Canonical models for fragments of the axiom of choice*

Paul Larson † Miami University
Jindřich Zapletal ‡ University of Florida

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Abstract

We develop a technology for investigation of natural forcing extensions of the model $L(\mathbb{R})$ which satisfy such statements as "there is an ultrafilter" or "there is a total selector for the Vitali equivalence relation". The technology reduces many questions about ZF implications between consequences of the axiom of choice to natural ZFC forcing problems.

1 Introduction

In this paper, we develop a technology for obtaining certain type of consistency results in choiceless set theory, showing that various consequences of the axiom of choice are independent of each other. We will consider the consequences of a certain syntactical form.

Definition 1.1. A Σ^2_1 sentence Φ is *tame* if it is of the form $\exists A \subset \omega^\omega$ $(\forall \vec{x} \in \omega^\omega\ \exists \vec{y} \in A\ \phi(\vec{x}, \vec{y})) \wedge (\forall \vec{x} \in A\ \psi(\vec{x}))$, where ϕ, ψ are formulas which contain only numerical quantifiers and do not refer to A anymore, and may refer to a fixed analytic subset of 2^ω as a predicate. The formula ψ are called the *resolvent* of the sentence.

This is a syntactical class familiar from the general treatment of cardinal invariants in [11, Section 6.1]. It is clear that many consequences of Axiom of Choice are of this form:

Example 1.2. The following statements are tame consequences of the axiom of choice:

1. there is a nonprincipal ultrafilter on ω. The resolvent formula is "$\bigcap \text{rng}(x)$ is infinite";

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2. there is an infinite maximal almost disjoint family of subsets of ω. The resolvent formula is "$x_0 \cap x_1$ is finite";

3. there is a maximal selector on a fixed Borel equivalence relation;

4. there is a Hamel basis for the space of real numbers;

5. there is an ω_1 sequence of distinct reals;

6. a fixed Borel hypergraph of finite arity has countable chromatic number.

A typical tame Σ^2_1 sentence with resolvent ψ is often associated with a natural partial order P_ψ of countable approximations. Let P_ψ be the poset of countable sets $a \subset \omega^\omega$ satisfying $\forall x \in a\ \psi(x)$ ordered by reverse inclusion. By definitions, the poset P_ψ is σ-closed and (as a union of the generic filter) adds a subset $A \subset \omega^\omega$. For a typical tame sentence Φ it is the case that P_ψ forces the generic set A to be a witness for Φ. Note that the poset P_ψ often forces a stronger property of the generic set A for which witnesses may no longer provably exist in ZFC. A generic ultrafilter forced with countable approximations is a Ramsey ultrafilter, a generic injection from ω_1 to 2^ω is a surjection etc. Also, note that the poset P_ψ depends only on the resolvent of the tame Σ^2_1 sentence.

In the presence of large cardinals, it becomes natural to investigate the model $L(\mathbb{R})[A]$ to see how large fraction of the axiom of choice holds in it. The present paper provides a technology for doing this. We show that the questions about the theory of the model $L(\mathbb{R})[A]$ frequently reduce to rather interesting ZFC forcing problems. As a result, we prove many consistency theorems regarding non-implications between the tame consequences of the axiom of choice, which are always verified by the rather canonical models of the form $L(\mathbb{R})[A]$.

Theorem 1.3. (ZFC+LC) *Let U be a Ramsey ultrafilter on ω. In the model $L(\mathbb{R})[U]$*

1. *there are no infinite MAD families;*

2. *the quotient space $2^\omega/E_0$ is linearly orderable, but the quotients $2^\omega/E_2$ and $(2^\omega)^\omega/=^+$ are not linearly orderable;*

3. *there are no Hamel bases for $\mathbb{R}$ and no transcendence bases for $\mathbb{C}$.*

Theorem 1.4. (ZFC+LC) *Let E be a pinned Borel equivalence relation on a Polish space X, and let $S \subset X$ be a generic total selector for E. In the model $L(\mathbb{R})[S]$,*

1. *there are no ω_1 sequences of reals;*

2. *there are no infinite MAD families;*

3. *there are no nonatomic measures on ω;*

4. *there are no Hamel bases for $\mathbb{R}$.*

Theorem 1.5. (ZFC+LC) *Let A be a generic improved maximal almost disjoint family. In the model $L(\mathbb{R})[A]$,*

1. *there are no ω_1 sequences of reals;*
2. *there are no nonatomic measures on ω;*
3. *there are no total selectors for E_0.*

The terminology used in the paper follows the set theoretic standard of [5]. The letters LC denote a "suitable large cardinal assumption"; in all cases a proper class of Woodin cardinals is sufficient. A hypergraph on a set X is a subset of X^n for some $n \leq \omega$. A coloring of a hypergraph $Z \subset X^n$ is a map $c\colon X \to Y$ whose fibers do not contain any edges of Z. The hypergraph has countable chromatic number if there is a coloring $c\colon X \to \omega$. We use the nomenclature of [3] concerning equivalence relations; that is, E_0 is the modulo finite equality on 2^ω, E_1 is the modulo finite equality on $(2^\omega)^\omega$, and $=^+$ is the equivalence relation on $(2^\omega)^\omega$ connecting two points if they have the same range. A selector for an equivalence relation E on X is a set which meets every equivalence class in at most one point; a selector is total if it meets every equivalence class in exactly one point. The E quotient space is the set of all E-equivalence classes. In several places we consider the set of finite binary strings $2^{<\omega}$ with coordinatewise binary addition as a group, which naturally acts on 2^ω by coordinatewise binary addition, and the action extends to an action on subsets of 2^ω as well.

2 Independence

The key to the technology is the following definition.

Definition 2.1. Let Φ_0, Φ_1 are tame Σ^2_1 sentences with respective resolvents ψ_0, ψ_1. Say that witnesses for Φ_1 are *independent of* witnesses for Φ_0 if for witnesses $A_0, A_1 \subset \omega^\omega$ for Φ_0 and Φ_1 respectively, for every poset Q collapsing $2^\mathfrak{c}$ to $\aleph_0$ and for all Q-names τ_0, τ_1 for witnesses to Φ_0 and Φ_1 extending A_0, A_1 there is a number $n \in \omega$ and (in some generic extension) filters $G_i \subset Q$ generic over V for each $i \in n$ so that $\forall \vec{x} \in \bigcup_i \tau_0/G_i\ \psi_0(\vec{x})$ holds and $\forall \vec{x} \in \bigcup_{i\in n} \tau_1/G_i\ \psi_1(\vec{x})$ fails. Similarly we define a notion of independence of a witness A_0 from A_1.

The definition of independence may appear awkward, but most of its instances are interesting ZFC problems which typically can be answered in ZFC. The answers can be applied to evaluate the theory of various choiceless generic extensions of the model $L(\mathbb{R})$ via the following central theorem.

Theorem 2.2. (ZFC+LC) *Suppose that Φ_0, Φ_1 are tame Σ^2_1 sentences with respective resolvents ψ_0, ψ_1. Let $A_0 \subset \omega^\omega$ be a P_{ψ_0}-generic instance of Φ_0. In $V[A_0]$, if witnesses for Φ_1 are independent of A_0, then $L(\mathbb{R})[A_0] \models \neg\Phi_1$.*

The conclusion of the theorem remains in force in the rather undesirable case when P_{ψ_0} does not force the generic set to be a witness to Φ_0. Such a situation will never appear in this paper.

Proof. Work in the model $V[A_0]$. Suppose for contradiction that $L(\mathbb{R})[A_0]$ does contain a witness $A_1 \subset \omega^\omega$ for Φ_1. In such a case, there must be a name $\eta \in L(\mathbb{R})$ such that $A_1 = \eta/A_0$. The name η can be easily coded as a set of reals. Let T, U be some trees such that T projects into η and in all forcing extensions the trees T and U project into complements–such trees exist by the large cardinal assumption and [6, Theorems 3.3.9 and 3.3.14]. There must be some condition in P_{ϕ_0} forcing that η is a witness for Φ_1; for definiteness assume that every condition in P_{ϕ_0} forces this.

By the assumptions, A_1 is independent of A_0. Let δ be a Woodin cardinal greater than the size of P and let Q be the countably based stationary tower at δ which certainly collapses $2^{\mathfrak{c}}$. Let $j\colon V[H_0] \to M$ be the Q-name for the usual generic elementary embedding, let $\tau = j(A_0)$ and $\sigma = j(A_1)$. In some generic extension $V[A_0][G]$, let $G_i \subset Q\colon i \in n$ be generic filters such that $\forall \vec{x} \in \bigcup_i \tau/G_i\ \psi_0(\vec{x})$ holds while $\forall \vec{x} \in \bigcup_i \sigma/G_i\ \psi_1(\vec{x})$ fails.

Now, the model $\langle L(\mathbb{R}), \in, p(T)\rangle$ of $V[A_0][G]$ is elementarily equivalent to all the models $\langle L(\mathbb{R}), \in, p(jT)\rangle$ of the various models M/G_i–[6, Theorem 3.3.17]. The condition $\bigcup_i \tau/G_i$ belongs to P_{ψ_0}, and it forces $\bigcup_i \sigma/G_i$ to be a subset of $j\sigma$. This is impossible though since there is $\vec{x} \in \bigcup_i \sigma/G_i$ such that $\psi_1(\vec{x})$ fails. □

3 Adding a Ramsey ultrafilter

The most commonly encountered model of the form $L(\mathbb{R})[A]$ is the one obtained by forcing an ultrafilter with countable approximations. It is not difficult to see that the poset used is equivalent to the quotient algebra $\mathcal{P}(\omega)$ modulo finite. The model $L(\mathbb{R})[U]$ has been studied in [4, 2], where the authors show that an ultrafilter U is generic over $L(\mathbb{R})$ if and only if it is Ramsey and that the model satisfies the perfect set theorem for all sets. It is also known that a number of compact groups have the automatic continuity property there. On the other hand, any nonprincipal ultrafilter immediately yields nonmeasurable sets and sets without the Baire property, so such sets will exist in $L(\mathbb{R})[U]$. To illustrate the amount of our ignorance about the properties of the model, we state a bold open question:

Question 3.1. Does the model $L(\mathbb{R})[U]$ collapse any cardinalities of $L(\mathbb{R})$? I.e. if $X, Y \in L(\mathbb{R})$ are sets such that there is no injection of X to Y in $L(\mathbb{R})$, does the same hold in $L(\mathbb{R})[U]$?

The question is particularly acute for the quotient spaces of countable Borel equivalence relations, as the usual techniques for discerning them in $L(\mathbb{R})$ cannot work in $L(\mathbb{R})[U]$ due to the existence of a nonmeasurable set there.

In order to apply the technology outlined above to the study of the model $L(\mathbb{R})[U]$, we first need information about how ultrafilters are preserved under multiple generic extensions.

Theorem 3.2. *Let U be a nonprincipal ultrafilter. Suppose that $n \in \omega$ and P_i are posets in V, $K_i \subset P_i$ are generic filters over V, and U still generates an ultrafilter in $V[K_i]$ for all $i \in n$. Suppose that $Q_i \in V[K_i]$ are posets and $\tau_i \in V[K_i]$ are Q_i-names for an ultrafilter extending U. Whenever $H_i \subset Q$ for $i \in n$ are filters mutually generic over $V[K_i \colon i \in n]$, then $\bigcup_i \tau/H_i$ generates a nonprincipal filter.*

Proof. By a genericity argument, it will be enough for every $m \in \omega$, every tuple $\langle q_i \colon i \in n\rangle \in \prod_i Q_i$ and every tuple $\langle \eta_i \colon i \in n\rangle$ of Q-names in the respective models $V[K_i]$ such that $q_i \Vdash \eta_i \in \tau_i$, to find a number $k > m$ and conditions $q'_i \leq q_i$ so that $q'_i \Vdash \check{k} \in \eta_i$ for each $i \in n$. To do this, for each $i \in n$ let $a_i = \{k \in \omega \colon \exists r \leq q_i\ r \Vdash k \in \eta_i\} \subset \omega$. The set a_i is in the model $V[K_i]$ and must belong to the ultrafilter U, since it is forced to be a superset of η_i, $\eta_i \in \tau$ and τ is an ultrafilter extending U. Thus, the set $\bigcap_i a_i$ must contain an element k greater than m. Pick conditions $q'_i \leq q_i$ witnessing the fact that $k \in a_i$; this completes the proof. □

The crudest features of the model $L(\mathbb{R})[U]$ can now be easily derived from Theorem 2.2.

Theorem 3.3. (ZFC+LC) *Let U be a Ramsey ultrafilter. In the model $L(\mathbb{R})[U]$,*

1. *there is no infinite MAD family;*
2. *there is no ω_1 sequence of distict reals.*

Much stronger statement than (2) was proved in [4]: $L(\mathbb{R})[U]$ and $L(\mathbb{R})$ in fact have the same sets of ordinals. We include a simple proof of (2) in order to use the same idea later.

Proof. To prove (1), we will show that infinite MAD families are independent of ultrafilters and then quote Theorem 2.2. The key feature of MAD families is they are not preserved by mutually generic extensions the way ultrafilters are.

Claim 3.4. *If $A \subset \mathcal{P}(\omega)$ is an infinite MAD family, Q is any poset collapsing $2^{\mathfrak{c}}$, τ is a Q-name for a MAD family extending A, and $G_i \subset Q$ for $i \in 2$ are mutually generic filters over V, the set $\tau/G_0 \cup \tau/G_1$ is not an AD family.*

Proof. Let U be a nonprincipal ultrafilter on ω with empty intersection with A, let P be the usual c.c.c. poset adding a set $\dot{x}_{gen} \subset \omega$ which has finite intersection with every set not in U. The poset P regularly embeds into Q and $\dot{x}_{gen}$ becomes a Q-name under the fixed embedding. There is a Q-name $\dot{y}$ such that $Q \Vdash \dot{y} \cap \dot{x}_{gen}$ is infinite and $\dot{y} \in \tau$. Note that by the choice of the ultrafilter U, $\dot{y}$ is forced not to be in V.

Let $G_0, G_1 \subset Q$ be mutually generic filters. It will be enough to show that $\dot{y}/G_0 \in \tau/G_0$ has infinite intersection with $\dot{y}/G_1 \in \tau/G_1$. To show this, go back to V and suppose that $q_0, q_1 \in Q$ are conditions and $n \in \omega$ is a number. By a genericity argument, it is enough to find $q_0' \leq q_0$, $q_1' \leq q_1$ and $m > n$ such that q_0', q_1' both force $\check{m} \in \dot{y}$. For this, consider the sets $a_0 = \{m \in \omega \colon \exists q \leq q_0\ q \Vdash \check{m} \in \dot{y}\}$ and $a_1 = \{m \in \omega \colon \exists q \leq q_1\ q \Vdash \check{m} \in \dot{y}\}$. Since $\dot{y}$ is forced to have an infinite intersection with the set $\dot{x}_{gen}$ which has a finite intersection with every set not in U, both sets a_0 and a_1 must be in U and so there is a number $m > n$ in their intersection. The proposition follows. □

(1) now follows from Theorem 3.2 in the case $V[K_0] = V[K_1] = V$ and Theorem 2.2.

The main point in (2) is that injections from ω_1 to 2^ω do not survive almost any simultaneous generic extensions at all.

Claim 3.5. *If Q_0, Q_1 are posets collapsing $2^{\mathfrak{c}}$, τ_0, τ_1 are Q_0, Q_1-names for an injection from ω_1 to 2^ω, then there are conditions $q_0 \in Q_0$ and $q_1 \in Q_1$ such that for any pair $G_0 \subset Q_0$, $G_1 \subset Q_1$ of filters separately generic over V and containing the conditions q_0, q_1 respectively, the set $\tau/G_0 \cup \tau/G_1$ is not a function.*

Proof. Note that the set of the ground model reals is forced to be countable and so it is possible to find an ordinal $\alpha \in 2^{\mathfrak{c}}$, a number $n \in \omega$, and conditions $q_0^0, q_0^1 \in Q_0$ such that $q_0^0 \Vdash \tau_0(\alpha)(n) = 0$ and $q_0^1 \Vdash \tau_0(\alpha)(n) = 1$. Let $q_1 \in Q_1$ be a condition deciding the value of $\tau_1(\alpha)(n)$; say that the value is forced to be 1. The conditions $q_0 = q_0^0 \in Q_0$ and $q_1 \in Q_1$ obviously work as desired. □

Theorem 3.2 in the case $V[K_0] = V[K_1] = V$ now implies that ω_1 sequences of distinct reals are independent of ultrafilters. (2) then follows immediately from Theorem 2.2. □

A great deal of more sophisticated information about the model $L(\mathbb{R})[U]$ can be extracted from the evaluation of chromatic numbers of Borel hypergraphs. This will be done using the following general theorem:

Theorem 3.6. (ZFC+LC) *Suppose that X is a Polish space, $n \leq \omega$, and $Z \subset X^n$ is a Borel hypergraph. Suppose that there is a poset P of size $\leq \mathfrak{c}$ and a P-name $\dot{x}$ for an element of X such that*

1. *P preserves Ramsey ultrafilters;*
2. *for every $p \in P$, in some generic extension there are filters $K_i \subset P$ for $i \in n$ which are separately generic over V, contain the condition p, and $\langle \dot{x}/K_i \colon i \in n \rangle \in Z$ is a sequence of distinct points.*

Let U be a Ramsey ultrafilter on ω. Then in $L(\mathbb{R})[U]$, the hypergraph Z has uncountable chromatic number.

Proof. We will show that the colorings of the hypergraph Z by ω colors are independent of Ramsey ultrafilters and then use Theorem 2.2. Suppose that U is a Ramsey ultrafilter, Q is a forcing collapsing $2^{\mathfrak{c}}$, τ is a Q-name for an ultrafilter extending U, and σ is a Q-name for a coloring of the graph Z with colors in ω. Note that P regularly embeds into Q, so $\dot{x}$ becomes Q-names via some fixed embedding of P.

Suppose that $q \in Q$ is an arbitrary condition. Strengthening q if necessary, we may assume that q decides the value $\sigma([\dot{x}]_E)$ to be some specific number $m \in \omega$. Let $p \in P$ be a condition stronger than the projection of q into P. Use (1) to find filters $K_i \subset P$ for $i \in n$ separately generic over V such that $p \in K_i$ and $\langle \dot{x}/K_i \colon i \in n\rangle \in Z$ is a sequence of distinct points. Let $H_i \subset Q/K_i$ for $i \in n$ be mutually generic filters over the model $V[K]$ containing the condition q/K_i. Let $G_i = K_i * H_i \subset Q$. These are generic filters over V; we claim that they work as desired.

First of all, it is clear that $\bigcup_i \sigma/G_i$ is not a coloring of the hypergraph Z: its domain contains the points $\dot{x}_{gen}/K_i$ for $i \in n$ which form a Z-edge but they are still assigned the same color. Second, the set $\bigcup_i \tau/G_i$ generates a nonprincipal filter. To see this, go to the model $V[K]$, note that U generates an ultrafilter in the models $V[K_i]$ by (1), and use Proposition 3.2 in $V[K]$. $\square$

The first observation about chromatic numbers in the model $L(\mathbb{R})[U]$ is that many simple graphs have uncountable chromatic number in $L(\mathbb{R})$ and countable one in $L(\mathbb{R})[U]$. The simplest example is the graph Z on 2^ω connecting binary sequences x, y if they differ in exactly one entry.

Observation 3.7. (ZF+DC) *If there is a nonprincipal ultrafilter on ω the the graph Z has chromatic number* 2.

Proof. This is well known. Let U be a nonprincipal ultrafilter on ω. For every $x \in 2^\omega$ and every $n \in \omega$ let $x[n]$ be the parity of the number $|\{m \in n \colon x(m) = 1\}|$. Let $f(x) = 0$ if the set $\{n \in \omega \colon x[n] = 0\}$ is in U, and $f(x) = 1$ otherwise. It is not difficult to see that no two elements of 2^ω connected by a graph edge have the same f-value. $\square$

Many other Borel graphs remain uncountably chromatic in the model $L(\mathbb{R})[U]$ though. This leads to a number of interesting results.

Theorem 3.8. ([2], ZFC+LC) *In $L(\mathbb{R})[U]$, there is no total selector on E_0.*

Proof. Let Z be the graph on 2^ω connecting points x, y if they are E_0-related and distinct. It is easy to observe in ZF+DC that if there is an E_0 selector then the graph Z has countable chromatic number. Thus, it is enough to show that the graph Z has uncountable chromatic number in $L(\mathbb{R})[U]$, and for this it is enough to produce a suitable partial ordering P and use Theorem 3.6.

Let P be the quotient partial ordering of Borel subsets of 2^ω positive with respect to the σ-ideal I generated by Borel E_0-selectors, with the inclusion ordering. Let $\dot{x}_{gen} \in 2^\omega$ be the P-name for its canonical generic point. The poset has been studied for example in [11, Section 4.7.1], where its combinatorial form is provided and several properties isolated.

Claim 3.9. *Every Ramsey ultrafilter generates an ultrafilter in the P-extension.*

Proof. The poset P is proper and bounding by the results of [11, Section 4.7.1]. It does not add independent real by [10, Proposition 4.5]. The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The claim abstractly follows from these properties by [11, Theorem 3.4.1]. □

Claim 3.10. *For every condition $p \in P$ there are filters $H_0, H_1 \subset P$ separately generic over V, containing the condition p such that $\dot{x}_{gen}/K_0 \; Z \; \dot{x}_{gen}/K_1$.*

Proof. Let $p \in P$ be a condition. There must be a nonempty finite binary string $s \in 2^{<\omega}$ such that $(s \cdot p) \cap p \notin I$ since otherwise $p \in I$ would hold. Note that the map $q \mapsto s \cdot q$ is an automorphism of the partial ordering P. Thus, if $K_0 \subset P$ is a filter generic over V, containing the condition $(s \cdot p) \cap p$, then the filter $K_1 = s \cdot K_0$ is also a filter generic over V and it contains the condition p. Also, $\dot{x}_{gen}/K_0 = s \cdot \dot{x}_{gen}/K_1$ and so the two generic points obtained from the two filters are E_0-related and distinct as required. □

A reference to Theorem 3.6 now concludes the proof. □

Corollary 3.11. (ZFC+LC) *There is no Hamel basis for $\mathbb{R}$ or a transcendental basis for $\mathbb{C}$ in the model $L(\mathbb{R})[U]$.*

Proof. This is easiest to show using the following observations of independent interest:

Observation 3.12. (ZFC+DC) *The following are equivalent:*

1. *there is a nonsmooth hyperfinite Borel equivalence relation with a total selector;*
2. *every hyperfinite Borel equivalence relation has a total selector.*

Proof. Only (1)→(2) requires a proof. Suppose that E is a nonsmooth hyperfinite Borel equivalence relation on a Polish space X with a total selector S. Since there is a Borel relation on X which orders each E-equivalence class in ordertype embeddable into $\mathbb{Z}$, one can use the selector S to produce a function $f\colon X \to \omega$ which is injective on every E-class. Now, let F be any hyperfinite equivalence relation on a Polish space Y. There is a Borel injective function $h\colon Y \to X$ which reduces F to E. Now let $T = \{y \in Y : f(h(y))$ is the smallest number in $f \circ h''[y]_F\}$ and observe that T is a selector for F. □

Observation 3.13. (Simon Thomas)(ZF+DC) *If there is a Hamel basis for $\mathbb{R}$ or a transcendental basis for $\mathbb{C}$, then there is a total E_0 selector.*

Proof. To treat the case of Hamel basis, we will show that its existence implies existence of a total selector for the Vitali equivalence relation on $\mathbb{R}$; this will complete the proof by Observation 3.12. Let $B \subset \mathbb{R}$ be such a basis. For every nonzero $r \in \mathbb{R}$ there is a unique set $b_r \subset B$ such that there is a linear combination of reals in b_r with nonzero coefficients whose result belongs to the

Vitali class of r. There is a unique such a combination c_r whose coefficients are all integers, and the coefficient at the smallest number in b_r is positive and in absolute value smallest possible. The choice of b_r and c_r does not depend on r itself but only on the Vitali class of r. Therefore, the results of the linear combinations c_r for nonzero values of $r \in R$ together form a selector for the Vitali equivalence relation.

To treat the case of a transcendental basis for $\mathbb{C}$, first write K for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let E be the equivalence relation on $\mathbb{C}$ connecting x and y if $x - y \in K$. Note that this is a nonsmooth hyperfinite equivalence relation as it is an orbit equivalence of a continuous action of the abelian group $\langle K, + \rangle$ on $\mathbb{C}$. We will show that the existence of a transcendental basis implies the existence of an E-selector. Assume that $B \subset \mathbb{C}$ is a transcendental basis. For every $r \in \mathbb{C}$ there is an inclusion smallest set $b_r \subset \mathbb{R}$ such that r belongs to the algebraic closure of b_r. In some fixed enumeration of terms for the agebraic closure, there must be a first term which, when applied to b_r, yields an element $c_r$of $[r_E]$. As before, the definition of c_r does not depend on r itself but only on its E-class, and therefore the set $\{c_r \colon r \in \mathbb{R}\}$ is an E-selector as desired. □

The proof is now concluded by the reference to Theorem 3.8. □

An interesting attempt to discern between Borel equivalence relations in the model $L(\mathbb{R})[U]$ is to discuss the linear orderability of their associated quotient spaces. In $L(\mathbb{R})$, the quotient space $2^\omega/E_0$ fails to be linearly orderable, and so the linear orderability of the quotient space fails for every Borel nonsmooth equivalence relation in $L(\mathbb{R})$. In the model $L(\mathbb{R})[U]$, the situation is more nuanced:

Observation 3.14. (ZF+DC) *If there is a nonprincipal ultrafilter on ω then the class of equivalence relations for which the quotient space is linearly orderable is closed under countable increasing unions.*

Proof. Let U be a nonprincipal ultrafilter on ω. Let $E = \bigcup_n E_n$ be an increasing union of equivalence relations on a Polish space X, and suppose that the quotient space of the relations E_n is linearly orderable for each $n \in \omega$. Let $\leq_n$ be a linear preordering on X such that the induced equivalence relation is exactly E_n. The sequence of linear orders can be found as we assume DC. Let $\leq$ be a preordering on X defined by $x \leq y$ if $\{n \in \omega \colon x \leq_n y\} \in U$. It is not difficult to verify that $\leq$ induces a linear ordering of E-classes. □

For example, for equivalence relations such as E_0 and E_1 the quotient space is linearly orderable in $L(\mathbb{R})[U]$ while no such linear orderings exist in $L(\mathbb{R})$.

To show that various quotient spaces cannot be linearly ordered in the model $L(\mathbb{R})[U]$, we will start with the summable equivalence relation E_2. Recall that this is an equivalence relation on 2^ω connecting binary sequences $x, y \in 2^\omega$ if $\sum\{\frac{1}{n+1} \colon x(n) \neq y(n)\} < \infty$.

Theorem 3.15. (ZFC+LC) *In $L(\mathbb{R})[U]$, the E_2 quotient space cannot be linearly ordered.*

Proof. Consider the graph Z on 2^ω connecting points x, y if $x\ E_2\ 1-y$.

Observation 3.16. (ZF) *If the E_2 quotient space is linearly orderable then the graph Z has chromatic number two.*

Proof. Let $\leq$ be a linear order on the E_2 quotient space. Define the coloring c on X by letting $c(x) = 0$ if for every $y \in X$ such that $x\ Z\ y$, $y < x$ holds; and $c(x) = 1$ otherwise. It is not difficult to see that c is a coloring of Z. □

Thus, it will be enough to use Theorem 3.6 to show that the chromatic number of Z is uncountable in the model $L(\mathbb{R})[U]$. For this, we need a suitable partial ordering P.

Let $\omega = \bigcup_n I_n$ be a partition of ω into successive intervals. Write $X_n = 2^{I_n}$ for every $n \in \omega$ and let $X = \prod_n X_n$; the space X is naturally identified with 2^ω via the bijection $\pi\colon x \mapsto \bigcup x$ from X to 2^ω. Let d_n be the metric on X_n given by $d_n(u, v) = \{\frac{1}{m+1}\colon u(m) \neq v(m)\}$. Let μ_n be the normalized counting measure on X_n multiplied by $n+1$. The concentration of measure computations as in [9, Theorem 4.3.19] show that the sequence $\langle I_n\colon n \in \omega\rangle$ can be chosen in such a way that for every $n > 0$ and every $a, b \subset X_n$ of μ_n-mass at least 1 there are binary strings $u \in a$ and $v \in b$ such that $d_n(u, v) \leq 2^{-n}$.

Let p_{ini} be the tree of all finite sequences t such that for all $n \in \text{dom}(t)$, $t(n) \in X_n$. Finally, let P be the poset all all trees $p \subset p_{\text{ini}}$ such that the numbers $\{\mu_{|s|}(\{u \in X_{|s|}\colon s^\frown u \in p\})\colon s \in p\}$ converge to ∞. The ordering is that of inclusion.

The forcing P is of the fat tree kind studied for example in [1, Section 7.3.B] or [11, Section 4.4.3]. It adds a generic point $\dot{x}_{gen} \in 2^\omega$ which is the union of the trunks of the trees in the generic filter. The following two claims are key.

Claim 3.17. *The poset P preserves Ramsey ultrafilters.*

Proof. The forcing properties of posets similar to P are investigated in [11, Section 4.4.3]. [11, Theorem 4.4.8] shows that P is proper, bounding, and does not add independent reals. The associated σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [11, Theorem 3.8.9]. Posets with these properties preserve Ramsey ultrafilters by [11, Theorem 3.4.1]. □

Claim 3.18. *For every condition $p \in P$, in some forcing extension there are filters $K_0, K_1 \subset P$ which are separately generic over the ground model, $p \in K_0 \cap K_1$, and $\dot{x}_{gen}/K_0\ E_2\ 1 - \dot{x}_{gen}/K_1$.*

Proof. Let $V[H]$ be a forcing extension in which $\mathcal{P}(\mathcal{P}(\omega))^V$ is a countable set. The usual fusion arguments for the forcing P as in [1, Section 7.3.B] show that in $V[H]$, there is a condition $p' \subset p$ in $P^{V[H]}$ such that all its branches yield P-generic filters over the ground model. Let $s_0 \in p'$ be a node such that all nodes of S extending s_0 have the set of immediate successors in S of submeasure at least 1. For simplicity of notation assme that $s_0 = 0$. By induction on $n \in \omega$ build nodes $s_n, t_n \in p'$ so that

- $t_0 = s_0 = 0$, t_{n+1} is an immediate successor of t_n and s_{n+1} is an immediate successor of s_n;
- writing $u_n, v_n \in X_n$ for the binary strings such that $s_n^\frown u_n = s_{n+1}$ and $t_n^\frown v_n = t_{n+1}$, it is the case that $d_n(u_n, 1 - v_n) \leq 2^{-n}$.

Once this is done, let $K_0 \subset P$ be the filter associated with $\bigcup_n s_n$ and let K_1 be the filter associated with $\bigcup_n t_n$. These are branches through the tree p', so the filters K_0, K_1 are generic over the ground model. The second item immediately implies that $\dot{x}_{gen}/K_0\ E_2\ 1 - \dot{x}_{gen}/K_1$ as desired.

The induction step of the construction above is obtained as follows. Suppose that $t_n, s_n \in S$ have been found. Let $a = \{u \in X_n : s_n^u \in p'\}$ and $b = \{v \in X_n : t_n^\frown(1 - v) \in S\}$. Then, $\mu_n(a)$, $\mu(b)$ are both numbers greater than 1, and therefore there are $u \in a$ and $v \in b$ such that $d_n(u, v) \leq 2^{-n}$. Setting $s_{n+1} = s_n^\frown u$ and $t_{n+1}^\frown(1 - v)$ completes the induction step. □

Now, in view of Theorem 3.6, colorings of the graph Z with countably many colors are independent of Ramsey ultrafilters. In view of Theorem 2.2, the graph Z has uncountable chromatic number in $L(\mathbb{R})[U]$ and so the E_2 quotient space cannot be linearly ordered inthis model. □

Our next example is the equivalence relation $=^+$ on $(2^\omega)^\omega$ connecting points x, y just in case $\mathrm{rng}(x) = \mathrm{rng}(y)$.

Theorem 3.19. (ZFC+LC) *In $L(\mathbb{R})[U]$, the $=^+$ quotient space cannot be linearly ordered.*

Proof. Consider the graph Z on $X = (2^\omega)^\omega$ connecting x, y if $\{x(n) : n \in \omega\} = \{1 - y(n) : n \in \omega\}$ and $x =^+ y$ fails.

Observation 3.20. (ZF) *If the $=^+$ quotient space is linearly orderable then the graph Z has chromatic number two.*

Proof. Let $\leq$ be a linear order on the $=^+$ quotient space. Define the coloring c on X by letting $c(x) = 0$ if for every $y \in X$ such that $x\ Z\ y$, $y < x$ holds; and $c(x) = 1$ otherwise. It is not difficult to see that c is a coloring of Z. □

Thus, it will be enough to use Theorem 3.6 to show that the chromatic number of the graph Z is uncountable in $L(\mathbb{R})[U]$. For this, we need to find a suitable partial order. Let P be the countable support product of ω_1 many Sacks reals, yielding an ω_1-sequence $\dot{x}_{gen}$. The following two claims are key:

Claim 3.21. *Any Ramsey ultrafilter generates an ultrafilter in the P-extension.*

Proof. The product of countably many copies Sacks forcing does not add an independent real by [7]. It is also well-known to be proper, bounding and definable, and so by [11, Theorem 3.4.1] every Ramsey ultrafilter generates a Ramsey ultrafilter in the countable product extension. Every subset of ω in the uncountable product extension comes from a countable product extension by a properness argument, proving the claim. □

Claim 3.22. *For every condition $p \in P$, in some generic extension there are filters $K_0, K_1 \subset P$ generic over over V, containing the condition p, such that* $\mathrm{rng}(\dot{x}_{gen}/K_1) = \{1 - z \colon z \in \mathrm{rng}(\dot{x}_{gen}/K_0)\}$.

Proof. First note that the involution $z \mapsto 1 - z$ on 2^ω generates an automorphism on the Sacks poset, sending every condition s (viewed as an uncountable Borel set) to the set $1 - s$ of complements of points in s. Any involution $\pi \colon \omega_1 \to \omega_1$ generates an automorphism of the poset P, sending any condition p to a condition $\pi(p)$ whose domain is $\pi''\mathrm{dom}(p)$ and for every $\alpha \in \mathrm{dom}(p)$, $\pi(p)(\pi(\alpha)) = 1 - p(\alpha)$. Finally, note that for this automorphism, if $K \subset P$ is a generic filter then $\mathrm{rng}(\dot{x}_{gen}/\pi''K) = \{1 - z \colon z \in \mathrm{rng}(\dot{x}_{gen}/K)\}$.

Now, suppose $p \in P$ is a condition. Write $a = \mathrm{dom}(p) \subset \omega_1$; this is a countable set. Let π be any involution of ω_1 such that $a \cap \pi''a = 0$. The conditions p and $\pi(p)$ are then compatible, with a lower bound q. Let $K_0 \subset P$ be a filter generic over V, containing the condition q. Let $K_1 = \pi''K_0$ and check that the filters K_0, K_1 work as desired. □

Note that the poset P does not literally add an element of the $=^+$-quotient space but only a code for an $=^+$-class in a $\mathrm{Coll}(\omega, \omega_1)$ extension. This does not change anything in the proof of Theorem 3.6 and we can conclude that the colorings of the graph Z with countably many colors are independent of Ramsey ultrafilters. By Theorem 2.2, it follows that in the model $L(\mathbb{R})[U]$ the graph Z has uncountable chromatic number and so the quotient space cannot be linearly ordered. □

We conclude this section with a natural question. [8] showed that there is a simple Borel graph which is of uncountable chromatic number in $L(\mathbb{R})$ and minimal in the sense that it can be continuously homomorphically embedded into any other uncountably chromatic graph in $L(\mathbb{R})$. Does this situation repeat in $L(\mathbb{R})[U]$?

Question 3.23. Is there a Borel graph Z_0 such that it has an uncountable chromatic number in $L(\mathbb{R})[U]$, and it can be continuously homomorphically embedded into every other Borel graph of uncountable chromatic number in $L(\mathbb{R})[U]$?

4 Adding selectors to equivalence relations

In this section, we investigate the model $L(\mathbb{R})[S]$ where $S \subset X$ is a total selector on a fixed equivalence relation E on a Polish space X, which is added generically by countable approximations. To prevent all of the axiom of choice from creeping into the model, we will restrict our attention to the class isolated by Kanovei:

Definition 4.1. An analytic equivalence relation E on a Polish space X is *pinned* if for mutually generic filters G, H, if C is an equivalence class represented in both $V[G]$ and $V[H]$ then it is represented in V.

The restriction on the complexity of the equivalence relation E is necessary. The standard example of an unpinned equivalence relation is $=^+$, the equivalence on $(2^\omega)^\omega$ connecting x and y if $\mathrm{rng}(x) = \mathrm{rng}(y)$. We have the following simple observation:

Observation 4.2. (ZF) *If there is an injection from the $=^+$ quotient space into the 2^ω, then there is an injection from ω_1 into 2^ω. If $S \subset (2^\omega)^\omega$ is a generic total $=^+$-selector, then the model $L(\mathbb{R})[S]$ satisfies the Axiom of Choice and the Continuum Hypothesis.*

Proof. For the first sentence, let h be an injection from the $=^+$ quotient space to 2^ω; we may assume that its range is not all of 2^ω, missing some point $y \in 2^\omega$. By transfinite recursion on $\alpha \in \omega_1$ define points $y_\alpha \in 2^\omega$ by $y_0 = y$ and $y_\alpha = h(x)$ for some (every) point $x \in (2^\omega)^\omega$ such that $\mathrm{rng}(x) = \{y_\beta \colon \beta \in \alpha\}$. It is not difficult to verify that the sequence $\langle y_\alpha \colon \alpha \in \omega_1\rangle$ is injective.

For the second sentence, first pick a Borel function $g \colon (2^\omega)^\omega \to 2^\omega$ such that for the g-image of any uncountable $=^+$ class is equal to 2^ω. Let P be the poset of partial countable $=^+$-selectors–a moment of thought will show that for example the function g defined by $g(x)(n) = x(n+1)(m)$, where $m \in \omega$ is the least number such that $x(0)(m) \neq x(1)(m)$ if such exists and $m = 0$ otherwise, satisfies this property. Pick a P-name $\dot f \in L(\mathbb{R})$ for an injective ω_1 sequence of elements of 2^ω in the P-extension. A simple genericity argument shows that the map $\dot e \colon \omega_1 \to 2^\omega$ defined by $e(\alpha) = g(z)$, where z is the unique point in the generic selector enumerating the set $\{\dot f(\beta) \colon \beta \in \alpha\}$, is forced to be a surjection, proving the second sentence. □

As in the previous section, the most important tool for the study of the model $L(\mathbb{R})[S]$ is a proposition showing how the selectors survive multiple forcing extensions.

Theorem 4.3. *Suppose that E is a pinned analytic equivalence relation on a Polish space X, $S \subset X$ is a total E-selector, Q is a poset and τ is a Q-name for an E-selector extending S. Whenever $n \leq \omega$ and $G_i \subset Q$ are pairwise mutually generic filters over V for each $i \in n$, then $\bigcup_i \tau/G_i$ is an E-selector.*

Proof. This is essentially immediate from the definition of a pinned equivalence relation, since there is no single E-equivalence class from which the different selectors τ/G_i could pick distinct elements. □

The crudest features of the model $L(\mathbb{R})[S]$ immediately follow:

Corollary 4.4. (ZFC+LC) *Let E be a pinned equivalence relation on a Polish space X and $S \subset X$ a generic total E-selector. In the model $L(\mathbb{R})[S]$,*

1. *there are no infinite MAD families;*
2. *there are no injective ω_1-sequences of reals.*

Proof. For (1), use Claim 3.4 and Theorem 2.2. For (2), use Claim 3.5 and Theorem 2.2. □

As in the previous section, the finer properties of the model $L(\mathbb{R})[S]$ follow from the investigation of the chromatic numbers of Borel hypergraphs there. This time, chromatic numbers of many graphs will be countable, and to make progress we need to reach for hypergraphs of higher finite dimension.

Theorem 4.5. (ZFC+LC) *Suppose that*

1. *E is a pinned Borel equivalence relation on a Polish space X;*
2. *$n \in \omega$ is a natural number, Y is a Polish space and $Z \subset Y^n$ is a Borel hypergraph;*
3. *there is a poset P of size $\leq \mathfrak{c}$ and a P-name $\dot{y}$ for an element of Y such that for every condition $p \in P$, in some generic extension there are filters $K_i \subset P$ for $i \in n$ containing p, pairwise mutually generic over V, and such that $\langle \dot{y}/K_i : i \in n\rangle \in Z$.*

Then in the model $L(\mathbb{R})[S]$, where S is the generic selector for E, the chromatic number of Z is uncountable.

Proof. We will first prove that colorings of Z with countably many colors are independent of total E-selectors and then apply Theorem 2.2. Thus, let $S \subset X$ be a total E-selector, Q a poset collapsing $2^{\mathfrak{c}}$ to $\aleph_0$, σ a Q-name for a total E-selector extending S, and τ a Q-name for a map from Y to ω which is a coloring of the hypergraph Z. Note that P is regularly embedded into Q and therefore $\dot{y}$ becomes a Q-name via this fixed embedding. Let $q \in Q$ be a condition deciding the value of $\tau(\dot{y})$ to be some specific number $m \in \omega$. Let $p \in P$ be a condition below the projection of q into P. Use the assumptions to find, in some generic extension, filters $K_i \subset P$ for $i \in n$ containing p, pairwise mutually generic over V, and such that $\langle \dot{y}/K_i : i \in n\rangle \in Z$. Let $H_i \subset Q/K_i$ be filters mutually generic over the model $V[K_i : i \in n]$, each containing the condition q. Write $G_i = K_i * H_i \subset Q$ for $i \in n$, and note that these filters are pairwise generic over the ground model and contain the condition Q. We claim that they work as desired.

First of all, the map $\bigcup_{i\in n} \tau/G_i$ is not a partial coloring of the hypergraph Z, since its m-th color contains the edge $\langle \dot{y}/G_i : i \in n\rangle$. Second, the set $\bigcup_{i\in n} \sigma/G_i$ is a partial E-selector by Proposition 4.3. This completes the proof. $\square$

Theorem 4.6. (ZFC+LC) *There are no nonatomic finitely additive probability measures on ω in the model $L(\mathbb{R})[S]$.*

Proof. Consider the hypergraph Z on $(\mathcal{P}(\omega))^{10}$ consisting of tuples $\vec{y}$ such that every number in ω (with finitely many exceptions) belongs to at least one of the sets $\vec{y}(i)$ for $i \in 3$ and at most two of the sets $\vec{y}(i)$ for $3 \leq i < 10$.

Observation 4.7. (ZF+DC) *If there is a nonatomic probability measure on ω then the hypergraph Z has chromatic number two.*

Proof. Let μ be the measure and consider the coloring c assigning a set $a \subset \omega$ color 0 if $\mu(a) < 1/3$ and color 1 otherwise. No edge in the hypergraph Z can contain only points of color 0 since the first three sets on the edge have co-finite union which has to have μ-mass 1. At the same time, no edge on the hypergraph Z can contain only points of color 1 since the last seven sets on the edge would contradict the Fubini theorem between μ and the evenly distributed probability measure on the set $10 \setminus 3$. □

Thus, it is enough to show that the hypergraph Z has uncountable chromatic number in the model $L(\mathbb{R})[S]$. To this end, consider the poset $P = 2^{<\omega}$ ordered by reverse extension and let $\dot{y}$ be the P-name for the set of all $n \in \omega$ such that for some condition p in the generic filter, $p(n) = 1$. Let $p \in P$ be an arbitrary condition. Pass to a generic extension in which $\mathfrak{c}$ is countable; we will produce filters $K_i \subset P$ for $i \in 10$ such that $\langle \dot{y}/K_i : i \in 10 \rangle \in Z$ and then apply Theorem 4.5. Let $D_k : k \in \omega$ be an enumeration of the open dense subsets of $P \times P$ in the ground model. Let $\pi : \omega \to \omega^3$ be a surjection. By induction on $n \in \omega$ build numbers $m_n \in \omega$ and maps $q_n : 10 \times m_n \to 2$ so that

1. $m_0 = \mathrm{dom}(p)$ and $q_0(i, j) = p(j)$ for every $i \in 10$;
2. $m_0 \leq m_1 \leq m_2 \leq \dots$ and $q_0 \subset q_1 \subset q_2 \subset \dots$;
3. for every $k \in m_{n+1} \setminus m_n$, $q(i, k) = 1$ for at least on $i \in 3$ and at most two $i \in 10 \setminus 3$;
4. if $\pi(n) = \langle k, i, j \rangle$ for some $k \in \omega$ and $i \neq j \in 10$, then q_{n+1} restricted to the i-th and j-th column belongs to D_k.

The induction process is immediate. In the end, for every $i \in 10$ let $K_i \subset P$ be the filter generated by the maps $q_n(i, \cdot)$ for $n \in \omega$ and observe that these filters work as required. □

Theorem 4.8. (ZFC+LC) *There are no Hamel bases for* $\mathbb{R}$ *in the model* $L(\mathbb{R})[S]$.

Proof. Consider the hypergraph Z on $\mathbb{R}^3$ consisting of triples $\langle y_0, y_1, y_2 \rangle$ of pairwise distinct real numbers such that $y_0 + y_1 + q = y_2$ for some rational number $q \in \mathbb{Q}$.

Observation 4.9. (ZF+DC) *If a Hamel basis for* $\mathbb{R}$ *exists then the hypergraph* Z *has countable chromatic number.*

Proof. Let B be a Hamel basis; rescaling, we may assume that it contains number 1. Each number $y \in \mathbb{R}$ can be expressed in a unique way as $y = q + \sum_{i \in n} q_i r_i$ for some rational numbers $q, q_i \neq 0$ and irrational numbers $r_0 < r_1 < \dots$ in B. Let f be a function on $\mathbb{R}$ defined by $f(y) = \langle q, q_0, p_0, q_1, p_1, \dots q_{n-1} \rangle$ where q, q_i are copied from the unique decomposition of y, and p_i is the least rational number (in some fixed enumeration) between the reals r_i and r_{i+1}. It is immediate that no edge in the graph can consist of three numbers with the same value of f. □

Thus, it is enough to show that the hypergraph Z has uncountable chromatic number in the model $L(\mathbb{R})[S]$. Towards this, we will find a suitable poset P and apply Theorem 4.5. Let P be the poset of nonempty open subsets of $\mathbb{R}$, ordered by inclusion; this is the Cohen poset with its associated name for a generic point $\dot{y} \in \mathbb{R}$. The following claim is immediate:

Claim 4.10. *If $y_0, y_1 \in \mathbb{R}$ are mutually generic points for P and $q \in \mathbb{Q}$ is a rational number, then the triple $y_0, y_1, y_0 + y_1 + q$ is pairwise mutually generic for P.*

Now, let $p \in P$ be an arbitrary condition. Find pairwise generic filters $K_0, K_1 \subset P$ containing the condition p, find a rational number $q \in \mathbb{Q}$ such that $y_2 = \dot{y}/K_0 + \dot{y}/K_1 + q \in p$, and let $K_2 \subset P$ be the filter of all open subsets of $\mathbb{R}$ containing the number y_2. It is clear that the filters $K_0, K_1, K_2 \subset P$ satisfy the assumptions of Theorem 4.5 and so the application of the theorem will complete the proof. □

An obvious question may be whether it is possible to discern between the existence of selectors for various equivalence relations. In general, it is not clear for which pairs E, F of Borel equivalence relations it is the case that in ZF+DC, the existence of a total selector for E implies the existence of a total selector for F. Already the case $E = E_0$ and $F = E_1$ appears to be open. We will prove one result in this direction concerning trim equivalence relations:

Definition 4.11. An analytic equivalence relation E on a Polish space X is *trim* if for in every forcing extension, whenever $V[G_0], V[G_1]$ are forcing extensions of the ground model such that $V[G_0] \cap V[G_1] = V$ and $x_0 \in V[G_0] \cap X$ and $x_1 \in V[G_1] \cap X$ are E-related points, then there is a point $x \in V$ which is E-related to both.

There is a rich supply of trim equivalence relations as exhibited in [12]; one interesting example is the equivalence relation E on $2^{\mathbb{Q}}$ connecting points x, y if $\{q \in \mathbb{Q} \colon x(q) \neq y(q)\}$ is nowhere dense.

Theorem 4.12. *Suppose that E is a Borel trim equivalence relation on a Polish space X and F is an orbit equivalence relation of a generically turbulent group action on a Polish space Y. The total F-selectors are independent of total E-selectors.*

Proof. The key forcing consideration is the following. Let $G \curvearrowright Y$ be the group action generating the equivalence relation E. Let P_Y be the Cohen poset of nonempty open subsets of Y ordered by inclusion, adding a generic element $\dot{y}_{gen} \in \dot{Y}$. Let P_G be the Cohen poset of nonempty open subsets of G ordered by inclusion, adding a generic element $\dot{g}_{gen} \in \dot{Y}$. The following is an alternative characterization of the turbulence of the action $G \curvearrowright Y$:

Claim 4.13. [12] $P_G \times P_Y \Vdash V[\dot{y}_{gen}] \cap V[\dot{g}_{gen} \cdot \dot{y}_{gen}] = V$.

Let S be a total E-selector, Q a poset collapsing $2^{\mathfrak{c}}$ to $\aleph_0$, let τ be a Q-name for an E-selector extending S, and let σ be a Q-name for a total F-selector. The poset P_Y regularly embeds into the poset Q and so $\dot{y}_{gen}$ becomes a Q-name. Use Claim 4.13 to find filters $K_0, K_1 \subset P_Y$ such that $V[K_0] \cap V[K_1] = V$ and $\dot{y}_{gen}/K_0 \ F \ \dot{y}_{gen}/K_1$. Let $H_0 \subset Q/K_0$ and $H_1 \subset Q/K_1$ be filters mutually generic over the model $V[K_0, K_1]$ and let $G_0 = K_0 * H_0$ and $G_1 = K_1 * H_1$. It is immediate that $G_0, G_1 \subset Q$ are filters separately generic over V, with $V[G_0] \cap V[G_1] = V$. We claim that these filters work as required.

First of all, the set $\sigma/G_0 \cup \sigma/G_1$ is not an F-selector: σ/G_0 contains some element of the class $[\dot{y}_{gen}/G_0]_F$ and σ/G_1 contains some element of this same class as well, these two elements belong to the models $V[G_0]$ and $V[G_1]$ respectively and so they cannot be equal. If they were equal, they would have to belong to the ground model, which contradicts the fact that the class $[\dot{y}_{gen}/K_0]_F$ has no ground model elements.

On the other hand, the set $\tau/G_0 \cup \tau/G_1$ is an E-selector: all E-equivalence classes represented in both $V[G_0]$ and $V[G_1]$ are represented already in V by the trimness of the equivalence relation E, and so already the common part S of τ/G_0 and τ/G_1 selected an element from this class and the two selectors cannot disagree on it. □

Corollary 4.14. (ZFC+LC) *Suppose that E is a Borel trim equivalence relation on a Polish space X and F is an orbit equivalence relation of a generically turbulent group action on a Polish space Y. Let S be a generic total E-selector. Then in $L(\mathbb{R})[S]$, the equivalence relation F has no total selector.*

5 Adding MAD families

From Claim 3.4, it appears to be difficult to preserve MAD families with the multiple generic extensions. We do not know how to handle the model $L(\mathbb{R})[A]$, where $A \subset \mathcal{P}(\omega)$ is a generic MAD family added with infinitely countable approximations. We have to resort to adding a more specific type of MAD family, which curiously enough has connections to the $\mathfrak{d} < \mathfrak{a}$ problem.

Definition 5.1. An *improved AD family* is a pair $\langle A, B \rangle$ such that

1. A is an infinite AD family in $\mathcal{P}(\omega)$;
2. B is a set consisting of pairs $\langle s, a \rangle$ such that s is a partition of ω into finite sets and $a \subset A$ is a countable set;
3. for every pair $\langle s, a \rangle \in B$ and every finite set $b \subset A \setminus a$, there are infinitely many sets $c \in s$ such that $\bigcup b \cap c = 0$.

An improved AD family $\langle A, B \rangle$ is *maximal* if A is a MAD family and for every partition s there is a with $\langle s, a \rangle \in B$.

The improved MAD families are naturally added by a poset of countable improved AD families ordered by coordinatewise inclusion. It is easy to verify that if G is a generic filter on the poset of countable improved AD families, then the coordinatewise union of conditions in G is an improved MAD family. Moreover, the second coordinate can be recovered from the first one by a genericity argument.

Proposition 5.2. *If the continuum hypothesis holds, then there is an improved MAD family. If $\mathfrak{d} < \mathfrak{a}$, then there is no improved MAD family.*

Proof. If the continuum hypothesis holds, it is easy to produce a filter on the poset of countable improved AD families which meets all the $\mathfrak{c} = \aleph_1$ open dense sets necessary to turn its union into an improved MAD family.

Towards the proof of the second sentence, it is enough to show that if $\langle A, B\rangle$ is an improved MAD family then $|A| \leq \mathfrak{d}$. To this end, let $\{s_\alpha \colon \alpha \in \mathfrak{d}\}$ be a collection of partitions of ω into finite sets such that for every other such partition t there is $\alpha \in \mathfrak{d}$ such that every element of s_α contains an element of t as a subset. For every ordinal $\alpha \in \mathfrak{d}$ pick a countable set $a_\alpha \subset A$ such that $\langle s_\alpha, a_\alpha\rangle \in B$ and use (3) in the definition of an improved MAD family to conclude that $A = \bigcup_{\alpha\in\mathfrak{d}} a_\alpha$. Thus, $|A| \leq \mathfrak{d}$ as desired. □

As in the previous sections, we must show how improved MAD families survive multiple forcing extensions. The following theorem will be sufficient for all of our purposes.

Theorem 5.3. *Suppose that $\langle A, B\rangle$ is an improved MAD family, $n \in \omega$, and $V[G_i]$ for $i \in n$ are bounding extensions of V. Suppose that $P_i \in V[G_i]$ are posets and $\langle \dot{A}_i, \dot{B}_i\rangle \in V[G_i]$ are P_i-names for an improved MAD family extending $\langle A, B\rangle$. Then, in some forcing extension, there are filters $H_i \subset P_i$, each generic over $V[G_i]$, such that $\langle \bigcup_{i\in n} \dot{A}_i/H_i, \bigcup_{i\in n} \dot{B}_i/H_i\rangle$ is an improved AD family.*

Proof. We will start with a key technical claim:

Claim 5.4. *For every $i \in n$, in the model $V[G_i]$ the following holds. If $p \in P_i$ is a condition and $\sigma_j \colon j \in m$ are names for elements of $\dot{A}_i$ such that $p \Vdash \sigma_j \notin V$, then there is $k \in \omega$ such that for every larger $l \in \omega$ there is a condition $q \leq p$ such that $q \Vdash \bigcup_{j\in m} \sigma_j \cap [k, l) = 0$.*

Proof. Suppose that the claim fails for some $i \in n$, condition $p \in P_i$ and names σ_j for $j \in m$. Then, in the model $V[G_i]$ there is a partition s of ω into finite sets such that $p \Vdash \bigcup_{j\in m} \sigma_j \cap b \neq 0$ for every $b \in s$. Since $V[G_i]$ is a bounding extension of V, there is a partition $t \in V$ such that every $c \in t$ contains some element of s as a subset. Since $p \Vdash \{\sigma_j \colon j \in m\} \cap V = 0$, there must be $c \in t$ and a condition $q \leq p$ such that $q \Vdash \bigcup_{j\in m} \sigma_j \cap \check{c} = 0$. If $b \in s$ is an element of s which is a subset of c, then $q \Vdash \bigcup_{j\in m} \sigma_j \cap \check{b} = 0$, and this contradicts the choice of the partition s. □

Let $V[G]$ be a forcing extension containing all the filters G_i for $i \in n$. Let $V[G][H]$ be a further generic extension collapsing a sufficiently large cardinal and work in $V[G][H]$. An inductive application of the claim makes it possible to find filters $H_i \subset P_i$, each generic over $V[G_i]$ for $i \in m$, and a partition $\omega = \bigcup_{i \in n, j \in \omega} a_{ij}$ of ω into finite sets such that

- for every $i \in \omega$ and every $x \in \tau_i/H_i$, either $x \in A$ or $x \subset \bigcup_j a_{ij}$ up to finitely many exceptions;
- for every $i \in \omega$ and every collection $\{b_k : k \in \omega\} \in V[G_i][H_i]$ of pairwise disjoint finite subsets of ω, there are $j, k \in \omega$ such that $b_k \subset a_{ij}$.

We claim that the filters work as required. Let $A' = \bigcup_i \dot{A}_i/H_i$ and $B' = \bigcup_i \dot{B}_i/H_i$; we must argue that $\langle A', B' \rangle$ is an improved AD family.

First, prove that A' is an almost disjoint family. To this end, suppose that $x, y \in A'$ are distinct points; we must show that they have finite intersection. The critical case is when there are numbers i, j both in n such that $x \in \dot{A}_i/G_i \setminus A$ and $y \in \dot{A}_j/G_j \setminus A$. But then, $x \subset \bigcup_k a_{ik}$ and $y \subset \bigcup_k a_{jk}$ with possibly finitely many exceptions, the sets $\bigcup_k a_{ik}$ and $\bigcup_k a_{jk}$ are disjoint, and so $x \cap y$ must be finite.

Second, suppose that $\langle s, a \rangle \in B'$ and $b \subset A' \setminus a$ is a finite set; we must find infinitely many sets $c \in s$ such that $\bigcup b \cap c = 0$. Let $i \in n$ be an index such that $\langle s, a \rangle \in \dot{B}_i/H_i$. There are infinitely many $c \in s$ such that $\bigcup (b \cap \dot{A}_i/H_i) \cap c = 0$ since $\langle \dot{A}_i, \dot{B}_i \rangle$ is forced to be an improved MAD family. By the second item above, there must be infinitely many $c \in s$ such that $\bigcup (b \cap \dot{A}_i/H_i) \cap c = 0$ and there is k such that $c \subset a_{ik}$. By the first item above, there must be infinitely many $c \in s$ such that $\bigcup (b \cap \dot{A}_i/H_i) \cap c = 0$ and there is k such that $c \subset a_{ik}$ and for all $x \in b \setminus \dot{A}_i/H_i$, $x \cap a_{ik} = 0$. This completes the proof. □

Theorem 5.5. *Injective maps from ω_1 to 2^ω are independent of improved MAD families.*

Proof. Suppose that $\langle A, B \rangle$ is an improved MAD family, Q is a poset which collapses $2^{\mathfrak{c}}$, and τ and σ are Q-names for an improved MAD family extending $\langle A, B \rangle$ and an injection from ω_1 to ω respectively. Use Claim 3.5 to find conditions $q_0, q_1 \in Q$ such that for any two filters $G_0, G_1 \subset Q$ generic over V and containing the respective conditions q_0, q_1 , the union $\sigma/G_0 \cup \sigma/G_1$ is not a map from ordinals to 2^ω. Use Theorem 5.3 to find generic filters $G_0, G_1 \subset Q$ such that $q_0 \in G_0$, $q_1 \in G_1$, and $\tau/G_0 \cup \tau/G_1$ is an improved AD family. This proves the theorem. □

Corollary 5.6. (ZFC+LC) *In the model $L(\mathbb{R})[A, B]$, where $\langle A, B \rangle$ is the generic improved MAD family, there is no injection from ω_1 to 2^ω.*

More sophisticated information about the model $L(\mathbb{R})[A, B]$ can be obtained by investigating chromatic numbers of Borel graphs.

Theorem 5.7. (ZFC+LC) *Let Z be a Borel hypergraph of finite dimension on a Polish space X. Then $L(\mathbb{R}) \models Z$ has countable chromatic number if and only if $L(\mathbb{R})[A, B] \models Z$ has countable chromatic number whenever $\langle A, B\rangle$ is a generic improved MAD family.*

Proof. The right-to-left implication is immediate as $L(\mathbb{R}) \subset L(\mathbb{R})[A, B]$ holds. For the left-to-right implication, fix a natural number d. [8] shows that there is a certain critical Borel graph Z_0 on d^ω that needs to be investigated. To obtain the graph Z_0, pick binary sequences $z_n \in d^n$ such that they are dense in $d^{<\omega}$ and let $\langle x_i : i \in d\rangle \in Z_0$ if there is $n \in \omega$ such that for every $i \in d$ $x_i(n) = i$ and $x_i \restriction n = z_n$ hold, and the functions $x_i \restriction \omega \setminus n + 1$ for $i \in d$ are all the same. It is known [8, Theorem 16] that in $L(\mathbb{R})$, the graph Z_0 has uncountable chromatic number and homomorphically continuously embeds into every other Borel hypergraph of dimension d and uncountable chromatic number. Thus, for the left-to-right implication it is only necessary to show that the graph Z_0 has uncountable chromatic number in the model $L(\mathbb{R})[A, B]$.

To this end, it will be enough to find a bounding proper poset P of size $\mathfrak{c}$ and a P-name $\dot{x}$ for an element of 2^ω such that for every condition $p \in P$, in some generic extension there are filters $K_i \subset P$ for $i \in d$, separately generic over V, containing the condition p and such that $\langle \dot{x}/K_i : i \in d\rangle \in Z_0$. Once this is done, the proof of Theorem 3.6 (with Theorem 5.3 replacing Theorem 3.2) shows that colorings of Z_0 with countably many colors are independent of improved MAD families. A reference to Theorem 2.2 then concludes the proof.

The construction of the requisite poset P is routine. By recursion on $k \in \omega$ build natural numbers $m_k \in \omega$ such that $0 = m_0 \in m_1 \in m_2 \in \dots$ and for every $t \in d^{m_k}$ there is $n \in m_{k+1}$ such that $t \subset z_n$. The poset P consists of all functions g whose domain is a coinfinite subset of ω and for each $k \in \mathrm{dom}(g)$, $g(k) \in d^{m_{k+1} \setminus m_k}$. The ordering is that of reverse inclusion. Thus, the poset P is a variation of the Silver forcing investigated in [1, Definition 7.4.11] It is well-known that the poset P is proper and bounding.

Now, if $\dot{G}$ is a P-name for the generic filter then $\dot{x} = \bigcup \mathrm{rng}(\bigcup \dot{G})$ is a point in 2^ω. We claim that the name $\dot{x}$ has the required properties. Indeed, whenever $p \in P$ is a condition then let $k = \min(\omega \setminus \mathrm{dom}(p))$, and let $n \in m_{k+1} \setminus m_k$ be a number such that $z = \bigcup_{l \in k} p(l) \subset z_n$. Let $u_i \in d^{m_{k+1} \setminus m_k}$ for $i \in d$ be strings such that z_n is an initial segment of $z \cup u_i$ and $u_i(n) = i$ and $u(i)(m) = 0$ for all $i \in d$ and all $n \in (m, m_{k+1})$. Let $K \subset P$ be a filter generic over V containing the condition p and get filters $K_i \subset P$ for $i \in d$ by adjusting the conditions in K to return the value u_i respectively at k. It is not difficult to see that the filters $K_i \subset P$ for $i \in d$ are as required. □

Corollary 5.8. (ZFC+LC) *If $\langle A, B\rangle$ is a generic improved MAD family, then in $L(\mathbb{R})[A, B]$*

1. *the E_0 quotient space is not linearly orderable;*
2. *there is no Hamel basis for $\mathbb{R}$;*
3. *there is no nonprincipal finitely additive measure on ω.*

Proof. For (1), consider the Borel graph Z on 2^ω connecting points x, y if $x \; E_0 \; 1-y$. It is clear that the Z-relation depends only on the E_0-classes of x, y, that modulo E_0 every node has degree exactly 1, and so a presence of a linear ordering on the E_0 quotient space would imply that a Z has chromatic number two. Now, the graph Z has uncountable chromatic number in $L(\mathbb{R})$ (say, by a Baire category argument), so it has uncountable chromatic number in $L(\mathbb{R})[A, B]$ by Theorem 5.7 and so the E_0 is not linearly orderable there. (2) follows from Observation 4.9. (3) follows from Observation 4.7 and Theorem 5.7. $\square$

We conclude this section with another natural question:

Question 5.9. Is there an ω-dimensional Borel hypergraph which is uncountably chromatic in $L(\mathbb{R})$ and countably chromatic in the model $L(\mathbb{R})[A, B]$?

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