On Σ–preorderings in HF($\mathbb{R}$)

A.S. Morozov

Sobolev Institute of Mathematics, Novosibirsk, Russia

morozov@math.nsc.ru

HF($\mathbb{R}$): what is it?

$\mathbb{HF}(\mathbb{R})$: hereditarily finite superstructure over $\mathbb{R}$ (similar for $\mathbb{HF}(\mathfrak{M})$, for $\mathfrak{M}$)

Basic set: all the elements of $\mathbb{R}$ and all the sets which can be explicitly written down using $\{, \}$, $\varnothing$, r ($r \in \mathbb{R}$).

Examples: $\varnothing$, $\{\varnothing, \sqrt{2}\}$, $\{7, \{\{\varnothing, 92\}, 3, \{\varnothing\}\}\}$, etc.

Signature of $\mathbb{HF}(\mathbb{R})$: $\langle +, \times, <, U, \in \rangle$

- U: a unary predicate, distinguishes $\mathbb{R}$ in $\mathbb{HF}(\mathbb{R})$ (urelements)
- $\in$ is a membership relation; elements of $\mathbb{R}$ contain no members (they are urelements!)
- $+$ and $\times$ are considered as predicates, i.e., graphs of the corresponding operations

Σ–formulas: a specific class of formulas that define analogs of c.e. sets in $\mathbb{HF}(\mathfrak{M})$.

Definability by means of Σ–formulas over $\mathbb{HF}(\mathbb{R})$ can be viewed as "computable enumerability" in a high level programming language in which we have **exact** realizations of the field $\mathbb{R}$ of real numbers together with operations on it, and, in addition, we can compute (and use in further computations) all the roots of polynomial equations from their coefficients.

"Computable" functions over $\mathbb{HF}(\mathfrak{M})$: functions whose graphs can be defined by Σ–formulas with parameters.

'Church thesis for $\mathbb{HF}(\mathbb{R})$'

If we assume the following to be computable

- compute sums and products of reals
- answer questions of kind $x < y$?, $x = y$? for any two reals x and y
- find sets of roots of polynomials and use them in further computations.

Then a function on $\mathbb{HF}(\mathbb{R})$ is intuitively computable if and only if it is Σ–definable.

Σ–presentability of structures

Computable structures: the basic set and all the operations and predicates are uniformly computable.
Replace c.e. with Σ–definability $\longrightarrow$ **Σ–presentable structures**.

A presentation of an algebraic structure $\mathfrak{M}$ of a finite predicate signature is any assignment of codes from some $A \subseteq \mathbb{HF}(\mathbb{R})$ to its elements, i.e., a mapping $\nu : A \subseteq \mathbb{HF}(\mathbb{R}) \xrightarrow{onto} |\mathfrak{M}|$.

- If ν is 1–1 then ν is said to be *simple*.
- If $D(\mathfrak{M}, \nu)$ is Σ–definable with parameters in $\mathbb{HF}(\mathbb{R})$ then ν is a *Σ–presentation of* $\mathfrak{M}$ over $\mathbb{HF}(\mathbb{R})$.
- If $D^+(\mathfrak{M}, \nu)$ (the positive diagram) is Σ–definable with parameters in $\mathbb{HF}(\mathbb{R})$ then ν is said to be a *positive Σ–presentation of* $\mathfrak{M}$ over $\mathbb{HF}(\mathbb{R})$.

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Theorem (Yu.L. Ershov, 1985, 1995)

- *$\mathbb{R}$ and $\mathbb{C}$ have no Σ–presentations in a hereditarily finite superstructure over an infinite set (i.e., infinite structure of empty signature).*
- *$\mathbb{C}$ has a Σ–presentation over any dense linearly ordered set of cardinality 2^ω.*
- *$\mathbb{R}$ has no Σ–presentations over such superstructures.*

(M.) Some structures **without** simple Σ–presentations over $\mathbb{HF}(\mathbb{R})$:

- the Boolean algebra of all subsets of ω and its quotient modulo the ideal of finite sets
- the group of all permutations on ω and its quotient modulo the subgroup of all finitary permutations
- the semigroup of all mappings from ω to ω
- the lattices of all open and all closed subsets of the reals
- the group of all permutations of $\mathbb{R}$ Σ–definable over $\mathbb{HF}(\mathbb{R})$
- the semigroup of all such mappings from $\mathbb{R}$ to $\mathbb{R}$
- the semigroup of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- some structures of nonstandard analysis (including ultrapowers of $\mathbb{R}$ modulo Fréchet ultrafilter with distinguished infinitesimal and standard elements)

The basic result

Theorem

Suppose that $\preccurlyeq$ and L are subsets of $\mathbb{HF}(\mathbb{R})$ definable by means of Σ–formulas with parameters and $\preccurlyeq$ is a preordering on L. Then there is no isomorphic embedding from ω_1 into $\langle L; \preccurlyeq\rangle$.

Remark Harrington and Shelah proved this property for *linear* Borel preorders.
The above result fails to be true for Borel preorders. A counterexample: $\langle P(\omega); \subseteq^*\rangle$.
It follows that this theorem does not follow from Shelah and Harrington's result.

Presentability of ordinals

Corollary

For any ordinal α the following conditions are equivalent:

1. α *has a simple* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
2. α *has a* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
3. α *has a positive* Σ*–presentation over* $\mathbb{HF}(\mathbb{R})$
4. $\alpha < \omega_1$.

Presentability of ordinals without parameters

Corollary

For any ordinal α the following conditions are equivalent:

1. α *has a simple* Σ*–presentation without parameters over* $\mathbb{HF}(\mathbb{R})$
2. α *has a* Σ*–presentation without parameters over* $\mathbb{HF}(\mathbb{R})$
3. $\alpha < \omega_1^{CK}$

Presentability of Gödel constructive sets

$$L_0 = \varnothing, \quad L_{\alpha+1} = \mathtt{Def}\,(L_\alpha), \quad L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha, \text{ for limit } \gamma$$

Corollary

For any ordinal α the following conditions are equivalent:

1. $\langle L_\alpha; \in \rangle$ *has a simple Σ–presentation over* $\mathbb{HF}(\mathbb{R})$
2. $\langle L_\alpha; \in \rangle$ *has a Σ–presentation over* $\mathbb{HF}(\mathbb{R})$
3. $\langle L_\alpha; \in \rangle$ *has a positive Σ–presentation over* $\mathbb{HF}(\mathbb{R})$
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3. $\langle L_\alpha; \in\rangle$ *has a positive Σ–presentation over* $\mathbb{HF}(\mathbb{R})$
4. $\alpha < \omega_1$.

Presentability of Gödel constructive sets without parameters

Theorem

For any ordinal α the following conditions are equivalent:

1. *The structure $\langle L_\alpha; \in\rangle$ has a simple Σ–presentation over $\mathbb{HF}(\mathbb{R})$ without parameters*
2. $\alpha \leqslant \omega$.

(Here we don't need the basic theorem)

Corollary

Assume that $\langle L; \leqslant\rangle$ is an arbitrary partially ordered set in which for any at most countable chain $C \subseteq L$ there exists an $x \in L \setminus C$ with the property $C \leqslant x$.
Then $\langle L; \leqslant\rangle$ has no positive Σ–presentations over $\mathbb{HF}(\mathbb{R})$ with parameters (it follows that it has no neither Σ–presentations nor simple Σ–presentations with parameters).

Nonpresentability of some degree structures

Theorem

The partially ordered sets of Turing, m–, 1–, and tt–degrees have no positive Σ–presentations over $\mathbb{HF}(\mathbb{R})$ with parameters.
(It follows that they have no neither Σ–presentations nor simple Σ–presentations with parameters).

Presentability over $\mathbb{C}$

Corollary

Let α be an ordinal. Then the following conditions are equivalent:

1. α *has a simple* Σ*–presentation over* $\mathbb{HF}(\mathbb{C})$
2. α *has a* Σ*–presentation over* $\mathbb{HF}(\mathbb{C})$
3. α *has a positive* Σ*–presentation over* $\mathbb{HF}(\mathbb{C})$
4. $\alpha < \omega_1^{CK}$

Contents of the talk

- Introduction and some early studies
- Non–embeddability of ω_1 into Σ–definable preorderings over $\mathbb{HF}(\mathbb{R})$ (basic result)
- Descriptions of Σ–presentable ordinals (with parameters and without them) over $\mathbb{HF}(\mathbb{R})$
- Description of Σ–presentable Gödel constructive sets (with parameters and without them) over $\mathbb{HF}(\mathbb{R})$
- Non–Σ–presentability of some degree structures (T–, m–, 1–, tt–) over $\mathbb{HF}(\mathbb{R})$
- Description of Σ–presentable ordinals over $\mathbb{HF}(\mathbb{C})$

Thank you!