Noetherian spaces in choiceless set theory*

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January 1, 2022

Abstract

I prove several independence results in the choiceless ZF+DC theory which separate algebraic and non-algebraic consequences of the axiom of choice.

1 Introduction

Geometric set theory [8] was developed in part to produce consistency results in choiceless ZF+DC set theory regarding various Σ^2_1 sentences. In this paper, I produce several such consistency results which separate Σ^2_1 sentences dealing with algebraic topics from those which do not. While the distinction may seem vague, the techniques of the paper show that there is in fact a clearly visible fracture line. As a first example of such a result, consider the following.

Theorem 1.1. *Let X be a K_σ Polish field with a countable subfield $F \subset X$. It is consistent relative to an inaccessible cardinal that ZF+DC holds, X has transcendence basis over F, and*

1. *there is no nonprincipal ultrafilter on ω;*
2. *if E is an orbit equivalence relation on a Polish space Y induced by a turbulent Polish group action, then every E-invariant subset of Y is either meager or co-meager;*
3. *the Lebesgue null ideal is closed under well-ordered unions.*

There are many similar consistency results which are difficult to subsume under a single heading. One possibility is the following.

Definition 1.2. Let X be a Polish space and $\Gamma \subset [X]^{<\aleph_0}$ be a hypergraph. Γ is *redundant* if for every finite set $e \subset X$ the set $\{x \in X : e \cup \{x\} \in \Gamma\}$ is countable.

*2000 AMS subject classification 03E35, 05C15, 05E14.

Definition 1.3. Let X be a Euclidean space. A hypergraph $\Gamma \subset [X]^{<\aleph_0}$ is *σ-algebraic* if there are algebraic sets A_n for $n \in \omega$ such that for every finite set $e \subset X$ with an injective enumeration $\vec{e} \in X^{|e|}$, $e \in \Gamma$ if and only if there is $n \in \omega$ such that $\vec{e} \in A_n$.

For example, the hypergraph of equilateral $n+1$-simplices in $\mathbb{R}^n$ is a redundant algebraic hypergraph of arity $n+1$. Still another example of a redundant algebraic hypergraph of arity three is the hypergraph of all real solutions to the polynomial $x^3 + y^3 + z^3 - 3xyz = 0$; similar polynomials work as well. The hypergraph of all right-angle triangles in $\mathbb{R}^2$ is algebraic but not redundant. Chromatic numbers of redundant algebraic hypergraphs on Euclidean spaces are always countable in the theory ZFC+CH, while some redundant hypergraphs are countably chromatic in ZFC alone. The reader is referred to the rich literature on algebraic (hyper)graphs for more detail [11, 7, 10, 2, 6]. In the choiceless context, I can prove the following.

Theorem 1.4. *Let Γ be a redundant σ-algebraic hypergraph on a Euclidean space. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ is countable, and*

1. *there is no nonprincipal ultrafilter on ω;*
2. *if E is an orbit equivalence relation on a Polish space Y induced by a turbulent Polish group action, then every E-invariant subset of Y is either meager or co-meager;*
3. *the Lebesgue null ideal is closed under well-ordered unions.*

There are other σ-algebraic (hyper)graphs which can be colored while items (1–3) are preserved [12, 14, 13, 15], but a general theorem remains elusive. As a simple delimitative result, let $\Gamma_{\triangleright}$ be the hypergraph on $\mathbb{R}^2$ of arity three consisting of those sets a whose projections to both coordinate axes have cardinality two. This is a sub-hypergraph of the hypergraph of right triangles in the plane. In the ZFC context, $\Gamma_{\triangleright}$ belongs to a rather large class of hypergraphs whose countable chromatic number is equivalent to the Continuum Hypothesis. However, in ZF a coloring of $\Gamma_{\triangleright}$ yields an object very close to a well-ordering of the reals:

Theorem 1.5. (ZF) *If the chromatic number of $\Gamma_{\triangleright}$ is countable then there is a total countable-to-one map from $\mathbb{R}$ to ω_1.*

Clearly, a countable-to-one map from $\mathbb{R}$ to ω_1 precludes (3) of Theorem 1.1 and 1.4. More importantly though, no such a map can exist in cofinally balanced extensions of the Solovay model [8, Section 9.1], and this completely disables the whole methodology used to prove these theorems.

In Section 2, I introduce analytic Noetherian topologies and several useful examples. Section 3 discusses the notion of a mutually Noetherian pair of generic extensions of a model of ZFC and its implications. This is an instrumental weakening of mutual genericity. In Section 4 I produce several interesting mutually Noetherian pairs of generic extensions, notably one induced by a turbulent action of a Polish group. Section 5 defines the notion of a Noetherian balanced Suslin forcing–this is a forcing in which conditions can be successfully amalgamated across mutually Noetherian pairs of generic extensions. There are several attendant preservation theorems for Noetherian balanced Suslin forcings. Section 6 lists some Noetherian balanced Suslin forcings and uses them with the preservation theorems to finally obtain independence results. In Section 7 I prove Theorem 1.5, which is independent of the rest of the paper.

The paper uses the notation standard of [3]. In matters pertaining to geometric set theory it follows the terminology and notation of [8]. The paper uses the Effros standard Borel space of closed subsets of a Polish space X, often denoted by $F(X)$ in the literature. DC denotes the Axiom of Dependent Choices. The inaccessible cardinal in the assumptions of Theorems 1.1 and 1.4 is used to start the method of balanced forcing as in [8]; I do not know if it is necessary.

2 Noetherian spaces

The technology of this paper rests on the following rather standard definition.

Definition 2.1. Let X be a K_σ-Polish space and let $\mathcal{T}$ be a topology on X different from the original Polish one. Say that $\mathcal{T}$ is an *analytic Noetherian topology* if

1. $\mathcal{T}$ is Noetherian. That is, there is no infinite strictly decreasing sequence of $\mathcal{T}$-closed sets;
2. $\mathcal{T}$ is analytic. That is, every $\mathcal{T}$-closed set is closed in the Polish topology, and the collection of $\mathcal{T}$-closed sets is analytic in the Effros Borel space on X.

Since there are two topologies on the space X, the word "closed" without a modifier denotes a set closed in the Polish topology, while the phrase "$\mathcal{T}$-closed" refers to a set closed in the Noetherian topology. There seems to be a good reason to restrict the considerations to Polish K_σ-spaces: the complexity computations do not seem to work out otherwise. The following fact stands at the root of all complexity computation of this paper [5, Section 12.C]. Let $F(X)$ denote the standard Effros Borel space.

Fact 2.2. *Let X be a Polish space. Then*

1. *the membership relation $\{\langle x, C\rangle \in X \times F(X) \colon x \in C\} \subset X \times F(X)$ is Borel;*
2. *the union relation $\{\langle C_0, C_1, C_2\rangle \in F(X)^3 \colon C_0 \cup C_1 = C_2\} \subset F(X)^3$ is Borel, and similarly for unions of any finite number of sets;*
3. *the subset relation $\{\langle C_0, C_1\rangle \in F(X)^2 \colon C_0 \subset C_1\} \subset F(X)^2$ is Borel.*

In addition, if the space X is K_σ, then

4. *the intersection relation $\{\langle C_0, C_1, C_2\rangle \in F(X)^3 \colon C_0 \cap C_1 = C_2\} \subset F(X)^3$ is Borel, and similarly for intersections of any finite number of sets;*
5. *the function $y \mapsto C_y$, whenever Y is a Polish space and $C \subset Y \times X$ is a closed set, is Borel.*

The last two items fail badly for non-K_σ spaces. It follows that for a K_σ Polish space X and an analytic set $\mathcal{T} \subset F(X)$, the statement "$\mathcal{T}$ is a Noetherian topology on X" is $\mathbf{\Pi}^1_2$ and therefore absolute throughout forcing extensions: it simply says that the collection of $\mathcal{T}$-closed sets is closed under finite unions and finite intersections and contains no infinite sequences strictly decreasing under inclusion. Note that the closure under intersection cannot be expressed in $\mathbf{\Pi}^1_2$ way unless the underlying space X is K_σ and intersection is a Borel function from $F(X)^2$ to $F(X)$ as per Fact 2.2(3).
This paper uses only the following basic algebraic example of a Noetherian topology.

Example 2.3. Let $\mathcal{T}$ be the topology on F^n generated by algebraic sets, where F is a K_σ Polish field. Then $\mathcal{T}$ is an analytic Noetherian topology. It is an analytic collection of closed sets, it is closed under finite unions and intersections, and the descending chain condition is verified via the Hilbert Basis Theorem.

3 Noetherian pairs of generic extensions

The following definition is the key tool for connecting Noetherian topologies with geometric set theory.

Definition 3.1. Let $\mathcal{T}$ be an analytic Noetherian topology on a K_σ-Polish space X. Let M be a transitive model of set theory containing a code for $\mathcal{T}$. Let $A \subset X$ be a set. The symbol $C(M, A)$ denotes the smallest $\mathcal{T}$-closed set coded in M which is a superset of A.

The most common situation for applying this definition is that neither the set A nor any of its elements belong to the model M. Note that the set $C(M, A)$ indeed exists by the Noetherian property of the topology $\mathcal{T}$. There is an obvious monotonicity property of this concept: If $M \subset N$ are two transitive models, then $C(N, A) \subseteq C(M, A)$ holds. In addition, $C(M, A) = C(M, \bar{A})$ where $\bar{A}$ is the closure of the set A in the Polish topology or in the Noetherian topology on the space X. The notation leaves out the dependence on the Noetherian topology, which is figured out from the context.

Definition 3.2. Let $V[G_0]$ and $V[G_1]$ be generic extensions of V inside an ambient generic extension. Say that $V[G_1]$ is *Noetherian over* $V[G_0]$ if for every K_σ Polish space X and an analytic Noetherian topology $\mathcal{T}$ on X coded in the ground model V and for every set $A \subset X$ in $V[G_1]$, $C(V[G_0], A) = C(V, A)$. Say that the extensions $V[G_0], V[G_1]$ are *mutually Noetherian* if each is Noetherian over the other.

Similar notions of perpendicularity always have a friendly relationship with product forcing, as recorded in the following routine proposition.

Proposition 3.3. *Let $n \geq 1$ be a number. Let $V[G_0], V[G_1]$ be generic extensions and $V[G_1]$ is Noetherian over $V[G_0]$. Suppose that $P_0 \in V[G_0]$ and $P_1 \in V[G_1]$ be posets and $H_0 \subset P_0$ and $H_1 \subset P_1$ be filters mutually generic over $V[G_0, G_1]$. Then $V[G_1][H_1]$ is Noetherian over $V[G_0][H_0]$.*

Proof. Work in the model $V[G_0, G_1]$ and consider the poset $P_0 \times P_1$. Let X be a Polish space and $\mathcal{T}$ an analytic Noetherian topology on it, both in V. Let $p_0 \in P_0$ and $p_1 \in P_1$ be conditions and let τ_0, τ_1 be respective P_0, P_1-names in the models $V[G_0], V[G_1]$ such that $p_0 \Vdash \tau_0$ is a $\mathcal{T}$-closed subset of X, $\tau_1 \subset X$ is a set, and $\langle p_0, p_1\rangle \Vdash \tau_0 = C(V[G_0][H_0], \tau_1)$; I must produce a ground model coded closed set $C \subset X$ such that $p_0 \Vdash \tau_0 = C$.

Working in $V[G_1]$, form the closed set $A \subset X$ as $A = X \setminus \bigcup\{O : O \subset X$ is open and $p_1 \Vdash O \cap \tau_1 = 0$. By the initial assumptions on the models $V[G_0]$ and $V[G_1]$, $C(V[G_0], A) = C(V, A)$ holds; write C for the common value. Observe that $p_1 \Vdash \tau_1 \subset C$. It will be enough to show that $p_0 \Vdash \tau_0 = C$.

Since $p_1 \Vdash \tau_1 \subset C$, the only way how the equality can fail is that there is a condition $p_0' \leq p_0$ forcing τ_0 to be a proper subset of C. Working in $V[G_0]$, let M_0 be a countable elementary submodel of some large structure containing τ_0, C, and p_0'. Let $h_0 \subset P_0 \cap M_0$ be a filter generic over the model M_0 and let $D = \tau_0/h_0$. This is a $\mathcal{T}$-closed set properly smaller than C, so $A \subseteq D$ fails. Thus, there must be a basic open set $O \subset X$ disjoint from D which contains some element of the set A. By the definitions, this means that there is a condition $p_0'' \leq p_0'$ in the filter h_0 which forces $\tau_0 \cap O = 0$, and a condition $p_1' \leq p_1$ which forces $\tau_1 \cap O \neq 0$. This contradicts the initial assumptions on the conditions p_0, p_1. □

Corollary 3.4. *Mutually generic extensions are mutually Noetherian.*

In the remainder of this section, I isolate several useful features of mutually Noetherian extensions.

Proposition 3.5. *Let $V[G_0], V[G_1]$ be mutually Noetherian extensions of V. Then $2^\omega \cap V[G_0] \cap V[G_1] = 2^\omega \cap V$.*

Proof. Let $\mathcal{T}$ be the topology on 2^ω whose closed sets are exactly the finite sets and 2^ω itself. It is not difficult to see that $\mathcal{T}$ is an analytic Noetherian topology on X. Now suppose that $x \in 2^\omega \cap V[G_1] \setminus V$ is a point. The set $2^\omega \setminus \{x\}$ is a $\mathcal{T}$-open set in $V[G_1]$ which covers $2^\omega \cap V$. By the mutual Noetherian assumption, it covers $2^\omega \cap V[G_0]$ as well, and therefore $x \notin V[G_0]$ as desired. □

Noetherian topologies are most common in algebra, and the following feature exploits standard algebraic facts about them.

Proposition 3.6. *Let $V[G_0], V[G_1]$ be mutually Noetherian extensions. Then*

1. $2^\omega \cap V[G_0] \cap V[G_1] = 2^\omega \cap V$;

2. *let X be a K_σ Polish field, $p(\bar{v}_0, v_1)$ a polynomial with all parameters in V and all free variables listed. Let $\bar{x}_0 \in V[G_0]$ and $\bar{x}_1 \in V[G_1]$ be tuples such that $X \models p(\bar{x}_0, \bar{x}_1) = 0$. Then there is a tuple $\bar{x}_0' \in V$ such that $X \models p(\bar{x}_0', \bar{x}_1) = 0$;*

3. *let $\phi(\bar{v}_0, \bar{v}_1)$ be a formula of the language of real closed fields with real parameters in V, with all free variables listed. Let $V[G_0], V[G_1]$ be mutually Noetherian extensions and $\bar{x}_0 \in V[G_0]$ and $\bar{x}_1 \in V[G_1]$ be tuples of reals such that $\mathbb{R} \models \phi(\bar{x}_0, \bar{x}_1)$ holds. Then there is a tuple $\bar{x}_0' \in V$ of reals such that $\mathbb{R} \models \phi(\bar{x}_0', \bar{x}_1)$ holds.*

Proof. For (1), let $\mathcal{T}$ be the topology on 2^ω whose closed sets are exactly the finite sets and 2^ω itself. It is not difficult to see that $\mathcal{T}$ is an analytic Noetherian topology on X. Now suppose that $x \in 2^\omega \cap V[G_0] \cap V[G_1]$ is a point. The set $\{x\}$ is $\mathcal{T}$-closed, coded in $V[G_0]$, containing x. By the mutual Noetherian assumption, it has to have a $\mathcal{T}$-closed subset coded in V which contains x, which is possible only if $x \in V$.

For(2), let $n = |\bar{v}_0|$ and observe that the topology of algebraic subsets of X^n is analytic and Noetherian by the Hilbert basis theorem. Consider the set $A = \{\bar{y} \in X^n \colon p(\bar{y}, \bar{x}_1) = 0\}$. This is an algebraic subset of X^n coded in $V[G_1]$ which contains $\bar{x}_0$ as an element. Thus, there must be an algebraic set $B \subseteq A$ coded in V which contains $\bar{x}_0$ as an element, in particular $B \neq 0$. Any element $\bar{x}_0' \in X^n \cap V$ will work as desired.

For (3), let $n_0 = |\bar{v}_0|$ and $n_1 = |\bar{v}_1|$. Use the quantifier elimination theorem for real closed fields [9, Theorem 3.3.15] to assume that ϕ is quantifier-free. Then ϕ is a boolean combination of statements of the form $p(\bar{v}_0, \bar{v}_1) > 0$ and $p(\bar{v}_0, \bar{v}_1) = 0$ for some polynomials p with coefficients in V. Let p_i for $i \in j$ be a list of all polynomials used in this boolean combination. Let $a \subset j$ be the set of all indices such that $p_i(\bar{x}_0, \bar{x}_1) = 0$ and let $A = \{\bar{y} \in \mathbb{R}^{n_0} \colon \Sigma_{i \in a} p_i(\bar{y}, \bar{x}_1)^2 = 0\}$. Now, the topology of algebraic subsets of $\mathbb{R}^{n_0}$ is analytic Noetherian by the Hilbert basis theorem, and by the initial assumptions on the models $V[G_0], V[G_1]$, there is an algebraic set $B \subseteq A$ coded in V containing $\bar{x}_0$. Let $O \subset \mathbb{R}^{n_0}$ be a rational open box containing $\bar{x}_0$ such that for each $i \in j \setminus a$, the values $p_i(\bar{y}, \bar{x}_1)$ have the same sign for all $\bar{y} \in O$. It is clear that $\mathbb{R} \models \phi(\bar{y}, \bar{x}_1)$ for all $\bar{y} \in B \cap O$. The set $B \cap O$ is nonempty, containing $\bar{x}_0$, it is also coded in V. Any tuple $\bar{y} \in V$ n $B \cap O$ works as required. ☐

It may seem difficult to verify that given two generic extensions are mutually Noetherian. In this paper, this is always done using the following *duplication criterion.*

Definition 3.7. Suppose that P is a poset, τ_0, τ_1 are P-names for subsets of the ground model. Say that τ_0 is *duplicable over* τ_1 in P if for every condition $p \in P$ there is a generic extension $V[K]$ and in it, a sequence $\langle H_\alpha \colon \alpha \in \kappa \rangle$ such that

1. κ is an uncountable ordinal in $V[K]$;

2. each $H_\alpha \subset P$ is a filter generic over the ground model containing the condition p;

3. τ_1/H_α is the same for all $\alpha \in \kappa$;

4. for disjoint finite sets $a, b \subset \kappa$, $V[\tau_0/H_\alpha \colon \alpha \in a] \cap V[\tau_0/H_\alpha \colon \alpha \in b] = V$.

Proposition 3.8. *Suppose that P is a poset, τ_0, τ_1 are P-names for subsets of the ground model, and τ_0 is duplicable over τ_1. Then P forces $V[\tau_0]$ to be Noetherian over $V[\tau_1]$.*

Proof. Suppose towards a contradiction that the conclusion fails. Then there must be a condition $p \in P$, a Polish space X, an analytic Noetherian topology $\mathcal{T}$ on it, a name σ for a subset of X in the model $V[\tau_1]$ such that p forces $C(V[\tau_0], \sigma)$ to be strictly smaller than $C(V, \sigma)$. Basic forcing theory shows that $V[\tau_1]$ is a generic extension of the ground model, and that we may assume that there is a poset Q_1 such that $p \Vdash \tau_1 \subset \check{Q}_1$ is a filter generic over the ground model. We also may assume that σ is in fact a Q_1-name.

Move to a generic extension $V[K]$ in which a sequence $\langle H_\alpha \colon \alpha \in \kappa \rangle$ satisfies the items of Definition 3.7. Write $A \subset X$ for the common value of $\sigma/(\tau_1/H_\alpha)$ for all ordinals $\alpha \in \kappa$ and write $C_\alpha = C(V[H_\alpha], A)$. For each ordinal $\beta \in \omega_1$ let $D_\beta = \bigcap_{\alpha \in \omega_1 \setminus \beta} C_\beta$. The sequence $\langle D_\beta \colon \beta \in \omega_1 \rangle$ is an inclusion increasing uncountable sequence of closed subsets of X, and as such it has to stabilize at some ordinal β_0. Write $D \subset X$ for the stable value. Use the Noetherian property of the topology $\mathcal{T}$ to find a finite set $b_0 \subset \omega_1 \setminus \beta_0$ such that $D = \bigcap_{\alpha \in b_0} C_\alpha$. Let $\beta_1 \in \omega_1$ be an ordinal larger than $\max(b_0)$ and find a finite set $b_1 \subset \omega_1 \setminus \beta_1$ such that $D = \bigcap_{\alpha \in b_1} C_\alpha$. By the intersection assumption on the sequence of the generic extensions, $V[G_\alpha \colon \alpha \in b_0] \cap V[G_\alpha \colon \alpha \in \beta_1] = V$ must hold. Thus, the set D is in fact coded in V. Now, for every ordinal $\alpha \in \omega_1 \setminus \beta_0$ it is the case that $A \subseteq D \subseteq C_\alpha$. The definition of the set C_α then shows that $D = C_\alpha$ and the proposiiton follows. $\square$

4 Examples I

This section contains several example of mutually duplicable names in forcing, which by Proposition 3.8 always lead to mutually Noetherian pairs of extensions.

Example 4.1. If Q_0, Q_1 are posets with the respective names τ_0, τ_1 for their generic filters and $P = Q_0 \times Q_1$, then τ_0, τ_1 are mutually duplicable in P. To see that τ_0 is duplicable over τ_1, let κ be the successor of the maximum of $|Q_0|$ and $|Q_1|$. For every condition $p = \langle q_0, q_1 \rangle \in P$, consider the product forcing of κ-many copies of $Q_0 \restriction q_0$ and a single copy of $Q_1 \restriction q_1$, yielding filters $G_0^\alpha \subset Q_0$ and $G_1 \subset Q_1$. Let $H_\alpha = G_0^\alpha \times H_1$ and observe that for finite disjoint sets $a, b \subset \kappa$, the intersection $V[G_0^\alpha \colon \alpha \in a] \cap V[G_0^\alpha \colon \alpha \in b]$ is equal to V by the product forcing theorem.

Example 4.2. Let P be the poset of all pairs $p = \langle s_p, t_p \rangle$ of sequences in $3^{<\omega}$ of the same length, such that $\forall i \in \mathrm{dom}(s_p)\ s_p(i) \neq t_p(i)$ holds; the ordering is that of coordinatewise reverse inclusion. Let τ_0, τ_1 be P-names for the unions of the first coordinates and the second coordinates of conditions in the generic filter respectively. Then τ_0, τ_1 are mutually duplicable names in P.

Proof. I will show that τ_0 is duplicable over τ_1; the example then follows by a symmetry argument. To start, for a finite set a let Q_a be the poset of all functions q such that $\mathrm{dom}(q) = a$, for some $n \in \omega$ for every $i \in a$ $q(i) \in 3^n$, and for every $j \in n$, the set $\{q(i)(j) \colon i \in a\} \neq 3$. The ordering is that of coordinatewise extension. The generic object is a tuple $\vec{y}_{gen} \in (3^\omega)^a$ which is the coordinatewise union of all conditions in the generic filter. Two easy claims are necessary to handle these posets.

Claim 4.3. *If $b \subset a$ then $Q_a \Vdash \vec{y}_{gen} \restriction b$ is generic over V for the poset $\check{Q}_b$.*

This is proved by an elementary density argument.

Claim 4.4. *If $b_0, b_1 \subset a$ are disjoint sets then $Q_a \Vdash V[\vec{y}_{gen} \restriction b_0] \cap V[\vec{y}_{gen} \restriction b_1] = V$.*

Proof. In view of Claim 4.3, it is enough to prove the following. Suppose that σ_0 is a Q_{b_0}-name for a set of ordinals not in the ground model and σ_1 is a Q_{b_1}-name for a set of ordinals, and $q \in Q_a$ is a condition. Then, there is a condition $r \leq q$ and an ordinal $\alpha \in \kappa$ such that $r \restriction b_0 \Vdash \check{\alpha} \in \sigma_0$ and $r \restriction b_1 \Vdash \check{\alpha} \notin \sigma_1$ or vice versa.

To this end, use the assumption on σ_0 to find an ordinal α and conditions $r_{00}, r_{01} \leq q \restriction b_0$ in the poset Q_{b_0} such that $r_{00} \Vdash \check{\alpha} \in \sigma_0$ and $r_{01} \Vdash \check{\alpha} \notin \sigma_1$. Strenghtening if necessary, assume that $n(r_{00}) = n(r_{01})$. For each $j \in n(r_{00}) \setminus n(q)$ find elements $k_{j0}, k_{j1} \in 3$ which are not in the sets $\{r_{00}(i)(j) \colon i \in b_0\}$ and $\{r_{01}(i)(j) \colon i \in b_0\}$ respectively. Consider the condition $r_{10} \leq q \restriction b_1$ in Q_{b_1} such that $n(r_{10}) = n(r_{00})$ and for all $j \in n(r_{00}) \setminus n(q)$ and every $i \in b_1$, $r_{10}(i)(j) \notin \{k_{j0}, k_{j1}\}$. Find a condition $r_{11} \leq r_{10}$ in Q_{b_1} which decides the membership of α in τ_1. For definiteness, assume that the decision is negative. Now, for all indices $i_0 \in b_0$ and $i_1 \in b_1$ and for every $j \in n(r_{00}) \setminus n(q)$, $k_{j0} \notin \{r_{00}(i_0)(j), r_{11}(i_1)(j)\}$ holds. Therefore, it is easy to find a condition $r \leq q$ such that $r \restriction b_0$ coordinatewise extends r_{00} and $r \restriction b_1$ coordinatewise extends r_{11}. This condition and nd the ordinal α are as required. $\square$

Now, write $\kappa = \omega_1$ and consider the poset R of conditions r such that r is a function, $\mathrm{dom}(r) \subset \kappa + 1$ is a finite set containing κ, for some number $n(r)$ for all ordinals $\alpha \in \mathrm{dom}(r)$ $r(\alpha) \in 3^{n(r)}$, and for every $\alpha \in \mathrm{dom}(r) \cap \kappa$ and every $j \in n(r)$, $r(\alpha)(j) \neq r(\kappa)(j)$ holds. The ordering is reverse coordinatewise inclusion. The generic object is a tuple $\vec{y}_{gen} \in ((3^\omega)^{\kappa+1}$ which is the coordinatewise union of all conditions in the generic filter. Again, a simple claim about this poset is needed; the proof is a simple density argument.

Claim 4.5. *For every finite set $a \subset \kappa$, $R \Vdash \vec{y}_{gen} \restriction a$ is $\check{Q}_a$-generic over V and for every ordinal $\beta \in \kappa$, the pair $\vec{y}_{gen}(\beta), \vec{y}_{gen}(\kappa)$ is $\check{P}$-generic over V.*

Now, suppose that $p \in P$ is a condition. Let $\vec{y}_{gen}$ be an R-generic sequence over V. Rewrite the initial segments of $\vec{y}_{gen}(\beta)$ for all $\beta \in \kappa$ with s_p and the initial segment of $\vec{y}_{gen}(\kappa)$ with s_p. While the resulting sequence $\vec{z}_{gen}$ is not R-generic over V, the conclusions of Claim 4.5 remain in force. In view of Claim 4.4, the filters $H_\alpha \subset P$ for $\alpha \in \kappa$ given by the pairs $\vec{z}_{gen}(\alpha), \vec{z}_{gen}(\kappa)$ show the duplicability of the name τ_0 over τ_1. □

Corollary 4.6. *The poset P of Example 4.2 forces $V[\tau_0]$, $V[\tau_1]$ to be mutually Noetherian extensions.*

The second example deals with turbulent actions of Polish groups as outlined in [4, Section 13.1].

Example 4.7. Let Γ be a Polish group acting turbulently on a Polish space X with dense meager orbits. Let $P = P_\Gamma \times P_X$, let $\dot\gamma_{\text{gen}}$ be the P-name for the P_Γ-generic point, τ_0 the P-name for the P_X-generic point, and $\tau_1 = \dot\gamma_{\text{gen}} \cdot \tau_0$. Then τ_0, τ_1 are mutually duplicable names in P.

Proof. By a symmetry argument, it is enough to show that τ_1 is duplicable over τ_1. Suppose that $p = \langle U, O\rangle$ is a condition in the poset P, where $U \subset \Gamma$ and $O \subset X$ are nonempty open sets. Let $\kappa = \omega_1$, let $x \in O$ be a point P_X-generic over V. Force with a finite support product of κ-many copies of the poset $P_\Gamma \restriction U$ to obtain a sequence $\langle g_\alpha \colon \alpha \in \kappa\rangle$ of points in U which are in finite tuples mutually P_Γ-generic over $V[x]$; write $x_\alpha = g_\alpha \cdot x$. Each of the filters $H_\alpha \subset P$ for $\alpha \in \kappa$ given by the pair $\langle g_\alpha, x\rangle$ is P-generic over V; I will show that the sequence $\langle H_\alpha \colon \alpha \in \kappa\rangle$ witnesses the duplicability of τ_1 over τ_0.

Claim 4.8. *Let $a \subset \kappa$ be a finite set. Then $V[x_\alpha \colon \alpha \in a] \cap V[x] = V$.*

Proof. Without loss, assume that the set a is nonempty, and write $\beta = \min(a)$. The point g_β^{-1} is P_Γ-generic over $V[x_\beta]$, and the tuple $t = \langle g_\alpha \cdot g_\beta^{-1} \colon \beta \in a \setminus \{\beta\}\rangle$ is generic over $V[x_\beta][g_\beta^{-1}]$ for a product of the posets P_Γ. Use the product forcing theorem to conclude that $V[x_\beta][g_\beta^{-1}] \cap V[x_\beta][s] = V[x_\beta]$ and therefore $V[x] \cap V[x_\alpha \colon \alpha \in a] \subseteq V[x_\beta]$. However, $V[x] \cap V[x_\beta] = V$ holds by the turbulence assumption and [8, Theorem 3.2.]. It follows that $V[x] \cap V[x_\alpha \colon \alpha \in a] = V$ as desired. □

Claim 4.9. *If $a, b \subset \kappa$ are disjoint finite sets then $V[x_\alpha \colon \alpha \in a] \cap V[x_\alpha \colon \alpha \in b] = V$.*

Proof. By the product forcing theorem, $V[x][g_\alpha \colon \alpha \in a] \cap V[x][g_\alpha \colon \alpha \in b] = V[x]$ holds, and therefore $V[x_\alpha \colon \alpha \in a] \cap V[x_\alpha \colon \alpha \in b] \subseteq V[x]$ holds. By the previous claim, $V[x_\alpha \colon \alpha \in a] \cap V[x] = V$ holds, and in consequence $V[x_\alpha \colon \alpha \in a] \cap V[x_\alpha \colon \alpha \in b] = V$ as desired. □

The duplicability follows. □

Corollary 4.10. *The poset P of Example 4.7 forces $V[\tau_0]$, $V[\tau_1]$ to be mutually Noetherian extensions.*

The following example uses the notion of Suslin forcing which occurs several times in this paper.

Definition 4.11. [1, Definition 3.6.1] A forcing Q is *Suslin* if there s an *ambient* Polish space X in which the conditions of Q form an analytic set, and the ordering and incompatibility relations on Q are analytic relations on X.

The following fact regarding Suslin forcing will be used repeatedly throughout the paper.

Fact 4.12. *Let Q be a c.c.c. Suslin forcing and let $V[G]$ be a forcing extension. Then*

1. *[1, Theorem 3.6.6] the reinterpretation $Q^{V[G]}$ is a c.c.c. Suslin forcing in $V[G]$;*
2. *[1, Corollary 3.6.5] if $H \subset Q^{V[G]}$ is a filter generic over $V[G]$, then $H \cap V \subset Q^V$ is a filter generic over V.*

Example 4.13. Let Q_0 be an arbitrary forcing and Q_1 be a Suslin c.c.c. forcing. Let P be the iteration $Q_0 * \dot{Q}_1$ where the definition of Q_1 is reinterpreted in the Q_0-forcing extension. Let τ_0 be the P-name for the filter on the first iterand and τ_1 be the P-name for the intersection of the filter on the second iterand with the ground model.Then τ_0, τ_1 are mutually duplicable names in P.

Note that by Fact 4.12(2) τ_1 is forced to be a filter on Q_1 generic over V.

Proof. To show that τ_1 is duplicable over τ_0, let $p = \langle q_0, \dot{q}_1 \rangle$ be a condition in the poset P. Let κ be a regular cardinal larger than $|Q_0|$. Let $G_0 \subset Q_0$ be a filter generic over V containing the condition q_0, and force with the finite support product of κ-many copies of the forcing $Q_1 \restriction \dot{q}_1/G_0$ to obtain filters $G_{1\alpha} \subset Q_1$ for $\alpha \in \kappa$. I claim that the filters $H_\alpha = G_0 * G_{1\alpha} \subset P$ for $\alpha \in \kappa$ witness the duplicability of τ_1 over τ_0. To see this, for each $\alpha \in \kappa$ write $K_\alpha = G_{1\alpha} \cap V = \tau_1/H_\alpha$. The following claim completes the proof by the product forcing theorem.

Claim 4.14. *If $a \subset \kappa$ is a finite set, then $\langle K_\alpha \colon \alpha \in a \rangle$ are filters on Q_1 which are mutually generic over V.*

Proof. Consider the poset R which is the product of a-many copies of Q_1. It is easy to check that R is Suslin, and by Fact 4.12(1), R is c.c.c. The filters $\langle G_{1\alpha} \colon \alpha \in a \rangle$ form an R-generic sequence over $V[G_0]$. By Fact 4.12(2), the sequence $\langle K_\alpha \colon \alpha \in a \rangle$ is an R-generic sequence over V as desired. □

To show that τ_0 is duplicable over τ_1, let $p = \langle q_0, \dot{q}_1 \rangle$ be a condition in the poset P. Let κ be a regular cardinal larger than $|Q_0|$. Let $s = \langle G_{0\alpha} \colon \alpha \in \kappa \rangle$ be a mutually generic sequence of filters on $Q_0 \restriction q_0$. In the model $V[s]$, consider the (reinterpretation of the) poset Q_1 and the conditions $r_\alpha = \dot{q}_1/G_{0\alpha}$ in it for $\alpha \in \kappa$. Since Q_1 is c.c.c. in $V[s]$ by Fact 4.12(1) there must be a condition

$r \in Q_1$ which forces that the set $\{\alpha \in \kappa \colon r_\alpha$ belongs to the generic filter$\}$ is cofinal in κ. Let $G_1 \subset Q_1$ be a filter generic over $V[s]$, containing the condition r. By Fact 4.12(2), for each ordinal $\alpha \in \kappa$, $G_{1\alpha} = G_1 \cap V[G_{0\alpha}]$ is a filter on Q_1 generic over $V[G_{0\alpha}]$. I claim that the filters $H_\alpha = G_{0\alpha} * G_{1\alpha}$ for $\alpha \in \kappa$ such that $r_\alpha \in G_1$ witness the duplicability of τ_0 over τ_1. This is an immediate corollary of the product forcing theorem applied to the models $V[G_{0\alpha}]$ for $\alpha \in \kappa$. $\square$

Corollary 4.15. *Let μ be a Borel probability measure on a Polish space X, and let $x_0, x_1 \in X$ be mutually μ-random elements of X. Then the models $V[x_0]$, $V[x_1]$ are mutually Noetherian.*

5 Preservation theorems

Any notion of perpendicularity similar to Definition 3.1 comes with a natural notion of balance for Suslin forcings.

Definition 5.1. Let P be a Suslin forcing.

1. A pair $\langle Q, \sigma\rangle$ is *Noetherian balanced* if $Q \Vdash \sigma \in P$ and for any pair $V[G_0]$, $V[G_1]$ of mutually Noetherian extensions of the ground model, every pair $H_0, H_1 \subset Q$ of filters generic over V and for every pair $p_0 \in V[G_0]$, $p_1 \in V[G_1]$ of conditions stronger than $\sigma/H_0, \sigma/H_1$ respectively and belonging to the respective models $V[G_0], V[G_1]$, the conditions $p_0, p_1 \in P$ have a common lower bound.
2. P is *Noetherian balanced* if for every condition $p \in P$ there is a Noetherian balanced pair $\langle Q, \sigma\rangle$ such that $Q \Vdash \sigma \leq \check{p}$.

The supply of mutually Noetherian pairs of extensions provided in the previous section now makes it possible to prove several preservation theorems. They are stated using the parlance of [8, Convention 1.7.18]. Thus, given an inaccessible cardinal κ, a Suslin poset P is Noetherian balanced cofinally below κ if for every generic extension $V[K_0]$ generated by poset of cardinality smaller than κ there is a larger generic extension $V[K_1]$ generated by a poset of cardinality smaller than κ such that $V_\kappa[K_1] \models P$ is Noetherian balanced.

Theorem 5.2. *Let κ be an inaccessible cardinal. Let W be the symmetric Solovay model derived from κ. In cofinally Noetherian balanced extensions of W, every nonmeager subset of 3^ω contains points y_0, y_1 such that the set $\{i \in \omega \colon y_0(i) = y_1(i)\}$ is finite.*

I do not know if the conclusion holds also for non-null sets for the usual Borel probability measure on 3^ω.

Proof. Let P be a Suslin forcing which is Noetherian balanced cofinally below κ. Work in W. Let $p \in P$ be a condition and let τ be a P-name such that $p \Vdash \tau \subset 3^\omega$ is a nonmeager set. I have to find two points y_0, y_1 such that the set $\{i \in \omega \colon y_0(i) = y_1(i)\}$ is finite a condition stronger than p which forces both to τ.

Both p, τ are definable from some elements of the ground model and an additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate extension obtained by a partial order of cardinality less than κ such that $z \in V[K]$, and such that $V[K] \models P$ is Noetherian balanced. Work in $V[K]$. Let $\langle Q, \sigma\rangle$ be a Noetherian balanced pair such that $Q \Vdash \sigma \leq p$. Let R be the Cohen poset of nonempty open subsets of 3^ω ordered by inclusion, adding a Cohen generic point $\dot{y}$. There must be a condition $q \in Q$, a condition $r \in R$, and a poset S of cardinality smaller than κ, and a $Q \times R \times S$-name η for a condition in P stronger than σ such that

$$q \Vdash_Q r \Vdash_R s \Vdash_S \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash \dot{y} \in \tau.$$

Otherwise, in the model W, if $H \subset Q$ is a filter generic over $V[K]$ then the condition σ/H would force in P that the comeager set of points R-generic over $V[K][H]$ to be disjoint from τ, contradicting the initial assumptions on p and τ.

In the model W, use Example 4.2 to produce points $x_0, x_1 \in 3^\omega$ which are are separately R-generic over $V[K]$, such that for all $i \in \omega$ $x_0(i) \neq x_1(i)$, and such that the models $V[K][x(0)]$, $V[K][x_1]$ are mutually Noetherian. Let $y_0 \in 3^\omega$ be a finite modification of x_0 which belongs to q and let $y_1 \in 3^\omega$ be a finite modification of x_1 which belongs to q. Let $H_0, H_1 \subset R \times S$ be filters mutually generic over $V[K][x_0][x_1]$ meeting the conditions $r \in R$ and $s \in S$. By Proposition 3.3, the models $V[K][y_0][H_0]$ and $V[K][y_1][H_1]$ are mutually Noetherian extensions of $V[K]$. Let $p_0 = \sigma/y_0, H_0$ and $p_1 = \sigma/y_1, H_1$. These are conditions in P in the respective models stronger than σ/H_0 and σ/H_1 respectively. By the balance assumption on $\langle Q, \sigma\rangle$, the conditions p_0, p_1 are compatible. By the forcing theorem applied in the respective models $V[K][y_0][H_0]$ and $V[K][y_1][H_1]$, the common lower bound of these two conditions forces $\check{y}_0, \check{y}_1 \in \tau$ as required. □

Corollary 5.3. *Let κ be an inaccessible cardinal. Let W be the symmetric Solovay model derived from κ. In closed, cofinally Noetherian balanced extensions of W, there are no nonprincipal ultrafilters on ω.*

Proof. If U is a nonprincipal ultrafilter on ω, then the map $c \colon 3^\omega \to 3$ defined by $c(x) = i$ if $\{n \in \omega \colon c(n) = i\} \in U$ partitions of 3^ω into three pieces neither of which contains points y_0, y_1 such that the set $\{i \in \omega \colon y_0(i) = y_1(i)\}$ is finite. One of these pieces must be non-meager. Theorem 5.2 concludes the proof. □

Theorem 5.4. *Let Γ be a Polish group acting continuously, turbulently and with dense meager orbits on a Polish space X. Let κ be an inaccessible cardinal. Let W be the symmetric Solovay model derived from κ. In cofinally Noetherian balanced extensions of W, every Γ-invariant subset of X is either meager or co-meager.*

Proof. Let P be a Suslin forcing which is Noetherian balanced cofinally below κ. Work in W. Let $p \in P$ be a condition and let τ be a P-name such that

$p \Vdash \tau \subset X$ is a set which is neither meager nor co-meager. I have to find an element $\gamma \in \Gamma$ and points $x_0, x_1 \in X$ such that $\gamma \cdot x_0 = x_1$, and a condition stronger than p which forces $\check{x}_0 \in \tau$ and $\check{x}_1 \notin \tau$.

Both p, τ are definable from some elements of the ground model and an additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate extension obtained by a partial order of cardinality less than κ such that $z \in V[K]$, and such that $V[K] \models P$ is Noetherian balanced. Work in $V[K]$. Let $\langle Q, \sigma \rangle$ be a Noetherian balanced pair such that $Q \Vdash \sigma \leq \check{p}$. Let R be the Cohen poset of all nonempty open subsets of X, adding a point $\dot{x}_{gen} \in X$. There must be conditions $q_0, q_1 \in Q$, $r_0, r_1 \in R$ and posets S_0, S_1 of cardinality smaller than κ, and $Q \times R_0 \times S_0$- and $Q \times R_1 \times S_1$-names η_0, η_1 for conditions in P stronger than σ such that

$$q_0 \Vdash_Q r_0 \Vdash_{R_0} s_0 \Vdash_{S_0} \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash_P \dot{x}_{gen} \in \tau$$

$$q_1 \Vdash_Q r_0 \Vdash_{R_1} s_0 \Vdash_{S10} \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash_P \dot{x}_{gen} \notin \tau.$$

Otherwise, in the model W, the poset P would force either τ_0 or τ_1 to be disjoint from the co-meager set of elements of X Cohen-generic over the model $V[K][G]$ where $G \subset Q$ is any filter generic over $V[K]$. This would contradict the initial assumptions on the name τ.

Now, since the group Γ acts on X continuously and with dense orbits, there are nonempty open sets $U \subset \Gamma$ and $O \subset r_0$ such that $U \cdot O \subset r_1$. In the model W, find points $g \in U$ and $x_0 \in O$ which are $P_\Gamma \times P_X$-generic over $V[K]$, and let $x_1 = g \cdot x_0$. By the turbulence assumption and Example 4.7, the models $V[K][x_0]$ and $V[K][x_1]$ are mutually Noetherian extensions of $V[K]$. Let $H_0 \subset Q_0 \times S_0$ and $H_1 \subset Q_1 \times S_1$ be filters mutually generic over $V[K][x_0][x_1]$ and containing the respective conditions q_0, s_0, q_1, s_1. By Proposition 3.3, the models $V[K][x_0][H_0]$ and $V[K][x_1][H_1]$ are mutually Noetherian as well.

Let $p_0 = \eta/x_0, H_0$ and $p_1 = \eta/x_1, H_1$. These are conditions in P in mutually Noetherian extensions extending the conditions σ/H_0 and σ/H_1. By the balance assumption on the pair $\langle Q, \sigma \rangle$, the conditions p_0, p_1 have a common lower bound in P. In the model W, the common lower bound forces both $\check{x}_0 \in \tau$ and $\check{x}_1 \notin \tau$, while it is also the case that $g \cdot x_0 = x_1$. This completes the proof of the theorem. □

In order to state the following preservation theorem in full generality, a standard definition will be useful.

Definition 5.5. Suppose that X is a Polish space and I is a σ-ideal on X. I is a *Suslin c.c.c. ideal* if there is a Suslin c.c.c. forcing R and an R-name $\dot{x}_{gen}$ for an element of the Polish space X such that I is generated by all those Borel sets $B \subset X$ for which $P \Vdash \dot{x}_{gen} \notin B$ holds.

In the last expression, the Borel set B has to be reinterpreted in the P-extension. In the usual situation, the name $\dot{x}_{gen}$ consists of maximal antichains of conditions deciding the membership of $\dot{x}_{gen}$ in basic open subsets of X. In such a case, in every generic extension the ideal I can be reinterpreted by first reinterpreting the Suslin poset R and the name $\dot{x}_{gen}$ as analytic sets and then using the same forcing definition to define the reinterpretation of I. A simple absoluteness argument irrelevant for the purposes of this paper shows that this reinterpretation of I in generic extension does not depend on the choice of the Suslin c.c.c. forcing R and the name $\dot{x}_{gen}$. If $B \subset X$ is a Borel set and $V[G]$ is a generic extension, then $V \models B \in I$ holds if and only if $V[G] \models$the reinterpretation of B belongs to the reinterpretation of I; this easily follows from Fact 4.12(2). Notorious examples of Suslin c.c.c. ideals include the ideal of meager and Lebesgue null sets on $\mathbb{R}$, but other examples are studied in the literature as well [1, Section 3.6].

Theorem 5.6. *Let κ be an inaccessible cardinal. Let W be the symmetric Solovay model derived from κ. In cofinally Noetherian balanced extensions of W, if X is a Polish space and I a Suslin c.c.c. ideal on it, then I is closed under well-ordered unions.*

Proof. Let P be a Suslin forcing which is Noetherian balanced cofinally below κ. Work in W. Let X be a Polish space and let I be a Suslin c.c.c. ideal on it obtained from some Suslin c.c.c. poset R and a name $\dot{x}_{gen}$. Let $p \in P$ be a condition, λ be an ordinal, and let τ be a P-name such that $p \Vdash \tau \colon X \to \lambda$ is a partial function. I have to find an ordinal $\alpha \in \kappa$ and a condition stronger than p which forces either $\mathrm{dom}(p) \in I$ or $\tau^{-1}\{\alpha\} \notin I$.

All of $p, \tau, X, R, \dot{x}_{gen}$ are definable from some elements of the ground model and an additional parameter $z \in 2^\omega$. Let $V[K]$ be an intermediate extension obtained by a partial order of cardinality less than κ such that $z \in V[K]$, and such that $V[K] \models P$ is Noetherian balanced. Work in $V[K]$. Let $\langle Q, \sigma\rangle$ be a Noetherian balanced pair such that $Q \Vdash \sigma \leq p$. There are two cases.

Case 1. $Q \times R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \dot{x}_{gen} \notin \mathrm{dom}(\tau)$ holds. In this case, work in W and let $H_0 \subset Q$ be a filter generic over $V[K]$ and set $p_0 = \sigma/H_0$. Consider the set $B = \{x \in X \colon \exists G \subset R \cap V[K]$ a filter generic over $V[K]\} \subset X$. This is a Borel set and its complement belongs to the ideal I. Thus, it will be enough to argue that $p \Vdash \mathrm{dom}(\tau) \cap B = 0$.

Suppose towards a contradiction that this fails. Then, there must be a point $x \in B$ and a condition $p_0' \leq p_0$ such that $p_0' \Vdash \check{x} \in \mathrm{dom}(\tau)$. Now, let $H_1 \subset Q$ be a filter generic over $V[K][x][H_0][p_0']$ and write $p_1 = \sigma/H_1$. By the forcing theorem applied in the model $V[K]$, $p_1 \Vdash \check{x} \notin \mathrm{dom}(\tau)$. By the balance assumption on the pair $\langle Q, \sigma\rangle$, the conditions p_0', p_1 are compatible in P. This is impossible as they force contrary statements.

Case 2. Case 1 fails. This means that there is a condition $q_0 \in Q$, a condition $r \in R$, and a poset S_0 of cardinality smaller than κ with a condition $s_0 \in S_0$, an ordinal $\alpha \in \lambda$, and a $Q \times R \times S_0$-name η_0 for a condition in P stronger than σ such that

$$q_0 \Vdash_Q r \Vdash_R s_0 \Vdash_{S_0} \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \eta_0 \Vdash \tau(\dot{x}_{gen}) = \check{\alpha}.$$

By the forcing theorem, it will now be enough to show that in the model $V[K]$, $Q \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \tau^{-1}\{\check{\alpha}\} \notin I$. Work in $V[K]$ and suppose towards a contradiction that this fails. Then, there must be a poset S_1 of cardinality smaller than κ and conditions $q_1 \in Q$ and $s_1 \in S_1$ and $Q \times S_1$-names ν for a Borel set in I and η_1 for a condition in P stronger than σ such that $q_1 \Vdash_Q s_1 \Vdash_{S_1} \Vdash \text{Coll}(\omega, < \kappa) \Vdash \eta_1 \Vdash \tau^{-1}\{\check{\alpha}\} \subset \nu$. Let $H_1 \subset Q \times S_1$ be a filter generic over $V[K]$, meeting the conditions q_1 and s_1. Let $G \subset R$ be a filter generic over $V[K][H_1]$, meeting the condition r. Example 4.13 now shows that the models $V[K][H_1]$ and $V[K][G \cap V[K]]$ are mutually Noetherian extensions of $V[K]$. Let $B = \nu/H_1$ and let $x = \dot{x}_{gen}/G$. Thus, $B \in I$ and $x \notin B$ holds by the definition of the ideal I.

Let $H_0 \subset Q \times S_0$ be a filter generic over the model $V[K][H_1][G]$, meeting the conditions q_0, s_0. By Proposition 3.3, the models $V[K][H_1]$ and $V[K][G \cap V[K]][H_0]$ are mutually Noetherian extensions of $V[K]$. Let $p_0 = \eta_0/G, H_0$ and $p_1 = \eta_1/H_1$. By the balance assumption on the pair $\langle Q, \sigma \rangle$, the conditions $p_0, p_1 \in P$ have a common lower bound. In the model W, this common lower bound forces in P that $\tau(\check{x}) = \check{\alpha}$ and $\tau^{-1}\{\alpha\} \subset B$, yet $x \notin B$ holds. This is a contradiction completing the proof of the theorem. □

6 Examples II

In this section, I produce several Noetherian balanced Suslin forcings. This is of course a necessary ingredient for any specific independence result.

Theorem 6.1. *Let X be a K_σ Polish field and F be a countable subfield. Let P be the partial order of countable subsets of X which are algebraically free over F. The ordering is reverse inclusion. Then P is σ-closed, Suslin and Noetherian balanced.*

Proof. The Suslinness and σ-closure of P are immediate. To argue for the Noetherian balance, let $p \in P$ be a condition. Let b be a transcendence basis for X over F containing p as a subset. It will be enough to argue that $\langle \text{Coll}(\omega, X), \check{b} \rangle$ is a Noetherian balanced pair.

Let $V[G_0], V[G_1]$ be mutually Noetherian generic extensions of V and let $p_0 \in V[G_0]$ and $p_1 \in V[G_1]$ be conditions in Psuch that $b \subset p_0, p_1$ holds. I must show that p_0, p_1 are compatible in P; in other words, $p_0 \cup p_1$ is algebraically free over F. Suppose towards a contradiction that this fails. Let r be a nonzero multivariate polynomial with coefficients in F, and let $\vec{x}_0, \vec{x}_1$ be tuples from p_0 and p_1 respectively such that $r(\vec{x}_0, \vec{x}_1) = 0$ holds. By Proposition 3.6(2), there must be a tuple $\vec{x}_0'$ in the ground model such that $r(\vec{x}_0', \vec{x}_1)$ holds. Note that all elements of the tuple $\vec{x}_0'$ are algebraic over b; let $a \subset b$ be a finite set such that elements of $\vec{x}_0'$ are algebraic over a. Let x_2 be any element of the tuple $\vec{x}_1$ and observe that x_2 is algebraic over a and the remainder of $\vec{x}_1$, contradicting the assumption that p_1 is an algebraically independent set. □

Theorem 6.2. *Let Γ be a redundant σ-algebraic hypergraph on a Euclidean space X. Then there is a σ-closed Suslin forcing which adds a countable coloring of Γ. In addition, under the Continuum Hypothesis, this forcing is Noetherian balanced.*

Proof. Let $X = \mathbb{R}^n$ for some number $n \geq 1$. To simplify the notation, I will assume that the algebraic sets in the σ-algebraic presentation of Γ are all obtained from polynomials with rational coefficients only. The main algebraic feature of the hypergraph is the following elementary proposition:

Proposition 6.3. *Suppose that $a \subset X$ is a finite set and $x \in X$ is an element such that $a \cup \{x\} \in \Gamma$. Then x is algebraic over a.*

Proof. Let A be an algebraic set in the σ-algebraic presentation of Γ such that $\vec{a}^\frown x \in A$. The vertical section $A_{\vec{a}}$ is a countable algebraic set by the redundancy assumption. By quantifier elimination for real closed fields [9, Theorem 3.3.15], this set is in fact finite and consists of points all of whose coordinates are algebraic over the set of real numbers used as coordinates of points in a. This proves the proposition. $\square$

The definition of the coloring poset depends on a Borel ideal I on ω which contains all singletons and which is not generated by countably many sets. All other properties of the ideal I are irrelevant; the summable ideal will do. The coloring poset P consists of all functions p such that there is a countable real closed subfield $\operatorname{supp}(p) \subset \mathbb{R}$ such that $\operatorname{dom}(p) = \operatorname{supp}(p)^n$ and p is a Γ-coloring with range consisting of natural numbers. The ordering is defined by $q \leq p$ if

(A) $p \subseteq q$;

(B) the set $\gamma(p, q) = \{m \in \omega\colon$ there is a Γ-hyperedge $e \subset \operatorname{dom}(p \cup q)$ such that the set $e \setminus \operatorname{dom}(p)$ is nonempty and q-monochromatic of color $m\}$ is empty;

(C) for every finite set $f \subset \operatorname{supp}(q)$, the set $\delta(p, q, f) = \{m \in \omega\colon$ there is a nonempty q-monochromatic finite set $a \subset \operatorname{dom}(q) \setminus \operatorname{dom}(p)$ of color m such that there is $x \in \operatorname{dom}(q)$ which is algebraic over $\operatorname{supp}(p) \cup f$ and $b \subset \operatorname{dom}(p)$ with $a \cup \{x\} \cup b \in \Gamma\}$ belongs to the ideal I.

Note that the definition of the coloring poset does not depend on the σ-algebraic presentation of Γ or on the redundancy of Γ.

Claim 6.4. *The relation $\leq$ is transitive and σ-closed.*

Proof. To see the transitivity, suppose that $r \leq q \leq p$ are conditions and argue that $r \leq p$ must hold. To verify (B), just note that $\gamma(p, r) \subseteq \gamma(p, q) \cup \gamma(q, r)$. To verify (C), let $f \subset \operatorname{supp}(r)$ be a finite set. Use a transcendence dimension argument to conclude that there is a finite set $f' \subset \operatorname{supp}(q)$ such that every real number in $\operatorname{supp}(q)$ which is algebraic over $\operatorname{supp}(p) \cup f$ is algebraic over $\operatorname{supp}(p) \cup f'$. It will then be enough to show that $\delta(p, r, f) \subseteq \delta(p, q, f') \cup \delta(q, r, f)$. To this end, suppose that $a \subset \operatorname{dom}(r)$ is a monochromatic finite set and there is a point $x \in X$ algebraic over $\operatorname{dom}(p) \cup f$ and a finite set $b \subset \operatorname{dom}(p)$ such that $a \cup b \cup \{x\} \in \Gamma$. There are two cases. Either, $a \subset \operatorname{dom}(q)$ holds. In such a case, $x \in \operatorname{dom}(q)$ holds by Proposition 6.3, consequently x is algebraic over $\operatorname{dom}(p) \cup f'$ and the color of a belongs to $\delta(p, q, f')$. Or, $a \setminus \operatorname{dom}(q) \neq 0$. In this case, the set $a \setminus \mathrm{dom}(q)$ witnesses that the color of a belongs to $\delta(q, r, f)$ as desired.

To see the σ-closure, argue that if $\langle p_m : m \in \omega\rangle$ is a descending sequence of conditions then $q = \bigcup_m p_m$ is their common lower bound. To see this, fix $m \in \omega$ and argue for (C) of $q \leq p_m$. Let $f \subset \mathrm{supp}(q)$ be a finite set and find a number $k \geq m$ such that $f \subset \mathrm{supp}(p_k)$; it will be enough to show that $\delta(p_m, q, f) \subseteq \delta(p_m, p_k, f)$. To see that, let $a \subset \mathrm{dom}(q) \setminus \mathrm{dom}(p)$ be a monochromatic finite set such that there is $x \in \mathrm{dom}(q)$ which is algebraic over $\mathrm{supp}(p_m) \cup f$ and $b \subset \mathrm{dom}(p_m)$ with $a \cup \{x\} \cup b \in \Gamma$. Then, since $x \in \mathrm{dom}(p_k)$, the set $a \setminus \mathrm{dom}(p_k)$ cannot be monochromatic if nonempty by (B). Therefore, $a \subset \mathrm{dom}(p_k)$ and the color of a must belong to $\delta(p_m, p_k, f)$ as desired. □

Other properties of the coloring poset depend on the following characterization of compatibility of conditions in P.

Claim 6.5. *Let $p_0, p_1 \in P$ be conditions. The following are equivalent:*

1. *p_0, p_1 have a common lower bound;*
2. *for every point $x_0 \in X$, p_0, p_1 have a common lower bound whose domain contains x;*
3. *$p_0 \cup p_1$ is a function and a Γ-coloring, the set $\gamma(p_0, p_1)$ and $\gamma(p_1, p_0)$ are both empty, and for finite sets $f_0 \subset \mathrm{supp}(p_0)$ and $f_1 \subset \mathrm{supp}(p_1)$, the sets $\delta(p_0, p_1, f_1)$ and $\delta(p_1, p_0, f_0)$ both belong to the ideal I.*

Proof. (2) implies (1) which in turn implies (3) by the definition of the ordering P. Thus, only the proof of (3) to (2) remains. Assume that (3) holds and let $x_0 \in X$ be an arbitrary point. Let $F \subset \mathbb{R}$ be a countable real closed subfield containing $\mathrm{supp}(p_0)$, $\mathrm{supp}(p_1)$, and all coordinates of the point x_0. I will find a lower bound of p_0, p_1 with support F. Let $d = F^n \setminus \mathrm{dom}(p_0 \cup p_1)$. For every point $x \in d$, let $\beta(x) = \{m \in \omega\colon$ there are finite sets $a_0 \subset \mathrm{dom}(p_0 \setminus p_1)$, $a_1 \subset \mathrm{dom}(p_1 \setminus p_0)$, and $b \subset \mathrm{dom}(p_0 \cap p_1)$ such that either a_0 or a_1 are monochromatic of color m and $b \cup a_0 \cup a_1 \cup \{x\} \in \Gamma\}$.

The key fact is that $\beta(x) \in I$ holds. To see this, pick finite sets $a_0 \subset \mathrm{dom}(p_0 \setminus p_1)$, $a_1 \subset \mathrm{dom}(p_1 \setminus p_0)$, and $b \subset \mathrm{dom}(p_0 \cap p_1)$ such that $b \cup a_0 \cup a_1 \cup \{x\} \in \Gamma$; if there happen to be none, then $\beta(x) = 0$. By (3) of the claim, it will be enough to show that $\beta(x) = \delta(p_0, p_1, f_1) \cup \delta(p_1, p_0, f_0)$ where $f_0 \subset \mathrm{supp}(p_0)$ and $f_1 \subset \mathrm{supp}(p_1)$ are finite sets which contain all coordinates of all points in a_0 or a_1 respectively. To this end, suppose that $a_0' \subset \mathrm{dom}(p_0 \setminus p_1)$, $a_1' \subset \mathrm{dom}(p_1 \setminus p_0)$, and $b' \subset \mathrm{dom}(p_0 \cap p_1)$ are such that a_0' is monochromatic of color some $m \in \omega$ and $b' \cup a_0' \cup a_1' \cup \{x\} \in \Gamma\}$. It will be enough to show that $m \in \delta(p_1, p_0, f_0)$. This, however, is immediate since the coordinates of the point x are algebraic in $f_0 \cup f_1$ by Proposition 6.3.

Now, use the initial choice of the ideal I to find a set $c \in I$ such that for every finite set $a \subset d$, $c \setminus \bigcup_{x \in a} \beta(x)$ is infinite. Let $g\colon d \to c$ be an injection such that $g(x) \notin \beta(x)$ holds for all $x \in d$. Let $q = p_0 \cup p_1 \cup g$ and argue that q is a common lower bound of the conditions p_0, p_1.

First of all, q is indeed a condition, i.e. a Γ-coloring. To see this, suppose that $e \subset \text{dom}(q)$ is a Γ-hyperedge. If $e \subset \text{dom}(p_0 \cup p_1)$ then e is not monochromatic by (3) of the proposition. If e contains more than one point in d then e is not monochromatic since $g \restriction d$ is an injection. Finally, if e contains exactly one point $x \in d$ then e is not monochromatic since $g(x) \notin \beta(x)$.

I must now show that $q \leq p_0$; the proof of $q \leq p_1$ is symmetric. To verify (B) of $q \leq p_0$, let $e \subset \text{dom}(q)$ be a hyperedge such that $e \setminus \text{dom}(p_0)$ is a nonempty set. If this set contains no points of d, then it is not monochromatic by (3) of the claim. If it contains more than one point of d, then it is not monochromatic as $g \restriction d$ is an injection. If it contains exactly one point x of d then it is not monochromatic either as $g(x) \notin \beta(x)$.

To verify (C) of $q \leq p_0$, suppose that $f \subset F$ is a finite set. A transcendence dimension argument produces a finite set $f' \subset \text{supp}(p_1)$ such that if a real number is algebraic over both $\text{supp}(p_0) \cup f$ and $\text{supp}(p_0) \cup \text{supp}(p_1)$, then it is algebraic over $\text{supp}(p_0) \cup f'$. It will be enough to show that $\delta(p_0, q, f) \subseteq c \cup \delta(p_0, p_1, f')$. To see this, let a be a q-monochromatic finite subset of $\text{dom}(q \setminus p_0)$ such that there is $x \in \text{dom}(q)$ which is algebraic over $\text{supp}(p_0) \cup f$ and $b \subset \text{dom}(p_0)$ with $a \cup \{x\} \cup b \in \Gamma$. If $a \cap d \neq 0$ then the color of a belongs to c and the proof is complete. Otherwise, $a \subset \text{dom}(p_1 \setminus p_0)$ must hold, in which case x is algebraic over $\text{supp}(p_0) \cup \text{supp}(p_1)$ by Proposition 6.3, by the choice of the set f' it is algebraic over $\text{supp}(p_0) \cup f'$, and the color of the set a belongs to $\delta(p_0, p_1, f')$ as desired. □

Corollary 6.6. *P is a Suslin forcing.*

Proof. It is clear that P is in a suitable presentation on a Polish space a Borel set and so is the ordering. Claim 6.5 shows that the compatibility relation is Borel as well. □

Corollary 6.7. *P forces the union of the generic filter to be a total Γ-coloring on X.*

Proof. The poset P is σ-closed and therefore adds no new points to X. Thus, by a density argument it is enough to show that for every point $x \in X$ the set of conditions containing x in its domain is open dense in P. This is an immediate consequence of Claim 6.5 applied to a pair of two identical conditions. □

Corollary 6.8. *(ZFC+CH) The poset P is Noetherian balanced.*

The Continuum Hypothesis assumption is necessary for some hypergraphs Γ, unnecessary for others.

Proof. Let $\{x_\alpha \colon \alpha \in \omega_1\}$ be an enumeration of all elements of X. Let $p \in P$ be an arbitrary condition. By transfinite recursion on $\alpha \in \omega_1$ build a descending sequence of conditions p_α so that $p_0 = p$ and $x_\alpha \in \text{dom}(p_{\alpha+1})$. This is immediately possible as the poset P is σ-closed. In the end, let $c = \bigcup_\alpha p_\alpha$; this is a total Γ-coloring of X. Let $Q = \text{Coll}(\omega, X)$; Q forces $\check{c}$ to be a condition in P

which is stronger than p. It will be enough to show that $\langle Q, \check{c} \rangle$ is a Noetherian balanced pair.

To do this, let $V[G_0], V[G_1]$ be mutually Noetherian generic extensions and $p_0, p_1 \in P$ be conditions in the respective models which are stronger than c; it must be proved that p_0, p_1 are compatible. It will be enough to verify Claim 6.5(3).

To verify that $\gamma(p_0, p_1) = 0$, suppose that $a_0 \subset \text{dom}(p_0)$ and $a_1 \subset \text{dom}(p_1 \setminus p_0)$ are sets such that $a_0 \cup a_1 \in \Gamma$ and argue that the set a_1, if monochromatic, is empty. Find an algebraic set A in the σ-algebraic presentation of Γ such that some enumeration of $a_0 \cup a_1$ belongs to A. By Proposition 3.6(3), there is a set $a_0' \subset X \cap V$ such that some enumeration of $a_0' \cup a_1$ belongs to A. Thus, $a_0' \cup a_1 \in \Gamma$, and since $p_1 \leq c$, the set a_1, if monochromatic, must be indeed empty.

The proof of $p_0 \cup p_1$ being a Γ-coloring is identical. Finally, let $f \subset \text{supp}(p_1)$ be a finite set; I must verify that $\delta(p_0, p_1, f) \in I$. Since $p_1 \leq c$ holds, $\delta(c, p_1, f) \in I$ holds by (C). Thus, it is enough to show that $\delta(p_0, p_1, f) \subseteq \delta(c, p_1, f)$. To this end, suppose that $a \subset \text{dom}(p_1 \setminus p_0)$ is a monochromatic set whose color belongs to $\delta(p_0, p_1, f)$, i.e. there is $b \subset \text{dom}(p_1)$ and point x algebraic over $\text{dom}(p_0) \cup f$ such that $a \cup b \cup \{x\} \in \Gamma$. Pick an algebraic set A from the σ-algebraic presentation of Γ such that some enumeration of $a \cup b \cup \{x\}$ belongs to A. By Proposition 3.6(3), there is a set $b' \subset (X \cap V)$ and a point $x' \in X$ algebraic over $(X \cap V) \cup f$ such that some enumeration of $a \cup b' \cup \{x'\}$ belongs to A. Thus, the color of A belongs to $\delta(c, p_1, f)$ as desired. □

The proof of the theorem is now complete. □

Now it is time to present the proofs of theorems from the introduction. Let κ be an inaccessible cardinal and let W be the symmetric Solovay model derived from it. For Theorem 1.1, consider the extension of the choiceless Solovay model by the poset from Theorem 6.1. Item (1) follows from Corollary 5.3, item (2) follows from Theorem 5.4, and item (3) follows from Theorem 5.6 applied to the Lebesgue null ideal. For Theorem 1.4, use the coloring poset from Theorem 6.2 and the preservation theorems as before.

7 Uncolorable hypergraphs

In this section I prove Theorem 1.5. The simple algebraic hypergraph $\Gamma_{\triangleright}$ on $\mathbb{R}^2$ of arity three consisting of those sets a whose projection to both coordinate axes has cardinality two has the following property. In ZF, if the chromatic number of $\Gamma_{\triangleright}$ is countable, then there is a countable-to-one map from $\mathbb{R}$ to ω_1. For the proof, argue in ZF. Let $c\colon \mathbb{R}^2 \to \omega$ be the $\Gamma_{\triangleright}$ coloring. For $x \in \mathbb{R}$ write M_x for the model of all sets hereditarily ordinally definable from x and c. Note that $c \restriction M_x \in M_x$ holds.

Case 1. There is a real x such that $\mathbb{R} \cap M_x$ is uncountable. In this case, we show that $\mathbb{R} \subset M_x$ and $M_x \models$CH, which will prove the theorem.

To show that $\mathbb{R} \subset M_x$ holds, suppose towards contradiction that it does not, and pick $z \in \mathbb{R} \setminus M_x$. By a counting argument, there are distinct points $y_0, y_1 \in \mathbb{R} \cap M_x$ such that $c(y_0, z) = c(y_1, z)$, the common value being some $n \in \omega$. Then z is not the unique point such that $c(y_1, z) = n$–otherwise it would be definable from c and y_1, and therefore from c and x, contradicting the choice of z. Let $u \in \mathbb{R}$ be a point different from y_1 such that $c(y_1, u) = n$. Then $\{\langle y_0, z\rangle, \langle y_1, z\rangle, \langle y_1, u\rangle\}$ is a monochromatic $\Gamma_{\triangleright}$-hyperedge of color n, a contradiction.

To show that $M_x \models$CH, suppose towards a contradiction that it fails. Work in M_x; observe that it is a model of AC. Let N_0 be an elementary submodel of some large structure containing $c \restriction M_x$ such that N_0 has cardinality $\aleph_1$; let $x_0 \in X \setminus N_0$. Let N_1 be an elementary submodel of a large structure containing $c \restriction M_x$, x_0, and N_0, such that N_1 is countable. Let $x_1 \in \mathbb{R} \cap N_1 \setminus N_0$. Let $n = c(x_0, x_1)$. By the elementarity of N_0, there must be $u \in N_0$ such that $c(u, x_1) = n$. By the elementarity of N_1, there must be $v \in N_0 \cap N_1$ such that $c(x_0, v)$. Note that $u \neq x_0$ and $v \neq x_1$ holds. Clearly, $\{\langle x_0, x_1\rangle, \langle u, x_1\rangle, \langle x_0, v\rangle\}$ is a monochromatic $\Gamma_{\triangleright}$-hyperedge of color n, a contradiction.

Case 2. Case 1 fails. Let $\pi \colon \mathbb{R} \to \omega_1$ be the map defined by $\pi(x) = \omega_1^{M_x}$. The case assumption shows that the range of this map is indeed a subset of ω_1. We will show that π is in fact countable-to-one. Suppose towards contradiction that it is not, and let $\alpha \in \omega_1$ be an ordinal such that the set $\{x \in \mathbb{R} \colon \pi(x) = \alpha\}$ is uncountable. By the case assumption, there have to be points x_0, x_1 in this set such that $x_1 \notin M_{x_0}$. We will reach a contradiction by a split into cases.

Suppose first that $x_0 \notin M_{x_1}$. Let L_0 be the line in $\mathbb{R}^2$ consisting of points whose 0-th coordinate is equal to x_0 and let L_1 be the line in $\mathbb{R}^2$ consisting of points whose 1-st coordinate is equal to x_1. Let $n = c(x_0, x_1)$. Then $\langle x_0, x_1\rangle$ is not the only point on L_0 which gets color n–otherwise x_1 would be definable from x_0. Let $\langle x_0, x_2\rangle \in L_0$ be a different point which gets color n. By the same argument, $\langle x_0, x_1\rangle$ is not the only point on L_1 which gets color n–otherwise x_0 would be definable from x_1. Let $\langle x_3, x_1\rangle \in L_1$ be a different point which gets color n. Then $\{\langle x_0, x_1\rangle, \langle x_0, x_2\rangle, \langle x_3, x_1\rangle\}$ is a monochromatic $\Gamma_{\triangleright}$-hyperedge of color n. A contradiction.

Assume now that $x_0 \in M_{x_1}$. The set $\mathbb{R} \cap M_{x_0}$ then belongs to M_{x_1} and must be uncountable there because the two models have the same ω_1. By a counting argument in M_{x_1}, there must be distinct points $y_0, y_1 \in \mathbb{R} \cap M_{x_0}$ such that $\langle y_0, x_1\rangle$ and $\langle y_1, x_1\rangle$ get the same c-color, say n. Now, x_1 cannot be the only point such that $\langle y_1, x_1\rangle$ gets the color n–otherwise x_1 would be definable from y_1 and then also from x_0. So, pick a point $z \in \mathbb{R}$ such that $c(y_1, z) = n$ and note that the set $\{\langle y_0, x_1\rangle, \langle y_1, x_1\rangle, \langle y_1, z\rangle\}$ is a c-monochromatic $\Gamma_{\triangleright}$-hyperedge of color n. This is a final contradiction proving Theorem 1.5.

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