Highness properties close to PA-completeness

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Joint work with Noam Greenberg and André Nies

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    - All three are intrinsically c.e. objects that are optimal in their classes,
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    - and they can all be approximated using PA degrees.
- We are motivated by the following question: in these and related examples, are the PA degrees necessary?

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- ($\varnothing'$ has PA degree.)

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Definition. A (super)martingale m *strongly succeeds* on $A \in 2^{\omega}$ if $\lim m(A \restriction n) = \infty$.

The optimal supermartingale

Fact. There is a c.e. supermartingale m that is *optimal* in the sense that for any other c.e. supermartingale d:

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Proposition (Franklin, Stephan, and Yu [2011])

Every PA degree computes a martingale that majorizes the optimal c.e. supermartingale.

Martingale domination and PA degrees

Theorem (Greenberg, M., Nies)

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Proof. Define M as follows:

If n enters $\varnothing'$ at stage s, find a $\sigma \in 2^s$ that still looks DNC_2 at stage s. Push 2^{-n} capital up σ (i.e., add 2^{-n} to the root and double along σ so 2^{s-n} is added to σ).

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Let d be a martingale majorizing M. Call $X \in 2^\omega$ an ε-atom if

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Case 2: d has no DNC_2 atoms.

Let $f(n) =$ first s such that if $\sigma \in 2^s$ still looks DNC_2 at stage s, then $d(\sigma) < 2^{s-n}$. By construction, f dominates the settling time function of $\varnothing'$. Therefore, $d \geqslant_T f \geqslant_T \varnothing'$ has PA degree. □

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Question. For some k, is there a uniform way to compute a DNC_k function from a martingale d majorizing m whose initial capital is bounded by 1?

Plain Kolmogorov complexity

Let $C\colon 2^{<\omega} \to \omega$ denote *(plain) Kolmogorov complexity*.

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- Conversely, from an f with these properties, it is easy to compute a C-compression function.

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Proposition (Greenberg, M., Nies). Depending on the choice of universal machine V, it may or may not be possible to take $k = 2$ in the theorem.

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- Conversely, given any such f, we can compute a corresponding K-compression function F (Kraft–Chaitin theorem).

Prefix-free complexity and PA degrees

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This implies, for example, that incomplete Martin-Löf random sequences cannot compute K-compression functions. (See Bienvenu and Porter's paper on *Deep* Π^0_1 *classes.*)

The strong continuous covering property

Next we consider a technical property that holds for PA degrees.

Definition

We say that X has the *strong continuous covering property* if for every Π^0_1 class $P \subseteq 2^\omega$ such that $\mu(P) > 0$, there is a nonempty X-computable tree $T \subseteq 2^{<\omega}$ such that $[T] \subseteq P$ and for all $\sigma \in T$, $\mu([T] \cap [\sigma]) > 0$.

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- There is a Π^0_1 class (of positive measure) that contains only Martin-Löf random sequences. So if X has the strong continuous covering property, then it computes a Martin-Löf random.

Properties “close” to PA

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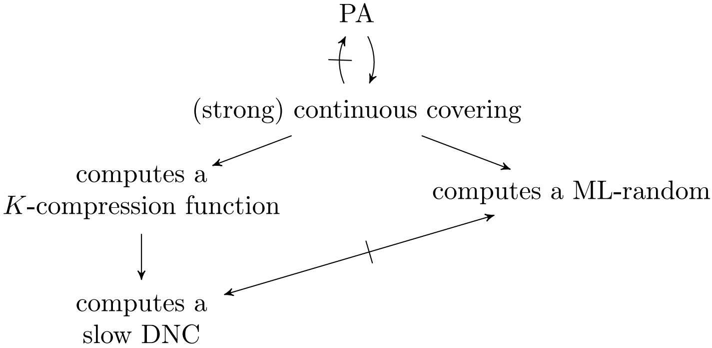
Putting our facts together:



PA

(strong) continuous covering

computes a K-compression function

computes a ML-random

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So we can adapt our results to show:

Theorem (Greenberg, M., Nies; Barmpalias and Wang?).
Over RCA_0,

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We also defined *weak strong weak weak Kőnig's lemma* (*WSWWKL*) to correspond to the continuous covering property.

THANK YOU