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# Applications of the ergodic iteration theorem

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I prove several natural preservation theorems for the countable support iteration. This solves a question of Rosłanowski regarding the preservation of localization properties and greatly simplifies the proofs in the area.



## 1 Introduction

The preservation theorems for the countable support iteration of proper forcing form a notoriously technical area of set theory. The expository book [1] includes several combinatorial schemes for such theorems, mostly originating from Shelah's work [12]. The book [14] offers a different approach based on descriptive set theory. This alternative has the disadvantage of being applicable to suitably definable forcings only; however, the resulting theorems are much more natural in their conclusions, and they are easier to use.

In this paper, I will describe several applications of the ergodic preservation theorem [14]. The statement of the theorem uses the notion of an ergodic ideal. A $\sigma$-ideal $J$ on a Polish space $X$ is *ergodic* if there is a countable Borel equivalence relation $E$ on $X$ such that every Borel $E$-invariant set either is in $J$, or its complement is in $J$. Thus for example the ideals of meager and null sets are both ergodic as witnessed by the Vitali equivalence relation. The main reason for considering these ideals is the following theorem.

**Fact 1.1** [14, Section 6.3.1] *Let $J$ be a suitably definable c.c.c. ergodic ideal. Suppose that suitable large cardinals exist and $P$ is a suitably definable proper forcing. If*

$$P \Vdash \text{“the set of the ground model points of } X \text{ is } J\text{-positive”},$$

*then its countable support iteration forces the same.*

Several words require explanation here. The necessary large cardinal assumptions depend on the complexity of the definitions of the ergodic ideal and the iterand. The most general version of the theorem uses iterands of the form $P_I$=Borel sets positive with respect to some $\sigma$-ideal $I$ on a Polish space $X$ such that, writing $A \subset 2^\omega \times X$ for a universal analytic set, the set $\{x \in 2^\omega : A_x \in I\}$ is universally Baire [2]. Many definable proper forcings adding a single real are of this form [14, Section 2.1.3]. The condition on the definability of the ergodic ideal is the same. The large cardinal assumption sufficient to carry the proof in this case is the existence of proper class many Woodin cardinals.

There is a very strong ZFC version of the theorem, used in all applications in this paper. Let me restrict attention to very nicely definable iterands:

**Definition 1.2** A poset $P$ is an *analytic CRN forcing* if $P$ is an analytic set of finitely branching trees on $\omega$ ordered by inclusion, closed under restriction, and such that for every $P$-name $\dot{y}$ and for every condition $p \in P$ there is a condition $q \leq p$ and a continuous function $f : [q] \longrightarrow 2^\omega$ such that $q \Vdash \dot{y} = \dot{f}(\dot{x}_{\mathrm{gen}})$ where $\dot{x}_{\mathrm{gen}} \in \omega^\omega$ is the name for the intersection of all conditions in the generic filter.

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This class should be compared with the snep forcings of [13]. Analytic CRN forcings are bounding. Most definable proper bounding forcings adding a single real can be represented as such. There are some unpleasant exceptions to this rule, such as the posets of [9, Section 2.2], and the methods of this paper cannot handle them directly. The $\sigma$-ideals associated with analytic CRN forcings are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [14, Theorem 3.8.9]. If the iterands are analytic CRN forcings and moreover the ergodic ideal is also $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then the ergodic preservation theorem can be proved without any large cardinal assumptions. Most applications of the theorem concern a situation with a single iterand repeating itself over and over. However, a statement allowing the iterand to vary is true too.

Thus for example if the iterand preserves the nonmeagerness of the set of ground model reals, even its countable support iterations are such. Other applications of the ergodic preservation theorem struggle with the lack of understanding of the collection of ergodic ideals. This paper identifies several forcing preservation properties that can be restated as "the set of the ground model points of $X$ is $I$-positive" for a suitable ergodic ideal $I$, or a conjunction of such statements. As a result, I obtain a number of preservation theorems for the countable support iteration of definable proper forcing.

As the first case, consider the $n$-localization property.

**Definition 1.3** [6] Let $n \in \omega$ be a natural number greater than one. An *$n$-tree* is a tree consisting of finite sequences of natural numbers in which every node has at most $n$ immediate successors. A forcing has the *$n$-localization property* if every function in $\omega^\omega$ in the extension is a branch through a ground model $n$-tree.

It is not difficult to see that the Sacks forcing has the 2-localization property, while the 3-branching variation of the Sacks forcing fails to have it. Several people [3, 4, 7, 8, 11] wondered about the preservation of the $n$-localization property in countable support iteration and product. The existing approaches yield awkward proofs applicable only in very special situations. Here, I will prove

**Theorem 1.4** *Let $n \in \omega$ be a number. The $n$-localization property is preserved under the countable support iterations of analytic CRN forcings.*

This solves some open questions of Rosłanowski [8]: for example, the countable support iteration of 2-Silver forcing does not add a 3-Silver generic. The theorem fails for arbitrary (undefinable) proper forcings already for iterations of length 2, as the following example shows.

**Example 1.5** The 4-Silver forcing does not have the 2-localization property. However, it can be decomposed into an iteration of two forcings, each of them with a 2-localization property.

As the second application, consider the weak Sacks property.

**Definition 1.6** A forcing has the *weak Sacks property* if for every function $f \in \omega^\omega$ in the extension there is a ground model infinite set $a \subset \omega$ and a ground model function $h$ with domain $a$ such that $|h(n)| \leq 2^n$ and $f(n) \in h(n)$ for all $n$ in $a$.

This property is the bounding variation of the weak Laver property of [15], which in conjunction with adding no independent reals is equivalent to the P-point preservation in definable proper forcings. Since P-point preservation is preserved in the countable support iteration by a theorem of Blass and Shelah [1, Section 6.1], it is natural to ask about the iteration status of the weak Sacks property.

**Theorem 1.7** *The weak Sacks property is preserved under the countable support iterations of analytic CRN forcings.*

The final application in this paper is included just to illustrate the power of the method. It concerns the ideal generated by partial Borel $E_0$-selectors.

**Definition 1.8** $E_0$ is the equivalence relation on $2^\omega$ defined by $xE_0y$ if $x\Delta y$ is finite. The $E_0$-ideal is the $\sigma$-ideal $\sigma$-generated by the partial Borel $E_0$ selectors.

The $E_0$-ideal has been investigated for example in [14, Section 4.7.1]. It is not difficult to see that every point in the Sacks extension belongs to a ground model coded Borel $E_0$ selector; however, the generic Silver real fails to have this property. It turns out that there is a natural preservation theorem.

**Theorem 1.9** *The conjunction "weak Sacks property and every point in the generic extension belongs to a ground model coded Borel partial $E_0$ selector" is preserved by the countable support iteration of analytic CRN forcings.*

The proof of the iteration theorems follows a pattern familiar from [14, Section 6.3.1], and uses the concept of Fubini properties of ideals [14, Section 3.2]. I will first identify some c.c.c. forcings, I will then show that their Fubini properties precisely characterize the preservation property, and then use [14, Theorem 6.3.3] to show that these Fubini properties are preserved under the countable support iteration of suitably definable forcings.

The notation used in this paper follows the set theoretic standard of [5]. If $t \in 2^{<\omega}$ is a finite binary sequence, then $O_t$ denotes the clopen subset of $2^\omega$ consisting of all infinite binary sequences containing $t$ as an initial segment. If $I$ is a $\sigma$-ideal on a Polish space $X$, then $P_I$ is the quotient poset of all Borel sets not in the ideal $I$ ordered by inclusion. This forcing adds a single element of the Polish space $X$, namely the point contained in all sets in the generic filter; the name for this point will be denoted by $\dot{x}_{\text{gen}}$. For a tree $T \subset \omega^{<\omega}$ the symbol $[T]$ stands for the set of all infinite branches of $T$. A subset of a Polish space is *universally Baire* [2] if its continuous preimages in Hausdorff spaces have the property of Baire. A $\sigma$-ideal $I$ on a Polish space $X$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{x \in 2^\omega : A_x \in I\}$ is coanalytic. If $M$ is a countable elementary submodel of a large structure, $P \in M$ is a poset, and $G \subset P \cap M$ is an $M$-generic filter, then the symbol $M[G]$ denotes the *transitive* model obtained from the transitive collapse of $M$, adjoining the image of the filter $G$ under the collapse. This abuse of notation should not cause any confusion. Finally, I will adopt the notation of [14, Section 3.2] for Fubini properties of $\sigma$-ideals: for $\sigma$-ideals $K, L$ on respective Polish spaces $X$ and $Y$, the symbol $K \perp L$ denotes the fact that there are a Borel $K$-positive set $B \subset X$, a Borel $L$-positive set $C \subset Y$, and a Borel set $D \subset B \times C$ such that the vertical sections of $D$ are $L$-small, while the horizontal sections of its complement are $K$-small.

## 2 The localization property

### 2.1 A c.c.c. forcing

The main tool in the proof of Theorem 1.4 is the $n$-localization forcing $P_n$:

**Definition 2.1** Let $n \in \omega$ be a natural number greater than one. The $n$-localization forcing $P_n$ consists of finite sets $a \subset \omega^\omega$ such that for every $t \in \omega^{<\omega}$ the set $\{m \in \omega : (\exists x \in a)\ t^\frown m \subset x\}$ has size at most $n$. The ordering is that of reverse inclusion.

It is not difficult to see that if $G \subset P_n$ is a generic filter, then $y_{\text{gen}} = \{t \in \omega^{<\omega} : (\exists a \in G)(\exists x \in a) t \subset x\}$ is an $n$-ary tree, and the generic filter $G$ can be recovered from $y_{\text{gen}}$ as $G = \{a \in P_n \cap V : a \subset [y_{\text{gen}}]\}$. Thus the poset $P_n$ can be viewed as adding a single point in the Polish space $Y_n$ of all $n$-ary trees on $\omega$, with topology inherited from the hyperspace $K(\omega^\omega)$ of compact subsets of the Baire space with the Vietoris topology. An obvious genericity argument shows that given a ground model function in the Baire space $\omega^\omega$, one can change finitely many values of it in such a way that the resulting function is a branch of the generic $n$-ary tree. A critical observation: the forcing $P_n$ satisfies a certain strengthening of the countable chain condition.

**Claim 2.2** *$P_n$ is $\sigma$-$n$-centered.*

P r o o f. I must show that $P_n = \bigcup_m A_m$ where every $n$ many elements of $A_m$ have a common lower bound. For every condition $a \in P_n$ let $t(a) \subset 2^{<\omega}$ be the inclusion-smallest finite tree such that for every terminal node of $t(a)$ there is exactly one element of $a$ extending it. Decompose the forcing $P_n$ into countably many pieces according to the value of $t(a)$. It is not difficult to see that for any collection $\{a_i : i \in n\} \subset P_n$ with a common value of $t(a_i)$ the union $\bigcup_i a_i$ is a condition in $P_n$ and a common lower bound. □

Let $J_n$ be the $\sigma$-ideal associated with the forcing $P_n$. That is, $J_n$ is the $\sigma$-ideal on the Polish space $Y_n$ generated by those Borel sets $B \subset Y_n$ such that $P_n \Vdash \dot{y}_{\text{gen}} \notin \dot{B}$.

**Claim 2.3** *The ideal $J_n$ is ergodic.*

P r o o f. Suppose that $k \in \omega$ is a number and $\pi$ is an automorphism of the tree $k^{\leq k}$. Extend $\pi$ to an automorphism $\hat{\pi}$ of the whole space $Y_n$ by setting

$$\hat{\pi}(y) = \{\pi(s)^\frown t : s^\frown t \in y \text{ and } s \text{ is the longest initial segment that belongs to } \mathrm{dom}(\pi)\}.$$

 

Note that the same definition also yields an automorphism of the forcing $P_n$. Let $E$ be the countable Borel equivalence relation on the space $Y_n$ generated by the graphs of all the countably many automorphisms obtained in this way. I claim that $E$ has the required properties.

Indeed, suppose that $B \subset Y_n$ is a Borel $E$-invariant set and assume for contradiction that neither $B$ nor its complement are in the ideal $J_n$. This means that there must be conditions $p, q \in P_n$ such that $p \Vdash \dot{y}_{\rm gen} \in \dot{B}$ and $q \Vdash \dot{y}_{\rm gen} \notin \dot{B}$. There is a sufficiently large number $k \in \omega$ and an automorphism $\pi$ of $k^{\leq k}$ such that the conditions $p$ and $\hat{\pi}(q)$ are compatible in $P_n$, with a lower bound $r$. Then $r$ forces that $\hat{\pi}^{-1}$-image of the generic filter is a generic filter containing the condition $q$, and by the forcing theorem $\dot{y}_{\rm gen} \in \dot{B}$ and $\hat{\pi}^{-1}(\dot{y}_{\rm gen}) \notin \dot{B}$. Thus the set $B$ is not $E$-invariant in the generic extension, and by an absoluteness argument, it is not invariant in the ground model either. Contradiction! □

To simplify several complexity computations and identify natural variations of the localization concept, I will use restricted versions of the above localization forcings. Suppose $f \in \omega^\omega$ is a function, and $n \in \omega$ is a number. The forcing $P_n \restriction f$ is defined in exactly the same way as $P_n$, except the conditions consist of functions dominated pointwise by $f$. The whole treatment transfers verbatim to the restricted versions. I will denote the space of all $n$-ary trees dominated by $f$ by $Y_n \restriction f$, and the $\sigma$-ideal on it generated by the forcing $P_n \restriction f$ will be denoted by $J_n \restriction f$. The main difference between the original forcings $P_n$ and their restricted versions is that the restricted $\sigma$-ideal $J_n \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

**Claim 2.4** *Let $f \in \omega^\omega$ and $n \in \omega$. The ideal $J_n \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

P r o o f. By [14, Proposition 3.8.11], it is enough to show that the set of maximal antichains of $P_n \restriction f$ is a Borel subset of $(P_n \restriction f)^\omega$ – in the language of [10], the poset is very Suslin. Fix a countable set $A \subset P_n \restriction f$. Pairwise incompatibility of elements of $A$ is certainly a Borel condition. The maximality of $A$ is equivalent to the statement $\forall t \bigcap_{a \in A} B_{t,a} = 0$, where $B_{t,a} = \{b \in P_n \restriction f : t = t(b) \wedge a \perp b\}$ and $t(b)$ is defined as in the proof of Claim 2.2. It is not difficult to check that the sets $B_{t,a}$ are closed subsets of the compact set $C_t \subset P_n \restriction f$ where $a \in C_t$ if and only if for every endnode of the tree $t$ there is exactly one element of $a$ extending it. Therefore they and their intersections are compact, and the statement that they are empty is Borel. □

While this definability property may seem mysterious, it has immediate forcing consequences.

**Corollary 2.5** *The forcings $P_n \restriction f$ do not add dominating reals.*

This follows immediately from [14, Proposition 3.8.15]. Note that the unrestricted forcings $P_n$ do add dominating reals and therefore the ideals $J_n$ are not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

### 2.2 Localization vs. Fubini property

This section is the heart of the proof. It contains just one key proposition connecting the $n$-localization property with the Fubini properties of the $\sigma$-ideal $J_n$ as defined in [14, Section 3.2] or in the final paragraph of the introduction.

**Proposition 2.6** *Let $I$ be a $\sigma$-ideal on a Polish space $X$ such that the quotient forcing $P_I$ is proper, and every analytic $I$-positive set has a Borel $I$-positive subset. Let $n$ be a natural number. The following are equivalent:*

(1) *$P_I$ has the $n$-localization property;*

(2) *$P_I$ is bounding and for every function $f \in \omega^\omega$, $I \not\perp J_n \restriction f$.*

Towards the proof of the proposition, first note that if the first item fails, then so does the other. If $P_I$ does not have the $n$-localization property, then either it is not bounding or else it adds a function $\dot{g} \in \omega^\omega$ forced to be dominated by some ground model function $f \in \omega^\omega$, and not covered by any ground model $n$-tree. In the former case (2) fails immediately. In the latter case find a Borel $I$-positive set $B \subset X$ and a Borel function $h : B \longrightarrow \omega^\omega$ such that $B \Vdash \dot{g} = \dot{h}(\dot{x}_{\rm gen})$ and observe that the Borel set

$$D = \{\langle x, T\rangle \in B \times Y_n \restriction f : h(x) \text{ is not modulo finite equal to any branch of the tree } T\}$$

has Borel $J_n \restriction f$-small vertical sections, and the horizontal sections of its complement are $I$-small, and (2) fails again.

For the reverse direction, let $n \in \omega$ be a natural number and suppose that the quotient forcing $P_I$ does have the $n$-localization property. Clearly, it has the Sacks property and so is bounding. Let $f \in \omega^\omega$ be a function; I must show that $I \not\perp J_n \restriction f$. Suppose that $B \subset X$ is an $I$-positive Borel set, and $D \subset B \times Y_n \restriction f$ is a Borel set whose vertical sections are $J_n \restriction f$-small. It will be enough to produce an $I$-positive horizontal section of the complement of the set $D$.

To simplify the notation, assume $X = 2^\omega$. Choose a countable elementary submodel $M$ of a large enough structure, and use the properness and the bounding property of the poset $P_I$ to find an $I$-positive compact set $C \subset B$ consisting of $M$-generic points, such that every subset of $X$ in the model $M$ has relatively clopen intersection with the set $C$. I will show that whenever $k \in \omega^\omega$ is a function that eventually dominates every function definable in the structure from parameters in $M \cup \{C\}$, and $C' \subset C$ is a compact set such that for all $j$ in $\omega$ we have $|\{t \in 2^{k(j)} : O_t \cap C' \neq 0\}| \leq 2^j$, then there is a point $y \in Y_n \restriction f$ such that $C' \times \{y\} \cap D = 0$. Note that it is possible to find an $I$-positive set $C' \subset C$ like that simply by using the Sacks property of the forcing $P_I$ to find a condition enclosing the sequence $\{\dot{x}_{\mathrm{gen}} \restriction k(j) : j \in \omega\}$ into a tunnel of thickness $2^j$. This will complete the proof.

The construction of the $n$-ary tree $y$ is the key step, and the following notion will be instrumental. A *wall* is a Borel function $h \in M$ with Borel $I$-positive domain and range consisting of conditions in $P_n \restriction f$ which *cohere*: $\bigcup \mathrm{rng}(h)$ is covered by branches of some $n$-tree, or equivalently, subsets of $\mathrm{rng}(h)$ of size $n+1$ all have lower bounds. The walls are ordered by $h' \leq h$ if $\mathrm{dom}(h') \subset \mathrm{dom}(h)$ and $h'(x) \leq h(x)$ for all $x$ in $\mathrm{dom}(h')$. Finally, consider the poset $Q$ of all walls $h$ such that $C' \subset \mathrm{dom}(h)$. I will show

**Claim 2.7** *Whenever $\dot{O} \in M$ is a $P_I$-name for an open dense subset of the poset $P_n \restriction f$, the collection of all walls $h$ such that $\mathrm{dom}(h) \Vdash \dot{h}(\dot{x}_{\mathrm{gen}}) \in \dot{O}$ is dense in $Q$.*

Once this claim is proved, the proposition follows: suppose that $g \subset Q$ is a filter meeting all the countably many open dense subsets of $Q$ described in this claim. For every point $x \in C'$, the set $\{h(x) : h \in g\} \subset P_n \restriction f$ is then $M[x]$-generic. The resulting $n$-ary tree $y$ does not depend on the choice of the point $x$, due to the coherence condition in the definition of a wall. Since the tree $y$ is $M[x]$-generic, it cannot belong to the $J_n \restriction f$-small set $D_x \subset Y_n \restriction f$. Thus $C' \times \{y\} \cap D = 0$ as required.

To prove the claim, fix a wall $h \in M$ and a $P_I$-name $\dot{O} \in M$ for an open dense set. Choose a number $m \in \omega$. I will show that there is a number $l = l(m, h, \dot{O}) \in \omega$ such that for every $m$-tuple $\langle t_i : i \in m\rangle$ of binary sequences of length $l$,

- either for some index $i \in m$, $O_{t_i} \cap \mathrm{dom}(h) \cap C = 0$
- or there is a wall $h' \leq h$ such that $C \cap \bigcup_{i \in m} O_{t_i} \subset \mathrm{dom}(h')$ and $\mathrm{dom}(h') \Vdash h'(\dot{x}_{\mathrm{gen}}) \in \dot{O}$. Note that neither $h$ nor $h'$ are required to be in the poset $Q$ at this point.

This will immediately prove the claim. If $h \in Q$ is a wall and $\dot{O} \in M$ is a name for an open dense set, then the fast growth of the function $k$ ensures that there will be a number $j \in \omega$ such that $k(j) > l(2^j, h, \dot{O})$. The set $\{t \in 2^{k(j)} : C' \cap O_t \neq 0\}$ has size $\leq 2^j$, and the second item above produces a wall $h' \in Q$, $h' \leq h$, and $\mathrm{dom}(h') \Vdash h'(\dot{x}_{\mathrm{gen}}) \in \dot{O}$ as required.

To produce the number $l = l(m, h, \dot{O})$, first investigate generic extensions of the model $M$. Suppose for $i \in m$ that $x_i$ are distinct points in the set $C \cap \mathrm{dom}(h)$. If they are not distinct just erase the repetitions. The set $p = \bigcup_{i \in m} h(x_i)$ is a condition in the poset $P_n \restriction f$ by the coherence condition in the definition of a wall. For every index $i \in m$, the point $x_i$ is $M$-generic, so the expression $\dot{O}/x_i$ makes sense and denotes an open dense subset of the forcing $P_n \restriction f \cap M[x_i]$. An analytic absoluteness argument shows that this set is in fact predense in the whole poset $P_n \restriction f$, and there must be conditions $q_i \in \dot{O}/x_i$, $q_i \leq h(x_i)$, such that the whole collection $\{p, q_i : i \in m\}$ has a lower bound. Creatively use the $n$-localization property to find an $n$-tree $y \in M$ such that $\bigcup_{i \in m} q_i \subset [y]$.

By the forcing theorem, this situation must be reflected in the model $M$. That is, there are pairwise disjoint sets $B_i$, $i \in m$, in $P_I \cap M$ and Borel functions $h_i : B_i \longrightarrow P_n \restriction f$, $i \in m$, in $M$ such that for every index $i \in m$, $x_i \in B_i$, $B_i \Vdash \dot{h}_i(\dot{x}_{\mathrm{gen}}) \in \dot{O}$, and for every point $x \in B_i$, $h_i(x) \leq h(x)$ and $h_i(x) \subset [y]$.

The point now is that the sets $\mathrm{dom}(h)$ and $B_i$, $i \in m$, are relatively clopen in the set $C$. Thus the compact set $(C \cap \mathrm{dom}(h))^m$ is covered by relatively open sets with certain properties. A compactness argument yields a finite subcover and the required number $l$.

 

### 2.3 The cinch

Suppose that $P$ is an analytic CRN forcing. Consider the ideal $I$ on $\omega^\omega$ generated by analytic sets $A$ such that there is no tree $p \in P$ such that $[p] \subset A$. [14, Proposition 2.1.6, Theorem 3.8.9] shows that this is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideal, every positive analytic set has a positive compact subset, and the forcing $P$ is naturally isomorphic to a dense subset of the quotient $P_I$. Theorem 1.4 then immediately follows from the conjunction of Proposition 2.6, [14, Theorem 6.3.3], and the fact that iterations of bounding forcings are bounding [1, Theorem 6.3.5].

### 2.4 Variations and limitations

The $n$-localization property implies the Sacks property, and therefore very few forcings actually exhibit it. A number of partial orders adding unbounded reals nevertheless possess a *bounded* 2-localization property: every function $x \in \omega^\omega$ in the extension bounded by some ground model function is in fact a branch of a ground model binary tree. In some cases, a straightforward generalization of the above approach yields a nice iteration theorem.

**Theorem 2.8** *The countable support iteration of Miller forcing has the bounded* 2*-localization property.*

P r o o f. Fix a function $f \in \omega^\omega$. The ideal $J_2 \restriction f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and therefore the forcing $P_2 \restriction f$ does not add a dominating real. Thus $P_2 \restriction f \Vdash \omega^\omega \cap V \notin I$, where $I$ is the $\sigma$-ideal associated with the Miller forcing: the ideal of $\sigma$-bounded sets. By [14, Proposition 3.2.2], this is equivalent to $I \not\perp J_2 \restriction f$. This Fubini property is preserved by the countable support iteration of Miller forcing by [14, Theorem 6.3.3], and therefore the countable support iteration of Miller forcing exhibits the bounded 2-localization property. $\square$

I conjecture that even the countable support iterations of Mathias forcing have the bounded 2-localization property. However, the approach of this paper cannot lead to such a result. Mathias forcing adds a reaping real while every suitably definable c.c.c. forcing adds a splitting real, leading to a failure of the requisite Fubini property.

The iteration theorems from the introduction deal with suitably definable forcings only. This is no accident, as 2-localization property is not preserved even under iterations of undefinable forcings of length 2. I will show that the 4-Silver forcing $Q_4$ can be decomposed into a two step iteration $Q_2 * \dot{R}$ such that $Q_2$ is the 2-Silver forcing (and so has 2-localization) and $Q_2 \Vdash \dot{R}$ has the 2-localization property as well. It is not difficult to see that the 4-Silver forcing fails the 2-localization–the generic point is not a branch of any ground model 2-tree, and therefore the general iteration theorem fails. The point of course is that the remainder forcing $\dot{R}$ does not have a definition to which Fact 1.1 can apply.

**Definition 2.9** Let $n \in \omega$. The *$n$-Silver forcing* $Q_n$ consists of partial functions $p : \omega \longrightarrow n$ with coinfinite domain, ordered by reverse inclusion.

**Theorem 2.10** *Let $n \in \omega$. The $n$-Silver forcing has the $n$-localization property.*

This result is optimal. Clearly, the $n$-Silver forcing fails the $n-1$-localization property, since the generic real cannot be enclosed by any ground model $n-1$-tree.

P r o o f. Suppose $p \Vdash \dot{y} \in \omega^\omega$ is a function; strengthening $p$ if necessary we may find a continuous function $f : n^\omega \longrightarrow \omega^\omega$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{\mathrm{gen}})$. For a point $x \in n^\omega$ and a finite partial function $u : \omega \longrightarrow n$ let $x \dot{\cup} u$ be the function obtained from $x$ by replacing $x \restriction \mathrm{dom}(u)$ with $u$. By a standard fusion argument find a condition $q \leq p$ such that, enumerating the infinite set $\omega \setminus \mathrm{dom}(q)$ by $\{n_i : i \in \omega\}$ in increasing order, the following holds:

(∗) For every $i \in \omega$ there is a number $m_i \geq n_i$ such that for every function $u : \{n_j : j \in i\} \longrightarrow n$, for every $x \in n^\omega$ with $q \subset x$ the initial segment $f(x \dot{\cup} u) \restriction m_i$ is the same sequence $g(u)$, and for two such functions $u, v$, $g(u) = g(v)$ if and only if $f(x \dot{\cup} u) = f(x \dot{\cup} v)$ for all $x \in n^\omega$ with $q \subset x$.

Now let $C = f\text{"}\{x \in n^\omega : q \subset x\}$. I will show that $C = [T]$ for some $n$-tree $T$; then clearly $q \Vdash \dot{y} \in [\check{T}]$ and the $n$-localization follows. Clearly $C$ is a compact set and as such it consists of all branches of some tree $T$. Suppose for contradiction that the tree $T$ branches into $n+1$ many immediate successors at some point, and let $\{x_l : l \in n+1\}$ be points in $n^\omega$ such that $q \subset x$ and such that the points $f(x_l)$, $l \in n+1$, split all at once at some natural number $k$.

Let $j \in \omega$ be the least number such that the set $a = \{x_l \restriction \{n_i : i \in j\} : l \in n+1\}$ has size greater than 1. Note that this set has size at most $n$. The key point: the sequences $\{g(u) : u \in a\}$ must be all the same. If two of them were different, then $m_j > k$, and since $\{f(x_l) \restriction m_j : l \in n+1\} = \{g(u) : u \in a\}$, this contradicts the fact that the set $\{f(x_l) \restriction k+1 : l \in n+1\}$ has size $n+1$.

This means that for every $l \in n+1$ and every $u \in a$, it is the case that $f(x_l) = f(x_l \dot\cup u)$, and it is possible to rewrite the sequences $\{x_l : l \in n+1\}$ in such a way that their restriction to the set $\{n_i : i \in j\}$ is any given single element $u \in a$, without changing the values $\{f(x_l) : l \in n+1\}$. One can repeat this procedure many times, pushing the first disagreement between the sequences $\{x_l : l \in n+1\}$ after the number $n_k$, but then the value $f(x_l)(k)$ will be the same for all numbers $l \in n+1$, contradiction. □

**Theorem 2.11** *The 4-Silver forcing $Q_4$ can be decomposed as $Q_2 * \dot{R}$, where $Q_2 \Vdash \dot{R}$ has the 2-localization property.*

The remainder forcing $\dot{R}$ clearly preserves $\aleph_1$ since $Q_4$ does. If the Continuum Hypothesis holds then the remainder will be in fact proper; I will avoid the awkward argument.

Proof. The decomposition is simple. Let $4 = a_0 \cup a_1$ be a partition into two disjoint sets of size 2. Suppose $x_4$ is a 4-Silver generic point. Let $x_2 \in 2^\omega$ be the point defined by $x_2(n) = i$ if $x_4(n) \in a_i$. It is rather obvious that $x_2$ is a 2-Silver generic. The forcing decomposition then follows the chain $V \subset V[x_2] \subset V[x_4]$ of generic extensions. I just have to verify that the second step has the 2-localization property, in other words, every point $y \in V[x_4] \cap \omega^\omega$ is a branch of a 2-tree in the model $V[x_2]$.

Back to $V$. Suppose $p \in Q_4$ is a condition and $\dot{y}$ is a $Q_4$-name for a point in $\omega^\omega$. Strengthening the condition $p$ if necessary find a continuous function $f : 4^\omega \longrightarrow \omega^\omega$ such that $p \Vdash \dot{y} = \dot{f}(\dot{x}_{\mathrm{gen}})$. Find a condition $q \leq p$ satisfying $(*)$ in the proof of the previous theorem. Now move to the model $V[x_2]$ and consider the set $C = f``\{x \in 4^\omega : (\forall i \in \omega) x(i) \in a_{x_2(i)} \wedge q \subset x\}$. The same argument as in the previous theorem shows that $C = [T]$ for some 2-tree $T \subset \omega^{<\omega}$. Clearly, $T \in V[x_2]$ is a 2-tree such that $y \in [T]$, and the theorem follows. □

## 3 The weak Sacks property

The whole approach for Theorem 1.7 is modeled after the previous section.

### 3.1 The c.c.c. forcing

Let $P$ be the forcing notion consisting of pairs $p = \langle a_p, b_p \rangle$ where $a_p$ is a finite partial function from $\omega$ to $[\omega]^{<\aleph_0}$ such that $|a_p(k)| \leq 2^k$ for all $k$ in $\mathrm{dom}(a_p)$, and $b_p \subset \omega^\omega$ is a finite set. The ordering is defined by $q \leq p$ if and only if $a_p \subset a_q$, $b_p \subset b_q$, and for every $k \in \mathrm{dom}(a_q \setminus a_p)$ and every $f \in b_p$ it is the case that $f(k) \in a_q(k)$. It is not difficult to see that $P$ is a $\sigma$-centered notion of forcing (conditions with the same first coordinate are mutually compatible), and the generic filter is determined by the union of the first coordinates of conditions in it. I will call this union $a_{\mathrm{gen}}$. It is a function with infinite domain, and for every ground model function $f \in \omega$, for all but finitely many numbers $n \in \mathrm{dom}(a_{\mathrm{gen}})$, $f(n) \in a_{\mathrm{gen}}(n)$. Thus the forcing $P$ is designed to perform a job perpendicular to the violation of the weak Sacks property. I will reserve the letter $J$ for the $\sigma$-ideal associated with the forcing $P$. The underlying Polish space $Y$ is the collection of all functions $a : \omega \longrightarrow [\omega]^{<\aleph_0}$ with infinite domain and such that for every $n \in \mathrm{dom}(a)$, $|a(n)| \leq 2^n$.

It is easy to see that the function $g \in \omega^\omega$ defined by $g(m) = \max(a_{\mathrm{gen}}(n))$ where $n = \min(\mathrm{dom}(a_{\mathrm{gen}}) \setminus m)$ modulo finite dominates all the ground model elements of $\omega^\omega$. Thus the poset $P$ adds a dominating real, but it is not equivalent to the Hechler forcing. In fact, the forcing $P$ is in a certain precise sense the most complicated definably $\sigma$-centered forcing, as this section shows.

In order to simplify certain complexity calculations and identify interesting variations, a restricted version of the forcing $P$ will be useful. Let $f \in \omega^\omega$ be a function. The forcing $P_f$ is defined just as $P$ is, except all the functions in $b_p$ must be pointwise dominated by $f$, and for every $p \in P_f$, $a_p(k) \subset f(k)$ for all $k$ in $\mathrm{dom}(a_p)$. The corresponding ideal on a Polish space $Y_f$ will be denoted by $J_f$. As in the previous section,

**Claim 3.1** *Let $f \in \omega^\omega$ be a function. The ideal $J_f$ is ergodic and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

### 3.2 The weak Sacks property in Fubini terms

In order to state the strongest possible theorem connecting the weak Sacks property with the Fubini properties of certain c.c.c. ideals, I will need the following notion.

**Definition 3.2** Call a forcing $Q$ *definably $\sigma$-centered* if

- there is a Polish space $Z$ such that $Q$ consists of nonempty closed subsets of $Z$ ordered by inclusion;

- $Q$ is a Suslin forcing; $Q$ is an analztic subset of the standard Borel space $C(Z)$ of all closed subsets of the Polish space $Z$ and the relations of compatibility and incompatibility are analytic;

- for every number $\varepsilon > 0$, the sets of radius $\leq \varepsilon$ form an open dense subset of $Q$;

- $Q$ is separated: for any two conditions $p, q \in Q$ either $p \cap q \in Q$ or there is a condition $p' \leq p$ such that $p' \cap q = 0$;

- there are countably many analytic sets $Q_n \subset C(Z)$, $n \in \omega$, such that $Q = \bigcup_n Q_n$ and each $Q_n$ is centered in $Q$.

The first four conditions are designed to ascertain that $Q$ is a Suslin forcing adding a single point in the space $Z$ such that a set is in the generic filter if and only if it contains this generic point. All Suslin forcings for adding a single real I know of are of this form, but I do not have a general theorem. The key condition is the last one. Clearly, the forcing $P$ together with all its restricted versions is definably $\sigma$-centered. On the other hand, there are definable forcings which are $\sigma$-centered but not definably $\sigma$-centered, such as the main c.c.c. forcing used in the next section under the assumption of the Continuum Hypothesis. This is a rather unusual situation though.

With this definition in hand, I can state the key lemma.

**Lemma 3.3** *Suppose that $I$ is a suitably definable $\sigma$-ideal on a Polish space $X$ such that the quotient forcing $P_I$ is proper. The following are equivalent:*

(1) *$P_I$ has the weak Sacks property;*

(2) *$I \not\perp J$;*

(3) *$P_I$ is bounding and $I \not\perp J_f$ for every function $f \in \omega^\omega$;*

(4) *for every definably $\sigma$-centered forcing and its associated $\sigma$-ideal $K$, $I \not\perp K$.*

P r o o f. The implications (4)→(2) and (4)→(3) are easy. To see why (4) implies the bounding condition, note that Hechler forcing is definably $\sigma$-centered. The other implications follow directlz from the fact that the posets $P$ and $P_f$ described above are definably $\sigma$-centered. To see why (2) implies (1), suppose that $P_I$ fails the weak Sacks property, let $B \in P_I$ be a condition and $f : B \longrightarrow \omega^\omega$ be a Borel function such that $B \Vdash \dot{f}(\dot{x}_{\mathrm{gen}})$ cannot be predicted on a ground model infinite set, and let $D \subset B \times Y$ be the Borel set defined by $\langle x, a\rangle \in D$ if and only if $x \restriction n \notin a(n)$ for infinitely many numbers $n \in \mathrm{dom}(a)$. It is not difficult to verify that the Borel set $D$ has $J$-small vertical sections, and its complement has $I$-small horizontal sections.

The key implication is (1)→(4). Suppose that $P_I$ has the weak Sacks property, and $K$ is a $\sigma$-ideal on a Polish space $Y$ obtained from a definably $\sigma$-centered forcing $Q$, as witnessed by the centered families $Q_n$, $n \in \omega$. Let $\dot{y}_{\mathrm{gen}}$ be the $Q$-name for the generic point in the space $Y$. Suppose that $B \subset X$ is an $I$-positive Borel set, $C \subset Y$ is a $K$-positive Borel set, and $D \subset B \times C$ is a Borel set with $K$-small vertical sections. I must find an $I$-positive horizontal section of the complement of the set $D$.

Let $M$ be a countable elementary submodel of a large structure. As in the previous section, a *wall* is a Borel function $f \in M$ such that $\mathrm{dom}(f) \subset B$ is a Borel $I$-positive set and (the coherence condition) $\mathrm{rng}(f) \subset Q_n$ for some number $n \in \omega$. Walls are ordered by $g \leq f$ if $\mathrm{dom}(g) \subset \mathrm{dom}(f)$ and $(\forall x \in \mathrm{dom}(g))\ g(x) \leq f(x)$. I will find a decreasing sequence $f_0 \geq f_1 \geq f_2 \geq \dots$ of walls such that $\bigcap_n \mathrm{dom}(f_n)$ is an $I$-positive set, and for every point $x$ in it the sequence $f_n(x)$, $x \in \omega$, is $M[x]$-generic for the poset $Q$. By the coherence condition, the generic point $y \in Y$ obtained from this generic sequence does not depend on the point $x$. Since the set $D \subset B \times C$ had $K$-small vertical sections, it must be the case that $\langle x, y\rangle \notin D$ for any point $x \in \bigcap_n \mathrm{dom}(f_n)$. Thus the horizontal section of the complement of the set $D$ corresponding to the point $y$ is $I$-positive as required.

Towards the construction of the decreasing sequence $f_n$, $n \in \omega$, of walls, first use the bounding property of the forcing $P_I$ to find a compact $I$-positive set $B_0 \subset B$ consisting of $M$-generic reals only, such that all Borel subsets of the space $X$ in the model $M$ are relatively clopen in it. This is possible by [14, Theorem 3.3.2]. The following is the key claim. In order to simplify the notation, I assume $X = 2^\omega$ and for a finite set $a \subset 2^{<\omega}$ I write $O_a = \{x \in 2^\omega : (\exists t \in a)\ t \subset x\}$.

**Claim 3.4** *Suppose that $f \in M$ is a wall, $\dot{O} \in M$ is a $P_I$-name for an open dense subset of $Q$ and $m \in \omega$ is a number. Then there is a number $l \in \omega$ such that for every set $a \subset 2^l$ of size $m$ there is a wall $g \leq f$ such that $\mathrm{dom}(g) \cap B_0 = \mathrm{dom}(f) \cap B_0 \cap O_a$ and $\mathrm{dom}(g) \Vdash g(\dot{x}_{\mathrm{gen}}) \in \dot{O}$.*

Once this is shown, first find a wall $f_0$ such that $\mathrm{dom}(f_0) \Vdash_{P_I} f_0(\dot{x}_{\mathrm{gen}}) \Vdash_Q \dot{y}_{\mathrm{gen}} \in \dot{C}$, enumerate $P_I$-names for open dense subsets of the poset $Q$ in the model $M$ by $\dot{O}_k : k \in \omega$ and then by induction on $m \in \omega$ construct numbers $l_m$, $m \in \omega$, and walls $f_a$, $a \in [2^{l_m}]^{\leq 2^m}$, such that

$$B_0 \cap O_a \subset \mathrm{dom}(f_a) \quad \text{and} \quad \mathrm{dom}(f_a) \Vdash \dot{f}_a(\dot{x}_{\mathrm{gen}}) \in \textstyle\bigcap_{k \in m} \dot{O}_k$$

where $m$ is such that $a \subset 2^{l_m}$, and whenever $b < a$ then $f_b \leq f_a$. Once this is done, use the weak Sacks property to find a set $B_1 \subset B_0$ in the poset $P_I$ and an infinite set $c \subset \omega$ such that $|\{t \in 2^{l_m} : O_t \cap B_1 \neq 0\}| \leq 2^m$ for all $k \in c$. The walls $f_a$, $a = \{t \in 2^{l_m} : O_t \cap B_1 \neq 0\}$, $m \in c$, will have the required properties.

Thus only the claim remains to be shown. Suppose $f, \dot{O}$ and $m$ are given. For every collection $x_i$, $i \in m$, of points in the compact set $\mathrm{dom}(f) \cap B_0$, repetitions allowed, consider the conditions $f(x_i)$, $i \in m$, in the poset $Q$. Since $f$ was a wall, these conditions are all in the same centered set, and they have a lower bound, say $p$. The sets $\dot{O}/x_i$ are all open dense in the poset $Q$, and therefore there is a condition $q \leq p$ belonging to all of them; say $q \in Q_j$ for some number $j$. By the forcing theorem, for every number $i \in m$ there is a condition $\bar{B}_i \subset B$ in the model $M$ and a Borel function $g_i : \bar{B}_i \longrightarrow Q_j$ such that $x_i \in \bar{B}_i$ and $\bar{B}_i \Vdash \dot{g}_i(\dot{x}_{\mathrm{gen}}) \in \dot{O}$. Note that the sets $\bar{B}_i$, $i \in m$, are all relatively clopen in $B_0$. It is clearly possible to choose the sets $\bar{B}_i$ in such a way that $x_i = x_j$ implies $\bar{B}_i = \bar{B}_j$ and $g_i = g_j$, and $x_i \neq x_j$ implies $\bar{B}_i \cap \bar{B}_j = 0$, and then it is possible to combine the functions $g_i$, $i \in m$, into a single wall. Thus the compact set $[B_0 \cap \mathrm{dom}(f)]^m$ is covered by relatively open sets for which there is a wall $g \leq f$ such that $\mathrm{dom}(g) \Vdash \dot{g}(\dot{x}_{\mathrm{gen}}) \in \dot{O}$. A compactness argument yields the required number $l$.

□

### 3.3 The cinch

Theorem 1.7 now quickly follows from the ergodic iteration theorem [14, Theorem 6.3.3]. For every function $f \in \omega^\omega$, the ideal $J_f$ is ergodic c.c.c. The Fubini property with respect to each $J_f$ is preserved by the countable support iteration of definable proper forcings by the ergodic iteration theorem, and so is the bounding condition by [1, Theorem 6.3.5]. The weak Sacks property is just a conjunction of these properties by the lemma.

## 4 The Borel $E_0$ selectors

The proof of Theorem 1.9 follows closely the pattern of the previous two sections, and I will only outline the main points.

**Definition 4.1** Let $P$ be the partial ordering of pairs $p = \langle u_p, a_p \rangle$ where $a_p \subset 2^\omega$ is a finite set consisting of pairwise $E_0$-nonequivalent points, and $u_p \subset \omega$ is a finite set such that for any pair of its elements $n \in m$ and any pair $x, y \in a_p$ either $x \restriction n = y \restriction n$ or $x \restriction (m \setminus n) \neq y \restriction (m \setminus n)$. The ordering is defined by $q \leq p$ if $a_p \subset a_q$ and $u_p \subset u_q$.

Let $G \subset P$ be a generic filter, let $C \subset 2^\omega$ be the closure of $\bigcup_{p \in G} a_p$, and let $U = \bigcup_{p \in G} u_p$. A density argument shows that $U$ is an infinite set. The set $C$ is an $E_0$-transversal, because whenever $x \neq y \in C$ are distinct points, then for any numbers $n \in m$ in the set $U$ above $x \Delta y$, $x \restriction (m \setminus n) \neq y \restriction (m \setminus n)$ holds; necessarily $\neg x E_0 y$. Another density argument shows that every ground model point $x \in 2^\omega$ has an $E_0$-equivalent in the set $C$, and thus in the $P$-extension, the set of the ground model points is covered by a set in the ideal $J$, the union of all the rational translates of the set $C$. I will need a couple of basic properties of the forcing $P$.

**Claim 4.2** *The forcing $P$ is c.c.c. and ergodic.*

Proof. I will show that $P$ has the Knaster property. Suppose $p_\alpha = \langle u_\alpha, a_\alpha \rangle$, $\alpha \in \omega_1$, are conditions in the poset $P$. Thinning out the collection if necessary we may assume that they share the same $u$-part, that the set $\{x \restriction \max(u_\alpha) : x \in a_\alpha\}$ is the same for all of them, and their $a$-parts form a $\Delta$-system with root $b$. Since $E_0$ is an equivalence relation with countably many classes, it is possible to find a club $C \subset \omega_1$ such that $(\forall \alpha \neq \beta \in C)(\forall x \in a_\alpha \setminus b)(\forall y \in a_\beta \setminus b) \neg x E_0 y$. It is easy to verify that the conditions $p_\alpha$, $\alpha \in C$, are pairwise compatible. $\square$

**Claim 4.3** *Suppose $P_I$ is a proper forcing with the weak Sacks property. The following are equivalent:*

1. $I \not\leq K$;
2. $P_I \Vdash 2^\omega = \bigcup(V \cap J)$.

Theorem 1.9 now follows from the ergodic iteration theorem in the usual fashion.

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