1 Syntax of modal logic

The symbols of modal logic consistute of an infinite countable set $P$ of propositional variables, logical connectives, parenthesization, and the modal operator $\Box$. The choice of logical connectives depends on the development of propositional logic one wants to follow; below I choose negation and implication.

The set of modal formulas is defined recursively as follows. Every propositional variable is a formula. If $\phi$ and $\psi$ are formulas then so are $\neg \phi, \phi \rightarrow \psi,$ and $\Box \phi$. All formulas are obtained by a repeated application of these constructions.

Example 1.1. $\Box(\Box \phi \rightarrow \psi) \rightarrow \neg \Box \phi$ is a beautiful modal formula when $\phi, \psi$ are propositional variables.

The formal proof system of modal logic includes the proof system for propositional logic as a subset. Here I choose to use the Hilbert system, with the following axioms:

- $\phi \rightarrow (\psi \rightarrow \phi)$;
- (implication distribution) $(\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta))$;
- $(\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi)$,

and the modus ponens inference rule: from $\phi$ and $\phi \rightarrow \psi$ infer $\psi$. The modal proof system also must include rules and axioms for the modal operator. There is only one extra inference rule, the generalization or necessitation or $\Box$ introduction: from $\phi$ infer $\Box \phi$. There are several possibilities for extra axioms, resulting in different modal logics.

- the logic K is obtained by adding the $\Box$ distribution axiom: $\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$;
- the logic K4 is obtained by adding the distribution axiom and the K4 axiom $\Box \phi \rightarrow \Box \Box \phi$;
- the provability logic or GL logic is obtained by adding the distribution and K4 axiom and the L"ob axiom: $\Box(\Box \phi \rightarrow \phi) \rightarrow \Box \phi$.

There are numerous other options. K stands for Kripke, GL stands for G"odel and L"ob.

Definition 1.2. Suppose that $\Gamma$ is a set of model formulas and $\phi$ is a modal formula.

1. $\Gamma \vdash_P \phi$ means that $\phi$ is provable from $\Gamma$ in propositional logic;
2. $\Gamma \vdash_K \phi$ indicates provability of $\phi$ in K logic from $\Gamma$;
3. similarly for $\Gamma \vdash_{K4} \phi$ etc.
We will need a couple of facts about formal proofs in K. For a theory \( \Delta \) let \( \Box \Delta \) be the set \( \{ \Box \theta : \theta \in \Delta \} \).

**Proposition 1.3.** Let \( \Delta \) be a theory and \( \phi \) a formula. If \( \Delta \models_K \phi \) then \( \Box \Delta \models_K \Box \phi \).

**Proof.** By induction on the length of the proof of \( \phi \) from \( \Delta \). Suppose that all proofs of length \( \leq n \) have been handled, and consider a proof of length \( n + 1 \). There are several possibilities for the formula on the last line of the proof (which is \( \phi \)):

**Case 1.** \( \phi \) is an axiom of K. Then \( \Box \phi \) is provable in K from no assumptions by an application of the \( \Box \) introduction rule to \( \phi \), and so it is also provable from \( \Box \Delta \).

**Case 2.** \( \phi \) is an element of \( \Delta \). In such a case, \( \Box \phi \) is provable from \( \Box \Delta \) since it is just an element of \( \Box \Delta \).

**Case 3.** \( \phi \) is obtained from some previous lines of the proof using modus ponens. So there is a formula \( \psi \) such that both \( \psi \) and \( \psi \rightarrow \phi \) appear earlier in the proof. By the induction hypothesis, \( \Box \psi \) and \( \Box (\psi \rightarrow \phi) \) are both provable from \( \Box \Delta \). To obtain the proof of \( \Box \phi \), quote an instance of \( \Box \) distribution: \( \Box (\psi \rightarrow \phi) \rightarrow (\Box \psi \rightarrow \Box \phi) \) and apply modus ponens twice to get a proof of \( \Box \phi \) from \( \Box \Delta \).

**Case 4.** \( \phi \) is obtained from some previous line of the proof using the \( \Box \) introduction rule. So there is a formula \( \psi \) appearing earlier in the proof such that \( \phi = \Box \psi \). By the induction hypothesis, \( \Box \psi \) is provable from \( \Box \Delta \). \( \Box \phi \) (which is just \( \Box \Box \psi \)) is obtained from \( \Box \psi \) by one application of the \( \Box \) introduction rule. So again, we produced a formal proof of \( \Box \phi \) from \( \Box \Delta \) in K.

For the following proposition, let \( \Gamma_0 \) be the set of all formulas provable in the K logic. The proof is the same as that of Proposition 1.3 except the last case disappears.

**Proposition 1.4.** Let \( \Delta \) be a theory and \( \phi \) a formula. If \( \Delta \vdash_P \phi \) then \( \Gamma_0 \cup \Box \Delta \vdash_P \Box \phi \).

One great distinction between modal logic and propositional logic is that the deduction theorem does not hold for modal logic. That is, \( \phi \vdash_K \psi \) is not generally equivalent to \( \vdash_K \phi \rightarrow \psi \). This is best observed on the case where \( \psi = \Box \phi \). Certainly \( \phi \vdash_K \Box \phi \) (the proof uses just one application of the \( \Box \) introduction rule), but \( \vdash_K \phi \rightarrow \Box \phi \) fails.

## 2 Semantics of modal logic

The models for modal logic were isolated by Kripke.

**Definition 2.1.** A Kripke frame is a pair \( (W, R) \) where \( W \) is a set (its elements are called worlds) and \( R \) is a binary relation on \( W \) called accessibility relation.
Definition 2.2. Let \( \langle W, R \rangle \) be a frame. Let \( V \) be a function from the product of \( W \) and the set of all modal formulas. \( V \) is a valuation if for all formulas \( \phi, \psi \) and all worlds \( w \in W \),

1. \( V(w, \neg \phi) = 1 - V(w, \phi) \);
2. \( V(w, \phi \rightarrow \psi) \) is the binary sum of \( (1 - V(w, \phi)) + V(w, \psi) \);
3. \( V(\Box \phi) = \Pi \{ V(v, \phi) : w R v \} \).

Definition 2.3. A model is a triple \( \langle W, R, V \rangle \) where \( \langle W, R \rangle \) is a frame and \( V \) is a valuation.

Thus, the truth value of a propositional variable in a model depends on the world in which it is evaluated.

Definition 2.4. The notation \( \langle W, R, V \rangle \models \phi \) means that for every world \( w \in W \), \( V(w, \phi) = 1 \). The notation \( \langle W, R \rangle \models \phi \) means that for every model \( \langle W, R, V \rangle \), it is the case that \( \langle W, R, V \rangle \models \phi \); it is verbalized as \( \phi \) holds on the frame \( \langle W, R \rangle \).

Note that for a modal formula \( \phi \) and a model \( \langle W, R, V \rangle \), neither of \( \langle W, R, V \rangle \models \phi \) and \( \langle W, R, V \rangle \models \neg \phi \) needs to hold since \( \phi \) may hold in one world and not in another; this is one of the differences between propositional and modal logic.

Theorem 2.5. (Soundness and completeness for K) Let \( \phi \) be a modal formula. The following are equivalent:

1. \( \phi \) is provable in K;
2. \( \phi \) holds on every frame.

Proof. The implication (1) \( \rightarrow \) (2) is the soundness. It is proved by induction on the length of the proof of \( \phi \). We have to verify that the inference rules of K are sound, and the axioms of K hold on every frame.

The more difficult implication (2) \( \rightarrow \) (1) is the completeness. To prove it, we introduce the canonical model for K. To construct it, let \( \Gamma_0 \) be the set of all formulas provable in K. Let \( W \) be the set of all maximal propositionally consistent theories containing \( \Gamma_0 \) as a subset. Let \( R \) be the relation on \( W \) defined by \( \Gamma R \Delta \) if for every formula \( \theta \), if \( \Box \theta \in \Gamma \) then \( \theta \in \Delta \). Let \( V \) be the function defined by \( V(\Gamma, \phi) = 1 \) if \( \phi \in \Gamma \). The triple \( \langle W, R, V \rangle \) is called the universal model for K. We have to verify that \( V \) in fact is a valuation, meaning that it satisfies all three clauses of Definition 2.2. The first two clauses are immediate. The verification of the \( \Box \) clause is more interesting, and it is split into two cases.

Case 1. Suppose first that \( \Box \phi \in \Gamma \). By the definition of the relation \( R \), for all theories \( \Delta \in W \) with \( \Gamma R \Delta \) it is the case that \( \phi \in \Delta \). This means that the \( \Box \) recursive demand of Definition 2.2 is satisfied in this case.

Case 2. Suppose now that \( \Box \phi \notin \Gamma \). In this case, we have to construct a theory \( \Delta \in W \) such that \( \Gamma R \Delta \) and \( \phi \notin \Delta \). Let \( \Delta_0 = \{ \theta : \Box \theta \in \Gamma \} \).
Claim 2.6. $\Gamma_0 \subset \Delta_0$.

Proof. If $\theta$ is provable in $K$ then so is $\Box \theta$: just add one more line to the proof of $\theta$ in $K$ using the $\Box$ introduction rule. Since $\Gamma_0 \subset \Gamma$, it follows that $\Box \theta \in \Gamma$ and so $\theta \in \Delta_0$ by the definition of $\theta_0$.

Claim 2.7. $\Delta_0$ does not propositionally prove $\phi$.

Proof. If $\Delta_0 \vdash_P \phi$, then $\Gamma_0 \cup \Box \Delta \vdash_P \Box \phi$ by Proposition 1.4. Now, $\Gamma_0 \cup \Box \Delta \subset \Gamma$ and so $\Gamma \vdash_P \Box \phi$. By the maximality of the theory $\Gamma$, this means that $\Box \phi \in \Gamma$, contradicting the case assumption.

The last claim shows that $\Delta_0 \cup \{\neg \phi\}$ is propositionally consistent. Let $\Delta$ be a maximal propositionally consistent theory containing $\Delta_0$. Then $\Gamma \models \Delta$, since for every formula $\Box \theta \in \Gamma$, $\theta$ is an element of $\Delta_0$ and therefore of $\Delta$. Moreover, $\phi \notin \Delta$ since $\neg \phi \in \Delta$ and $\Delta$ is propositionally consistent. This completes the verification of the $\Box$ clause of Definition 2.2. Thus, the triple $\langle W, R, V \rangle$ is a model.

The completeness immediately follows. If $\phi$ is a formula not derivable in $K$, then $\Gamma_0 \cup \{\neg \phi\}$ is a propositionally consistent theory. Extend this theory to a maximal consistent theory $\Gamma$. Then $V(\Gamma, \phi) = 0$, showing that $\phi$ does not hold in the canonical model.

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