More σ-ideals with the Laczkovich-Komjáth property*

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Abstract

I consider the Laczkovich-Komjath property of sigma-ideals concerning countable sequences of analytic sets and I prove or disprove it for various sigma-ideals. Connections with definable forcing appear.

1 Introduction

The starting point of this paper is a fairly old result of Laczkovich.

Fact 1.1. *[8] Given a Polish space and an infinite sequence $\vec{B}$ of its Borel subsets, then*

1. *either there is an infinite set $a \subset \omega$ such that $\limsup_a \vec{B}$ is countable,*
2. *or there is an infinite set $a \subset \omega$ such that $\liminf_a \vec{B}$ is uncountable.*

In the spirit of the work of Balcar [1], one can view this statement in terms of the σ-algebra of Borel sets modulo the ideal of countable sets: every countable sequence of elements in this algebra contains either a subsequence converging to zero, or a subsequence bounded away from zero. Later, Komjáth [7], improved this to include sequences of analytic sets. Following further work of Balcerzak and Głab [2], I define

Definition 1.2. Let I be a σ-ideal on a Polish space X, the ideal has the *Laczkovich-Komjáth*, or LK property, if for every infinite sequence $\vec{B}$ of analytic sets, either there is an infinite set $a \subset \omega$ with $\limsup_a \vec{B} \in I$ or there is an infinite set $a \subset \omega$ such that $\liminf_a \vec{B} \notin I$.

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In this paper, I will verify or disprove this property for many ideals on Polish spaces. I will show that the property closely corelates with the forcing properties of the quotient poset P_I of all Borel sets not in I ordered by inclusion. Thus the LK property holds for the σ-ideal generated by sets of finite packing measure, and for the σ-ideal generated by sets of finite Davies-Rogers Hausdorff measure. The property fails for the σ-ideal generated by compact subsets of ω^ω, or for the σ-ideal generated by those Borel sets of reals that meet every Vitali equivalence class in at most one point. This improves the results of [2]. There is a number of general results and open questions.

The notation follows the set theoretic standard of [4]. If $\vec{B}$ is a sequence of sets and $a \subset \omega$ is inifnite, then $\liminf_a \vec{B} = \{x : \exists n \forall m > n\ m \in a \to x \in \vec{B}(m)\}$, and $\limsup_a \vec{B} = \{x : \forall n \exists m > n\ m \in a \land x \in \vec{B}(m)\}$. If I is a σ-ideal on a Polish space then P_I stands for the partial order of Borel I-positive sets ordered by inclusion.

2 Negative results

The first concern: is the LK property for sequences of analytic sets truly stronger than the formulation with just sequences of Borel sets? It turns out that the answer is negative for a large and well-researched class of σ-ideals:

Definition 2.1. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic.

For example, the ideals of countable, meager or Lebesgue null sets are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$[6, Section 29.E]. [11, Section 3.8] gives many more examples and relates this property to forcing properties of the poset of Borel I-positive sets ordered by inclusion.

Proposition 2.2. *Suppose that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. If the LK property fails for a sequence of analytic sets, then it fails for a sequence of Borel sets.*

Proof. Note that the property $\phi(\vec{B}) = \forall a \subset \omega\ \liminf_a B_n \in I$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property for countable sequences of sets. By the first reflection theorem [6], whenever $\vec{B}$ is a sequence of analytic sets with $\phi(\vec{B})$, then there is a sequence $\vec{C}$ of their Borel supersets with $\phi(\vec{C})$. Clearly, if $\vec{B}$ witnessed the failure of LK-property, so does the sequence $\vec{C}$. □

The question of further reduction of Borel rank of the offending sequence of Borel sets remains open. In all specific cases discussed in this paper, these sets can be chosen to be closed.

The remainder of the paper concerns the connections between the status of the LK property of a σ-ideal I and the forcing properties of the poset P_I of Borel I-positive sets ordered by inclusion. The key concern is the properness of the quotient P_I [10], [11, Section 2.2]. While the properness is not easy to check, or even to define, for many σ-ideals appearing naturally in mathematical analysis this has been done in [11]. While properness in itself may not have much to do with the status of the LK property, if it is assumed then many other forcing features of the quotient turn out to be directly related to the LK property.

Proposition 2.3. *Suppose that I is a σ-ideal on a Polish space X such that the quotient forcing P_I is proper. If P_I adds an independent real then I fails the LK property.*

Proof. Suppose that $B \in P_I$ is a condition and $\dot{y} \in 2^\omega$ is a P_I-name. Use the properness assumption to strengthen B if necessary to find a Borel function $f : B \to 2^\omega$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Consider the sets $B_n = \{x \in B : f(x)(n) = 0\}$ for $n \in \omega$. By the LK property, there are two cases. Either there is an infinite set $a \subset \omega$ such that $C = \limsup_{n \in a} B_n \notin I$; in this case $C \Vdash$ for all but finitely many numbers $n \in a$, $\dot{y}(n) = 0$. Or, there is an infinite set $a \subset \omega$ such that $C = B \setminus \limsup_{n \in a} B_n \notin I$; in this case $C \Vdash$ for all but finitely many numbers $n \in a$, $\dot{y}(n) = 1$. In either case, the sequence $\dot{y}$ is not independent. □

This proposition shows that ideals such as the meager sets or the null sets do not have the LK property. One forcing adding no independent reals is the Sacks forcing associated with the σ-ideal of countable sets. The LK property of this ideal is exactly the contents of the results of Laczkovich and Komjáth. Another forcing adding no independent reals is the Miller forcing, associated with the σ-ideal generated by the compact subsets of the Baire space ω^ω. There, the LK property fails since the forcing adds an unbounded real:

Proposition 2.4. *Suppose that I is a σ-ideal on a Polish space X such that the quotient forcing P_I is proper and adds an unbounded real.*

1. *If there is a perfect antichain consisting of pairwise disjoint sets below any condition, then I fails the LK-property.*
2. *If suitable large cardinals exist, I is a universally Baire σ-ideal and the forcing P_I preserves Baire category, then I fails the LK property.*

The list of assumptions is peculiar and perhaps can be improved. In the first item, I require that for every I-positive Borel set $B \subset X$, there is a Borel function $f : B \to 2^\omega$ such that the preimages of singletons are all I-positive. Perhaps a remark is in order. If the ideal I has LK-property and is suitably definable, the quotient cannot be c.c.c. by the previous result and [1]. It does not necessarily follow that there must be perfect antichains in the ordering P_I by the result of [5]. However, in practice such antichains do exist, as in the case of the Miller forcing.

Regarding the second item, the large cardinals and definability assumptions are used to secure determinacy in a certain infinite game; in such cases as the Miller forcing, the necessary winning strategy can be easily constructed manually.

Proof. For (1), suppose that $B \Vdash \dot{y} \in \omega^\omega$ is an unbounded increasing function. Use the properness assumption to strengthen B if necessary and find a Borel function $f : B \to \omega^\omega$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Let $B_a : a \in [\omega]^{\aleph_0}$ be a perfect collection of pairwise disjoint I-positive subsets of B. Consider the function $g : \bigcup_a B_a \to \omega^\omega$ defined by $g(x)(n) = f(x)(n)$-th element of a when $x \in B_a$. Note that $\dot{g}(\dot{x}_{gen})$ is still a name for an unbounded real, since it is above $\dot{f}(\dot{x}_{gen})$. Let $B_n = \{x \in \bigcup_a B_a : n \in \text{rng}(g(x))\}$. It is not difficult to see that for every $a \in [\omega]^{\aleph_0}$, $B_a \subset \limsup_a B_n$ and therefore the latter set is I-positive. On the other hand, $\bigcap_a B_n$ cannot be an I-positive set for any infinite set a since it would force $\dot{g}(\dot{x}_{gen})$ to be bounded by the enumeration function of $\dot{a}$.

For (2), the assumptions imply [11, Section 3.10.9]that Player I has a winning strategy σ in the following two player infinite game: in it, Player I and II alternate to produce Borel sets $C_n \in P_I$ and $D_n \in P_I$ respectively with the demand that $D_n \subset C_n$. Player II wins if the *result* of the play, the set $\limsup_n D_n$, does not belong to I.

Now suppose that $\dot{y}$ is a P_I-name for an unbounded real; I must produce a failure of the LK property. By induction on $n \in \omega$ build finite sets T_n of partial finite plays of the game according to the strategy σ in which Player II makes the last move and

- $T_0 = \{0\}$;
- the last move of every play in T_{n+1} decides $\dot{y} \restriction n$;
- every play in $\bigcup_{m \in n} T_m$ has a one move extension in T_n.

Let $B_n \subset X$ be the union of the last moves of Player II in all plays in the set T_n, for every number $n \in \omega$. Obviously, these are Borel sets. If $a \subset \omega$ is an infinite set, then the first item shows that the set $\liminf_{n \in a} B_n$, if I-positive, would force a ground model bound on the function $\dot{y}$, which is impossible. And the second item shows that there is an infinite play τ according to the strategy σ such that $\tau \restriction i \in T_{n_i}$ for every number $i \in \omega$, where n_i is the i-th element of the set a. Since the result of the play τ must be I-positive, so must $\limsup_{n \in a} B_n$. Thus the sequence $B_n : n \in \omega$ witnesses the failure of the LK property.

□

In search for bounding proper partial orders that do not add independent reals one immediately encounters iterations and products of Sacks forcing. It turns out that σ-ideals associated with such posets also never have the LK property:

Proposition 2.5. *Suppose that I is a universally Baire σ-ideal on a Polish space X such that the quotient forcing P_I is proper and adds more than one generic real degree.*

1. *If there is a perfect antichain consisting of pairwise disjoint sets below every condition then I fails the LK-property.*

2. *If suitable large cardinals exist, the ideal is universally Baire and the quotient forcing preserves Baire category then I fails the LK-property.*

The list of assumptions again seems to contain some unnecessary items.

Proof. Suppose that $B \in P_I$ is a condition and $\dot{y} \in 2^\omega$ is a P_I-name. such that $\dot{y} \notin V$ and $\dot{x}_{gen} \notin V[\dot{y}]$. Passing to a stronger condition if necessary I may find a Borel function $f : B \to 2^\omega$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Look at the σ-ideal J on 2^ω of all Borel sets C such that $B \Vdash \dot{y} \notin C$. If there is a J-positive singleton, then its f-preimage is a Borel I-positive set on which the function f is constant, and this set would force $\dot{y} \in V$. Thus all singletons are in J and proceed to extract a perfect set of conditions in P_I such that their f-images are pairwise disjoint.

If the ideal J were c.c.c. then P_J is a definable c.c.c. forcing and therefore adds an independent real by the work of Balcar and ? Moreover, $\dot{y}$ is a P_I-name for P_J-generic. Thus P_I adds an independent real, contradicting the LK property of I by Proposition 2.3. Now, if the forcing P_I adds unbounded real then we are done by the previous proposition. Thus the forcing P_I is bounding. In the case (2) follow the proof of [11, Proposition 3.7.7] to find a perfect collection of compact I-positive sets $C_b : b \in [\omega]^{\aleph_0}$ whose f-images are pairwise disjoint.

For the remainder of the proof, fix an enumeration $\{t_i : i \in \omega\}$ of $2^{<\omega}$ without repetition. For every number $n \in \omega$, let B_n be the set of points $x \in X$ such that there is $b \in [\omega]^{\aleph_0}$ such that $x \in C_b$, and for this unique b, $t_i \subset f(x)$ where i is the index of the largest number of b which is $\leq n$ in the increasing enumeration of b. It is not difficult to see that $B_n : n \in \omega$ are Borel sets. For every infinite set $c \subset \omega$, $\limsup_c B_n$ contains the set C_c and therefore it is I-positive. By the LK property, there must be an infinite set $c \subset \omega$ such that $C = \bigcup_{n \in c} B_n \notin I$. It is not difficult to check that $C \subset B$ is a Borel I-positive set meeting each $C_b : b \in [\omega]^{\aleph_0}$ in exactly one point. Therefore $f \restriction C$ is one-to-one as desired. □

The above propositions do not cover all reasons for which the LK property may fail; see the following example obtained directly from the definitions:

Example 2.6. Let $B_n : n \in \omega$ be an independent collection of clopen subsets of the Cantor space 2^ω. Consider the σ-ideal generated by all sets $C \subset 2^\omega$ such that there is $m \in \omega$ such that for every $k \in \omega$ there are m many sets in the collection, indexed by numbers greater than k, whose union covers the set C. This construction fits into the Hausdorff submeasure scheme of [11], and so the quotient forcing P_I is proper, bounding, adds no independent reals, and adds one generic real degree. In addition, the σ-ideal is generated by closed sets and therefore the quotient P_I also preserves Baire category.

It is quite obvious that the sequence $B_n : n \in \omega$ witnesses the failure of the LK property. Let $a \subset \omega$ be infinite and consider the sets $\limsup_a B_n$, and $\liminf_a B_n$. The latter belongs to the ideal I by the definitions. I must prove that the former is I-positive. Suppose that $C = \bigcup_m C_m$ is a set in the ideal I, written as a countable union of sets such that for every $k \in \omega$ there is a set $b_{m,k} \subset \omega \setminus k$ of size m such that $C_m \subset \bigcup_{n \in b_{m,k}} B_n$. By induction on $i \in \omega$ choose numbers $n_i \in a$ such that $b_{i,n_i+1} \subset n_{i+1}$, and find a point $x \in \bigcap_i B_{n_i} \setminus \bigcup \{B_n : n \in b_{i,n_{i+1}}, i \in \omega\}$. Then $x \in \limsup_a B_n \setminus C$ and the set $\limsup_a B_n$ is I-positive as desired.

Another negative example of a quite different flavor:

Example 2.7. Let E_0 be the equivalence on 2^ω defined by $x E_0 y$ iff $x \Delta y$ is finite, and let I be the σ-ideal generated by those Borel sets that meet every E_0 equivalence class in at most one point. Then I does not have the LK property.

In order to prove this, fix an enumeration $t_i : i \in \omega$ of $2^{<\omega}$ and a Borel set $C \subset [\omega]^{\aleph_0} \times 2^\omega$ such that its vertical sections are I-positive and pairwise non-E_0-connected. Define sets $B_n : n \in \omega$ by letting $x \in B_n$ if there is a (unique) $a \in [\omega]^{\aleph_0}$ such that $x \in B_a$ and, writing i for the number such that n is between i-th and $i+1$-st element of a, $t_i \subset x$.

Clearly, if $a \in [\omega]^{\aleph_0}$ then $\liminf_a \vec{B}$ chooses at most one point from each vertical section C_b. Thus, $\liminf_a \vec{B}$ meets every E_0 class in at most one point, and must be in I. On the other hand, $\limsup_a \vec{B}$ is positive, since it contains all elements of the set C_a.

Question 2.8. Suppose $n < m$ are natural numbers. Does the ideal of sets of σ-finite n-dimensional Hausdorff measure on $\mathbb{R}^m$ have the LK property?

3 Positive results

There are two classes of ideals for which I can confirm the LK property, both studied in [11]. The verification proceeds through the Mathias forcing [9] consisting of pairs $p = \langle c_p, a_p \rangle$ such that $c, a \subset \omega$ are a finite and infinite set respectively, and $q \leq p$ if $c_p \subset c_q$, $a_q \subset a_p$, and $c_q \setminus c_p \subset a_p$. This forcing is proper and adds a generic infinite set $\dot{a}_{gen} \subset \omega$ which is the union of the first coordinates in the generic filter.

Proposition 3.1. *Suppose that I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X. The following are equivalent.*

1. *the LK property of I;*
2. *for every sequence $\vec{B}$ of analytic sets, either there is a condition forcing $\limsup_{\dot{a}_{gen}} \vec{B} \in I$, or there is a condition forcing $\liminf_{\dot{a}_{gen}} \vec{B} \in I$.*

Proof. (2) implies (1) easily through a standard absoluteness argument. Let M be a countable elementary submodel of a large structure, and let $a \subset \omega$ be an M-generic set for the Mathias forcing compatible with the condition p. By the forcing theorem, $M \models \limsup_a \vec{B} \in I$ or $\liminf_a \vec{B} \notin I$. The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and therefore this statement carries over to V by analytic absoluteness.

The proof of the converse is a little more difficult. Suppose I is a σ-ideal such that (2) fails, and let $\vec{B}$ be the offending sequence of analytic sets. Thus the Mathias forcing outright forces $\limsup_{\dot{a}_{gen}} \vec{B} \notin I$ and $\liminf_{\dot{a}_{gen}} \vec{B} \in I$. Let M be a countable elementary submodel of a large structure and let $a \subset \omega$ be an M-generic Mathias real. The geometric genericity criterion of [9] implies that every infinite subset $b \subset a$ is an M-generic Mathias real as well. Now $M[b] \models \limsup_b \vec{B} \notin I \wedge \liminf_b \vec{B} \in I$ by the forcing theorem. Since the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, this statement transfers to V without change. It follows that the sequence $\vec{B} \restriction a$ violates the LK property.

□

Definition 3.2. A σ-ideal I on a Polish space X is generated by a σ-compact collection of compact sets if there are compact sets $K_n : n \in \omega$ in the hyperspace $K(X)$ such that the elements of $\bigcup_n K_n$ σ-generated the ideal I.

It turns out that in this situation one can find a single compact set $K \subset K(X)$ whose elements generate the σ-ideal I [**?**]. A typical example of ideals in this class is the ideal of countable sets; a more sophisticated example is the ideal of sets of σ-finite packing measure mass in a compact metric space. The quotient forcings P_I arising from the ideals in this class have been studied in [11, Theorem 4.1.8]. The ideals are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [11, Theorem 3.8.9].

Proposition 3.3. *Every σ-ideal generated by a σ-compact collection of compact subsets of a Polish space X has the LK property.*

Proof. Before we embark on the proof, I will review several properties of Mathias forcing, all coming essentially directly from [9]. First of all, given a formula ϕ of the forcing language and a condition p, ϕ can be decided by a *direct extension* of p, that is, a condition $q \leq p$ with $c_p = c_q$. This has the following consequence. Whenever $K \subset K(X)$ is a compact set in the hyperspace closed under subsets, and $\dot{C}$ a name for an element of K, then I can pass to a direct extension $q \leq p$ which *almost decides $\dot{C}$ in the direction of D* in the sense that for every basic open set $O \subset X$ there is a tail $a_O \subset a_q$ such that $\langle c_q, a_O \rangle$ decides the statement $\dot{C} \cap O = 0$, and $D = X \setminus \bigcup\{O \subset X : O$ is basic open and the decision was negative$\}$. A compactness argument shows that necessarily $D \in K$.

Another observation: whenever $\vec{B}$ is a sequence of analytic subsets of the space X, the sets $\liminf_{\dot{a}_{gen}} \vec{B}$ and $\limsup_{\dot{a}_{gen}} \vec{B}$ do not depend on finite changes of the set $\dot{a}_{gen}$, and therefore if a condition forces a statement about these two sets and perhaps some ground model parameters, then so do all finite variations of this condition. This means that if ϕ is a formula of the forcing language using these two sets and perhaps some ground model parameters and $p \in P$ is a condition forcing $\exists n\ \phi(n)$, then there is a direct extension of the condition p deciding the number n: first, find an arbitrary extension $q \leq p$ deciding the value of n and then replace c_q with c_p.

Suppose that $\vec{B}$ is a sequence of analytic sets. I will show that either there is a condition $p \in P$ forcing $\limsup_{\dot{a}_{gen}} \vec{B} \in I$, or there is a condition $p \in P$ forcing $\liminf_{\dot{a}_{gen}} \vec{B} \notin I$. The proposition will then follow from the previous proposition.

Suppose then for contradiction that the empty condition forces $\limsup_{\dot{a}_{gen}} \vec{B} \notin I$ and $\liminf_{\dot{a}_{gen}} \vec{B} \in I$. Let $K_i : i \in \omega$ be the compact subsets of the hyperspace $K(X)$ whose elements generate the σ-ideal I; without loss of generality, the sets K_i are closed under subsets, and increase with respect to inclusion. I can find names $\dot{C}_i : i \in \omega$ such that $P \Vdash \forall i\ \dot{C}_i \in \dot{K}_i$ and $\liminf_{\dot{a}_{gen}} \vec{B} \subset \bigcup_i \dot{C}_i$. Fix also continuous functions $f_n : \omega^\omega \to X$ such that $B_n = \mathrm{rng}(f_n)$. Now, by induction on $i \in \omega$ build

- numbers n_i and infinite sets $a_i \subset \omega$ such that $n_0 < n_1 < \dots$, $a_0 \supset a_1 \supset \dots$ and $n_{i+1} \in a_i$;
- finite sequences $t^j_i : j \leq i$ of natural numbers such that for fixed j, the sequences t^j_i increase with respect to inclusion;
- basic open sets O_i;

so that the condition $\langle 0, \{n_j : j \in i\} \cup a_i\rangle \in I$ forces $O_i \cap \dot{C}_i = 0$ and $\limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t^j_i} \notin I$. If this can be done, in the end there will be a unique point $x \in \bigcap_{i,j} f''_{n_j} O_{t^j_i}$, and the condition $\langle 0, \{n_i : i \in \omega\}\rangle$ will force $\check{x} \in \liminf_{\dot{a}_{gen}} \vec{B} \setminus \bigcup_i \dot{C}_i$, contradicting the choice of the names $\dot{C}_i$.

The induction process is easy. Start with setting $a_0 = \omega$. Suppose that the numbers $n_j : j \in i$, sequences $t^j_i : j \in i$, the open sets $O_j : j \in i$, and the set a_i have been constructed. Find an infinite set $b \subset a_i$ such that for every set $c \subset \{n_j : j \in i\}$, the condition $\langle c, b\rangle$ almost decides the set $\dot{C}_i$ in the direction of a set $D_c \in K_i$. Thinning out the set b if necessary, I can find a basic open set O_i disjoint from all the sets D_c and sequences $t^j_{i+1} : j \in i$ properly extending $t^j_i : j \in i$ such that the condition $\langle 0, b\rangle$ forces $\limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i+1} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t^j_{i+1}} \notin I$. Passing to a tail of b, I can make sure that for every set $c \subset \{n_j : j \in i\}$, $\langle c, b\rangle \Vdash \dot{O}_i \cap \dot{C}_i = 0$. Finally, thinning out b to some further infinite set a_{i+1}, I can find a number $n_i \in b$ such that $\langle c, a_{i+1}\rangle \Vdash \limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i+1} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t^j_{i+1}} \cap \vec{B}(n_i) \notin I$. Let $t^i_{i+1} = 0$ and proceed with the induction process.

□

Definition 3.4. A capacity ϕ on a compact metric space X is *Ramsey* if for every $\varepsilon > 0$ and $\delta > 0$ and every sequence $B_n : n \in \omega$ of Borel subsets of X of ϕ-mass $< \varepsilon$, there are distinct numbers $n \neq m$ such that $\phi(B_n \cup B_m) < \varepsilon + \delta$.

Examples of Ramsey capacities are not so easy to come by. Clearly, the outer Lebesgue measure is not Ramsey, as any stochastically independent sequence of sets of measure 1/2 shows. The arguments of [11, Sections 4.3.5, 4.3.6] construct a number of Ramsey capacities. It turns out that the Hausdorff content in the Davies-Rogers example of Hausdorff measure with only zero and infinite values is a Ramsey capacity.

Proposition 3.5. *If ϕ is a Ramsey capacity on a compact metric space, then the ideal of sets of ϕ-mass zero has the LK property.*

Proof. First observe that the ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$–[6, Exercise 30.16(ii)]. Then, note [11, Theorem 4.3.13(5)]: in the Mathias forcing extension, every set can be covered by a ground model open set of arbitrarily close ϕ-mass.

To prove the proposition, suppose for contradiction that $\vec{B}$ is a sequence of analytic sets violating the LK property. In particular, the Mathias poset P forces $\phi(\liminf_{\dot{a}_{gen}} \vec{B}) = 0$ and $\phi(\limsup_{\dot{a}_{gen}} \vec{B}) > 0$. Passing to a stronger condition $p \in P$, I may find a real number $\varepsilon > 0$ and a ground model open set O of mass $< \varepsilon$ such that $p \Vdash \liminf_{\dot{a}_{gen}} \vec{B} \subset O \wedge \phi(\limsup_{\dot{a}_{gen}} \vec{B}) > \varepsilon$. Let $a \subset \omega$ be a Mathias generic set consistent with the condition p, and work in $V[a]$. Let $x \in \limsup_a \vec{B} \setminus O$ be a point, and let $b \subset a$ be an infinite set consistent with the condition p such that $x \in \liminf_b \vec{B}$. Now $V[a] \models \liminf_b \vec{B} \setminus O \neq 0$, and by analytic absoluteness, $V[b] \models \liminf_b \vec{B} \setminus O \neq 0$. However, the set b is also Mathias generic by the geometric criterion, and so the latter statement contradicts what was forced by the condition p!

□

Another class of examples is associated with fat tree forcings of [11]:

Definition 3.6. Suppose that $u_n : n \in \omega$ are pairwise disjoint finite sets, and $\phi_n : n \in \omega$ are submeasures on each, such that $\liminf \phi_n(u_n) = \infty$. Let $I_{\vec{\phi}}$ be the σ-ideal generated by sets $B \subset \Pi_n u_n$ for which there is an $l \in \omega$ such that for every $m \in \omega$ there are sets $v_k \subset a_k : k > m$ such that $B \subset \{x \in \Pi_n a_n : \exists k > m\ x(k) \in v_k$. In other words, a set $A \subset \Pi_n u_n$ is in the ideal $I_{\vec{\phi}}$ if there are sets $v(l, m, k)$ for $l, m \in \omega$ and $k > m$ such that $v(l, m, k) \subset v_k$ is a set of ϕ_k-mass $< k$, and $\forall x \in A \exists l \forall m \exists k > m\ x(k) \in v(l, m, k)$.

The fat tree forcing associated with the sets u_n and submeasures $\phi_n : n \in \omega$ consists of those trees $T \subset \Pi_n u_n$ such that $\liminf_{t \in T} \phi_{|t|}\{i : t^\frown i \in T\} = \infty$. The ordering is that of inclusion. It follows from [11, Section 4.4.3] that the ideal $I_{\vec{\phi}}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

Fact 3.7. *If $A \subset \Pi_n u_n$ is an analytic set then either $A \in I_\phi$ or A contains all branches of some fat tree.*

Proposition 3.8. *If each ϕ_n is a counting measure, then the ideal $I_{\vec{\phi}}$ fails the LK-property.*

Proof. Choose numbers $n_i : i \in \omega$ such that for every $i \in \omega$, $|u_{n_i}| > i$. Choose a collection $c^i_j : j < i$ of distinct elements of u_{n_i}, this for every $i \in \omega$. Let $B_j = \{x \in \Pi_n u_n : \exists i > k\ x(n_i) = c^i_k\}$ for every $j \in \omega$. I claim that this sequence of sets violates the LK property.

First of all, whenever $a \subset \omega$ is an infinite set then $\liminf_a B_j \in I_{\vec{\phi}}$. Revisiting the definition of the ideal $I_{\vec{\phi}}$, it is clear that $\liminf_a B_j$ is in fact one of the generating sets of the σ-ideal as witnessed by $l = 1$. On the other hand, if $a \subset \omega$ is infinite, then $\limsup_a B_j$ is I-positive. To see this, suppose $v(l, m, k) : l, m \in \omega, k > m$ are sets such that $v(l, m, k) \subset u_k$ is of size at most l; we must find a point $x \in \limsup_a B_j$ with $\forall l \exists m \forall k > m\ x(k) \notin v(l, m, k)$. To construct

x, for every l find a number m_l such that $\forall k > m_l\ |v_k| > l^2$, and then choose elements $x(k) \in v_k \setminus \bigcup\{v(l, m_l, k) : m_l < k\}$. $\square$

Proposition 3.9. *If for every $i \in \omega$ there is m such that for all $k > m$, the ϕ_k-mass of union of $< i$ many sets is not bigger than the maximum of their mass $+1$, then the ideal $I_{\vec\phi}$ satisfies the LK-property.*

Proof. The argument follows closely the previous proofs, and I will only outline it. We will need a fact proved essentially in [11, Claim 4.4.4]. It does not use the assumptions on the sequence of submeasures $\vec\phi$.

Claim 3.10. *If B is a positive set and $l \in \omega$ is a number, then there is $m = m(l, B)$ such that for every sequence $v_k \subset u_k : k \in \omega$ of sets of respective ϕ_k mass $< k$, the set $\{x \in B : \forall k > m\ x(k) \notin v_k\}$ is still I-positive. Moreover, if $B = \bigcup_m B_m$ then there is m such that $m(l, B_m) \leq m$.*

Proof. If the first part failed for B, l, then for every number $m \in \omega$ there would be sets $v(m, k) : k > m$ of ϕ_k-mass $< l$ such that the set $B_m = \{x : \forall k > m\ x(k) \notin v(m, k)\}$ is in the σ-ideal $I_{\vec\phi}$. But then $B = \bigcup_m B_m \cup \{x : \forall m \exists k > m x(k) \in v(m, k)\}$ would be in the σ-ideal as well, contradiction.

For the second part, if $m(l, B_m) > m$ then one can find sets $v(m, k) : k > m$ witnessing this, i.e. $v(m, k) \subset u_k$ is of ϕ_k-mass at most l and $C_m = B_m \setminus \{x \in \Pi_n u_n : \exists k > m\ x(k) \in v(m, k)\} \in I_{\vec\phi}$. But then $B = \bigcup_m C_m \cup \{x \in \Pi_n u_n : \forall m \exists k > m\ x(k) \in v(m, k)\} \in I_{\vec\phi}$! $\square$

We will need again a notion of almost decision. If p is a condition in Mathias forcing and $\dot v_k : k \in \omega$ are names for subsets of $u_k : k \in \omega$, then I can find a direct extension $q \leq p$ which *almost decides* $\dot v_k : k \in \omega$ in the direction of $w_k : k \in \omega$ if for every number $k \in \omega$, the condition q after perhaps removing finitely many numbers from its infinite part, forces $\dot v_k = \check w_k$.

Suppose that $\vec B$ is a sequence of analytic subsets of $\Pi_n v_n$. For contradiction assume that the largest condition in the Mathias forcing forces $\limsup_{\dot a_{gen}} \vec B \notin I_{\vec\phi}$ and $\liminf_{\dot a_{gen}} \vec B \in I_{\vec\phi}$. Fix names $\dot v(l, m, k)$ for $l, m \in \omega$, $k > m$ such that it is forced that $\dot v(l, m, k) \subset \check u_k$ is a set of ϕ_k mass $< \check l$ and $\liminf_{\dot a_{gen}} \vec B \subset \{x \in \Pi_k u_k : \exists l \in \omega \forall m \in \omega \exists k > m\ x(k) \in \dot v(l, m, k)\}$. Fix continuous functions $f_j : \omega^\omega \to \Pi_n u_n$ such that $\vec B(j) = \mathrm{rng}(f_j)$. By induction on $i \in \omega$ build

- numbers n_i and infinite sets $a_i \subset \omega$ such that $n_0 < n_1 < \dots$, $a_0 \supset a_1 \supset \dots$ and $n_{i+1} \in a_i$;
- finite sequence $t_i^j : j \leq i$ of natural numbers such that for fixed j, the sequences t_i^j increase with respect to inclusion;
- numbers m_i, and sets $w(k) \subset u_k : m_0 < k \leq m_i$

so that the condition $\langle 0, \{n_j : j \in i\} \cup a_i \rangle$ forces $\dot v(j, m_j, k) \subset \check w(k)$ for all $j \in i$ and all $m_j < k \leq m_i$ and $\dot C_i = \limsup_{\dot a_{gen}} \vec B \cap \bigcap_{j \in i} f''_{n_j} O_{t_i^j} \cap \{x \in \Pi_n u_n : \forall m_0 <$

$k \leq m_i\ x(k) \notin w(k)\} \notin I_{\vec{\phi}}$ and $m(\dot{C}, i^2) < m_i$. If this can be done, in the end there will be a unique point $x \in \bigcap_{i,j} f''_{n_j} O_{t_i^j}$ and the condition $\langle 0, \{n_i : i \in \omega\}\rangle$ will force $\check{x} \in \liminf_{\dot{a}_{gen}} \vec{B}$ and at the same time $\forall j \forall k > m_j\ \check{x}(k) \notin \dot{v}(j, m_j, k)$, contradicting the choice of the names $\dot{v}(l, m, k)$.

The induction process is not difficult. Start with setting $a_0 = \omega$. For the induction step, suppose that the set a_i, the numbers $n_j, m_j : j \in i$, the nodes $t_i^j : j \in i$, and sets $y(k) : m_0 < k \leq m_{i-1}$ have been found. Find an infinite set $b \subset a_i$ such that for every set $c \subset \{n_j : j \in i\}$ the condition $\langle c, b\rangle$ almost decides the sequences $\dot{v}(j, m_j, k)$ in the direction of some $w(j, m_j, k, c)$ and let $w(k) = \bigcup_{c,j} v(j, m_j, k, c)$ for all $k > m_{i-1}$. Note that the ϕ-masses of these sets are $< i^2$ by the assumption on the subadditivity properties of submeasures on the sequence $\vec{\phi}$. Thus it is forced that the set $\dot{C}'_i = \dot{C}_i \cap \{x \in Pi_n u_n : \forall k > m_{i-1}\ x(k) \notin w(k)\}$ is $I_{\vec{\phi}}$-positive. Find an infinite set $\{n_j : j \in i\} \subset c \subset b$ such that the condition $\langle 0, c\rangle$ identifies proper extensions $t_{i+1}^j : j \leq i$ of nodes t_i^j such that the set $\dot{C}''_i = \dot{C}'_i \cap \bigcap_{j \leq i} f''_{n_j} O_{t_i^j}$ is positive. Use the second part of the claim in the Mathias extension to find an infinite subset a_{i+1} and a number $n_i \in c$ such that $\dot{C}''_i \cap \vec{B}(n_i)$ is $I_{\vec{\phi}}$-positive, and moreover for some number m_i, $m(\dot{C}''_i \cap \vec{B}(n_{i+1}), i^2) = m_i$ and at the same time the condition $\langle 0, \{n_j : j \in i+1\} \cup (b \setminus n_{i+1})\rangle$ still forces $\forall j \in i \forall m_{i-1} < k \leq m_i\ \dot{v}(j, m_j k) \subset w(k)$. Let $t_i^i = 0$ and continue with the induction.

□

With such a variety of positive results, it is perhaps natural to wonder which operations over ideals preserve the LK property. I will mention the union and intersection.

Proposition 3.11. *Suppose that $I_n : n \in \omega$ are σ-ideals on a Polish space X and each of them has the LK property. Then $I = \bigcap_n I_n$ has the property as well.*

Proof. Suppose that $\vec{B}$ is a sequence of analytic sets. Either there is an infinite set $a \subset \omega$ and a number n such that $\liminf_a \vec{B} \notin I_n$, in which case $\liminf_a \vec{B} \notin I$ and $\vec{B}$ is not a counterexample to the LK property of I. Otherwise, one can use the LK property of I_n's inductively to build a decreasing sequence $a_n : n \in \omega$ of infinite sets such that for every number $n \in \omega$, $\limsup_{a_n} \vec{B} \in I_n$. Let $a \subset \omega$ be any infinite diagonalization of the sequence $a_n : n \in \omega$. Since $\limsup_n a\vec{B} \subset \bigcap_n \limsup_{a_n} \vec{B}$, this set belongs to the ideal I and again, $\vec{B}$ is not a counterexample to the LK property. The proposition follows. □

Thus, properness of the quotient P_I is not necessary for the LK property of the ideal I. [11, Section 4.3.7] constructs a decreasing sequence of Ramsey capacities on the Cantor space. The σ-ideal I of sets simultaneously null for all of them has the LK property by the previous proposition and Proposition 3.5, while the quotient is not proper by [11, Proposition 2.2.6].

The operation of union (and generation) of σ-ideals is much more slippery, and I will state an open question.

Question 3.12. Suppose that I, J are σ ideals on a Polish space X with LK property. Does the ideal geenrated by $I \cup J$ have the LK property?

4 Parametric LK property

In[3], Szymon Głab introduces a parametric LK property. It turns out that a strong version of such parametrization follows directly from the LK property itself. I will need a definition.

Definition 4.1. Let I be a σ-ideal on a Polish space X, and J a σ-ideal on a Polish space Y. The *J-parametrized LK property* of I is the following statement: For every sequence $\vec{B}$ of analytic subsets of $Y \times X$, there is a Borel J-positive subset $C \subset Y$ and an infinite set $a \subset \omega$ such that either $\forall y \in B \limsup_a \vec{B}_y \in I$, or $\forall y \in B \ \liminf_a \vec{B}_y \notin I$.

Proposition 4.2. *Suppose that I, J are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideals on Polish spaces X, Y, and ZFC proves that P_J is proper, bounding, and does not add independent reals, and I has LK property. Then ZFC proves that I has the J-parametrized LK property.*

Proof. I will include only a sketch of the argument. Let $\vec{B}$ be a countable sequence of analytic subsets of $Y \times X$. Consider the partition of $Y \times [\omega]^{\aleph_0}$ into three parts, $D_0 = \{\langle y, a\rangle : \limsup_a \vec{B}_y \in I\}$, $D_1 = \{\langle y, a\rangle : \liminf_a \vec{B}_y \notin I\}$, and $D_2 = Y \times [\omega]^{\aleph_0} \setminus (D_0 \cup D_1)$. The assumptions on the ideal J imply that J together with the Mathias null ideal has the rectangular Ramsey property, as in [11, Theorem 3.4.1]. Thus there is a Borel J-positive set $D \subset Y$ and an infinite set a such that $B \times [a]^{\aleph_0}$ is wholly contained in either D_0 or D_1 or D_2. It cannot be contained in D_2, since then, for any point $y \in C$, the sequence $\vec{B}_y \restriction a$ would contradict the LK property of the ideal I. Thus the rectangle has to be contained either in D_0 or in D_1, which completes the proof. □

This improves the results of [3], which proved this in the special case of $I = J =$ the ideal of countable sets.

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