

Theorem 0.1. *There are Σ_1 sets $A, B \subset \omega$ such that $A \not\leq_m B$ and $B \not\leq_m A$*

Proof. Fix a universal set $W \subset \omega \times (\omega^2)$ with a stratification $\bigcup_{s \in \omega} W_s$. Fix a partition $\omega = \bigcup_{e \in \omega} a_e$ of natural numbers into infinite sets.

By recursion on $s \in \omega$ build finite sets $A_s, B_s \subset \omega$ and finite partial functions $w_s: \omega \rightarrow \omega^2$ (the *work*) such that, among other things,

- A_s, B_s increase with respect to inclusion;
- for every $e \in \omega$, if $2e \in \text{dom}(w_s)$ then $f(2e) = \langle n, m \rangle$ for a pair $\langle n, m \rangle \in W_e$ such that $n \in A_s \not\leftrightarrow m \in B_s$;
- for every $e \in \omega$, $2e + 1 \in \text{dom}(w_s)$ then $f(2e + 1) = \langle n, m \rangle$ for a pair $\langle n, m \rangle \in W_e$ such that $n \in B_s \not\leftrightarrow m \in A_s$.

The idea is that the whole construction will be performed by a Turing machine. This together with (1) will guarantee that $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ will be Σ_1 sets. Moreover, the work w_s will converge to some partial function $w: \omega \rightarrow \omega^2$ such that for every e such that W_e is a graph of a total function, both $2e$ and $2e + 1$ will be in the domain of w . It then has to be the case that A is not many-one reducible to B and vice versa: every putative recursive reduction is realized as W_e for some $e \in \omega$, and then the second and third item show that $w(2e)$ and $w(2e + 1)$ are pairs witnessing that W_e does not reduce A to B and vice versa.

One important thing to understand is that the work w_s will not monotonically increase to w , but converge to w in some unpredictable fashion. As a result, w will not be a Σ_1 function.

To describe the recursion, I need some provisional definitions. If $e \in \omega$, I will say that the *requirement* $2e$ *needs attention at stage* s if $2e \notin \text{dom}(w_s)$ and there is a pair $\langle n, m \rangle \in (W_s)_e$ such that $n \in a_{2e}$, $n \notin A_s$, and n is not equal to any number on the pairs in the set $\text{rng}(w_s \upharpoonright 2e)$. Similarly, the *requirement* $2e + 1$ *needs attention* if $2e + 1 \notin \text{dom}(w_s)$ and there is a pair $\langle n, m \rangle \in (W_s)_e$ such that $n \in a_{2e+1}$, $n \notin B_s$, and n is not equal to any number on the pairs in the set $\text{rng}(w_s \upharpoonright 2e + 1)$.

Now, the recursion can be succinctly described as follows. Start with $A_0 = B_0 = w_0 = 0$. Suppose that A_s, B_s, w_s have been found. Look for the least requirement d (the requirement of *highest priority*) which needs attention at stage s . This is a finite search as the set W_s is finite. If the search comes up empty, set $A_{s+1} = A_s$, $B_{s+1} = B_s$, $w_{s+1} = w_s$. Otherwise, the description divides according to whether d is even or odd; let me describe the case when d is even, $d = 2e$. Find the least pair $\langle n, m \rangle$ which witnesses the fact that $2e$ needs attention; again, this is a finite search. Let $w_{s+1} = (w_s \upharpoonright 2e) \cup \{\langle 2e, \langle n, m \rangle \rangle\}$. If $m \in B_s$ then let $A_{s+1} = A_s$, $B_{s+1} = B_s$; if $m \notin B_s$ then let $A_{s+1} = A_s \cup \{n\}$ and $B_{s+1} = B_s$. If d is odd, $d = 2e + 1$, exchange the role of A, B .

The first important observation is that the recursion demands listed in the bulleted items above survive each stage of the recursion. The second observation is that $w_s \subset w_{s+1}$ may fail; the tail of w_s past d is erased, and the work done there is said to have been *injured*.

We now have to verify the expectations set out initially about the limit w of w_s .

Claim 0.2. *For every $e \in \omega$ there is a stage $s_e \in \omega$ such that for all $s \geq s_e$, $w_s \upharpoonright e = w_{s_e} \upharpoonright e$.*

Proof. By induction on e . $s_0 = 0$ works. Suppose the claim is known for e . If for all $s \geq s_e$, $e \notin \text{dom}(w_s)$, then $s_{e+1} = s_e$. Otherwise, pick the least $s \geq s_e$ such that $e \in \text{dom}(w_s)$ and let $s_{e+1} = s$. In this latter case, to see that s works as desired, observe that the work done at e is never injured past stage s , since such an injury would require that some work of priority higher than e is done past stage s , and by induction hypothesis this is not the case. \square

The claim lets us define the limit $w = \bigcup_e w_{s_e} \upharpoonright e$.

Claim 0.3. *$A \subset \bigcup_e a_{2e}$, $B \subset \bigcup_e a_{2e+1}$, and for each e , the sets $A \cap a_{2e}$ and $B \cap a_{2e+1}$ are finite.*

Proof. The only time a number n gets placed in the set A , it is for a requirement $2e$ of highest priority, and one of the demands there is that $n \in a_{2e}$. Thus $A \subset \bigcup_e a_{2e}$. Also, by the previous claim, the requirement $2e$ has highest priority only finitely many times, so $A \cap a_{2e}$ is finite. \square

Claim 0.4. *For every $e \in \omega$ such that W_e is a total function, both $2e$ and $2e+1$ are in $\text{dom}(w)$.*

Proof. Let me deal with case of $2e$. Look at the set $a_{2e} \cap A$ together with set of all numbers that appear in the pairs in the range of $w_{s_{2e+1}} \upharpoonright 2e$. This is a finite set. Thus, there has to be a number $n \in a_{2e}$ which is not in it. Since W_e is a total function, there is a number m such that $\langle n, m \rangle \in W_e$. Suppose towards contradiction that $2e \notin \text{dom}(w_{s_{2e+1}})$. Then, the triple $\langle e, \langle n, m \rangle \rangle$ will appear in W_s for some $s > s_{2e+1}$ and at that stage the pair $\langle n, m \rangle$ stands witness to the fact that the requirement $2e$ has the highest priority and will be placed in $\text{dom}(w_{s+1})$, which contradicts the choice of s_{2e+1} . \square

The proof of the theorem is now complete. \square