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Cofinalities of Borel ideals

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We study the possible values of the cofinality invariant for various Borel ideals on the natural numbers. We introduce the notions of a fragmented and gradually fragmented F_σ ideal and prove a dichotomy for fragmented ideals. We show that every gradually fragmented ideal has cofinality consistently strictly smaller than the cardinal invariant $\mathfrak{b}$ and produce a model where there are uncountably many pairwise distinct cofinalities of gradually fragmented ideals.

1 Introduction

This paper concerns the possibilities for the cofinalities of Borel ideals on ω. Here, an *ideal* is a subset of $\mathcal{P}(\omega)$ closed under subsets and unions; in order to avoid trivialities, we shall always assume that the ideal contains all finite sets and is not generated by a countable collection of sets. The space $\mathcal{P}(\omega)$ is equipped with the usual Polish topology, and therefore it makes sense to speak about descriptive set theoretic complexity of ideals on ω. Finally, the *cofinality* of an ideal $\mathcal{I}$, $\mathrm{cof}(\mathcal{I})$, is the least cardinality of a collection $A \subset \mathcal{I}$ such that every set in the ideal has a superset in the collection A; thus our ideals will always have uncountable cofinality. The cofinality of an ideal is a cardinal number less or equal to the continuum. The comparison of these numbers with traditional cardinal invariants and with each other in various models of set theory carries information about the structure of the underlying ideals. A survey of known results will generate several natural questions and hypotheses, of which we address two.

Question 1.1 What are the possible cofinalities of Borel ideals?

Only four possible uncountable values of standard ideals were known: $\mathfrak{d} = \mathrm{cof}(\mathrm{Fin} \times \mathrm{Fin})$, $\mathrm{cof}(\mathsf{meager}) = \mathrm{cof}(\mathrm{nwd}(\mathbb{Q}))$, $\mathrm{cof}(\mathsf{null}) = \mathrm{cof}(Z)$ and $\mathfrak{c} = \mathrm{cof}(\mathrm{ED})$, where $\mathrm{nwd}(\mathbb{Q})$ is the ideal of nowhere dense subsets of the rationals, Z is the ideal of sets of natural numbers of asymptotic density 0 and ED is the ideal on the square $\omega \times \omega$ generated by vertical sections and graphs of functions. A possible conjecture that these are the only values fails badly, we shall produce many F_σ ideals such that the inequalities between their cofinalities can be manipulated arbitrarily in various generic extensions.

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Question 1.2 What is the smallest cofinality of a Borel ideal?

It is not difficult to argue that every F_σ ideal has cofinality larger or equal to cov(meager), and a result of Louveau and Velickovic [4] shows that every non-F_σ Borel ideal has cofinality at least $\mathfrak{d}$. In view of known examples, the natural conjecture was that $\mathfrak{d}$ is, in fact, the smallest possible cofinality of a Borel ideal. We shall identify a whole array of F_σ ideals whose cofinality is equal to $\aleph_1$ in the Laver model. Since in that model, $\aleph_2 = \mathfrak{b} = \mathfrak{d}$, this refutes the conjecture.

The results of this paper were announced in [3], which also serves as a good source of background and literature. Our notation is standard and follows [1]. For a tree $T \subset (\omega \times \omega)^{<\omega}$, the symbol $[T]$ stands for its set of cofinal branches as a subset of $\omega^\omega \times \omega^\omega$, and $\mathrm{p}[T]$ is the projection of this set into the first coordinate.

2 The smallest possible cofinality

Our F_σ-ideals with very small cofinality will be of a quite special form that sets them apart from the analytic P-ideals.

Definition 2.1 An ideal $\mathcal{I}$ on ω is *fragmented* if there is a partition of $\omega = \bigcup_j a_j$ into finite sets and submeasures φ_j on each of them such that $\lim_j \varphi_j(a_j) = \infty$ and

$$\mathcal{I} = \{b \subset \omega : \exists k\ \forall j\ \varphi_j(a_j \cap b) < k\}.$$

The ideal $\mathcal{I}$ represented as in the previous sentence is *gradually fragmented* if for every k there is an m such that for all l, for all but finitely many j and for any $B \subset \mathcal{P}(a_j)$, if $|B| = l$ and $\varphi_j(b) < k$ (for each $b \in B$), then $\varphi_j(\bigcup B) < m$.

Note that every fragmented ideal is F_σ. The ideal of sets of *polynomial growth* $\mathcal{P} = \{A \subseteq \omega : (\exists k \in \omega)(\forall n \in \omega)\ |A \cap 2^n| \leq n^k\}$ introduced in [4] is a typical example of a gradually fragmented ideal. Many ideals which in retrospect are gradually fragmented were also considered by K. Mazur in [5].

Next we show that any proper forcing notion having the *Laver property* [1] preserves cofinalities of gradually fragmented ideals. As a corollary we get the following:

Theorem 2.2 *In the iterated Laver model,* $\mathrm{cof}(\mathcal{I}) = \aleph_1 < \mathfrak{b} = \mathfrak{c} = \aleph_2$ *for every gradually fragmented ideal* $\mathcal{I}$.

Recall that a forcing notion has the *Laver property* if for every function $f \in \omega^\omega$ in the extension which is dominated by a ground model function, there is a ground model function $g : \omega \to [\omega]^{<\aleph_0}$ such that for every $i \in \omega$, $|g(i)| \leq i + 1$ and $f(i) \in g(i)$. As the terminology suggests, the Laver forcing as well as its countable support iterations have the Laver property (cf. [1]).

Proposition 2.3 *Let $\mathbb{P}$ be a proper forcing notion having the Laver property and let $\mathcal{I}$ be a gradually fragmented ideal. Then in the $\mathbb{P}$-extension, $\mathcal{I} \cap V$ is cofinal in $\mathcal{I}$.*

Proof. Let $\mathcal{I}$ be an ideal gradually fragmented via $\langle a_j : j \in \omega\rangle$ and $\varphi = \sup_j \varphi_j$. Let $\dot{a}$ be an $\mathbb{P}$-name and $p \in \mathbb{P}$ a condition such that $p \Vdash \dot{a} \in \mathcal{I}$. Find $p' \leq p$ and $k \in \omega$ such that $p' \Vdash \varphi(a) < k$. Use the gradual fragmentation to find a number $m \in \omega$ as well as numbers $0 = l_0 < l_1 < l_2 < \cdots$ so that for every $i \in \omega$ and for every l, if $l_i \leq l < l_{i+1}$ and $B \subset \mathcal{P}(a_l)$ is a collection of size $\leq i + 1$ consisting of sets of ϕ_l-mass $< k$, then $\varphi_l(\bigcup B) < m$. Use the Laver property of $\mathbb{P}$ to find a function $g : \omega \to [\omega]^{<\omega}$ in the ground model and a condition $q \leq p'$ such that for all $i \in \omega$ and every $l_i \leq l < l_{i+1}$ it is the case that $g(l) \subset \mathcal{P}(a_l)$ is a set of size at most $i + 1$ consisting of sets of ϕ_l-mass $< k$, and $q \Vdash \forall i \in \omega\ \dot{a} \cap a_l \in g(i)$. Let $b = \bigcup_i \bigcup g(i)$. The properties of the sequence $\langle l_i : i \in \omega\rangle$ imply that $\varphi(b) < m$, so $b \in \mathcal{I}$ and clearly $q \Vdash \dot{a} \subset b$. □

The previous result should be contrasted with the provably high cofinality of fragmented ideals which are not gradually fragmented. Recall (e.g., from [4]) that a subset P of an ideal $\mathcal{I}$ is *strongly unbounded* if P contains no infinite bounded subset, i.e the union of every infinite subset of P is $\mathcal{I}$-positive. Clearly, every ideal $\mathcal{I}$ which contains a strongly unbounded subset of size $\mathfrak{c}$ has $\mathrm{cof}(\mathcal{I}) = \mathfrak{c}$.

Theorem 2.4 *If $\mathcal{I}$ is a fragmented ideal then either*

1. *$\mathcal{I}$ is gradually fragmented, or*
2. *$\mathcal{I}$ contains a perfect strongly unbounded subset.*

P r o o f . Let $\mathcal{I}$ be fragmented (via $\langle a_j : j \in \omega\rangle$ and $\varphi = \sup_j \varphi_j$) which is not gradually fragmented. If $k \in \omega$ is where graduality fails, then there is an infinite set $C \subset \omega$, a sequence $\langle B_j : j \in C\rangle$ (with $B_j \subset \mathcal{P}(a_j)$ and $\varphi(b) < k$ for all $b \in B_j$) and a partition $\{C_m : m \in \omega\}$ of C into infinite sets, such that for each $m \in \omega$ there is an $l_m \in \omega$ such that:

$$j \in C_m \Rightarrow |B_j| = l_m \text{ and } \varphi\left(\bigcup B_j\right) > m.$$

For $j \in C_m$ write $B_j = \{K_i^j : i < l_m\}$. Now, for each $m \in \omega$, let $\left\{C_m^n : n \in \omega\right\}$ be a partition of C_m into infinite sets, and set:

$$C^n = \bigcup_{m\in\omega} C_m^n, \quad X_n = \bigcup\{a_j : j \in C^n\} \text{ and } X = \bigcup_{n\in\omega} X_n.$$

We shall use the following simple fact:

Claim 2.5 *For all $N \in \omega$, there is a sequence of functions $\langle f_n : n \in \omega\rangle$ from ω to N such that:*

$$\left(\forall A \in [\omega]^N\right)(\exists M \in \omega)([0, N) \subseteq \{f_n(M) : n \in A\}).$$

P r o o f . Fix $N \in \omega$, for each $t \in N^{<\omega}$ define A_t an infinite subset of ω by recursion on the length of t as follows: Let $A_\varnothing = \omega$, if A_t has been defined for all $t \in N^n$, let $\left\{A_{t^\frown\langle j\rangle} : j < N\right\}$ be a partition of A_t into infinite sets. Let $f_0 : \omega \to N$ be the function such that $f_0{\restriction}A_{\langle j\rangle} = j$ (for each $j \in N$). Define $f_{n+1} : \omega \to N$ by: $f_{n+1}{\restriction}A_{t^\frown\langle j\rangle} = j$ (for each $t \in N^n$ and $j \in N$). The sequence $\langle f_n : n \in \omega\rangle$ has the desired property: If $A = \{n_0, \ldots, n_{N-1}\} \in [\omega]^N$ is such that $n_i < n_j$, let $t \in N^{n_{N-1}+1}$ such that $t(n_i) = i$, then for $M \in A_t$ and for $i < N$, $f_{n_i}(M) = i$. □

Apply the claim to each C_m^n and $N = l_m$, in order to obtain a sequence of functions $\langle f_p^{\langle n,m\rangle} : p \in \omega\rangle$ from C_m^n to l_m. Then, define a sequence of functions $\langle f_p : p \in \omega\rangle$ from C to ω by:

$$f_p = \bigcup_{n,m\in\omega} f_p^{\langle n,m\rangle}$$

and a sequence $\langle J_p : p \in \omega\rangle$ of subsets of ω:

$$J_p = \bigcup_{j\in C} K^j_{f_p(j)}$$

Clearly $\varphi(J_p) < k$, as each $K^j_{f_p(j)} \subset a_j$ is of φ-mass less than k. For $n, p \in \omega$, let $J_p^n = X_n \cap J_p$.

Claim 2.6 *For each $n, m \in \omega$ and $A \in [\omega]^{l_m}$,*

$$\varphi\left(\bigcup_{p\in A} J_p^n\right) > m.$$

In particular, for each n, the sequence $\left\langle J_p^n : p \in \omega\right\rangle$ is strongly unbounded.

Fix $n, m \in \omega$ and $A \in [\omega]^{l_m}$. By the choice of the sequence $\left\langle f_p^{\langle n,m\rangle} : p \in \omega\right\rangle$ (Claim 2.5), there is $M \in C_m^n$ such that $[0, l_m) = \left\{f_p^{\langle n,m\rangle}(M) : p \in A\right\}$. So

$$\bigcup B_M = \bigcup_{p\in A} K^M_{f_p(M)} \subset \bigcup_{p\in A} J_p^n$$

and $\varphi(\bigcup B_M) > m$.

We now define the perfect strongly unbounded subset of $\mathcal{I}$: Let $\mathcal{A} \subset \omega^\omega$ be a perfect family of eventually-different functions. Define $G : \mathcal{A} \to \mathcal{I}$ by

$$G(g) = \bigcup_{n\in\omega}\left(X_n \cap J_{g(n)}\right).$$

It is clear that $\varphi(G(g)) < k$ and that G is a well defined continuous injection.

Claim 2.7 *The set G"$\mathcal{A}$ is strongly unbounded.*

Let $\langle G_r = G(g_r) : r \in \omega\rangle$ be an infinite subset of G"$\mathcal{A}$. First, observe that, since $\mathcal{A}$ is an eventually-different family of functions, for each $m \in \omega$ there is $L \in \omega$ such that for each $n \geq L$, the set $\{g_r(n) : r < l_m\}$ has cardinality l_m. Now, set $m \in \omega$, $n \geq L$ and $A = \{g_r(n) : r < l_m\}$. By Claim 2.6, $\varphi\big(\bigcup_{r<l_m} J^n_{g_r(n)}\big) > m$. Hence $\varphi(\bigcup_{r\in\omega} G_r) = \infty$. □

While the cofinality of gradually fragmented ideals is consistently small, it is also true that their cofinality is consistently quite large in comparison to traditional cardinal invariants.

There is a natural forcing associated to every Borel ideal $\mathcal{I}$, which adds a new element of $\mathcal{I}$ not contained in any ground model set in $\mathcal{I}$.

Definition 2.8 Let $\mathcal{I}$ be a Borel ideal, not countably generated. Let $\mathcal{J}$ be the σ-ideal on $\mathcal{I}$ generated by the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$. Denote by $\mathbb{P}_\mathcal{I}$ the forcing $\mathrm{Borel}(\mathcal{I})/\mathcal{J}$.

The forcing $\mathbb{P}_\mathcal{I}$ falls naturally into the scope of [7]. Formally, one should define $\mathcal{J}$ as the σ-ideal on $\mathcal{P}(\omega)$ generated by singletons and the sets in the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$, hence dealing with the quotient $P_\mathcal{J} = \mathrm{Borel}(\mathcal{P}(\omega))/\mathcal{J}$. The Borel ideal $\mathcal{I}$ is then itself a condition in $P_\mathcal{J}$ and $\mathbb{P}_\mathcal{I}$ is just a restriction of $P_\mathcal{J}$ below $\mathcal{I}$. General theorems of [7, Section 4.1] and simple genericity arguments give:

Proposition 2.9 *Let $\mathcal{I}$ be a Borel ideal and let $\mathbb{P}_\mathcal{I}$ be the corresponding forcing. Then*:

1. $\mathbb{P}_\mathcal{I}$ *is proper.*
2. $\mathbb{P}_\mathcal{I}$ *preserves* $\mathrm{non}(\mathsf{meager})$.
3. $\mathbb{P}_\mathcal{I}$ *preserves* $\mathrm{cof}(\mathsf{meager})$ *and preserves P-points, provided that $\mathcal{I}$ is F_σ.*
4. $\mathbb{P}_\mathcal{I}$ *adds an unbounded element of $\mathcal{I}$.*

P r o o f . Items 1 and 2 follow directly from the fact that the ideal J is σ-generated by compact sets [7, Theorem 4.1.2], item 4 is a straightforward genericity argument (here we use the restriction to $\mathcal{I}$). To see item 3, one only needs to realize that if $\mathcal{I}$ is an F_σ ideal on ω, then the σ-ideal J is, in fact, σ-generated by a σ-compact collection of compact sets. By [7, Theorem 4.1.8] $\mathbb{P}_\mathcal{I}$ is ω^ω-bounding (does not add unbounded reals) which together with (2) implies that $\mathrm{cof}(\mathsf{meager})$ is preserved. The fact that $\mathbb{P}_\mathcal{I}$ preserves P-points is proved yet not stated in [7, Theorem 4.1.8]. □

As a corollary one gets the following:

Theorem 2.10 *It is consistent that* $\mathrm{cof}(\mathsf{meager}) = \aleph_1 < \mathrm{cof}(\mathcal{I}) = \mathfrak{c} = \aleph_2$ *for all uncountably generated* F_σ *ideals $\mathcal{I}$ at once.*

P r o o f . To construct the model witnessing the statement of the theorem, start with a model of CH and use a suitable bookkeeping tool to set up a countable support iteration of forcings of the form $\mathbb{P}_\mathcal{I}$ defined above, as $\mathcal{I}$ varies over all possible F_σ ideals in the extension. Suitable iteration theorems show that the iteration is proper, bounding, preserves Baire category (and also preserves P-points). Thus, in the resulting model the desired statement holds. □

Another property of the forcing $\mathbb{P}_\mathcal{I}$ used heavily in the next section is the *continuous reading of names*: For every $\mathcal{J}$-positive Borel subset B of $\mathcal{I}$ and a Borel function $f : B \to 2^\omega$ there is a $\mathcal{J}$-positive Borel subset C of B such that f restricted to C is continuous (cf. [7, Theorem 4.1.2]).

3 Nonclassification of possible cofinalities

This section aims to produce many F_σ ideals whose cofinality invariants can take quite independent values in various generic extensions. These will be gradually fragmented ideals with an additional weak boundedness property.

Definition 3.1 Let $g, h \in \omega^\omega$ be increasing functions such that $g(0), h(0) > 0$, $\log_{h(i)} g(i) \geq i$ and $(\prod_{j<i} g(j))(\sum_{j<i} g(j)) \log_{h(i)} 2 < 2^{-i}$. Let $\omega = \bigcup_i a_i$ be a partition of ω into successive intervals of respective lengths $g(i)$, and let φ_i be the submeasure on a_i defined by $\varphi_i(b) = \log_{h(i)}(|b|)$ if $b \neq \emptyset$ and $\varphi_i(\emptyset) = 0$.

Finally, for each infinite set $u \subset \omega$ define $\mathcal{I}_u$ to be the ideal on the countable set $\bigcup_{i\in u} a_i$ given by

$$\mathcal{I}_u = \{b : \varphi(b) < \infty\},$$

where $\varphi(b) = \sup_{i\in u} \varphi_i(b \cap a_i)$.

The choice of the functions g, h is not particularly relevant to this paper. The following two lemmas encapsulate all the arithmetic properties of the submeasures used below.

Lemma 3.2 *Whenever $i \in \omega$, $k \le \left|\prod_{j<i} \mathcal{P}(a_j)^{\prod_{j<i} a_j}\right|$ and $\{b_l : l \in k\}$ are subsets of a_i, then $\varphi_i(\bigcup_l b_l) < \max_l \varphi_i(b_l) + 2^{-i}$.*

Proof. Let $b = \bigcup_l b_l$. Thus, $|b| \le k \cdot \max_l |b_l|$ and by the properties of the functions $g, h \in \omega^\omega$, $\varphi_i(b) \le \log_{h(i)} k + \max_l \varphi_i(b_l) < \max_l \varphi_i(b_l) + 2^{-i}$. □

Lemma 3.3 *If $b_i \subset a_i$ for $i \in \omega$ are nonempty sets such that $\lim_i \varphi_i(b_i) = \infty$, and $f : \prod_i b_i \to \prod_i \mathcal{P}(a_i)$ is a continuous function, then there are sets $c_i \subset b_i$ such that $\lim_i \varphi_i(c_i) = \infty$ and for $x \in \prod_i c_i$ and $j \in \omega$, $f(x)\restriction j$ depends only on $x\restriction j$.*

Proof. First argue that for every $i \in \omega$ and every map $f : \prod_{j\le i} b_j \to \prod_{j<i} \mathcal{P}(a_j)$ there is a nonempty set $b'_i \subset b_i$ such that $\varphi_i(b'_i) > \varphi_i(b_i) - 2^{-i}$ and for every $x \in \prod_{j<i} b_j \times b'_i$, the value $f(x)$ depends only on $x\restriction i$. This is immediate by Lemma 3.2, as there are certainly fewer than $|\prod_{j<i} \mathcal{P}(a_j)^{\prod_{j<i} a_j}|$ many maps from $\prod_{j<i} b_j$ to $\prod_{j<i} \mathcal{P}(a_j)$.

Now argue that for every $i \le k$ and every map $f : \prod_{j\le k} b_j \to \prod_{j<i} \mathcal{P}(a_j)$ there are nonempty set $b'_j \subset b_j$ for each $i \le j \le k$ such that $\varphi_j(b'_j) > \varphi_j(b_j) - 2^{-j}$ and for every $x \in \prod_{j<i} b_j \times \prod_{i\le j\le k} b'_j$, the value $f(x)$ depends only on $x\restriction i$. This is proved by a straightforward downward induction on k for every fixed i using the previous paragraph.

Finally, towards the proof of the lemma, suppose that $f : \prod_i b_i \to \prod_i \mathcal{P}(a_i)$ is a continuous function. By a compactness argument and the continuity of the function f, for every number $i \in \omega$ there is a number $k_i > i$ such that the value of $f(x)\restriction i$ depends only on $x\restriction k_i$. By induction on $i \in \omega$, build nonempty sets b^i_j for $j \in \omega$ so that

1. $b^0_j = b_j$, $b^{i+1}_j = b^i_j$ if $j < i$, and $b^{i+1}_j \subset b^i_j$ and $\varphi_j(b^{i+1}_j) > \varphi_j(b^i_j) - 2^{-j}$ if $j \ge i$;
2. for every $x \in \prod_j b^i_j$, the value $f(x)\restriction i$ depends only on $x\restriction i$.

The induction step is performed easily by the previous paragraph applied to $i \le k = k_i$. In the end, let $c_i = b^i_i$. It is clear that for $x \in \prod_i c_i$ and every $j \in \omega$, $f(x)\restriction j$ depends only on $x\restriction j$. Also, $\varphi_i(c_i) > \varphi(b_i) - i2^{-i}$ for each $i \in \omega$; as $\lim_i i2^{-i} = 0$, it follows that $\lim_i \varphi_i(c_i) = \infty$ as required. □

We shall show that whenever $u, v \subset \omega$ are almost disjoint infinite sets then the inequalities $\mathrm{cof}(\mathcal{I}_u) > \mathrm{cof}(\mathcal{I}_v)$ and $\mathrm{cof}(\mathcal{I}_v) > \mathrm{cof}(\mathcal{I}_u)$ are both consistent, and this effect can be reached in both iteration-type and product-type extensions. The product method even leads to the consistency of the cofinalities of many of these ideals being mutually distinct at the same time (a somewhat similar result has been proved in [2]).

Theorem 3.4 *It is relatively consistent with* ZFC *that there are uncountably many distinct cofinalities of ideals of the form $\mathcal{I}_u$.*

The basic forcing $\mathbb{P}_\mathcal{I}$ to achieve this has already been introduced in Proposition 2.9 as the forcing $P_\mathcal{J} = \mathrm{Borel}(\mathcal{P}(\omega))/\mathcal{J}$, where $\mathcal{J}$ is the σ-ideal on $\mathcal{P}(\omega)$ σ-generated by singletons and the sets in the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$ restricted to $\mathcal{I}$ (considered as a condition of $P_\mathcal{J}$). Here we shall strengthen the initial condition and give a different presentation of the forcing for the case of the fragmented ideals $\mathcal{I}_u$.

Let $u \subset \omega$ be an infinite set. Set $T = \bigcup_j \prod_{i\in j\cap u} a_i$ and let $\mathcal{J}_u$ be the σ-ideal on $\prod_{i\in u} a_i = [T]$ generated by all products $\prod_{i\in u} b_i$ of sets b_i whose φ_i-masses are uniformly bounded by some real number. This is equivalent to generating the ideal by sets $A \subset \prod_{i\in u} a_i$ such that $\bigcup_{f\in A} \mathrm{rng}(f) \in \mathcal{I}_u$. So the quotient forcing $P_{\mathcal{J}_u}$ of Borel $\mathcal{J}_u$-positive subsets of $[T]$ ordered by inclusion is a proper, bounding forcing preserving Baire category and adding an unbounded element of $\mathcal{I}_u$ ([7, Section 4.1] and Proposition 2.9) .

Identifying functions in the product with their ranges, it is quite clear that, in fact, $P_{\mathcal{J}_u}$ is equivalent to the forcing $\mathbb{P}_{\mathcal{I}_u}$ below the set of all selectors on the sets $a_i : i \in u$.

We shall give a combinatorial form of the quotient forcing $P_{\mathcal{J}_u}$. Say that a tree $S \subset T$ is a *large tree* if for every real number r, every node of T can be extended to a splitting node s at some level $i \in u$ such that the φ_i-mass of the set of immediate successors of the splitting node is at least r. As in [7, Claim 4.1.9] the following lemma holds.

Lemma 3.5 *Every analytic $\mathcal{J}_u$-positive set contains all branches of a large tree.*

Thus, the poset of large trees ordered by inclusion is naturally densely embedded in $P_{\mathcal{J}_u}$ by the embedding $S \mapsto [S]$.

Proof. Suppose that $A \subset \prod_{i\in u} a_i$ is an analytic $\mathcal{J}_u$-positive set, a projection of some tree $S \subset (\omega \times \omega)^{<\omega}$. Thinning out the tree S if necessary we may assume that for every node $t \in S$, $\mathrm{p}[S\restriction t] \notin \mathcal{J}_u$. By recursion on $n \in \omega$ build finite trees U_n as well as functions f_n so that

1. $0 = U_0$ and U_{n+1} is an end-extension of U_n. The tree $U = \bigcup_n U_n$ will be the large tree we are looking for;
2. $f_0 \subset f_1 \subset \dots$ are functions such that $\mathrm{dom}(f_n) \subset U_n$ is a set including all endnodes of U_n, and $f_n(t) \in S$ is a pair of finite sequences of which t is the first, for every $t \in \mathrm{dom}(f_n)$. Thus, for every point $x \in [U]$, the union $\bigcup_n f_n(x\restriction n)$ witnesses the fact that $x \in A$ and therefore $[U] \subset A$;
3. for every endnode $t \in U_n$ there is an extension $s \in U_{n+1}$ such that, writing $i = \min(u \setminus \mathrm{dom}(s))$, $\varphi_i(\{j \in a_i : s^\frown\langle i, j\rangle \in U_{n+1}\}) > n$. This guarantees the largeness of the tree U.

The recursion is straightforward: Suppose that U_n, f_n have been constructed, fix an endnode $t \in U_n$ and construct the part of U_{n+1} and f_{n+1} above t in the following way. There must be a finite sequence s extending t such that writing $i = \min(u \setminus \mathrm{dom}(s))$, $\varphi_i(\{j \in a_i : \exists x \in \mathrm{p}[S\restriction f_n(t)]\ s^\frown\langle i, j\rangle \subset x\}) > n$. For if such a sequence s did not exist, Lemma 3.2 would imply that for every $i \in u \setminus \mathrm{dom}(t)$, $\varphi_i\{j \in a_i : \exists x \in \mathrm{p}[S\restriction f_n(t)]\ x(i) = j\} < n + 1$ and therefore the set $\mathrm{p}[S\restriction f_n(t)]$ would be in the ideal $\mathcal{I}_u$. Pick such a finite sequence s, write $i = \min(u \setminus \mathrm{dom}(s))$, for every number $j \in a_i$ such that $\exists x \in \mathrm{p}[S\restriction f_n(t)]\ s^\frown\langle i, j\rangle \subset x$ put the sequence $s^\frown\langle i, j\rangle$ into U_{n+1} and pick a node $f_{n+1}(s^\frown\langle i, j\rangle)$ in the tree $S\restriction f_n(t)$ whose first coordinate is this sequence, and proceed to another endnode of U_n. □

The following simple definition and lemma will be useful in the fusion arguments below.

Definition 3.6 A *good map* is a continuous map $G : \prod_{i\in w} b_i \to [T]$ ($b_i \subset a_i$) such that the numbers $\varphi_i(b_i)$ tend to infinity and for every choice of sets $c_i \subset b_i$ such that the numbers $\varphi_i(c_i)$ tend to infinity, the image G'' $\prod_{i\in w} c_i$ contains all branches of some large tree.

Lemma 3.7 *If S is a large tree whose splitting nodes occur only at levels in some set $w \subset \omega$, then there is a good map $G : \prod_{i\in w} b_i \to [S]$.*

Proof. Thin out S if necessary so that every level of S contains at most one splitting node, and writing $b_i \subset a_i$ for the set of all immediate successors of the splitting node at level i, the numbers $\varphi_i(b_i)$, $i \in w$ tend to infinity. The function $G : \prod_{i\in w} b_i \to [S]$ is then defined in such a way that $G(x)$ is the unique path y through the tree S such that whenever $i \in w$ is such that $x\restriction i$ is a splitting node of S then $x(i) = y(i)$. It is easy to verify the required properties of the function G. □

We shall show that if $v \subset \omega$ is an infinite set with finite intersection with u, then both countable support iterations and countable support products of quotient forcing $P_{\mathcal{J}_u}$ preserve the cofinality of $\mathcal{I}_v$.

Lemma 3.8 *In the $P_{\mathcal{J}_u}$ extension, every set in $\mathcal{I}_v$ can be covered by a ground model set in $\mathcal{I}_v$ with arbitrarily close φ-mass.*

Proof. Suppose that $B \in P_{\mathcal{J}_u}$ is a condition forcing that $\dot{a} \in \mathcal{I}_v$ is a set of φ-mass $< r$, and let $\epsilon > 0$ be a real. Find a large tree S and a continuous function f such that $[S] \subset B$, $f : [S] \to \mathcal{I}_v$ and $B \Vdash \dot{f}(\dot{x}_{\mathrm{gen}}) = \dot{a}$. Find a number $m \in \omega$ large enough so that $u \cap v \subset m$ and $2^{-m} < \epsilon$. Thinning out the tree S we may assume that the range of the function f consists only of sets of mass $< r$, and for every $i \in v \cap m$, $[S]$ decides the value $\dot{a} \cap a_i$ to be some specific set $H(i) \subset a_i$.

Let w be the set of splitting levels of S and let $G_0 : \prod_{i\in w} b_i \to [S]$ be a good function as in Lemma 3.7. Extend G_0 to a continuous function G on some product $\prod_{i\in\omega} b_i$ by setting $G(x) = G_0(x\restriction w)$. Use Lemma 3.3 to find sets

$c_i \subset b_i$ such that $\lim_i \varphi_i(c_i) = \infty$ and for every $i \in v \setminus m$, $f \circ G(x) \cap a_i$ depends only on $x \restriction i+1$. Since $i \notin u$, the value $G(x)$ does not depend on $x(i)$. Thus, there are at most $|\prod_{j<i} c_j|$ many possibilities for the value of $f \circ G(x) \cap a_i$, each of φ_i-mass $\leq r$. By Lemma 3.2, their union $H(i)$ has φ_i-mass $< r + 2^{-i}$. Let $b = \bigcup_{i \in v} H(i)$ and observe that $G\text{''} \prod_i c_i \Vdash \dot{a} \subset \check{b}$. □

For the treatment of the product, we need a slight strengthening of this argument. Let K be an arbitrary set, let u_k for $k \in K$ be infinite subsets of ω, let $P_{u_k} = P_{\mathcal{J}_{u_k}}$, and consider the countable support product $P = \prod_k P_{u_k}$.

Lemma 3.9 *Let $u \subset \omega$ be an infinite set. P forces that every set in I_u is covered by a set of arbitrarily close φ-mass that belongs to the model given by $\prod\{P_{u_k} : u_k \cap u$ is infinite$\}$.*

P r o o f . Let us set up some useful standard notation for the product. A condition in the product is a function p with a countable domain $\mathrm{dom}(p) \subset K$ such that for each $k \in \mathrm{dom}(p)$ the value $p(k)$ is a tree in the poset P_{u_k}. The set $[p]$ is defined as the subset of $(2^\omega)^{\mathrm{dom}(p)}$ consisting of those sequences $\vec{x}$ such that for every $k \in \mathrm{dom}(p)$, $\vec{x}(k)$ is a branch through the tree $p(k)$. A splitting node of p is a splitting node of one of the trees in $\mathrm{rng}(p)$. The generic object for the product is identified with the sequence $\vec{x}_{\mathrm{gen}} : K \to 2^\omega$ such that for every condition p in the generic filter, $\vec{x}_{\mathrm{gen}} \restriction \mathrm{dom}(p) \in [p]$.

Let $p \in P$ and let $\dot{a}$ be a P-name for a set in $\mathcal{I}_u$ of φ-mass $< r$ and let $\epsilon > 0$ be a real number. Find $m \in \omega$ such that $2^{-m} < \epsilon$ and strengthen the condition p if necessary to decide the values of $\dot{a} \cap a_j$ for all $j \in m$. The usual countable support product fusion arguments yield a condition $q \leq p$ and a continuous function $f : [q] \to \prod_i \mathcal{P}(a_i)$ such that $q \Vdash \forall i \in u\ \dot{a} \cap a_i = f(\vec{x}_{\mathrm{gen}} \restriction \mathrm{dom}(q))(i)$, and for every $\vec{x} \in [q]$ and every $i \in u$, $\varphi_i(f(\vec{x})(i)) < r$. For $k \in \mathrm{dom}(q)$ let $v_k \subset u_k$ be the set of splitting levels of the tree $q(k)$. Thinning out the condition q if necessary we may assume that the sets $\{v_k : k \in \mathrm{dom}(q)\}$ are pairwise disjoint, and if $u \cap u_k$ is finite then $v_k \cap u = 0$.

For each $k \in \mathrm{dom}(q)$, Lemma 3.7 yields a good map $G_k : \prod_{i \in v_k} b_i \to [q(k)]$. The product of these maps yields a map $H : \prod_i b_i \to [q]$. Use Lemma 3.3 to find sets $c_i \subset b_i$ such that $\lim \varphi_i(c_i) = \infty$ and $f \circ H(y) \cap a_j$ depends only on $y \restriction j+1$ for $y \in \prod_i c_i$ and $j \in u$. Let $q' \leq q$ be some condition with $\mathrm{dom}(q') = \mathrm{dom}(q)$ and for every $k \in \mathrm{dom}(q')$, $q'(k)$ is some large tree below the set $G_k\text{''} \prod_{i \in v_k} c_i$.

Let $L = \{k \in \mathrm{dom}(q) : u_k \cap u$ is infinite$\}$ and let $P_L = \prod_{k \in L} P_{u_k}$. Consider the P_L-name $\dot{b}$ for the set defined as follows: for every $k \in L$ choose some $y_k \in \prod_{i \in v_k} c_i$ such that $\vec{x}_{\mathrm{gen}}(k) = G_k(y_k)$, and for each $j \in u$ let $\dot{b} \cap a_j$ be the union of all possible values of $f \circ H(y) \cap a_j$ for $y \in \prod_i c_i$ extending all $y_k : k \in L$. There are only fewer than $\prod_{j<i} |c_i|$ many such values, and so by Lemma 3.2 this union has size φ_j-mass less than $r + \epsilon$. It is also clear that $q' \Vdash \dot{a} \subset \dot{b}$, and the lemma follows. □

P r o o f o f T h e o r e m 3 . 4 . Let V be a model of CH and let $u_\alpha, \alpha < \omega_1$ be an almost disjoint family of infinite subsets of ω. Let $\mathbb{P}_\alpha$ be a countable support product of $\omega_{\alpha+1}$ copies of the forcing $P_{\mathcal{J}_{u_\alpha}}$ and let $\mathbb{P}$ be a countable support product of the $\mathbb{P}_\alpha$, $\alpha < \omega_1$. Then:

1. $\mathbb{P}$ is proper and ω_2-c.c., hence it does not collapse cardinals.
2. Each $\mathbb{P}_\alpha$ forces $\mathrm{cof}(\mathcal{I}_{u_\alpha}) = \omega_{\alpha+1}$.
3. $\mathbb{P}$ forces $\mathrm{cof}(\mathcal{I}_{u_\alpha}) = \omega_{\alpha+1}$ for every $\alpha < \omega_1$.

As V is a model of CH and each forcing in the product has size $\mathfrak{c}$ the ω_2-c.c. follows from a standard Δ-system argument. The properness of $\mathbb{P}$ easily follows from a standard Sacks-type fusion argument.

By a simple genericity argument all of the generic reals added by $\mathbb{P}_\alpha$ are mutually independent elements of $\mathcal{I}$ each unbounded over the rest. If the ground model is a model of CH, then $\mathbb{P}_\alpha$ forces $\mathfrak{c} = \omega_{\alpha+1}$.

To see (3) first note that $\mathbb{P}$ forces $\mathrm{cof}(\mathcal{I}_{u_\alpha}) \geq \omega_{\alpha+1}$ by (2). On the other hand, the fact that $\mathbb{P}$ forces $\mathrm{cof}(\mathcal{I}_{u_\alpha}) \leq \omega_{\alpha+1}$ follows directly from Lemma 3.9. □

As mentioned before, also countable support iteration can be used to separate the cofinalities of the ideals $\mathcal{I}_u$.

Theorem 3.10 *Let u, v be infinite almost disjoint subsets of ω, let $\mathbb{P}$ be a countable support iteration of length ω_2 of the forcing $P_{\mathcal{J}_u}$, and let G be $\mathbb{P}$-generic over a model of CH. Then $V[G] \models \mathrm{cof}(\mathcal{I}_v) < \mathrm{cof}(\mathcal{I}_u)$.*

P r o o f . Say that a forcing P *strongly preserves* the ideal $\mathcal{I}_v$, if every set $a \in \mathcal{I}_v$ in the extension can be covered by a ground model set of an arbitrarily close φ-mass. We shall show that the countable support iteration of proper posets strongly preserving $\mathcal{I}_v$ also strongly preserves $\mathcal{I}_v$, using the first preservation theorem [1, Theorem 6.1.13]. The theorem then follows from the fact that the forcing $P_{\mathcal{J}_u}$ strongly preserves $\mathcal{I}_v$, Lemma 3.8.

The following easy claim, which follows directly from the very slow fragmentation property of the submeasure, is the starting point.

Claim 3.11 *Suppose that $\{b_n : n \in \omega\}$ are sets in $\mathcal{I}_v$ and $r > 0$ is a real number such that $\varphi(b_n) \leq r$ holds for every number n. Suppose $\epsilon > 0$. Then there is an infinite set $W \subset \omega$ such that $\varphi(\bigcup_{n\in W} b_n) < r + \epsilon$.*

Proof. By a compactness argument we may assume that the sets b_n converge to some c, and then $\varphi(c) \leq r$. Find $i \in \omega$ such that for every $j > i$, $j2^{-j} < \epsilon$. For every $j > i$, pick some n_j so that b_{n_j}, c have the same intersection with the set $\bigcup_{k\in j} a_k$, and so that the numbers n_j are pairwise distinct. The set $W = \{n_j : j > i\}$ works as desired by Lemma 3.2. □

Fix positive rationals $r, \epsilon > 0$. Let X be the space of all sequences $x = \langle r_x, x(0), x(1), \dots\rangle$ where $r_x \in \mathbb{Q}$ is a positive rational smaller than r and $(\forall i \in \omega)(x(i) \subset a_{j_i} \wedge \varphi_i(x(i)) \leq r_x)$, where j_i is the i-th element of the set v. Let Y be the set of all sequences $y = \langle y(0), y(1), \dots\rangle$ such that $(\forall i \in \omega)(y(i) \subset a_{j_i} \wedge \varphi_i(y(i)) \leq r + \epsilon)$. Let $\sqsubseteq_n$ be the relation on $X \times Y$ defined by: $x \sqsubseteq_n y$ if for every $i > n$, $x(i) \subset y(i)$. Let $\sqsubseteq = \bigcup_n \sqsubseteq_n$.

These relations fall into the framework of [1, Definition 6.1.6]: (0) Both X and Y are naturally homeomorphic to closed subsets of ω^ω (treating the first coordinate of X as a discrete set), (1) for every $y \in Y$ the set $\{x \in X : x \sqsubseteq_n y\}$ is obviously closed in X, (2) the domain of the relation $\sqsubseteq$ is all of X, hence it is closed in X, (3) For every countable set $A \subseteq X$ there is a $y \in Y$ such that for every $x \in A$ $x \sqsubseteq y$ (This follows directly from very slow fragmentation: if $A = \langle x_n : n \in \omega\rangle$ and $\langle i_n : n \in \omega\rangle$ is an increasing sequence of numbers such that for every $j > i_n$ and sets $B, C \subset a_j$, if $\varphi_j(B), \varphi_j(C) < r + \epsilon - \epsilon/(n+1)$, then $\varphi_j(B \cup C) < r + \epsilon - \epsilon/(n+2)$, then the sequence $y = \langle \bigcup_{n:i_n<i} x_n(i) : i \in \omega\rangle \in Y$ $\sqsubseteq$-dominates all the points in $\{x_n : n \in \omega\}$), and (4) all of the closed sets mentioned have absolute definition.

We shall show that if the forcing P strongly preserves $\mathcal{I}_v$, then it preserves $\sqsubseteq$ in the sense of [1, Definition 6.1.10]. The preservation theorem [1, Theorem 6.1.13] then completes the proof.

Suppose that M is a countable elementary submodel of a large structure, $\langle \dot{x}_l : l \in k\rangle$ are finitely many names for elements of the space X in the model M, suppose $\langle p_n : n \in \omega\rangle$ is a decreasing collection of conditions in the model M such that p_n decides $\dot{x}_l \restriction n$, yielding sequences $\langle \bar{x}_l : l \in k\rangle$ in $X \cap M$, and suppose that $y \in Y$ is a point such that $\forall x \in X \cap M$, $x \sqsubseteq y$. We must find a condition $q \leq p_0$ such that

1. q is M-master for P;
2. $q \Vdash \forall x \in X \cap M[G]\ x \sqsubseteq y$; and
3. for all $l \in k$ for all $n \in \omega$ $q \Vdash \bar{x}_l \sqsubseteq_n y \to \dot{x}_l \sqsubseteq_n y$.

To find the condition q, first work in the model M. Fix a rational $r' < r$ greater than all the numbers $r_{\bar{x}_l}$. By assumption, for each n there is a condition p'_n and a set $b_{ln} \in \mathcal{I}_v$ such that $(\forall i \geq n)(\varphi_i(b_{ln} \cap a_{j_i}) < r')$, $p'_n \Vdash \dot{x}_l(i) \subset (b_{ln} \cap a_{j_i})$ and for $i \in n$, $a_{ln} \cap a_{j_i} = \bar{x}_l(i)$.

Use Claim 3.11 to find an infinite set $d \subset \omega$ such that for each l less than or equal to k, $\forall i \varphi_i(\bigcup_{n\in d} b_{ln} \cap a_{j_i}) < r$. Set $b = \bigcup_{n\in d, l\in k} b_{ln}$, then by the very slow fragmentation, there is $i_0 \in \omega$ such that $(\forall i > i_0)$ $\varphi_i(b \cap a_{j_i}) < r$. Since y $\sqsubseteq$-dominates all elements of $X \cap M$, there must be $j > i_0$ such that for every $i > j$, $b \cap a_{j_i} \subset y(i)$. Let $n > j$ be a number in the set d, and use the properness of the forcing P to find a master condition $q \leq p'_n$. The last item holds by the choice of the condition p'_n: for $l \leq k$,

$$p'_n \Vdash \dot{x}_l(i) \subset (b_{ln} \cap a_{j_i}) \subset (b \cap a_{j_i}) \subset y(i)$$

for all $i > n$. The first item holds by the choice of the condition q. The second item is an immediate consequence of the first, and the fact that the forcing P strongly preserves the ideal $\mathcal{I}_v$: If $\dot{x}$ is a name in M for an element of $X \cap M[G]$, such that $\varphi(\dot{x}) < r$, by assumption, there is $a \in M$ such that $\varphi(a) < r$ and $\dot{x} \subset a$. But y bounds M, therefore $\dot{x} \sqsubseteq y$. □

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