What generic automorphisms of the random poset look like

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Thus, a **generic automorphism** of $\mathbf{K}$ is a generic element of $\mathrm{Aut}(\mathbf{K})$.

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- If $\mathfrak{K}$ is a class of L-structures, $\mathbf{K}$ is **universal** for $\mathfrak{K}$ if every structure in $\mathfrak{K}$ embeds into $\mathbf{K}$.

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Remark

Generic automorphisms of $(\mathbb{Q}, <)$ and the random graph admit similar kinds of descriptions.

What about the random poset?

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The random poset $\mathbf{P}$ *admits generic automorphisms.*

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Goal

Find an explicit description of generic automorphisms of $\mathbf{P}$.

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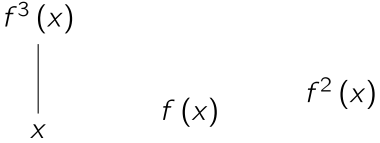
- The **spiral length** of x, denoted $\mathsf{sp}(x, f)$, is the least $n \geq 1$ for which x and $f^n(x)$ are comparable, or ∞ if no such n exists.
- The **parity** of x is given by:

$$\mathsf{par}(x, f) := \begin{cases} +1 & \text{if } \mathsf{sp}(x, f) = n < \infty \text{ and } x < f^n(x); \\ -1 & \text{if } \mathsf{sp}(x, f) = n < \infty \text{ and } x > f^n(x); \\ 0 & \text{otherwise.} \end{cases}$$

Tools for describing automorphisms' actions



Example: suppose $\mathsf{sp}(x, f) = 3$ and $\mathsf{par}(x, f) = +1$.

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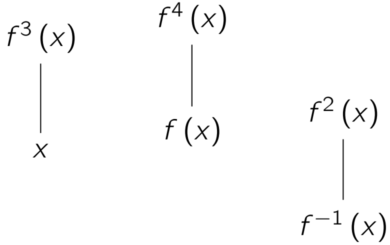

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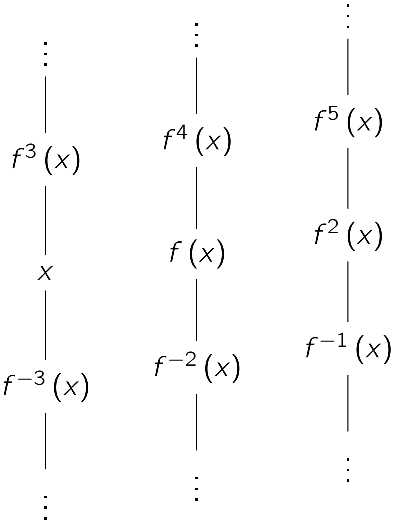

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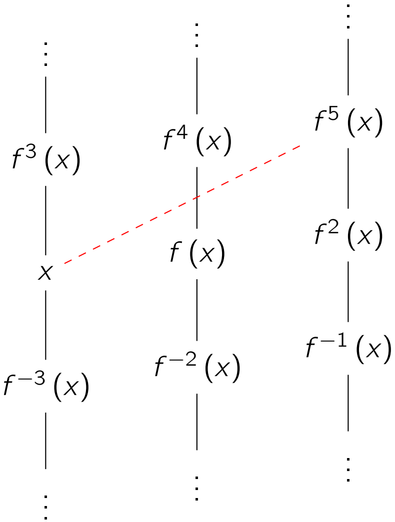

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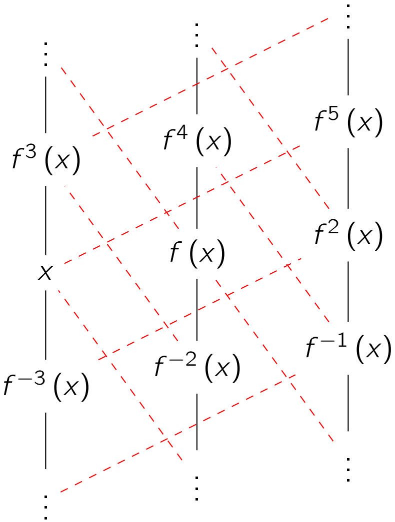

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- The quotient $\mathcal{O}_f[P]$ is called the **orbital quotient**.

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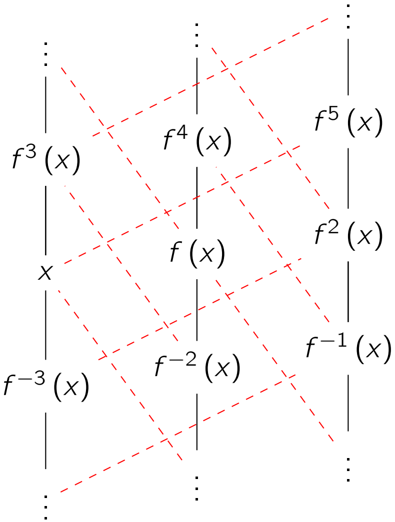

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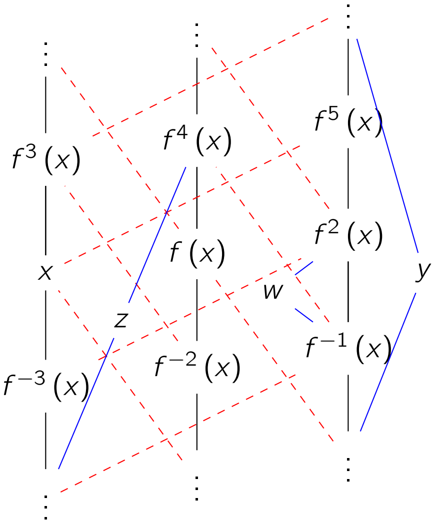

Orbitals are the "convex hulls" of orbits.

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- Parity is orbital-invariant; that is, $x \sim_f y$ implies $\mathsf{par}(x, f) = \mathsf{par}(y, f)$.
- Spiral length *need not be* orbital-invariant (unless $\mathsf{par}(x, f) = 0$).

Example in $\mathbb{Q}$:

Let $$f(x) := \begin{cases} 2x+3 & x \le -2, \\ \frac{x}{2} & -2 \le x \le 2, \\ 2x-3 & 2 \le x. \end{cases}$$

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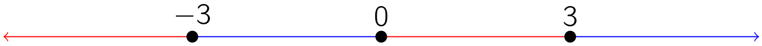

Then there are seven orbitals: three of parity 0, two each of parity -1 and $+1$.

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Remark

If P is linearly ordered, these orders agree and are linear orders themselves.

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Question

Is an analogous statement true for $\mathbf{P}$?

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Remark

This is a *partial* answer to our goal because we do not know if the converse holds: whether this property implies genericity.

A different tactic

Note

Equip $\mathbf{P}$ with a relation symbol for the graph of a unary function, and consider the resulting structures $(\mathbf{P}, f)$ and $(\mathbf{P}, g)$ for $f, g \in \operatorname{Aut}(\mathbf{P})$.

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Yes and no. Finite substructures in this language don't "remember" enough.

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- Let L be the language consisting of binary relations b_i, for $i \in \mathbb{Z}$.
- For each poset P and each $f \in \mathsf{Aut}(P)$, let P_f be the L-structure obtained by letting:

$$b_i^f(x, y) \overset{\text{def}}{\iff} x \leq f^i(y) .$$

A first-order language

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- Let L be the language consisting of binary relations b_i, for $i \in \mathbb{Z}$.
- For each poset P and each $f \in \mathrm{Aut}(P)$, let P_f be the L-structure obtained by letting:

$$b_i^f(x, y) \overset{\text{def}}{\iff} x \leq f^i(y).$$

- We identify $b_i^f(x, y)$ with its truth value in $\{0, 1\}$, and we consider the bi-infinite sequence $\mathbf{b}^f(x, y) \in 2^{\mathbb{Z}}$.

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$P_f \cong P_g$ iff f and g are conjugate in $\mathsf{Aut}(P)$. But also, since L is infinite, finite substructures can encode a lot more information.

How to control $\mathbf{b}^f$ sequences

Remark

Controlling behavior of $\mathbf{b}^f$ sequences with finitary configurations is the name of the game here.

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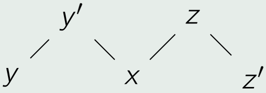
Suppose the following is a substructure of $\mathbf{P}$:



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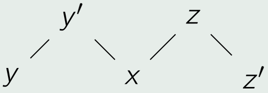

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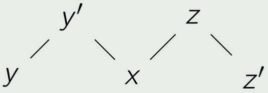

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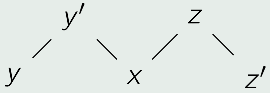

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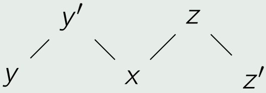

Let $p := \{(y, y'), (z, z')\} \in \mathsf{Aut}_{<\omega}(\mathbf{P})$. Then $\mathsf{sp}(x, f) = \infty$ for every $f \in [p]$, i.e. $b_i^f(x, x) = 0$ for all $i \neq 0$. It turns out for generic $f \in \mathsf{Aut}(\mathbf{P})$, the converse holds: this condition on the b_i^f's must be witnessed by an "M" configuration.

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(D) A technical condition — illustrated on the next slide — that forces $\mathbf{b}^f(x, y)$ to be eventually periodic on both sides for all $x, y \in \mathbf{P}$.

(...what's with condition (D)?)

Let $x, y \in \mathbf{P}$
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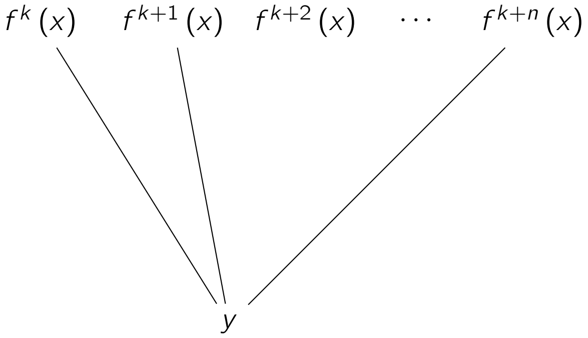

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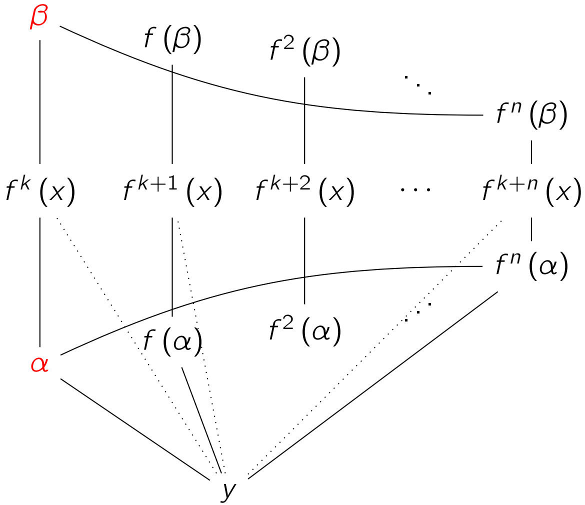

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where this configuration exists.

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- $\mathbf{P}_f$ is not ω-saturated.
- The relation $\mathrm{sp}(x, f) = \infty$ is definable in $\mathbf{P}_f$, but not quantifier-freely. Thus, the L-theory of $\mathbf{P}_f$ does not have QE.

Tack så mycket!