Classifying open subgroups of non-archimedean Polish groups*

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Abstract

We show that for many structures X, the open subgroups of the automorphism group $\mathrm{Aut}(X)$ can be classified in great detail. This has implications to the theory of the associated permutation models of ZF.

1 Introduction

Given a topological group, its open subgroups form a quite special company, which normally has only a few, select members. The Polish groups for which this collection is potentially larger than for others are the non-archimedean ones, equivalently closed subgroups of S_∞, or the automorphisms of structures with countable universe and countable language. In this paper, I will show that for many countable structures X, the open subgroups of the group $\mathrm{Aut}(X)$ of all automorphisms of X with its usual Polish topology admit a very detailed classification. This leads to an interesting and novel collection of Fraissé classes whose limit has this desirable property, and an application to the theory of permutation models of ZF.

The initial definitions are standard:

Definition 1.1. Let X be a countable structure and $\mathrm{Aut}(X)$ its group of automorphisms.

1. If $x \in X$, I write $\mathrm{stab}(x) = \{\gamma \in \mathrm{Aut}(X) \colon \gamma \cdot x = x\}$. If $A \subset X$ is a set, write $\mathrm{pstab}(A) = \bigcap_{x\in A} \mathrm{stab}(x)$;
2. if $A \subset X$ is a set and $x \in X$ is an element, say that x is *algebraic over A* if the set $\{g(x) \colon g \in \mathrm{pstab}(A)\}$ is finite;
3. the *algebraic closure* of a set $A \subset X$ is the set of all elements of x algebraic over A;

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4. a set is *algebraically closed* if it is equal to its algebraic closure;
5. the structure X is locally finite if algebraic closure of any finite set is finite.

It is well known that the collection of algebraic sets is closed under arbitrary intersections, and the algebraic closure of any set is algebraically closed. Now, for the central definition of this paper:

Definition 1.2. Let X be a countable structure and $\mathrm{Aut}(X)$ its group of automorphisms with its usual Polish group topology. We say that X *admits classification of open subgroups* if for any two finite algebraically closed sets $a, b \subset X$, the union $\mathrm{pstab}(a) \cup \mathrm{pstab}(b)$ generates the group $\mathrm{pstab}(a \cap b)$.

The terminology is justified by the following nearly trivial proposition.

Proposition 1.3. *Let X be a countable, locally finite structure and $\mathrm{Aut}(X)$ its group of automorphisms. The following are equivalent:*

1. *X allows classification of open subgroups;*
2. *for every open subgroup $\Delta \subset \mathrm{Aut}(X)$ there is a finite set $c \subset X$ and a subgroup G of the symmetric group on c such that $\Delta = \{g \in \mathrm{Aut}(X) \colon g \restriction c \in G\}$.*

Proof. Suppose first that (1) fails, as witnessed by finite algebraically closed sets $a, b \subset X$. Let Δ be the subgroup of $\mathrm{Aut}(X)$ generated by $\mathrm{pstab}(a) \cup \mathrm{pstab}(b)$; since the set of generators is open, this is an open subgroup. I will prove that Δ witnesses the failure of (2). Let $c \subset X$ be a finite set with a subgroup G of the symmetric group on c, put $\Gamma = \{g \in \mathrm{Aut}(X) \colon g \restriction c \in G\}$, and work to show that $\Delta = \Gamma$ fails.

If $c \subseteq a$ fails, then pick an element $x \in c \setminus a$ and use the algebraic closure of a to find an element $g \in \mathrm{pstab}(a)$ such that $g(x) \notin c$; then $g \in \Delta \setminus \Gamma$. If $c \subseteq b$ fails, then there is an element in $\Delta \setminus \Gamma$ for a symmetric reason. Finally, if $c \subseteq a \cap b$ holds, then use the failure of (1) to find an element $g \in \mathrm{pstab}(a \cap b)$ which is not in Δ, and observe that $g \in \mathrm{pstab}(c)$ so $g \in \Gamma \setminus \Delta$ holds. These three cases cover all possible configurations of a, b and c, and failure of (2) follows.

Suppose now that (1) holds, and let $\Delta \subset \mathrm{Aut}(X)$ be an open subgroup. The set $\{a \subset X \colon a \text{ is finite, algebraically closed, and } \mathrm{pstab}(a) \subseteq \Delta\}$ is nonempty, it is closed under intersections by (1), and therefore contains an inclusion-smallest element c. Now, let G be the set of all permutations π of c such that there is $g \in \Delta$ such that $g \restriction c = \pi$, set $\Gamma = \{g \in \mathrm{Aut}(X) \colon g \restriction c \in G\}$ and argue that $\Delta = \Gamma$.

For the right-to-left inclusion, if $g \in \Delta$ then a diagram-chasing argument shows that $g''c$ is an algebraically closed set and $\mathrm{pstab}(g''c) \subseteq \Delta$; the minimal choice of the set c then implies that $c \subseteq g''c$, so $c = g''c$. Therefore, g permutes the set c and $g \in \Gamma$ holds by the definition of Γ. For the left-to-right-inclusion, suppose that $g \in \Gamma$ holds. Use the definition of Γ to find a group element $h \in \Delta$ such that $g \restriction c = h \restriction c$. Then $h^{-1} \circ g \in \mathrm{pstab}(c) \subseteq \Delta$, in consequence $g = h \circ (h^{-1} \circ g)$ is in Δ and the proof is complete. $\square$

In Section 2, I will show that for a natural collection of relational Fraissé classes F (the *bootstrapping* ones, Definition 2.4), their limit structure admits classification of open subgroups (Theorem 2.15). There are also examples of structures which are very far from admitting any type of open subgroup classification (the Urysohn rational metric space of Example 2.2) and very many examples of structures for which I do not know the status of the classification (Question 2.18). In Section 3, I deal with the impact of open subgroup classification on the theory of the derived permutation model of ZF. It turns out that the classification has been used in a non-systematic way to analyze such models for a long time. Theorem 3.1 includes some of its abstract consequences, which are quite difficult to obtain otherwise.

The notation of the paper is standard and follows the textbook [1].

2 Examples of structures

It is not difficult to find simple structures which do not admit classification of open subgroups.

Example 2.1. Let X be a structure with a single equivalence relation E with at least two infinite equivalence classes. Let $A \subset X$ be one of these classes, and consider the group $\Delta \subset \mathrm{Aut}(X)$ of all automorphisms which permute the equivalence class A. It is not difficult to check that this is an open subgroup of $\mathrm{Aut}(X)$. If $x \in A$ is any element then $\mathrm{pstab}(\{x\}) \subset \Delta$, but $\mathrm{pstab}(0) = \mathrm{Aut}(X) \neq \Delta$ holds, so classification of open subgroups fails.

In the above example, the number of open subgroups is countable. This very weak consequence of classification is violated in the following example.

Example 2.2. Let X be the countable rational Urysohn ultrametric space. Then $\mathrm{Aut}(X)$ has uncountably many distinct open subgroups.

Proof. Let d be the ultrametric on X. Let Y be the rational ultrametric space defined on the positive rationals by setting $d(x, y)$ =the larger of the numbers x, y whenever x and y are distinct. Using the universality of X, Y is isomorphic to a metric subspace of X; to simplify the notation, I assume that $Y \subset X$ holds. For a positive real number $r > 0$, consider the group $\Delta_r \subset \mathrm{Aut}(X)$ generated by $\mathrm{pstab}(\{y\})$ for $y < r$. Since this group is generated by open groups, it is open itself, and it will be enough to show that distinct positive real numbers give distinct open subgroups.

To this end, let $r < s$ be distinct positive reals, and let $y \in Y$ be a rational number between them. Let $x \in X$ be a point such that it has distance y from y and also from some element $z \in Y$ which is smaller than r. Such a point has to exist by the universality properties of the space X. Let $g \in \mathrm{Aut}(X)$ be an isometry such that $g(y) = y$ and $g(z) = x$ and $g(x) = z$; such an isometry has to exist by the universality properties of X. It will be enough to show that $g \in \Delta_s \setminus \Delta_r$.

It is clear that $g \in \Delta_s$ holds, since g fixes the point y. To show that $g \notin \Delta_r$ holds, consider the set $A = \{u \in X\colon$ for some rational $v < r$, $d(u,v) > r\}$, which by the ultrametric condition is equal to $\{u \in X\colon$ for all rational $v < r$, $d(u,v) > r\}$. The generators of the group Δ_r all permute the set A, since they are isometries; as a result, all elements of Δ_r permute the set A. However, g does not permute the set A as $x \in A$ and $g(x) = z \notin A$. $\square$

It seems to be much harder to identify examples of structures which admit classification of open subgroups. To this end, I isolate a novel collection of Fraissé classes.

Definition 2.3. Let F be a relational Fraissé class.

1. Structures $A, B \in F$ form an *amalgamation pair* if $A \restriction (\mathrm{dom}(A) \cap \mathrm{dom}(B)) = B \restriction (\mathrm{dom}(A) \cap \mathrm{dom}(B)$;
2. for an amalgamation pair $A, B \in F$, a *minimal amalgamation* of A and B is a structure C on $\mathrm{dom}(A) \cup \mathrm{dom}(B)$ such that $C \restriction \mathrm{dom}(A) = A$ and $C \restriction \mathrm{dom}(B) = B$.

Definition 2.4. Let F be a relational Fraissé class. Let $A, B \in F$ be an amalgamation pair and C, D be minimal amalgamations of A and B.

1. A *walk* from C to D is a finite sequence $\langle C_i \colon i \leq n\rangle$ of minimal amalgamations of A and B such that $C_0 = C$, $C_n = D$, and for every $i \in n$ there is a point $x_i \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$ such that $C_i \restriction \mathrm{dom}(A) \cup \mathrm{dom}(B) \setminus \{x_i\} = C_{i+1} \restriction \mathrm{dom}(A) \cup \mathrm{dom}(B) \setminus \{x_i\}$;
2. the point $x_i \in B \setminus A$ as above is a *focus of difference* between C_i and C_{i+1};
3. F is a *bootstrapping class* if it has disjoint amalgamation and for any amalgamation pair $A, B \in F$ and any two minimal amalgamations C, D of A and B, there is a walk from C to D.

All proofs showing that a certain relational Fraissé class is a bootstrapping class are quite natural: among all minimal amalgamations of A and B, they identify an optimal one and then show how an arbitrary minimal amalgamation can be transformed to the optimal one one point at a time. An important issue to keep in mind is that the points all have to be on the same side of the amalgamation diagram.

To kick off the list of examples, recall that a relational Fraissé class F is *hereditary* if it is closed under subsets in the sense that if A, B are structures for its relational language with the same domain, $A \in F$, and all relations in B are subsets of the corresponding relations in A, then $B \in F$ holds.

Proposition 2.5. *If F is a relational, hereditary Fraissé class with finite language and disjoint amalgamation, then F is a bootstrapping class.*

Proof. Let A, B be an amalgamation pair of structures in F. Let D be the minimal disjoint amalgamation of A and B obtained by taking the union of the corresponding relations in A and B. This is indeed a structure in F since its relations are subsets of any minimal disjoint amalgamation of A and B. Now, if C is any minimal disjoint amalgamation of A and B, it is possible to produce a walk from C to D by setting $C_0 = C$ and C_{i+1} to be the structure obtained from C_i by erasing (if it exists) one tuple $\bar{u}_i$ which appears in some relation of the structure C_i but not in D. Note that it must be the case that $\bar{u}_i$ contains some points from both $\mathrm{dom}(A) \setminus \mathrm{dom}(B)$ and $\mathrm{dom}(B) \setminus \mathrm{dom}(A)$–otherwise, the tuple would be in the optimal amalgamation D. Picking any point $x_i \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$ in the tuple $\bar{u}_i$, it is clear that x_i is a focus of difference between C_i and C_{i+1}. It is easy to prove by induction on i that all the structures C_i are in fact in the Fraissé class F as F is hereditary. By the finiteness of the language of F, for some number $n \in \omega$ it will be the case that $C_n = D$. The proof is complete. $\square$

Example 2.6. The class of all graphs is bootstrapping.

Example 2.7. For any number $k > 2$, the class of all graphs not containing a clique of cardinality k is bootstrapping.

The non-hereditary bootstrapping classes are much more interesting.

Example 2.8. The class of linear orderings is bootstrapping.

Proof. Let A and B be an amalgamation pair of linear orderings. There is an optimal amalgamation D of A and B, in which if $a \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$ and $b \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$ are consecutive points, then $b \leq a$ holds. There is only one amalgamation satisfying this property: inside every interval of $A \restriction \mathrm{dom}(A) \cap \mathrm{dom}(B)$, first the appropriate elements of $\mathrm{dom}(B)$ are all smaller than the appropriate elements of $\mathrm{dom}(A)$. Now, given an arbitrary minimal disjoint amalgamation C of A and B, it is possible to produce a walk from C to D by setting $C_0 = C$ and C_{i+1} to be the structure obtained from C_i by finding (if it exists) a pair $\langle a_i, b_i \rangle$ such that $a_i \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$, $b_i \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$, and $a_i \leq b_i$ are consecutive in C_i. The linear ordering obtained by switching a_i and b_i will then be the amalgamation C_{i+1}. Note that b_i is the focus of difference between amalgamations C_i and C_{i+1}. The walk has to end after finitely many steps: for every element $a \in A$, the sets $B(a, i) = \{b \in B \colon b \leq a \text{ in } C_i\}$ are nondecreasing in i, and the only way how $B(a, i) = B(a, i + 1)$ can occur for all $a \in \mathrm{dom}(A)$ simultaneously is that the pair $\langle a_i, b_i \rangle$ has not been found. The final element of the walk must be the amalgamation D. $\square$

Example 2.9. The class of rational metric spaces is bootstrapping.

Proof. Let A and B be an amalgamation pair of rational metric spaces, with metrics d_A and d_B. There is an optimal minimal amalgamation D of A and B, in which if $a \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$ and $b \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$ are points then the distance of a and b is the minimum of the set $\{d_A(a, c) + d_B(c, b) \colon c \in$

$\mathrm{dom}(A) \cap \mathrm{dom}(B)\}$. In view of the triangle inequality, this is the largest metric available. Now, given an arbitrary minimal amalgamation C of A and B, it is possible to find a walk from C to D in the following way.

Find a positive rational number q such that every distance used in the amalgamation C is an integer multiple of q. Note that then also every distance used in D is an integer multiple of q. To get a walk from C to D, start with $C_0 = C$, and construct C_{i+1} from C_i by finding (if it exists) a pair $\langle a_i, b_i \rangle$ such that $a_i \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$, $b_i \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$, and adding q to the distance between a_i and b_i in C_i still results in a metric space; this new metric space will be called C_{i+1}. Note that b_i is the focus of difference between C_i and C_{i+1}. By induction on i it can easily be proved that C_i is a minimal disjoint amalgamation of A and B, and every distance in C_i is an integer multiple of q. The walk has to end after finitely many steps: the sum of all distances used in C_i is an integer multiple of q, it cannot overtake the sum of all distances used in D, and the only way how it can stop increasing is that the pair $\langle a_i, b_i \rangle$ has not been found.

The final element C_n of the walk must be the amalgamation D. To see this, towards a contradiction assume otherwise, and find a pair $\langle a, b \rangle$ such that $a \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$, $b \in \mathrm{dom}(B) \setminus \mathrm{dom}(A)$ such that $d_{C_n}(a, b) < d_D(a, b)$, and $d_{C_n}(a, b)$ is the smallest possible. Since adding q to the distance between a and b will not result in a metric space, there has to be a triangle in which the triangle inequality will not hold after this operation. Call the third vertex c, and without loss assume that $c \in \mathrm{dom}(A)$. Since $d_{C_n}(a, b) + q > d_{C_n}(a, c) + d_{C_n}(c, b)$ and all numbers in question are integer multiples of q, it must be the case that $d_{C_n}(a, b) = d_{C_n}(a, c) + d_{C_n}(c, b)$. Now, if $c \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$, this contradicts the assumption that $d_{C_n}(a, b) < d_D(a, b)$. If, on the other hand, $c \in \mathrm{dom}(A) \setminus \mathrm{dom}(B)$ holds, then by the minimal choice of the points a and b it has to be the case that $d_{C_n}(c, b) = d_D(c, b)$. Pick an element $e \in \mathrm{dom}(A) \cap \mathrm{dom}(B)$ such that $d_{C_n}(c, b) = d_A(c, e) + d_B(e, b)$, By the triangle inequality in d_A, $d_A(a, e) \leq d_A(a, c) + d_A(c, e)$. The strict inequality is impossible here, since it would violate the triangle inequality in C_n in the triangle with vertices a, b, e. However, the equality leads to the conclusion that $d_{C_n}(a, b) = d_A(a, e) + d_B(e, b)$, violating the choice of the pair $\langle a, b \rangle$. □

Example 2.10. The class of selectors is bootstrapping.

This is the class of all finite structures A equipped with a function f_A which, to each nonempty subset $\mathrm{dom}(A)$, assigns one of its elements. It is not difficult to restate this class as one with an infinite relational language.

Proof. The proof is slightly different from the previous ones in that one does not need to find an optimal amalgamation to streamline the argument. Let A and B be an amalgamation pair and C and D two of its minimal disjoint amalgamations. To get a walk from C to D, start with $C_0 = C$, and construct C_{i+1} from C_i by finding (if it exists) a finite set $u_i \subset \mathrm{dom}(A) \cup \mathrm{dom}(B)$ such that $f_{C_i}(u_i) \neq f_D(u_i)$, and switching this only value to get $f_{C_{i+1}}(u_i) = f_D(u_i)$. Note that the set u_i must contain some elements of both $\mathrm{dom}(A) \setminus \mathrm{dom}(B)$ and

$\text{dom}(B) \setminus \text{dom}(A)$, and any element $x_i \in u_i \cap \text{dom}(B) \setminus \text{dom}(A)$ is a focus of difference between C_i and C_{i+1}.

Since the number of disagreements between C_i and D is an integer and it keeps decreasing, the walk has to end after finitely many steps, reaching $C_n = D$. The proof is complete. $\square$

An essentially identical argument provides the following:

Example 2.11. The class of tournaments is bootstrapping.

Finally, there is a simple but powerful proposition regarding superposition of bootstrapping classes. If F and G are two Fraissé classes in disjoint languages L_F and L_G respectively, then $F + G$, the *superposition* of F and G, is the class in the language $L_F \cup L_G$ consisting of all structures A such that $A \restriction L_F \in F$ and $A \restriction L_G \in G$.

Proposition 2.12. *The superposition of two bootstrapping Fraissé classes is bootstrapping again.*

Proof. Suppose that F, G are bootstrapping classes, and A and B is an amalgamation pair of structures in the class $F + G$, and C and D are two minimal disjoint amalgamations of A and B. First, consider the F-structures $C \restriction L_F$ and $D \restriction L_F$. These are minimal disjoint amalgamations of $A \restriction L_F$ and $B \restriction L_F$. By the bootstrapping property of the class F, there is a walk $\langle C_i^F : i \leq n\rangle$ from $C \restriction L_F$ to $D \restriction L_F$. Similarly, there is a walk $\langle C_j^G : j \leq m\rangle$ from $C \restriction L_G$ to $D \restriction L_G$. Now, a walk from C to D will consist first of superpositions of the structures C_i^F and $C \restriction L_G$ for $i \leq n$ and then from superpositions of the structures $D \restriction L_F$ and C_j^G for $j \leq m$. $\square$

Example 2.13. The class of all linearly ordered graphs is bootstrapping, as it is a superposition of graphs and linear orderings.

Example 2.14. The class of all linearly ordered rational metric spaces is bootstrapping, as it is a superposition of rational metric spaces and linear orderings.

Theorem 2.15. *Let F be a relational bootstrapping Fraissé class with limit X. Then X admits classification of open subgroups.*

Proof. Note that as F has the disjoint amalgamation property, every finite subset of X is algebraically closed.

Now, let $a, b \subset X$ are finite sets. It will be enough to show that $\text{pstab}(a) \cup \text{pstab}(b)$ generate a group $\Gamma \subset \text{Aut}(X)$ which is dense in the set $\text{pstab}(a \cap b)$. Then, since $\Gamma \subset \text{Aut}(X)$ is open, it is also closed (???), it will follow that $\Gamma = \text{pstab}(a \cap b)$ holds, proving the theorem.

Thus, suppose that $c, d \subset X$ are finite sets containing $a \cap b$ and $h\colon c \to d$ is an isomorphism of $X \restriction c$ and $X \restriction d$ which is equal to the identity on $a \cap b$. We must find an element $g \in \Gamma$ such that $h \subset g$.

Case 1. $c \cap d = a \cap b$ and $(c \cup d)$ is disjoint from $a \Delta b$. In this case, let $A = X \restriction c$ and $B = X \restriction a \cup b$. This is an amalgamation pair, and there are two minimal amalgamations: $C = X \restriction a \cup b \cup c$ and D which is the preimage of $X \restriction a \cup b \cup d$ under the function which is equal to the identity on $a \cup b$ and to h^{-1} on d. The bootstrapping assumption yields a walk $\langle C_i \colon i \leq n \rangle$ from C to D, with foci of difference $\langle x_i \colon i \in n \rangle$; these are all points in $a \Delta b$. By the universality properties of the structure X, there are finite sets $c_i \subset \mathrm{dom}(X)$ disjoint from $a \cup b$ and functions h_i for $i \leq n$ so that

- $h_i \colon a \cup b \cup c \to a \cup b \cup c_i$ is a bijection and an isomorphism between C_i and $X \restriction a \cup b \cup c_i$ which is the identity on $a \cup b$;
- $c_0 = c$ and h_0 is the identity;
- $c_n = d$ and h_n is the function which is equal to the identity on $a \cup b$ and to h on c.

Now, if $i \in n$ and x_i is the focus of difference between C_i and C_{i+1} then $h_{i+1} \circ h_i^{-1}$ is an isomorphism between $X \restriction a \cup b \cup c_i \setminus \{x_i\}$ and $X \restriction a \cup b \cup c_{i+1} \setminus \{x_i\}$. The point x_i must belong to either a or to b, but not to both. In the former case, the universality properties of X imply that there is an automorphism $\gamma_i \in \mathrm{Aut}(X)$ which extends $h_{i+1} \circ h_i^{-1} \restriction b \cup c_i$. In the latter case, the universality properties of X imply that there is an automorphism $\gamma_i \in \mathrm{Aut}(X)$ which extends $h_{i+1} \circ h_i^{-1} \restriction a \cup c_i$. In either case, $\gamma_i \in \Gamma$. A review of the definiions shows that the composition g of γ_i for $i \in n$ in decreasing order extends $h \colon c \to d$.
Case 2. The general case. Here, the following claim is useful in reducing Case 2 to Case 1:

Claim 2.16. *Let $e \subset X$ be a finite set. Then there is an element $g \in \Gamma$ such that $g''e$ is disjoint from $a \Delta b$.*

Proof. First, find an element $g_a \in \mathrm{pstab}(a)$ such that $g_a''(e \setminus a)$ is disjoint from b. Then, find an element $g_b \in \mathrm{pstab}(b)$ such that $g_b''((e \setminus b) \cup g_a''(e \setminus a)$ is disjoint from $a \cup b$. Then $g = g_b \circ g_a$ works. □

Now, use the claim to find $g \in \Gamma$ such that $g''(c \cup d)$ is disjoint from $a \Delta b$. Let $c' = g''c$, $d' = g''d$, and $h' \colon c' \to d'$ be the composition $g \circ h \circ g^{-1}$. Apply Case 1 to find an element $g' \in \Gamma$ such that $h' \subset g'$. Then, $h \subset g^{-1} \circ h' \circ g$ and the latter expression is an element of Γ.

□

Finally, I want to produce an example of a structure which allows classification of open subgroups even though it is not a limit of a relational bootstrapping Fraissé class.

Example 2.17. Let X be an infinitely branching acyclic graph with a graph relation G, and a unary predicate A for one of the two classes of the equivalence relation on X connecting two vertices of the tree if they have even distance in the graph G. Then X allows classification of open subgroups.

Proof. It is clear that algebraically closed finite sets are exactly those finite sets which are connected in X. Now, suppose that $a, b \subset X$ are connected finite sets; we need to show that the clopen group $\Gamma \subset \mathrm{Aut}(X)$ generated by $\mathrm{pstab}(a) \cup \mathrm{pstab}(b)$ is dense in $\mathrm{pstab}(a \cap b)$. This is proved via a discussion of several cases.

Case 1. $a \cap b \neq 0$. ???

Case 2. $a = \{x_0\}$ and $b = \{x_1\}$ for distinct points $x_0, x_1 \in X$. In this case, we need to show that $\Gamma = \mathrm{Aut}(X)$. This was proved by Anton Bernshteyn in private communication; his argument is the following. Write c for the path between x_0 and x_1, including both of these vertices.

First, let x be any vertex at even distance from x_0. We will show that some element of Γ moves x to x_0. The proof proceeds by induction on the distance of x from x_0. If the distance is smaller or equal to the length of b, first find an automorphism π_0 in $\mathrm{stab}(x_0)$ moving x to some point $y \in c$. Then, let d be the path from x_0 to y and find an automorphism $\pi_1 \in \mathrm{stab}(x_0)$ which fixes exactly the first half of the path d. This is possible as the length of d is even. Then, the distance of x_1 from x_0 and from $\pi_1(y)$ is the same, so there is an automorphism $\pi_2 \in \mathrm{stab}(x_1)$ such that $\pi_2\pi_1(y) = x_0$. As a result, the composition $\pi_2 \circ \pi_1 \circ \pi_0 \in \Delta$ moves x to x_0.

Now, if the distance of x to x_0 is larger than the length of c, first find an automorphism $\pi_0 \in \mathrm{stab}(x_0)$ so that c is the part of the path from x_0 to $\pi_0(x)$. Then, find an automorphism $\pi_1 \in \mathrm{stab}(x_1)$ so that either (if the distance between x_1 and $\pi_0(x)$ is at most equal to the length of b) $\pi_1\pi_0(x)$ is on the path b, or (if the distance is longer) b is a subset of the path from x_1 to $\pi_1\pi_0(x)$. This way, $\pi_1\pi_0(x)$ is closer to x_0 than x. By the induction hypothesis, there is $\delta \in \Gamma$ such that $\delta \circ \pi_1 \circ \pi_0 \in \Gamma$ moves x to x_0, as desired.

Now, suppose that $\gamma \in \mathrm{Aut}(X)$ be any automorphism. Let x be any vertex at even distance from x_0; then $\gamma(x)$ is also at even distance from x_0. Let $\delta_0 \in \Gamma$ be any automorphism such that $\delta_0(x) = x_0$; let $\delta_1 \in \Gamma$ be any automorphism such that $\delta_1(\gamma(x)) = x_0$. These automorphisms exist by the previous two paragraphs. Now $\delta_1 \circ \gamma \circ \delta_0^{-1}$ belongs to $\mathrm{stab}(x_0)$ and therefore to Γ. It follows that $\gamma = \delta_1^{-1} \circ (\delta_1 \circ \gamma \circ \delta_0^{-1}) \circ \delta_0$ belongs to Γ, completing the proof of Case 2.

Case 3. $a \cap b = 0$, the general case. Let $x_0 \in a, x_1 \in b$ be the closest points in a, b and let c be the path connecting and including them. Then $\mathrm{stab}(x_0)$ is the group generated by $\mathrm{pstab}(a) \cup \mathrm{pstab}(b \cup c)$ by Case 1; in particular, $\mathrm{stab}(x_0)$ is a subset of the group generated by $\mathrm{pstab}(a) \cup \mathrm{pstab}(b)$. For the same reason, $\mathrm{stab}(x_1)$ is a subset of the group generated by $\mathrm{pstab}(a) \cup \mathrm{pstab}(b)$. Apply Case 2 to see that $\mathrm{stab}(x_0) \cup \mathrm{stab}(x_1)$ generates the whole group $\mathrm{Aut}(X)$; the proof is complete. □

There are many locally finite structures for which the classification of open subgroups is an open questions. Most importantly, the techniques of this paper do not address the case of structures with functions on them.

Question 2.18. Do the following structures admit classification of open subgroups?

1. countable atomless Boolean algebra;
2. the countably infinite dimensional vector space over a given finite field;
3. same as above but with additional structure such as a generic linear ordering.

3 Impact on permutation models of ZFA

Classification of open subgroups has immediate and thorough impact on the theory of permutation models derived from the structures in question. I will discuss a rather special case of permutation models, leaving the general exposition to a different paper.

Let X be a countable structure and Γ its group of automorphisms, with the usual Polish topology. Build a model $V[[X]]$ of ZFC with atoms by using elements of $\mathrm{dom}(X)$ as atoms (declaring for example that each of them is an element of itself and has no other elements) and then building a cumulative hierarchy over $\mathrm{dom}(X)$ in the usual way. Note that the application action of Γ on $\mathrm{dom}(X)$ extends to an action on $V[[X]]$ by $\in$-automorphisms. Now, let the *permutation model* $W[[X]]$ be the transitive part of the class $\{A \in V[[X]]\colon$ there is a finite set $a \subset \mathrm{dom}(X)$ such that $\mathrm{pstab}(a) \subset \mathrm{stab}(A)\}$. It is a standard fact ??? that X belongs to $W[[X]]$, $W[[X]]$ is a model of ZF with atoms, and $W[[X]]$ is a Γ-invariant subclass of $V[[X]]$.

Theorem 3.1. *Suppose that the structure X is locally finite and admits classification of open subgroups. Then the permutation model $W[[X]]$ derived from X satisfies the following:*

1. *the class of well-orderable sets is closed under increasing unions;*
2. *if X is linearly orderable then so is every set;*
3. *if the set of nonempty finite subsets of X has a selector then so does every set of nonempty finite sets.*

For the proof of the theorem, fix a countable locally finite structure X admitting classification of open subgroups. The following definition is central.

Definition 3.2. For every set $A \in W[[X]]$ define $\mathrm{supp}(A)$ to be the smallest algebraically closed finite subset $a \subset X$ such that $\mathrm{pstab}(a) \subseteq \mathrm{stab}(A)$.

The assumption that X is locally finite and admits classification of open subgroups implies that the definition is sound: the set $\{a \subset X\colon$ a is finite, algebraically closed, and such that $\mathrm{pstab}(a) \subseteq \mathrm{stab}(A)\}$ is nonempty (as $A \in W[[X]]$) and closed under intersections (as the classification of open subgroups shows), and therefore contains an inclusion-smallest element. A diagram chasing argument shows that the map $A \mapsto \mathrm{supp}(A)$ is invariant under the group action. Therefore, whenever $B \in W[[X]]$ is a set, the function with domain B, assigning to each $A \in B$ its support, is an element of $W[[X]]$.

Claim 3.3. *A set $B \in W[[X]]$ is well-orderable in $W[[X]]$ if and only if the set $\bigcup_{A\in B} \operatorname{supp}(A)$ is finite.*

Proof. For the left-to-right direction, if $\leq \in W[[X]]$ is a well-ordering on B and $a = \operatorname{supp}(\{B, \leq\})$, then by transfinite induction along $\leq$ one can show that every element of $\operatorname{pstab}(a)$ fixes every element of B. Thus, a must be a superset of all sets $\operatorname{supp}(A)$ for $A \in B$ and $\bigcup_{A\in B} \operatorname{supp}(A)$ is a subset of a and therefore finite. For the right-to-left direction, if $\bigcup_{A\in B} \operatorname{supp}(A)$ is finite, just choose any well-ordering $\leq$ on B in the model $V[[X]]$ whatsoever, let $b \subset X$ be any finite algebraically closed superset of $\bigcup_{A\in B} \operatorname{supp}(A)$, observe that every element of $\operatorname{pstab}(b)$ fixes $\leq$, and conclude that $\leq \in W[[X]]$ holds as required. □

For (1) of the theorem, work in $W[[X]]$. Let C be a set consisting of well-orderable sets which are in addition linearly ordered by inclusion; we must show that $\bigcup C$ is well-orderable. By the claim, this means to show that $\bigcup_{A\in\bigcup C} \operatorname{supp}(A)$ is finite. Towards a contradiction, assume that this set is infinite. Consider the set $D = \{\bigcup_{A\in B} \operatorname{supp}(A) \colon B \in C\}$. By the claim and the initial assumptions on C, this is a set consisting of finite sets, and it is linearly ordered by inclusion. By the contradictory assumption, the union of D is infinite. This means that D can be listed in a strictly increasing order as $L = \langle a_n \colon n \in \omega\rangle$ and this list is indeed infinite. Let $b = \operatorname{supp}(L)$. Let $n \in \omega$ be so large that $a_n \not\subseteq b$, and find an element $\gamma \in \operatorname{pstab}(b)$ and an element $x \in a_n$ such that $\gamma(x) \notin a_n$. It follows that $\gamma \cdot a_n \neq a_n$, therefore γ does not fix the list L. This is in contradiction with the choice of the set b.

For (2) and (3) of the theorem, another claim will be useful.

Claim 3.4. *In $W[[X]]$, assume that there is a selector function on the set of all finite nonempty subsets of $\operatorname{dom}(X)$. Then every set can be injected into $[\operatorname{dom}(X)]^{<\aleph_0} \times \alpha$ for some ordinal α.*

Proof. In $W[[X]]$, find a selector f on the set of all finite nonempty subsets of $\operatorname{dom}(X)$. Let B be an arbitrary set; I need to produce an injection from B into $[\operatorname{dom}(X)]^{<\aleph_0} \times \alpha$ for some ordinal α. To this end, let $a = \operatorname{supp}(\{f, B\})$ and move to the model $V[[X]]$.

Observe that $\operatorname{pstab}(a)$ permutes the set B. Consider the resulting orbit equivalence relation E and any injection g from the set of all E-classes into an ordinal α. Note that E as well as each of its orbits is $\operatorname{pstab}(a)$-invariant, and so is g. In consequence, $g \in W[[X]]$. It will be enough to show that inside any given E-class, the support function is injective, because then the map $h \colon B \to [\operatorname{dom}(X)]^{<\aleph_0} \times \alpha$ defined by $h(A) = \langle \operatorname{supp}(A), g([A]_E)\rangle$ is the desired injection in $W[[X]]$.

To prove the injectivity, suppose that $A_0, A_1 \in B$ are elements of the same $\operatorname{pstab}(a)$-orbit with the same support, and work to show that $A_0 = A_1$. Let $\gamma \in \operatorname{pstab}(a)$ be the group element such that $\gamma \cdot A_0 = A_1$. Then $\gamma \cdot \operatorname{supp}(A_0) = \operatorname{supp}(A_1)$ by the invariance of the support function. By the initial assumption then, $g \cdot \operatorname{supp}(A_0) = \operatorname{supp}(A_0)$ holds. It will be enough to show that γ fixes $\operatorname{supp}(A_0)$ pointwise, because then $\gamma \in \operatorname{pstab}(\operatorname{supp}(A_0))$, and $\gamma \cdot A_0 = A_0$ and $A_0 = A_1$ follows.

Let $b \subseteq \text{supp}(A_0)$ be the set $\{x \in \text{supp}(A_0) \colon \gamma(x) \neq x\}$; it will be enough to show that the set b is empty. Towards a contradiction, assume that it is nonempty. By its definition, the set b is permuted by γ. Since γ fixes the injection f, it must be that $\gamma \cdot (f(b)) = (\gamma \cdot f)(\gamma \cdot b) = f(b)$ and $f(b)$ is fixed by γ. Since $f(b) \in b$, this contradicts the definition of the set b. □

For (2) of the theorem, I will use a nearly trivial ZF claim.

Claim 3.5. *(ZF) The class C_2 of all linearly orderable sets is closed under the following operations:*

1. *injective preimages;*
2. *product;*
3. *taking the set of all finite subsets.*

Proof. (1) is trivial and (2) and (3) use the lexicographic ordering. To elaborate on (3), if Y is a set with a linear ordering $\leq$ then on $[Y]^{<\aleph_0}$ define the relation $b \leq^* c$ if either $c \subseteq b$ or else the $\leq$-first element of $b \Delta c$ belongs to the set b. It is not hard to check that $\leq^*$ is a linear ordering on $[Y]^{<\aleph_0}$. □

Now, work in $W[[X]]$ and suppose that there is a linear ordering on $\text{dom}(X)$. Note that there is a selector on the set of all nonempty finite subsets of $\text{dom}(X)$– namely, the map choosing the smallest element in a given linear ordering of $\text{dom}(X)$. By Claim 3.4, every set can be injectively mapped into $[\text{dom}(X)]^{<\aleph_0} \times \alpha$ for some ordinal α. By Claim 3.5, every set can be linearly ordered. (2) follows.

For (3) of the theorem, I will use another tnearly trivial ZF claim.

Claim 3.6. *(ZF) The class C_3 of all sets Y such that there is a selector on the set of all nonempty finite subsets of Y is closed under the following operations:*

1. *injective preimages;*
2. *product;*
3. *taking the set of all finite subsets.*

Proof. The first item is trivial. For the second item, let Y, Z be sets and g, h be their associated selectors. Write π_Y, π_Z for the projections from $Y \times Z$ to Y and Z respectively. To produce a selector f on the set of all nonempty finite subsets of $Y \times Z$, for every such a set $a \subset Y \times Z$ let $f(a) = \langle y, z \rangle$ where $y = g(\pi_Y'' a)$ and $z = h(\pi_Z''((\{y\} \times Z) \cap a)$. The last item is somewhat more involved. Let Y be a set and let f be a selector on the set of all nonempty finite subsets of Y. Consider the set $Z = [Y]^{<\aleph_0}$; we need to find a selector g on the set of all nonempty finite subsets of Z. For every nonempty finite set $b \subset Z$, first define a linear ordering $\leq_b$ on the finite set $\bigcup b \subset Y$ as the unique one satisfying the condition $\forall x \in \bigcup b \; x = f(\{y \in \bigcup b \colon x \leq_b y\})$. Then, define a linear ordering $\leq_b^*$ on $\mathcal{P}(\bigcup b)$ as in the proof of Claim 3.5, and define $g(b)$ to be the first element of b in the linear ordering $\leq_b^*$. □

Now, work in $W[[X]]$ and suppose that there is a setector on the set of all nonempty finite subsets of X. By Claim 3.4, every set can be injectively mapped into $[\mathrm{dom}(X)]^{<\aleph_0} \times \alpha$ for some ordinal α. By Claim 3.6, for every set B there is a selector on a set of all nonempty finite subsets of B. (3) follows.

References

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