.*

Proof. Consider the σ-ideal U_0 of sets of extended uniqueness on $\mathbb{T}$. The deep results of Debs and Saint-Raymond [3] show that this is a σ-ideal σ-generated by closed sets and it is polar. The collection of closed sets in U_0 is coanalytic, in fact $\mathbf{\Pi}^1_1$-complete by a result of Solovay and Kaufman [5, Section IV.2], and so by [13, Theorem 3.8.9] the σ-ideal U_0 is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Thus, the quotient P_{U_0} is proper, bounding, preserves Baire category, and outer Lebesgue measure by [13, Theorem 3.6.2]. Moreover, every set of uniqueness is a set of extended uniqueness, and so the poset P_{U_0} forces its generic real not to belong to any ground model coded closed sets of uniqueness. Ergo, starting with a model of the Continuum Hypothesis and iterating P_{U_0} ω_2 many times, a model of the statement of the theorem is achieved as the various preservation theorems of [13, Section 6.3] or [1] show.

Note that the poset does not have the Sacks property–by the results of the previous section, it would imply a particularly strong version of overspill, and the σ-ideal U_0 does not have the overspill property since it includes U_σ and with it the analytic collection of compact sets of uniqueness discovered by Loomis. Thus, in the extension, the cofinality of the Lebesgue null ideal is $\aleph_2$. I do not know if the products of the poset P_{U_0} preserve outer Lebesgue measure, and therefore I do not know if it is possible to push the continuum beyond $\aleph_2$. $\square$

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