1 Syntax of modal logic

The symbols of modal logic consistute of an infinite countable set P of propositional variables, logical connectives, parenthesization, and the modal operator $\Box$. The choice of logical connectives depends on the development of propositional logic one wants to follow; below I choose negation and implication.

The set of *modal formulas* is defined recursively as follows. Every propositional variable is a formula. If ϕ and ψ are formulas then so are $\neg\phi$, $\phi \to \psi$, and $\Box\phi$. All formulas are obtained by a repeated application of these constructions.

Example 1.1. $\Box(\Box\phi \to \psi) \to \neg\Box\phi$ is a beautiful modal formula when ϕ, ψ are propositional variables.

The formal proof system of modal logic includes the proof system for propositional logic as a subset. Here I choose to use the Hilbert system, with the following axioms:

- $\phi \to (\psi \to \phi)$;
- (implication distribution) $(\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))$;
- $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$,

and the *modus ponens* inference rule: from ϕ and $\phi \to \psi$ infer ψ. The modal proof system also must include rules and axioms for the modal operator. There is only one extra inference rule, the *generalization* or *necessitation* or $\Box$ *introduction*: from ϕ infer $\Box\phi$. There are several possibilities for extra axioms, resulting in different modal logics.

- the logic K is obtained by adding the $\Box$ distribution axiom: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
- the logic K4 is obtained by adding the distribution axiom and the K4 axiom $\Box\phi \to \Box\Box\phi$;
- the provability logic or GL logic is obtained by adding the distribution and K4 axiom and the Löb axiom: $\Box(\Box\phi \to \phi) \to \Box\phi$.

There are numerous other options. K stands for Kripke, GL stands for Gödel and Löb.

Definition 1.2. Suppose that Γ is a set of model formulas and ϕ is a modal formula.

1. $\Gamma \vdash_P \phi$ means that ϕ is provable from Γ in propositional logic;
2. $\Gamma \vdash_K \phi$ indicates provability of ϕ in K logic from Γ;
3. similarly for $\Gamma \vdash_{K4} \phi$ etc.

We will need a couple of facts about formal proofs in K. For a theory Δ let $\Box\Delta$ be the set $\{\Box\theta \colon \theta \in \Delta\}$.

Proposition 1.3. *Let Δ be a theory and ϕ a formula. If $\Delta \Vdash_K \phi$ then $\Box\Delta \Vdash_K \Box\phi$.*

Proof. By induction on the length of the proof of ϕ from Δ. Suppose that all proofs of length $\leq n$ have been handled, and consider a proof of length $n+1$. There are several possibilities for the formula on the last line of the proof (which is ϕ):
Case 1. ϕ is an axiom of K. Then $\Box\phi$ is provable in K from no assumptions by an application of the $\Box$ introduction rule to ϕ, and so it is also provable from $\Box\Delta$.
Case 2. ϕ is an element of Δ. In such a case, $\Box\phi$ is provable from $\Box\Delta$ since it is just an element of $\Box\Delta$.
Case 3. ϕ is obtained from some previous lines of the proof using modus ponens. So there is a formula ψ such that both ψ and $\psi \to \phi$ appear earlier in the proof. By the induction hypothesis, $\Box\psi$ and $\Box(\psi \to \phi)$ are both provable from $\Box\Delta$. To obtain the proof of $\Box\phi$, quote an instance of $\Box$ distribution: $\Box(\psi \to \phi) \to (\Box\psi \to \Box\phi)$ and apply modus ponens twice to get a proof of $\Box\phi$ from $\Box\Delta$.
Case 4. ϕ is obtained from some previous line of the proof using the $\Box$ introduction rule. So there is a formula ψ appearing earlier in the proof such that $\phi = \Box\psi$. By the induction hypothesis, $\Box\psi$ is provable from $\Box\Delta$. $\Box\phi$ (which is just $\Box\Box\psi$) is obtained from $\Box\psi$ by one application of the $\Box$ introduction rule. So again, we produced a formal proof of $\Box\phi$ from $\Box\Delta$ in K. $\square$

For the following proposition, let Γ_0 be the set of all formulas provable in the K logic. The proof is the same as that of Proposition 1.3 except the last case disappears.

Proposition 1.4. *Let Δ be a theory and ϕ a formula. If $\Delta \vdash_P \phi$ then $\Gamma_0 \cup \Box\Delta \vdash_P \Box\phi$.*

One great distinction between modal logic and propositional logic is that the deduction theorem does not hold for modal logic. That is, $\phi \vdash_K \psi$ is not generally equivalent to $\vdash_K \phi \to \psi$. This is best observed on the case where $\psi = \Box\phi$. Certainly $\phi \vdash_K \Box\phi$ (the proof uses just one application of the $\Box$ introduction rule), but $\vdash_K \phi \to \Box\phi$ fails.

2 Semantics of modal logic

The models for modal logic were isolated by Kripke.

Definition 2.1. A *Kripke frame* is a pair $\langle W, R\rangle$ where W is a set (its elements are called *worlds*) and R is a binary relation on W called *accessibility relation.*

Definition 2.2. Let $\langle W, R\rangle$ be a frame. Let V be a function from the product of W and the set of all modal formulas. V is a *valuation* if for all formulas ϕ, ψ and all worlds $w \in W$,

1. $V(w, \neg\phi) = 1 - V(w, \phi)$;
2. $V(w, \phi \to \psi)$ is the binary sum of $(1 - V(w, \phi)) + V(w, \psi)$;
3. $V(\Box\phi) = \Pi\{V(v, \phi) \colon w\ R\ v\}$.

Definition 2.3. A *model* is a triple $\langle W, R, V\rangle$ where $\langle W, R\rangle$ is a frame and V is a valuation.

Thus, the truth value of a propositional variable in a model depends on the world in which it is evaluated.

Definition 2.4. The notation $\langle W, R, V\rangle \models \phi$ means that for every world $w \in W$, $V(w, \phi) = 1$. The notation $\langle W, R\rangle \models \phi$ means that for every model $\langle W, R, V\rangle$, it is the case that $\langle W, R, V\rangle \models \phi$; it is verbalized as ϕ *holds on the frame* $\langle W, R\rangle$.

Note that for a modal formula ϕ and a model $\langle W, R, V\rangle$, neither of $\langle W, R, V\rangle \models \phi$ and $\langle W, R, V\rangle \models \neg\phi$ needs to hold since ϕ may hold in one world and not in another; this is one of the differences between propositional and modal logic.

Theorem 2.5. (Soundness and completeness for K) *Let ϕ be a modal formula. The following are equivalent:*

1. *ϕ is provable in K;*
2. *ϕ holds on every frame.*

Proof. The implication (1)→(2) is the soundness. It is proved by induction on the length of the proof of ϕ. We have to verify that the inference rules of K are sound, and the axioms of K hold on every frame.

The more difficult implication (2)→(1) is the completeness. To prove it, we introduce the *canonical model* for K. To construct it, let Γ_0 be the set of all formulas provable in K. Let W be the set of all maximal propositionally consistent theories containing Γ_0 as a subset. Let R be the relation on W defined by $\Gamma\ R\ \Delta$ if for every formula θ, if $\Box\theta \in \Gamma$ then $\theta \in \Delta$. Let V be the function defined by $V(\Gamma, \phi) = 1$ if $\phi \in \Gamma$. The triple $\langle W, R, V\rangle$ is called the universal model for K. We have to verify that V in fact is a valuation, meaning that it satisfies all three clauses of Definition 2.2. The first two clauses are immediate. The verification of the $\Box$ clause is more interesting, and it is split into two cases.

Case 1. Suppose first that $\Box\phi \in \Gamma$. By the definition of the relation R, for all theories $\Delta \in W$ with $\Gamma\ R\ \Delta$ it is the case that $\phi \in \Delta$. This means that the $\Box$ recursive demand of Definition 2.2 is satisfied in this case.

Case 2. Suppose now that $\Box\phi \notin \Gamma$. In this case, we have to construct a theory $\Delta \in W$ such that $\Gamma\ R\ \Delta$ and $\phi \notin \Delta$. Let $\Delta_0 = \{\theta \colon \Box\theta \in \Gamma\}$.

Claim 2.6. $\Gamma_0 \subset \Delta_0$.

Proof. If θ is provable in K then so is $\Box\theta$: just add one more line to the proof of θ in K using the $\Box$ introduction rule. Since $\Gamma_0 \subset \Gamma$, it follows that $\Box\theta \in \Gamma$ and so $\theta \in \Delta_0$ by the definition of θ_0. $\Box$

Claim 2.7. Δ_0 *does not propositionally prove* ϕ.

Proof. If $\Delta_0 \vdash_P \phi$, then $\Gamma_0 \cup \Box\Delta \vdash_P \Box\phi$ by Proposition 1.4. Now, $\Gamma_0 \cup \Box\Delta \subset \Gamma$ and so $\Gamma \Vdash_P \Box\phi$. By the maximality of the theory Γ, this means that $\Box\phi \in \Gamma$, contradicting the case assumption. $\Box$

The last claim shows that $\Delta_0 \cup \{\neg\phi\}$ is propositionally consistent. Let Δ be a maximal propositionally consistent theory containing Δ_0. Then $\Gamma\ R\ \Delta$, since for every formula $\Box\theta \in \Gamma$, θ is an element of Δ_0 and therefore of Δ. Moreover, $\phi \notin \Delta$ since $\neg\phi \in \Delta$ and Δ is propositionally consistent. This completes the verification of the $\Box$ clause of Definition 2.2. Thus, the triple $\langle W, R, V\rangle$ is a model.

The completeness immediately follows. If ϕ is a formula not derivable in K, then $\Gamma_0 \cup \{\neg\phi\}$ is a propositionally consistent theory. Extend this theory to a maximal consistent theory Γ. Then $V(\Gamma, \phi) = 0$, showing that ϕ does not hold in the canonical model. $\Box$