Cardinal invariants of closed graphs[*]

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April 28, 2017

Abstract

We study several cardinal characteristics of closed graphs G on compact metrizable spaces. In particular, we address the question when it is consistent for the bounding number to be strictly smaller than the smallest size of a set not covered by countably many compact G-anticliques. We also provide a descriptive set theoretic characterization of the class of analytic graphs with countable coloring number.

1   Introduction

The theory of Borel and analytic graphs on Polish spaces is currently a fast growing field [12, 8]. In this paper, we contribute to the study of cardinal invariants associated with such graphs. Consider the following invariant:

Definition 1.1. Let G be a graph on a Polish space X.

1. A set $A \subset X$ is a *G-anticlique* if no two distinct points of A are G-connected;

2. $\kappa(G)$ is the minimum cardinality of a subset of X which is not covered by countably many compact G-anticliques.

If the whole space X is covered by countably many compact anticliques, then let $\kappa(G) = \infty$.

Clearly, $\kappa(G)$ is just the uniformity of the σ-ideal generated by compact G-anticliques. We consider the problem of comparing the invariant $\kappa(G)$ for various closed graphs G to the standard cardinal invariant $\mathfrak{b}$, the minimum cardinality of a subset of ω^ω which cannot be covered by countably many compact subsets of ω^ω. While the problem may sound somewhat arbitrary, it in fact connects in an elegant way with various known combinatorial and descriptive set theoretic problems.

Question 1.2. Characterize those closed graphs G for which the inequality $\mathfrak{b} < \kappa(G)$ is consistent with ZFC.

In order to resolve this question, we introduce another cardinal invariant, the *loose number* of a topological graph (Definition 3.2), and prove the main result of the paper:

Theorem 1.3. *Let G be a closed graph on a compact metrizable space X. If G has countable loose number, then in some generic extension $\mathfrak{b} < \kappa(G)$ holds.*

It may seem that the theorem just replaces one difficult concept with another. However, plenty of information is available on the loose number. It sits neatly between the known combinatorial characteristics of the graph, the chromatic and coloring numbers by Theorem 3.4. This immediately yields many informative examples such as locally countable graphs and acyclic graphs; the pre-existing work of [13, 10] provides some other natural examples connected with Euclidean spaces.

Fully characterizing the closed or analytic graphs with countable chromatic or loose numbers seems to be a very difficult problem. However, in the case of coloring number, there is a full characterization and a minimal analytic graph of uncountable coloring number:

Theorem 1.4. *There is a closed graph G_1 on a Polish space such that for every analytic graph G on a Polish space X, exactly one of the following occurs:*

1. G has countable coloring number;

2. there is a continuous injective homomorphism of G_1 to G.

As for the anatomy of the paper, in Section 2 we introduce the single step forcing to increase the cardinal invariant $\kappa(G)$; Section 3 connects its forcing properties with combinatorial properties of the graph G. Section 4 deals with the rather thorny question of iterating the single step forcing. In Section 5, we discuss numerous concrete examples. Section 6 contains the proof of the dichotomy theorem for the coloring number of analytic graphs.

We use the standard set theoretic notation of [7]. For a subset A of a topological space X, the symbol $\bar{A}$ stands for its closure. If $t \in 2^{<\omega}$ is a finite binary string, then the symbol $[t]$ stands for the basic clopen set $\{x \in 2^\omega \colon t \subset x\}$ of the Cantor space. The phrase "large enough structure" identifies the collection of all sets whose transitive closure has size $< 2^{2^{\mathfrak{c}}}$, equipped with the membership relation. Our graphs are non-oriented and do not contain multiplicities or loops; i.e. a graph G on a set X is a symmetric relation on X which has empty intersection with the diagonal. If the set X is equipped with a Polish topology, we say that the graph is closed, analytic etc. if it is a closed or analytic relation of the Polish space $(X \times X)\setminus$ the diagonal with the topology inherited from the product. An orientation of the graph G is an antisymmetric relation o on X whose symmetrization is equal to G. The o-outflow of any element $x \in X$ is the set $\{y \in X \colon \langle x, y \rangle \in o\}$.

Many of the results of the present paper appeared in the first author's Ph. D. thesis [1].

2 Single step forcing

A forcing notion representing a natural try to increase the cardinal invariant $\kappa(G)$ has been known for quite some time to several authors [6, Definition 3.3]:

Definition 2.1. Let X be a Polish space and G a closed graph on it. The poset P_G consists of all pairs $p = \langle a_p, o_p \rangle$ where $a_p \subset X$ is a finite G-anticlique and $o_p \subset X$ is an open set containing a_p as a subset. The ordering is defined by $q \leq p$ just in case $a_p \subset a_q$ and $o_q \subset o_p$.

The poset P_G has a canonical generic object: the closure $\dot{K}_{gen}$ of the union of the sets a_p where p ranges over all conditions in the generic filter.

Proposition 2.2. *Let X be a Polish space and G a closed graph on it. Then $P_G \Vdash \dot{K}_{gen} \subset X$ is a compact G-anticlique.*

Proof. Let d be a compatible metric for the space X. To show that the set $\dot{K}_{gen} \subset X$ is forced to be compact, it will be enough to show that it is totally bounded. For each real number $\varepsilon > 0$ let $D_\varepsilon = \{q \in P_G \colon$ there is a number n such that there is no collection of n many points of $\bar{o}_q$ which are pairwise at distance greater than ε from each other$\}$.

Claim 2.3. *The set $D_\varepsilon \subset P_G$ is open dense.*

Proof. It is immediate that the set D_ε is open. For the density, given any condition $p \in P_G$, for each $x \in a_p$ select an open set o_x containing x of d-diameter $< \varepsilon$, and let $q = \langle a_p, o_p \cap \bigcup_{x \in a_p} o_x \rangle \leq p$. To see that the condition q belongs to the set D_ε, note that every set of points of pairwise distance $> \varepsilon$ which is a subset of $\bar{o}_q$ can have size at most $|a_p|$. □

To show that $\dot{K}_{gen}$ is forced to be totally bounded, note that for each $\varepsilon > 0$ there must be a condition $p \in D_\varepsilon$ in the generic filter. Such a condition clearly forces that $\dot{K}_{gen} \subset \bar{o}_q$ and therefore $\dot{K}_{gen}$ cannot contain any infinite collection of points which are pairwise at distance greater than ε from each other.

Now we need to show that $\dot{K}_{gen}$ is forced to be a G-anticlique. For every real number $\varepsilon > 0$ write $D_\varepsilon = \{p \in P_G \colon$ any two points in $\bar{o}_p$ which are G-related must be at a distance less than ε of each other$\}$.

Claim 2.4. *D_ε is open dense in P_G.*

Proof. It is immediate that the set D_ε is open. For the density, given any condition $p \in P_G$, use the fact that the graph G is closed to find open neighborhoods o_x of each point $x \in a_p$ which are pairwise disjoint, of diameter $< \varepsilon$, and such that $x \neq y \in a_p$ implies $(o_x \times o_y) \cap G = 0$. It is not difficult to see that the condition $q = \langle a_p, o_p \cap \bigcup_{x \in a_p} o_x \rangle \leq p$ belongs to the set D_ε. □

Now suppose that $H \subset P_G$ is a generic filter and $x \neq y \in \dot{K}_{gen}$ are distinct points, at a distance $> \varepsilon$ from each other for some positive rational ε. By the claim and a genericity argument, there is a condition $p \in P$ in the filter H which

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belongs to the set D_ε. Then, $\dot{K}_{gen} \subset \bar{o}_p$, and by a Mostowski absoluteness argument between V and $V[H]$, no two points of $\bar{o}_p$ (in particular, x and y) which are at a distance greater than ε from each other can be G-connected. Since the points x, y were arbitrary, this shows that the set $\dot{K}_{gen}$ is forced to be a G-anticlique. $\square$

The poset P_G is uniquely qualified to resolve our motivating Question 1.2. This follows from the following theorem, together with the absoluteness and iteration results of Section 4:

Theorem 2.5. *Let G be a closed graph on a compact metrizable space X. If $\mathfrak{b} < \kappa(G)$, then the poset P_G adds no dominating reals.*

Proof. Suppose towards a contradiction that $p \Vdash \dot{z} \in \omega^\omega$ modulo finite dominates all ground model elements of ω^ω for some condition $p \in P_G$ and a P_G-name $\dot{z}$ for an element of ω^ω. Let $F \subset \omega^\omega$ be an unbounded set of size $\mathfrak{b}$. Let M be an elementary submodel of a large enough structure of size $\mathfrak{b}$ containing $p, \dot{z}, F$ as elements and F as a subset. Let b be a countable set of compact G-anticliques such that $X \cap M \subset \bigcup b$. Let N be a countable elementary submodel of a large structure containing $p, \dot{z}, b$ as elements. Let $y \in F$ be a function which is not modulo finite dominated by any element of $\omega^\omega \cap N$. By the elementarity of the model M, there is a condition $q \in P_G \cap M$ and a natural number m such that $q \Vdash \forall k > m \ \dot{z}(k) > \check{y}(k)$.

Now, choose a one-to-one enumeration $a_q = \{x_i : i \in j\}$ of the anticlique in the condition q. There must be P_i, O_i, K_i for $i \in j$ such that

- P_i, O_i are basic open subsets of X and $K_i \in b$ are compact G-anticliques;

- $o_q \supset P_i \supset \bar{O}_i$, the sets $\bar{O}_i$ are pairwise disjoint and G-disconnected;

- $x_i \in O_i \cap K_i$.

Note that this sequence of objects belongs to the model N. Now, for each number $k > m$, consider the set c_k of those numbers $l \in \omega$ such that for some condition $r_l \in P_G$, $r_l \Vdash \dot{z}(k) > l$ holds, $\bigcup_i P_i \subset o_{r_l}$, and the set a_{r_l} can be listed as $\{x_i^l : i \in j\}$ so that for each $i \in j$, $x_i^l \in O_i \cap K_i$ holds. The key claim:

Claim 2.6. *The set c_k is finite.*

Proof. Suppose towards a contradiction that the set c_k is infinite, and for each number $l \in c_k$ select a condition $r_l \in P_G$ witnessing the membership in c_k, and let x_i^l for $i \in j$ denote the unique element of $a_r \cap O_i \cap K_i$. Using the compactness of the space X and thinning out the set c_k if necessary, we may assume that the points x_i^l converge to some $\hat{x}_i \in X$ for each $i \in j$. Note that as the G-anticliques K_i are closed, $\hat{x}_i \in K_i$ holds.

Now, consider the condition $s \in P_G$ given by the demands $a_s = \{\hat{x}_i : i \in j\}$ and $o_s = \bigcup_{i \in j} P_i$. This is indeed a condition: the set a_s is a G-anticlique by the choice of the basic open sets O_i. We will reach the contradiction by showing that s forces infinitely many of the conditions r_l into the generic filter. This

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is of course impossible since then s forces that there is no value that the name $\dot{z}(k)$ can attain.

Suppose that $t \leq s$ is a condition and $\hat{l}$ is some natural number; we must find a number $l > \hat{l}$ and a lower bound of t and r_l. To find this number l, use the fact that the graph G is closed to find respective neighborhoods $\hat{O}_i$ of points $\hat{x}_i$ for $i \in j$ such that $\hat{O}_i \subset o_t$ and for all points $x \neq \hat{x}_i$, x has no G-neighbors in the set $\hat{O}_i$. Find a number $l > \hat{l}$ so large that for each $i \in j$ the points x_i^l belong to the sets $\hat{O}_i$ for each $i \in j$. Observe that the set $a_t \cup a_{r_l}$ then must be a G-anticlique: if $i \in j$ and $x \neq \hat{x}_i$ is a point in a_t, then x_i^l is not G-connected to x since $x_i^l \in \hat{O}_i$, and x_i^l is not G-connected to $\hat{x}_i$ since both belong to the same compact anticlique K_i. It now follows immediately that $\langle a_t \cup a_{r_l}, o_t \cap o_{r_l} \rangle$ is a condition in the poset P_G and a lower bound of t, r_l. □

Now note that the sequence $\langle c_k \colon k > m \rangle$ belongs to the model N by elementarity, and so the model N contains some function $z \in \omega^\omega$ such that for each $k > m$ returns a value larger than $\max(c_k)$. Note also that $y(k) \in c_k$ holds as the condition q witnesses the membership. This means that the function $z \in N$ modulo finite dominates the function y, contradicting the choice of y. □

3   Combinatorics

Theorem 2.5 does not shed any light on how to actually evaluate the critical forcing properties of the poset P_G in any specific case. It turns out though that the forcing properties of the poset P_G faithfully reflect certain combinatorial cardinal invariants of the graph G. In order to state the interesting correspondence theorem, we must introduce the relevant invariants and forcing features.

Definition 3.1. [5] Let G be a graph on a set X. The *chromatic number* $\chi(G)$ of the graph G is the smallest cardinality of a collection of G-anticliques covering the space X.

If the set X is equipped with a topology and the graph G is open then the closure of any anticlique is again an anticlique. However, in most interesting closed graphs, anticliques cannot be in general enclosed by closed, Borel, or analytic anticliques. Thus, constructing a cover of the space by G-anticliques becomes a process in which the axiom of choice must be considered.

Definition 3.2. Let G be a graph on a topological space X. A *G-loose set* is a set $A \subset X$ such that for every point $x \in X$ there is an open neighborhood O of x containing no elements of the set A which are G-connected to x. The *loose number* $\lambda(G)$ of the graph G is the smallest cardinality of a collection of G-loose sets covering the space X.

A rather primitive example of a G-loose set is a closed G-anticlique. Not every G-loose set needs to be an anticlique, but every G-loose set is a union of countably many anticliques. On the other hand, one can find graphs in which

there are G_δ-anticliques which are not unions of countably many G-loose sets, see Example 5.2.

The last relevant cardinal invariant of a graph G is the coloring number. There are several equivalent ways to define it. To shorten the arguments, we use a definition which may give different values from others in the case these values are finite. This wrinkle is inconsequential for this paper.

Definition 3.3. [5] Let G be a graph on a set X. The *coloring number* $\mu(G)$ of G is the smallest cardinal κ such that there is an orientation of the edges of G such that the outflow of each vertex has size $< \kappa$.

A good example of a graph with countable coloring number is a locally countable graph. To construct the orientation, choose an enumeration of each connected component by natural numbers, and within the component, orient the edges towards the vertex with a smaller index in that enumeration. Another example of a graph with countable coloring number (in fact, coloring number equal to 2) is an acyclic graph. In each of it connected components, choose a single vertex and orient the edges within the component towards the chosen vertex.

While the chromatic and coloring numbers have been studied for many years, the loose number is a new concept. Note that unlike the chromatic and coloring numbers it depends on the topology of the underlying space. The following theorem shows the important implications among the three concepts.

Theorem 3.4. *Let G be a graph on a Polish space X. Then $\mu(G) \leq \aleph_0$ implies $\lambda(G) \leq \aleph_0$, which in turn implies $\chi(G) \leq \aleph_0$.*

The implications cannot be reversed even in the case of closed graphs on Polish spaces, which is clear from the examples in Section 5. The theorem makes the loose number look suspiciously close to the list-chromatic number, but this is a red herring: Corollary 6.4 below shows that for analytic graphs G, the list-chromatic number is countable just in case its coloring number is countable, and therefore the list-chromatic number is not useful for the present discussion.

Proof. Suppose first that the coloring number of the graph G is countable. Let o be an orientation of the edges of G such that the o-outflow of any point $x \in X$ is finite. To each point $x \in X$ assign a basic open set $f(x) \subset X$ containing x such that its closure contains none of the points in the finite o-outflow of x. For every basic open set $O \subset X$ let $A^O = \{x \in X : f(x) = O\}$.

Claim 3.5. *The set A^O is G-loose.*

Proof. If this failed, there would be a point $y \in X$ such that each neighborhood of y contains some point of A^O not equal to y and G-connected to y. In particular, $y \in \bar{A}^O \subset \bar{O}$. Let P be an open neighborhood of y containing none of the points in the finite o-outflow of y, and let $x \in A^O \cap P$ be a point G-connected to y. The edge $\{x, y\} \in G$ cannot be oriented towards the point x by the choice of the neighborhood P. It also cannot be oriented towards the point y since $y \in \bar{O}$ and no points in $\bar{O}$ are in the outflow of the point x. A contradiction. $\square$

Clearly, the G-loose sets A^O, as O varies over some fixed countable basis of the space X, cover the whole space, and so the loose number of the graph G is countable.

Now, suppose that the loose number of G is countable, and let $X = \bigcup_n A_n$ be a cover of the space X by countably many G-loose sets. For each $n \in \omega$ and each point $x \in X$, let $f_n(x) \subset X$ be some basic open set containing x and no points of A_n which are G-connected to x. For every $n \in \omega$ and every basic open set $O \subset X$, let $A_n^O = \{x \in A_n \colon f_n(x) = O\}$.

Claim 3.6. *The set A_n^O is a G-anticlique.*

Proof. Suppose that $x \neq y$ are distinct points in the set A_n^O. Both of the points belong to both A and O. By the definition of the function f_n, no elements of $A \cap O$ are connected to x, in particular y is not connected to x. The claim follows. $\square$

Clearly, for each number $n \in \omega$ the sets A_n^O cover the set A_n as O varies over some fixed countable basis of the space X. Thus, $X = \bigcup_{n,O} A_n^O$ is a countable cover of the whole space by countably many G-anticliques and so the chromatic number of G is countable. $\square$

The forcing properties of the poset P_G connected to the combinatorial concepts listed above are the following:

Definition 3.7. Let $\langle P, \leq \rangle$ be a partial ordering. Let $A \subset X$. The set A is *centered* if for every finite set $b \subset A$ there is a condition $q \in P$ such that for every $p \in b$, $q \leq p$. The poset P is σ-*centered* if it can be written as a union of countably many centered sets.

Definition 3.8. Let $\langle P, \leq \rangle$ be a partial ordering. Let $A \subset X$. The set A is *liminf centered* if for every sequence $\langle p_i \colon i \in \omega \rangle$ of elements of A, there is a condition $q \in P$ such that for every $r \leq p$ the set $\{i \in \omega \colon p_i$ is compatible with $r\}$ is infinite. The poset P is σ-*liminf-centered* if it can be written as a union of countably many liminf centered sets.

Both σ-centeredness and σ-liminf-centeredness imply c.c.c. since no centered set can contain two incompatible elements, and no liminf-centered set can contain an infinite antichain. In general, there are no implications between σ-centeredness and σ-liminf-centeredness: for example, the Hechler forcing is σ-centered but not σ-liminf-centered, while the random forcing is σ-liminf-centered but not σ-centered.

Theorem 3.9. *Let G be a closed graph on a compact metrizable space X.*

1. *[6, Lemma 3.6], [15, Theorem 19.6] P_G is c.c.c. iff G contains no perfect clique;*

2. *P_G is σ-centered iff $\chi(G) \leq \aleph_0$;*

3. *P_G is σ-liminf-centered iff $\lambda(G) \leq \aleph_0$.*

Proof. To see the left-to-right implication of (2), let $P_G = \bigcup_n A_n$ be a countable union of centered sets, and for every number $n \in \omega$ let $B_n = \{x \in X : \langle\{x\}, X\rangle \in A_n\}$. It is immediate that $X = \bigcup_n B_n$ holds; we must show that each set B_n is a G-anticlique. This, however, is immediate since any edge in B_n would result in a pair of incompatible conditions in P_G.

To see the right-to-left implication of (2), suppose that $X = \bigcup_n B_n$ be a countable union of countably many G-anticliques. Say that a finite sequence $t = \langle P_i, O_i, n_i : i \in j\rangle$ is *good* if P_i, O_i are basic open subsets of X, $\bar{O}_i \subset P_i$, the sets $\bar{O}_i$ are pairwise disjoint and G-disconnected, and $n_i \in \omega$. For each good sequence t, let $A_t \subset P_G$ be the set of those conditions $p = \langle a_p, o_p\rangle$ such that a_p can be listed as $\{x_i : i \in j\}$ with $x_i \in \bar{O}_i \cap B_{n_i}$, and $o_p \supset \bigcup_{i \in j} P_i$. Since there are only countably many good sequences, the proof will be complete if we show that the set A_t is centered.

Indeed, if $b \subset A_t$ is a finite set, then $a = \bigcup_{p \in b} a_b$ is a G-anticlique: if $x_0, x_1 \in a$ then either both of them belong to the same set $O_i \cap B_{n_i}$ and they are G-disconnected as B_{n_i} is an anticlique, or the belong to distinct such sets, and they are again G-disconnected since $\bar{O}_{i_0}$ is G-disconnected from $\bar{O}_{i_1}$ if $i_0 \neq i_1$. It follows that the pair $\langle a, \bigcup_{i \in j} P_i\rangle$ is a condition in P_G which is a lower bound of the set b.

To see the left-to-right implication of (3), let $P_G = \bigcup_n A_n$ be a countable union of liminf-centered sets, and for every number $n \in \omega$ let $B_n = \{x \in X : \langle\{x\}, X\rangle \in A_n\}$. It is immediate that $X = \bigcup_n B_n$ holds; we must show that each set B_n is G-loose. Indeed, suppose towards a contradiction that there is a sequence $\langle x_m : m \in \omega\rangle$ of points in some set B_n which converge to some point $y \in X$ and at the same time are connected to y. Since the set A_n is liminf-centered, there must be a condition $p \in P_G$ which forces infinitely many points on the sequence to belong to the generic compact anticlique $\dot{K}_{gen}$. However, then $p \Vdash \check{y} \in \dot{K}_{gen}$ as well, contradicting the fact that $\dot{K}_{gen}$ is forced to be a G-anticlique.

For the right-to-left implication of (3), suppose that $X = \bigcup_n B_n$ is a countable union of countably many G-loose sets. Say that a finite sequence $t = \langle P_i, O_i, n_i : i \in j\rangle$ is *good* if P_i, O_i are basic open subsets of X, $\bar{O}_i \subset P_i$, the sets $\bar{O}_i$ are pairwise disjoint and G-disconnected, and $n_i \in \omega$. For each good sequence t, let $A_t \subset P_G$ be the set of those conditions $p = \langle a_p, o_p\rangle$ such that a_p can be listed as $\{x_i : i \in j\}$ with $x_i \in \bar{O}_i \cap B_{n_i}$, and $o_p \supset \bigcup_{i \in j} P_i$. Since there are only countably many good sequences, the proof will be complete if we show that the set A_t is liminf-centered.

To this end, suppose that $\langle p_l : l \in \omega\rangle$ is a countable sequence of conditions in the set A_t. List a_{p_l} as $\{x_i^l : i \in j\}$ so that $x_i^l \in \bar{O}_i \cap B_{n_i}$. Use the compactness of the space X and thin out the countable sequence of conditions if necessary to make sure that the sequence $\langle x_i^l : l \in \omega\rangle$ converges to a point $\hat{x}_i \in X$, this for each index $i \in j$. Since each of the sets B_{n_i} is G-loose, there is a basic open neighborhood R_i of $\hat{x}_i$ such that no points in B_{n_i} in this neighborhood are connected to $\hat{x}_i$. Consider the condition $q \in P_G$ given by the following demands: $a_q = \{\hat{x}_i : i \in j\}$ and $o_q = \bigcup_{i \in j} P_i \cap R_i$. Since each point $\hat{x}_i$ belongs to $\bar{O}_i$ and

the sets $\bar{O}_i$ are pairwise G-disconnected, it is clear that a_q is a G-anticlique and so q is indeed a condition in P_G. It will be enough to show that q forces the set of all l for which p_l is in the generic filter to be infinite.

To this end, let $r \leq q$ be a condition and $\hat{l} \in \omega$ be a number. We need to produce a number $l > \hat{l}$ and a lower bound of the conditions r, p_l. To this end, for each $i \in j$ find open neighborhoods $S_i \subset X$ of $\hat{x}_i$ such that no point $x \in a_r$ with $x \neq \hat{x}_i$ has any G-neighbors in the set S_i. Find $l > \hat{l}$ so large that for each $i \in j$, the point x_i^l belongs to $S_i \cap R_i$. Now note that the set $a_r \cup a_{p_l}$ is a G-anticlique: a point x_i^l is not G-connected to any point $x \neq \hat{x}_i$ in a_r because $x_i^l \in S_i$ holds, and it is not G-connected to $\hat{x}_i$ either since $x_i^l \in R_i \cap B_{n_i}$ holds. It follows that the pair $\langle a_r \cup a_{p_l}, o_r \rangle \in P_G$ is the requested lower bound of conditions r and p_l. $\qquad\square$

For the purposes of this paper, the σ-liminf-centered property of posets has the following central implication:

Theorem 3.10. *Let P be a σ-liminf-centered poset. Then P does not add a dominating real. In fact, P preserves unboundedness of all subsets of ω^ω.*

Proof. Suppose towards a contradiction that $F \subset \omega^\omega$ is an unbounded set, $p \in P$ and $\dot{z}$ is a P-name for an element of ω^ω such that $p \Vdash \dot{z}$ modulo finite dominates all elements of F. Let $P = \bigcup\{A_n : n \in \omega\}$ be a union of liminf-centered sets. Let M be a countable elementary submodel of a large enough structure containing $F, p, \dot{z}$ and A_n for $n \in \omega$ as elements. Let $y \in F$ be a function which is not modulo finite dominated by any element of $\omega^\omega \cap M$, and let $q \leq p$ be a condition and $m \in \omega$ be a number such that $q \Vdash \forall k > m \ \dot{z}(k) > \check{y}_k$.

Let $n \in \omega$ be such that $q \in A_n$. For each number $k \in \omega$, let $c_k \subset \omega$ be the set of all numbers l such that there is a condition $r_{lk} \in A_n$ such that $r_{lk} \Vdash \dot{z}(k) > \check{l}$. The set c_k must be finite, since otherwise by the liminf-centeredness of the set A_n there would be a condition $r \in P$ which forces infinitely many of the conditions r_{lk} into the generic filter, which means that r forces that there is no value the name $\dot{z}(k)$ can attain. Now, the sequence $\langle c_k : k \in \omega \rangle$ belongs to the model M by elementarity, and so is the function $\hat{y} \in \omega^\omega$ given by $\hat{y}(k) = \max(c_k)$. By the definitions, for all $k > m$, the number $y(k)$ belongs to the set c_k. This means that $\hat{y}$ dominates y modulo finite, contradicting the choice of the function y. $\quad\square$

The reward for all the work in this section is the following corollary, which shows that closed graphs with countable loose number behave well in an important respect:

Corollary 3.11. *Suppose that G is a closed graph on a compact metrizable space X. If X can be written as a countable union of G-loose sets, then the poset P_G is c.c.c. and adds no dominating real.*

4 Iteration

This section is devoted to the problem of iterating the poset P_G without adding dominating reals, which we find quite tricky. The main difficulty is that in the

absence of a characterization of countable loose number of closed or analytic graphs in descriptive theoretic terms, even if the graph G has countable loose number in the ground model, there is no guarantee that it will maintain this property in the intermediate generic extensions arising in the iteration. Failing that, it could occur that the poset P_G destroys some unbounded sequences in the intermediate extensions, and the iteration adds a dominating real after all. We could not find any example of such a situation as all closed graphs of countable loose number we know of possess this property in all generic extensions. Still, the difficulty forced us to consider a rather unlikely workaround. It starts with the following natural definition:

Definition 4.1. A pair $\langle P, \leq \rangle$ is a Borel partial order if the relation $\leq$ is a partial ordering on P, there is a Polish space X such that $P \subset X$ is Borel, and the relations $\leq$, compatibility, and incompatibility are Borel subsets of $X \times X$.

As a particularly relevant example, if G is a closed graph on some Polish space X, then the poset P_G can be easily viewed as a Borel partial ordering. Note that every Borel partial ordering has a natural interpretation in every generic extension, as a straightforward application of the Shoenfield absoluteness shows. An interesting question appears, which of the forcing properties of the poset P are absolute throughout all forcing extensions? The most important general fact in this direction is the following classical absoluteness theorem:

Fact 4.2. [3, Corollary 3.6.9] *Let $\langle P, \leq \rangle$ be a Borel partial order. If in some generic extension P is c.c.c. then in all generic extensions P is c.c.c.*

For our main theorem, the key point is the absoluteness of adding no dominating reals. This is handled by the following:

Fact 4.3. [16] *Let $\langle P, \leq \rangle$ be a Borel c.c.c. partial order. If in some generic extension the poset P adds no dominating reals, then in all generic extensions, the poset P preserves all unbounded sequences of elements of ω^ω which are modulo finite increasing, consist of increasing functions, and have regular uncountable length greater than ω_1. If there is a proper class of Woodin cardinals, then the condition on the length of the unbounded sequences can be dropped.*

The conclusion of the fact is rather odd. In the presence of a proper class of Woodin cardinals, the demand on the length of the unbounded sequence disappears, as it does for large, more specific classes of Borel c.c.c. partial orders in ZFC. Still, the conclusion is good enough to dovetail with the following iteration theorem:

Fact 4.4. [3, Lemma 6.5.3] *Let $F \subset \omega^\omega$ be a unbounded family such that every countable subset of F has an upper bound in F in the modulo finite domination ordering. Let $\langle P_\alpha \colon \alpha \leq \lambda, \dot{Q}_\alpha \colon \alpha \in \lambda \rangle$ be a finite support iteration of c.c.c. forcings such that each iterand preserves the unboundedness of F. Then the whole iteration preserves the unboundedness of F.*

Now we are ready to prove Theorem 1.3. Suppose that G is a closed graph on a compact metrizable space X with countable loose number. The poset P_G is σ-liminf-centered by Theorem 3.9, therefore c.c.c. and does not add dominating reals. By Facts 4.2 and 4.3, the poset P_G remains c.c.c. and preserves all increasing unbounded sequences of regular length $> \omega_1$ in all forcing extensions. Pick regular cardinals $\omega_1 < \kappa < \lambda$. First, use a c.c.c. forcing of size κ to add a sequence $\vec{y} = \langle y_\alpha : \alpha \in \kappa \rangle$ which consists of increasing functions in ω^ω, is modulo finite increasing and unbounded. Then, iterate the poset P_G λ-many times with finite support. By Fact 4.2, the iterands are c.c.c. and so is the whole iteration. By Fact 4.3, the iterands preserve the unboundedness of the sequence $\vec{y}$, and so does the whole iteration by Fact 4.4.

In the resulting model, the bounding number $\mathfrak{b}$ is $\leq \kappa$, since the sequence $\vec{y}$ is unbounded there. It is also true that $\kappa(G) \geq \lambda$: whenever $A \subset X$ is a set of size $< \lambda$, by a chain condition argument there is an ordinal $\alpha < \lambda$ such that the set A belongs to the model obtained after the α-th stage of the iteration. Then, look at the generic compact G-anticliques $K_n \subset X$ for $n \in \omega$ obtained at the respective $\alpha + n$-th stages of the iteration. A genericity argument shows that $A \subset \bigcup_n K_n$ must hold, and so the set A is not a witness to $\kappa(G) < \lambda$.

The method of proof brings up the following question:

Question 4.5. Suppose that G is a closed graph on a compact metrizable space X. Let λ be a regular cardinal such that $\lambda^{\aleph_0} = \lambda$. If $\mathfrak{b} < \kappa(G)$ holds in some extension, does $\aleph_1 = \mathfrak{b} < \kappa(G) = \lambda$ hold in some extension?

In view of Theorem 2.5, this question really asks whether the preservation of unbounded sequences of length ω_1 by Borel c.c.c. forcings is suitably absolute in ZFC.

5   Examples

It is not entirely easy to come up with closed graphs for which the concepts introduced in Section 2 exhibit nontrivial interplay. This section is devoted to a number of examples that illustrate the various fault lines.

One class of closed graphs is generated by sequences of continuous functions. If X is a compact metrizable space and $f_n \colon X \to X$ are continuous functions such that for each point $x \in X$ the values $f_n(x)$ for $n \in \omega$ converge to x, one can consider the *associated graph* G on X which connects points $x \neq y$ just in case there is $n \in \omega$ such that $f_n(x) = y$ or $f_n(y) = x$. The assumptions on the sequence of functions easily imply that the graph G is closed. Such graphs have coloring number $\leq \aleph_1$ (just orient the edges from x to $f_n(x)$) and as such cannot contain cliques of size $\aleph_2$, and by an absoluteness argument they cannot contain any perfect cliques. Our first example, separating the uncountable chromatic number of G from the existence of perfect cliques in G, belongs to this class:

Example 5.1. [14] Let $\langle a_n : n \in \omega \rangle$ be pairwise disjoint subsets of ω, for each $n \in \omega$ let $g_n \in \omega^\omega$ be the increasing enumeration of $a_n \cup n$, and let $f_n \colon 2^\omega \to 2^\omega$

be the continuous function defined by $f_n(x) = x \circ g_n$. The associated closed graph G has no perfect cliques and uncountable chromatic number. Thus the poset P_G is c.c.c. but not σ-centered.

We do not know if $\kappa(G) \leq \mathfrak{b}$ holds in ZFC for the above graph.

Proof. Let B_n for $n \in \omega$ be G-anticliques; we must find a point $x \in 2^\omega \setminus \bigcup_n B_n$. For this purpose, by induction on $n \in \omega$ build binary strings $t_n \in 2^{<\omega}$, numbers $m_n \in \omega$ and points $x_n, y_n \in 2^\omega$ such that

- $0 = t_0 \subset t_1 \subset \ldots$ and $t_n \subset y_n, x_n$;

- either $B_n \cap [t_{n+1}] = 0$ or else $y_n \in B_n$;

- there is $x_n \in [t_n]$ such that for every $i \in n$, $f_{m_i}(x_n) = y_n$.

Once the induction is performed, let $x = \bigcup_n t_n \in 2^\omega$. By the continuity of the functions f_m for $m \in \omega$ it follows from the third item of the induction hypothesis that $f_{m_i}(x) = y_i$ for all $i \in \omega$. From the second item of the induction hypothesis, it follows that $x \notin B_i$ for any $i \in \omega$ as required.

To perform the induction, start with $t_0 = 0$ and $x_0 \in 2^\omega$ arbitrary. Now suppose that t_n, x_n as well as y_i and m_i for $i \in n$ have been found. Find a natural number m_n greater than all of m_i for $i \in n$ and $|t_n|$ and let $t_{n+1} = x \upharpoonright m_n$. The construction now splits into two cases. If $B_n \cap [t_{n+1}] = 0$ then let $x_{n+1} = x_n$ and $y_n = f_{m_n}(x)$; this successfully completes the induction step in this case. Otherwise, pick a point $y_n \in B_n \cap [t_{n+1}]$ and let $x_{n+1} \in 2^\omega$ be the point which is equal to x_n except at the entries in the set a_{m_n} where it satisfies $y_n = x_{n+1} \circ g_{m_n}$. This completes the induction step. $\square$

Our second example is again generated by a countable collection of continuous maps. This time, it separates the countable chromatic number from the countable loose number:

Example 5.2. For every natural number $n \in \omega$, let $f_n \colon 2^\omega \to 2^\omega$ be the function defined by $f_n(x)(i) = x(i)$ if $i \in n$ and $f_n(x)(i) = 0$ if $i \notin n$. The associated closed graph G on 2^ω has countable chromatic number, its loose number is equal to $\mathfrak{d}$ and $\kappa(G) = \mathfrak{b}$, provably in ZFC.

Proof. To see that the chromatic number of the graph G is countable, let $A \subset 2^\omega$ be the countable set of all binary sequences which are eventually zero. The set $2^\omega \setminus A$ is a G-anticlique by the definition of the graph G, and so 2^ω can be written as a countable union of (even Borel) G-anticliques: $2^\omega = A \cup \bigcup_{x \in A} \{x\}$. For the evaluation of the other cardinal invariants, a claim will be helpful:

Claim 5.3. *If $B \subset 2^\omega$ is a G-loose set then $\bar{B} \setminus A \subset 2^\omega$ is a closed subset of 2^ω.*

Proof. If the conclusion fails, there must be a point $x \in A$ and points $y_n \in B$ for $n \in \omega$ such that $y_n \neq x$ and $\lim_n y_n = x$. A review of the definition of the graph G shows that all but finitely many points y_n are G-connected with x, showing that B is not G-loose. $\square$

Now, to see that $\lambda(G) \geq \mathfrak{d}$, suppose that $2^\omega = \bigcup_{i \in I} B_i$ is a union of G-loose sets. By Claim 5.3, $2^\omega \setminus A = \bigcup_{i \in I} \bar{B}_i \setminus A$ is a union of compact sets. Since the set $2^\omega \setminus A$ is homeomorphic to the Baire space, $|I| \geq \mathfrak{d}$ immediately follows. To see that $\lambda(G) \leq \mathfrak{d}$, just write $2^\omega \setminus A$ as a union of $\mathfrak{d}$ many compact sets. All of these sets are G-anticliques and therefore also G-loose sets. Adding the singletons from the countable set A to the cover, we get a cover of 2^ω by $\mathfrak{d}$ many G-loose sets.

To see that $\kappa(G) \leq \mathfrak{b}$, use the fact that $2^\omega \setminus A$ is homeomorphic to the Baire space again to find a set $B \subset 2^\omega \setminus A$ of size $\mathfrak{b}$ which cannot be covered by countably many compact subsets of $2^\omega \setminus A$. By Claim 5.3, it cannot be covered by countably many loose G-sets, in particular by countably many compact G-anticliques. To see that $\kappa(G) \geq \mathfrak{b}$, if $B \subset 2^\omega$ cannot be covered by countably many compact G-anticliques, then $B \cap (2^\omega \setminus A)$ cannot be covered by countably many compact subsets of $2^\omega \setminus A$ as desired. $\square$

Another interesting family of closed graphs arises from metrics on compact metrizable spaces. Let X be a compact metrizable space with a compatible metric d. Let $\langle r_n \colon n \in \omega \rangle$ be a sequence of positive real numbers converging to 0. The *associated graph* connects points $x, y \in X$ if $d(x, y) = r_n$ for some $n \in \omega$. In this class of closed graphs, perfect cliques are possible in zero-dimensional spaces. For example, if d is the usual least difference metric on 2^ω, then the set of all possible values of d forms a sequence converging to 0 and so even the whole space may be a clique in this case.

The most interesting representatives of the metric generated graphs are connected with the Euclidean metrics:

Example 5.4. [10, Theorem 7] Let $n \leq 3$ be a natural number, let d be the Euclidean metric on $[0, 1]^n$, let $\langle r_n \colon n \in \omega \rangle$ be a sequence of positive real numbers converging to 0, and let G be the associated closed graph on $[0, 1]^n$. The graph G has countable coloring number.

The dimensions higher than 3 surprisingly yield an example separating the countable coloring number from the countable loose number:

Example 5.5. [13] Let $n > 3$ be a natural number, let d be the Euclidean metric on $X = [0, 1]^n$, let $\langle r_n \colon n \in \omega \rangle$ be a sequence of positive real numbers converging to 0, and let G be the associated closed graph on $[0, 1]^n$. The graph G has uncountable coloring number but countable loose number.

Proof. To see that the coloring number is uncountable, choose a number $r \in \mathbb{R}$ such that for some $n \in \omega$ $\sqrt{2r} = r_n$ and $r < 1$, and consider the sets $A_0 = \{\langle x, y, 0, 0 \ldots \rangle \in X \colon x^2 + y^2 = r\}$ and $A_1 = \{\langle 0, 0, x, y, 0, 0 \ldots \rangle \in X \colon x^2 + y^2 = r\}$. These are uncountable sets such that each point of A_0 is G-connected with each point in A_1. Such sets cannot exist in graphs of countable coloring number, say by Theorem 6.2 below.

To see that the loose number of G is countable, note that [13] found a well-ordering $\leq$ of X such that for every point $x \in X$, the set $\{y \in X \colon y \leq x \wedge \langle x, y \rangle \in G\}$ is bounded away from x. For every number $m \in \omega$, let $A_m = \{x \in X \colon \forall y \leq$

$x \langle x, y \rangle \in G \to d(x, y) > 2^{-m}\}$. The choice of the well-ordering $\leq$ implies that $\bigcup_m A_m = X$; thus, it will be enough to argue that each set A_m is G-loose. To see this, suppose towards a contradiction that $z \in X$ is a point and $\langle x_i : i \in \omega \rangle$ is a sequence of points in A_m G-connected to some point z. Let $\varepsilon > 0$ be a real number such that for all $y \leq z$ which is G-connected to z, $d(y, z) > \varepsilon$. Then, for every $i \in \omega$, if $x_i \leq z$ then $d(x_i, z) > \varepsilon$, and if $x_i \geq z$ then $d(x_i, z) > 2^{-m}$, showing that the sequence is bounded away from the point z. $\qquad \square$

The metric spaces of finite dimension behave quite differently in this respect than the infinite dimensional ones [4]. The dimension fault line appears in the following example:

Example 5.6. Let X be a strongly infinite-dimensional compact metrizable space and let d be a compatible metric. Let $\langle r_n : n \in \omega \rangle$ be a sequence of positive reals converging to 0, and let G be the associated metric graph. The graph G has uncountable chromatic number.

In fact, in the usual metrizations of the Hilbert cube for example, this graph contains perfect cliques. We do not know if it is possible to have a strongly infinite-dimensional space with a compatible metric such that the associated metric graph has no perfect cliques.

Proof. The argument starts with an auxiliary claim:

Claim 5.7. *Let $K \subset X$ be a set, $\varepsilon > 0$, and $B \subset X$ be a G-anticlique. There is an open set $O \subset X$ such that $K \subset O$, every point of O is within ε-distance from some point in K, and the boundary of O has empty intersection with B.*

Proof. Replacing K with its closure we may assume that K is compact. Every point $x \in K$ is contained either (1) in some open ball of radius $< \varepsilon/2$ whose closure contains no elements of B, or (2) in some open ball of radius $< \varepsilon/2$ whose center is in B and whose radius belongs to the set $\{r_n : n \in \omega\}$. By a compactness argument, the whole set K is covered by finitely many balls of this type. Let O be the union of the finitely many balls. It is clear that $K \subset O$ and every point of O is within ε-distance from some point in X. Finally, the boundary of the set O is a subset of the union of the boundaries of the finitely many balls in the union, and none of them contain any elements of the anticlique B: in case (1) this occurs because the closure of the whole ball contains no elements of the set B, and in case (2), this occurs because B is a G-anticlique and the points on the boundary of the ball are G-related to the center of the ball which belongs to B. $\qquad \square$

Now, suppose that B_n for $n \in \omega$ are G-anticliques; we must produce a point $x \in X \setminus \bigcup_n B_n$. Use the infinite dimensionality of the space X to find an essential sequence $\langle K_n^0, K_n^1 : n \in \omega \rangle$; that is, K_n^0, K_n^1 are disjoint nonempty subsets of X for each n, and whenever $O_n \subset X$ for $n \in \omega$ are open sets such that $K_n^0 \subset O_n$ and $\bar{O}_n \cap K_n^1 = 0$ then the intersection of the boundaries of the sets O_n is nonempty. Now, use the claim to find, for each number $n \in \omega$, an

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open set $O_n \subset X$ such that $K_n^0 \subset O_n$ and $\bar{O}_n \cap K_n^1 = 0$ and the boundary of the set O_n has empty intersection with the anticlique B_n. By essentiality, the intersection of the boundaries of the sets O_n is nonempty, and any point in it belongs to $X \setminus \bigcup_n B_n$ as desired. $\square$

Finally, we owe the reader an example showing that the conclusion of Theorem 1.3 fails if one considers graphs only slightly more complicated than closed:

Example 5.8. Let G be the graph on 2^ω connecting sequences x, y if they differ on at most finitely many entries. The graph G is locally countable and therefore has countable coloring number. At the same time, $\kappa(G) \leq \mathfrak{b}$ holds in ZFC.

It is well-known that each measurable G-anticlique must have zero μ-mass, where μ is the Haar probability measure on 2^ω. Thus, no set of positive outer μ-mass can be covered by countably many G-anticliques. This shows that $\kappa(G) \leq \text{non(null)}$ holds in ZFC, in particular $\kappa(G) < \mathfrak{b}$ is consistent with ZFC.

Proof. Let $A \subset 2^\omega$ be the countable set of binary sequences which are eventually zero, so that $2^\omega \setminus A$ is homeomorphic to the Baire space. Let $F \subset 2^\omega \setminus A$ be a set of size $\mathfrak{b}$ which cannot be covered by countably many compact subsets of $2^\omega \setminus A$. We claim that F cannot be covered by countably many G-anticliques.

Indeed, if $K \subset 2^\omega$ is a compact G-anticlique, then it intersects the set A in at most one point, and so $K \setminus A$ is a union of countably many compact sets. It follows that if F were covered by countably many G-anticliques, it would be covered by countably many compact subsets of $2^\omega \setminus A$, which is impossible by the choice of the set F. $\square$

6 A dichotomy

The purpose of this section is to characterize those analytic graphs on Polish spaces for which the coloring number is countable. The evaluation of the coloring number for finite or infinite graphs is a fairly involved business, see [2]. Even in the case of a closed or analytic graph, the orientations witnessing the coloring number are typically obtained through a heavy use of the axiom of choice. However, the existence of such an orientation can be characterized by a simple formula. It turns out that there is a minimal analytic graph of uncountable coloring number, which is in addition a clopen graph on a (noncompact) σ-compact Polish space:

Definition 6.1. Let Y be the Polish space which is the disjoint union of $2^{<\omega}$, viewed as a discrete space, and 2^ω with its usual topology. G_1 is the graph on Y given by $G_1 = \{\{y \restriction n, y\} : y \in 2^\omega, n \in \omega\}$.

Theorem 6.2. *Let G be an analytic graph on a Polish space X. The following are equivalent:*

1. the coloring number of G is countable;

2. *for every countable set $a \subset X$, the set $\{x \in X : \exists^\infty y \in a \; x \; G \; y\}$ is countable;*

3. *there is no continuous injective homomorphism of G_1 to G.*

If there is a proper class of Woodin cardinals then the conclusion holds for all universally Baire graphs.

Proof. We will deal with the case of analytic graphs; the case of universally Baire graphs under a large cardinal assumption is left to the reader, as it needs no additional tricks.

The (1)→(2) direction does not use any definability assumptions on the graph G. If (1) holds and $a \subset X$ is a countable set, let o be an orientation of G in which the outflow of every point is finite, and let M be a countable elementary submodel of a large structure containing both a and o as elements. Since the set a is countable, it is also a subset of M by elementarity, and it will be enough to show that for every point $x \in X \setminus M$, x is connected with only finitely many elements of $X \cap M$. Indeed, if x were connected with infinitely many points in M, then there would be a point $y \in M$ which is not in the finite o-outflow of x and is connected with x. It follows that x is in the outflow of y; but, as the outflow of y is finite, this implies that $x \in M$ by the elementarity of the model M. A contradiction.

For the (2)→(1) direction, assume that (2) holds and prove the following claim:

Claim 6.3. *For every submodel M of a large enough structure containing the graph G and the space X as elements, for every point $x \in X \setminus M$, x is G-connected with only finitely many elements of the model M.*

Note that there is no cardinality restriction on the submodel M.

Proof. Suppose that M is an elementary submodel of some large structure containing the space X and the graph G. Let $H \subset \mathrm{Coll}(\omega, X \cap M)$ be a filter generic over V. One can then form the model $M[H]$, since $\mathrm{Coll}(\omega, X \cap M) = \mathrm{Coll}(\omega, X) \cap M$. Comparison of the models concerned yields the following:

- $M[H] \cap V = M$. This follows from the genericity of the filter H.

- $M[H] \models$ for every countable set $a \subset X$, the set $\{x \in X : \exists^\infty y \in a \; x \; G \; y\}$ is countable. This is because the given statement is coanalytic, true in V and one can apply the Mostowski absoluteness to the wellfounded model $M[H]$.

- $M[H] \models X \cap M$ is countable, and by the previous item there is an ω-sequence $z \in X^\omega$ in $M[H]$ such that $M[H] \models \forall x \notin \mathrm{rng}(z) \; \{y \in X \cap M : x \; G \; y\}$ is finite;

- $V[H] \models \forall x \notin \mathrm{rng}(z) \; \{y \in X \cap M : x \; G \; y\}$ is finite, since this property of the countable set $X \cap M$ and the sequence z is coanalytic, and by the Mostowski absoluteness it can be transfered from $M[H]$ to $V[H]$.

Now, if a point $x \in X$ in $V \setminus M$ is an arbitrary point, then $x \notin M[H]$ by the first item, so $x \notin \mathrm{rng}(z)$, and by the last item x is connected to only finitely many elements of $X \cap M$ as required in the claim. $\qquad\square$

Now, by induction on the cardinality of a set $A \subset X$ argue that the graph G restricted to A can be oriented so that the outflow of every point in A is finite. This is immediate if $|A| = \aleph_0$. Now suppose that $|A| = \kappa$ and the statement has been proved for all sets of size $< \kappa$. Choose a continuous increasing sequence $\langle M_\alpha \colon \alpha \in \mathtt{cf}(\kappa) \rangle$ of elementary submodels of a large structure such that $f, A \in M_0$, $|M_\alpha| < \kappa$ for every α, and $A \subset \bigcup_\alpha M_\alpha$. Use the inductive assumption to find an orientation o_α of the graph $G \restriction A \cap M_\alpha$ for each ordinal $\alpha \in \beta$ such that the outflow of any node is finite. Define an orientation o of $G \restriction A$ by orienting an edge $\langle x, y \rangle$ towards y just in case either the smallest ordinal α for which y appears in M_α is smaller that the smallest ordinal α for which x appears in M_α, or in the case that these two ordinals are equal to some α, then $\langle x, y \rangle$ is oriented in the same way by o as it is in o_α. Claim 6.3 immediately implies that the o-outflow of any node in A is finite as required.

Now, the negation of (3) implies the negation of (2) since in the graph G_1, all the non-isolated points of the underlying space are connected to infinitely many elements of the countable set of isolated points. To see how the negation of (2) implies the negation of (3), fix a countable set $a \subset X$ such that the set $B = \{x \in X \colon \exists^\infty y \in a \colon x\, G\, y\}$ is uncountable. Since the set B is analytic, it contains a perfect subset $C \subset B$ disjoint from a. Now, by induction on $t \in 2^{<\omega}$ build perfect sets $C_t \subset C$ and points $x_t \in a$ such that

- $t \subset s$ implies $C_s \subset C_t$, the set C_t has diameter $2^{-|t|}$ in some fixed complete compatible metric on X, and it is either disjoint from or a subset of the $|t|$-th basic open subset of X in some fixed enumeration of a countable topology base for X;

- for each $t \in 2^{<\omega}$, the sets $C_{t^\frown 0}$ and $C_{t^\frown 1}$ are pairwise disjoint;

- the points $x_t \in a$ are pairwise distinct and all points in C_t are G-connected with x_t.

To perform the induction step, suppose that the sets C_t and points x_t for $t \in 2^{\leq n}$ have been constructed. First, use standard arguments to find sets C'_s for each $s \in 2^{n+1}$ which satisfy the first two items above. Now, enumerate 2^{n+1} as $\{s_i \colon i \in j\}$ and by subinduction on i find points $x_{s_i} \in a$ such that they are pairwise distinct and also distinct from the points x_t for $t \in 2^{\leq n}$, and such that there are uncountably many elements of the set C'_{s_i} which are G-connected to the point x_{s_i}. The subinduction is easy to perform given the fact that the set a is countable and every element of C'_{s_i} is connected to infinitely many of its members. In the end, use the perfect set theorem to find a perfect set $C_{s_i} \subset C'_{s_i}$ of points connected to x_{s_i}. This completes the induction step.

Once the induction has been performed, consider the function $f \colon Y \to X$ given by the following description. If $t \in 2^{<\omega}$ then $f(t) = x_t$ and if $y \in 2^\omega$ then

$f(y)$ is the unique point in $\bigcap_n C_{y\restriction n}$. It is clear from the induction assumptions that f is a continuous injection from Y to X which is moreover a homomorphism of the graph G_1 to G. $\qquad\square$

Corollary 6.4. *For an analytic graph G on a Polish space X, the coloring number of G is countable if and only if the list-chromatic number of G is countable.*

Here, the list-chromatic number of G [5] is the smallest cardinal κ such that for every function F which assigns each point $x \in X$ a set of size $\geq \kappa$, there is a function f which assigns each point x and element of $F(x)$ such that for any two G-connected points $x_0, x_1 \in X$, $f(x_0) \neq f(x_1)$ holds. Komjáth [11] showed that consistently, the list-chromatic number of infinite graphs can be equal to the coloring number, and consistently, for all graphs of size $\aleph_1$, if the chromatic number is countable then so is the list-chromatic number. The theorem shows that no such antics appear in the case of analytic graphs and countable list-chromatic number.

Proof. Since for any graph G, the list-chromatic number is not greater than the coloring number, it is enough to show that the graph G_1 on the space $Y = 2^{<\omega} \cup 2^\omega$ has uncountable list-chromatic number. For this, let E be a (Borel) bijection between 2^ω and the set of all maps $g\colon 2^{<\omega} \to \omega$ such that $\forall t \in 2^{<\omega}\ g(t) > |t|$. Let F be the function which to each $t \in 2^{<\omega}$ assigns the set $\{n \in \omega\colon |t| < n\}$, and to each point $y \in 2^\omega$ assigns the infinite set $\{E(y)(y \restriction n)\colon n \in \omega\}$. The function F stands witness to the fact that the list-coloring number of the graph G_1 is uncountable.

To see this, if f is a function on the space Y which for each $y \in Y$ selects an element of $F(y)$, then there must be a point $y \in 2^\omega$ such that $E(y) = f \restriction 2^{<\omega}$. But then, there must be $n \in \omega$ such that $f(y) = E(y)(y \restriction n) = f(y \restriction n)$. Since $y \restriction n$ is G_1-connected to y, this completes the proof. $\qquad\square$

The equivalence of items (1) and (2) in Theorem 6.2 fails for graphs which are not definable, as the following example shows:

Example 6.5. There is a graph G on ω_1 such that the coloring number of G is uncountable, while for every countable set $a \subset X$, the set $\{x \in X\colon \exists^\infty y \in a\ x\ G\ y\}$ is countable.

Proof. For every countable limit ordinal α choose a set $c_\alpha \subset \alpha$ which is cofinal in it and of ordertype ω. The graph G on ω_1 consists of all pairs $\{\beta, \alpha\}$ such that α is a limit ordinal and $\beta \in c_\alpha$.

First of all, if $a \subset \omega_1$ is a countable set and α is any successor ordinal larger than $\sup(a)$, then no ordinal $\beta \in \omega_1$ is connected with infinitely many elements of α, verifying that the set $\{\beta \in \omega_1\colon \exists^\infty \gamma \in a\ \beta\ G\ \gamma\}$ is countable. At the same time, the coloring number of the graph G is uncountable: if o is some orientation of G, consider the regressive function f on ω_1 which assigns to each limit ordinal α an element of c_α which does not belong to the o-outflow of α. Apply Fodor's theorem to find a stationary set $S \subset \omega_1$ and an ordinal β such that $f(\alpha) = \beta$

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for all $\alpha \in S$. Now, if $\alpha \in S$ is any ordinal which is not in the outflow of β, there is no way of orienting the edge $\{\beta, \alpha\} \in G$ in a way consistent with the assumptions. $\qquad \square$

It is immediate from Theorem 6.2 that the concept of countable coloring number is $\mathbf{\Pi}_1^1$ on $\mathbf{\Sigma}_1^1$ in the following sense [9, Section 29.E]. If X is a Polish space and $A \subset \omega^\omega \times X \times X$ is an analytic set whose vertical sections are symmetric and reflexive relations, then the set $\{y \in \omega^\omega : A_y$ has countable coloring number$\}$ is coanalytic. The final computation in this paper shows that this is an optimal complexity bound:

Proposition 6.6. *The collection of closed graphs on 2^ω with countable coloring number is a complete coanalytic subset of the space $K((2^\omega)^2)$.*

Proof. It is enough to work with any zero-dimensional compact space in place of 2^ω as such a space is homeomorphic to a closed subspace of 2^ω. Let $X = \bigcup_n 2^n \times 2^n \cup (2^\omega \times 2^\omega)$ and its topology is generated by sets $O_{t,s}$ where for some $n \in \omega$ $t, s \in 2^n$ and $O_{t,s} = \{\langle u, v \rangle \in X : t \subset u, s \subset v\}$. For every tree $T \subset 2^{<\omega}$ consider the graph G_T connecting $\langle t, s \rangle \in X$ with $\langle u, v \rangle \in 2^\omega \times 2^\omega$ if $t \in T$, the last entry on t is 1, and $t \subset u, s \subset v$. It is not difficult to check that the map $T \mapsto G_T$ is a continuous map from the space of all trees to $K(X^2)$. I claim that the coloring number of G_T is countable just in case the tree T has no infinite branch with infinitely many unit entries. This will complete the proof of the theorem since the set of binary trees which have an infinite branch with infinitely many unit entries is a complete analytic set [9, Exercise 27.3].

Indeed, if T has no infinite branch with infinitely many unit entries, then orient edges in G_T so that they point from elements of $2^\omega \times 2^\omega$ to the pairs of finite binary sequences connected with them. The lack of infinite paths in T shows that the outflow of every node in this orientation of the graph G_T is finite, and so the coloring number of G_T is countable. On the other hand, if $x \in 2^\omega$ is an infinite path through the tree T which contains infinitely many units, then every point of the form $\langle x, y \rangle$ for $y \in 2^\omega$ is connected with infinitely many nodes in the countable collection $\bigcup_n 2^n \times 2^n$, showing that the coloring number of the graph G_T is uncountable by Theorem 6.2. $\qquad \square$

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