Intersections of Algorithmically Random Closed Sets

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Introduction

The study of algorithmically random closed sets was initiated by a working group led by Doug Cenzer at the 2006 AIM meeting on algorithmic randomness.

The primary notion of random closed set that emerged from this work is sometimes referred to as *Florida random* (or less elegantly as *BBCDW*-random).

This has proven to be a fruitful area of study, with interesting connections to topics such as

- Galton-Watson processes,
- the hit-or-miss topology on the space of closed subsets of 2^ω, and
- Choquet capacity and potential theory.

Work of Cenzer and Weber

In a 2013 paper, Cenzer and Weber considered the intersection and union of various types of random closed sets with respect to different underlying probability measures.

In particular, these underlying measures are derived from Bernoulli measures on 3^ω, which we will explain shortly.

Here we will focus on their results on intersections of random closed sets.

The main Cenzer/Weber result on intersections

Their main result obtained by Cenzer and Weber on the intersection of randomness closed sets is this:

Theorem (Cenzer, Weber - Informal Version)

The intersection of Bernoulli random closed sets that are random relative to each other is itself a Bernoulli random closed set.

Our interest in this work stems from the fact that they left the converse open, which we derive.

We also investigate multiple intersections of relatively random closed sets (with respect to some underlying Bernoulli measure)

Outline

1. Background
2. The intersection of random closed sets
3. Multiple intersections

Part 1: Background

Computable Probability Measures on 3^ω

Definition

A probability measure μ on 3^ω is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

Hereafter, for $i \in \{0,1,2\}$ and $\sigma \in 3^{<\omega}$, I will use the shorthand

$$\mu(\sigma i \mid \sigma) = \frac{\mu(\llbracket\sigma i\rrbracket)}{\mu(\llbracket\sigma\rrbracket)}.$$

Let us consider some examples.

Bernoulli measures on 3^ω

Let $p, q \in [0,1]$ satisfy $p + q \leq 1$. Then the measure $\mu_{\langle p,q \rangle}$ defined by the conditional probabilities

- $\mu_{\langle p,q \rangle}(\sigma 0 \mid \sigma) = p$
- $\mu_{\langle p,q \rangle}(\sigma 1 \mid \sigma) = q$
- $\mu_{\langle p,q \rangle}(\sigma 2 \mid \sigma) = 1 - p - q$

for $\sigma \in 3^{<\omega}$ defines a *Bernoulli measure* on 2^ω.

$\mu_{\langle p,q \rangle}$ is a computable measure if and only if p and q are both computable.

Symmetric Bernoulli measures on 3^ω

Let $p \in (0, \frac{1}{2})$. Then the measure μ_p defined by the conditional probabilities

- $\mu_p(\sigma 0 \mid \sigma) = p$
- $\mu_p(\sigma 1 \mid \sigma) = p$
- $\mu_p(\sigma 2 \mid \sigma) = 1 - 2p$

for $\sigma \in 3^{<\omega}$ defines a *symmetric Bernoulli measure* on 2^ω.

μ_p is a computable measure if and only if p is computable.

Martin-Löf randomness

Let μ be a computable measure on 3^ω.

Definition

A *μ-Martin-Löf test* is a uniformly Σ^0_1 sequence $(\mathcal{U}_i)_{i\in\omega}$ of subsets of 3^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

A sequence $X \in 3^\omega$ *passes the μ-Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

$X \in 3^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes *every* μ-Martin-Löf test.

Note that we can relative this definition to any oracle.

Algorithmically random closed sets

Let $\mathcal{K}(2^\omega)$ be the collection of closed subsets of 2^ω.

One way to define an algorithmically random closed subset of 2^ω, due to Barmpalias, Brodhead, Cenzer, Dashti, and Weber:

- A closed set $\mathcal{C} \subseteq 2^\omega$ is random if it can be coded by an algorithmically random sequence $X \in 3^\omega$ as shown by the following example.

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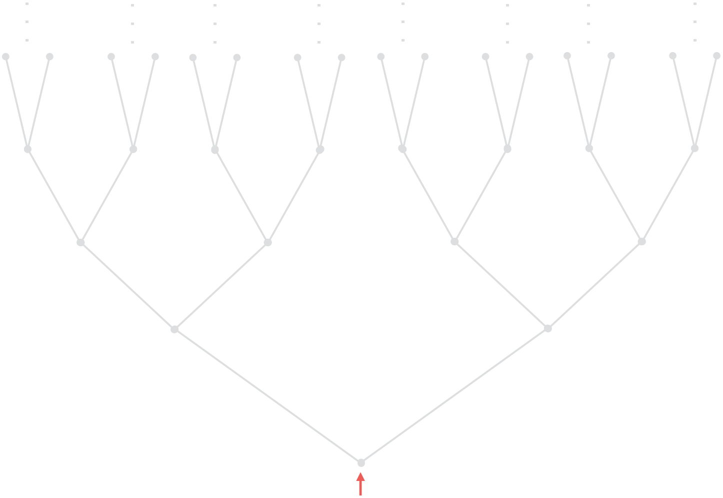

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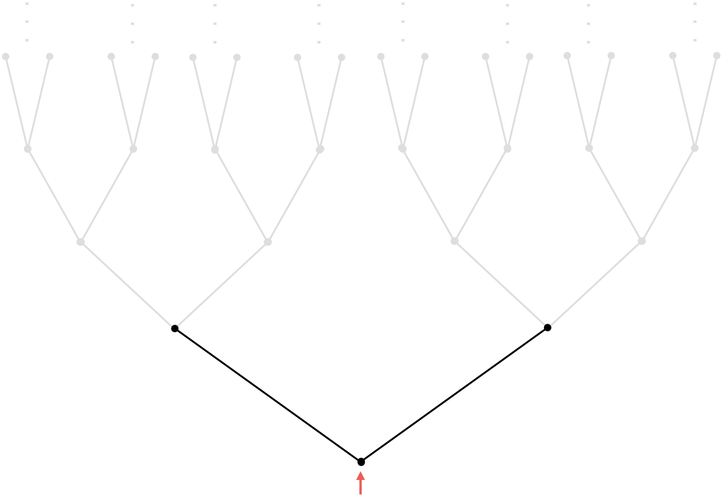

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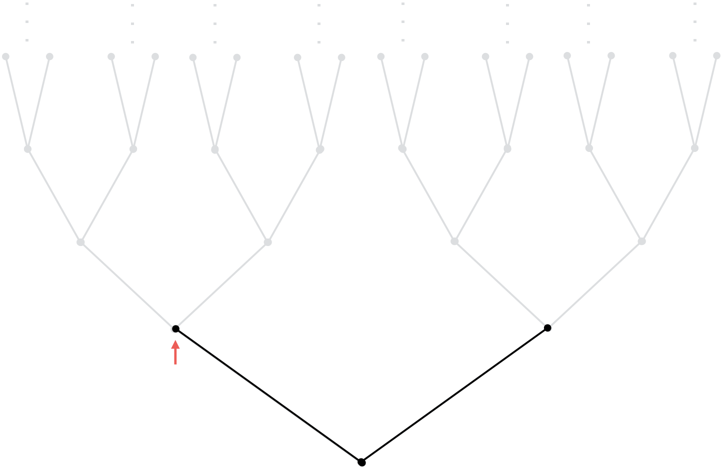

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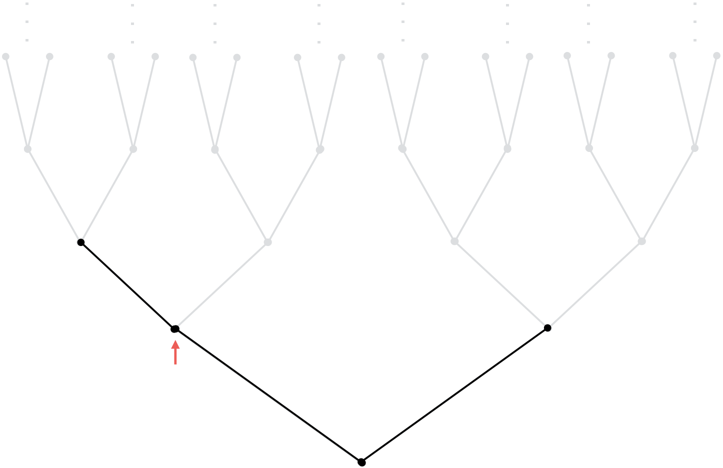
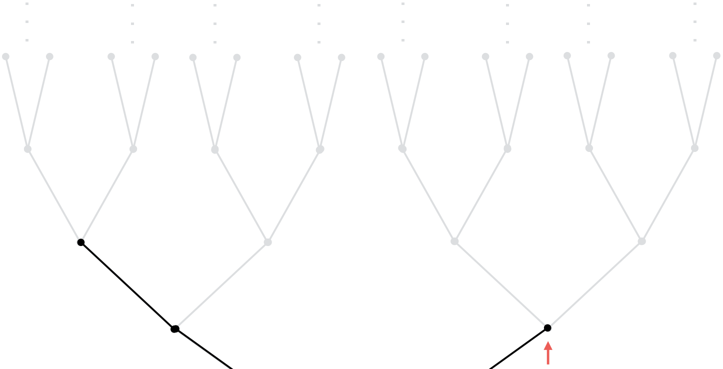


$$X = 2\textcolor{red}{0}21122012\ldots$$



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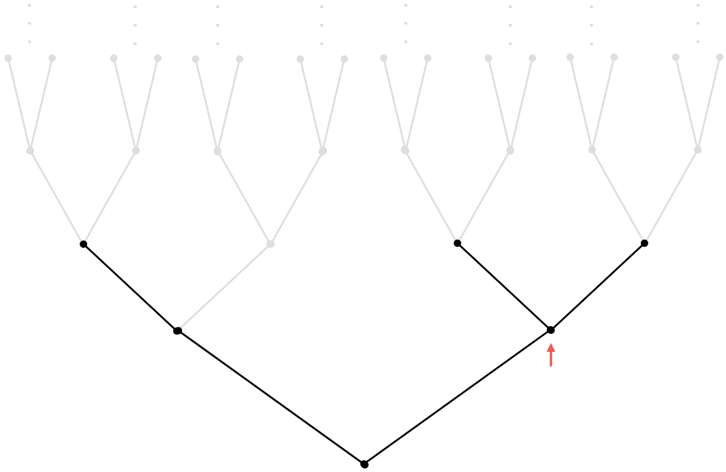
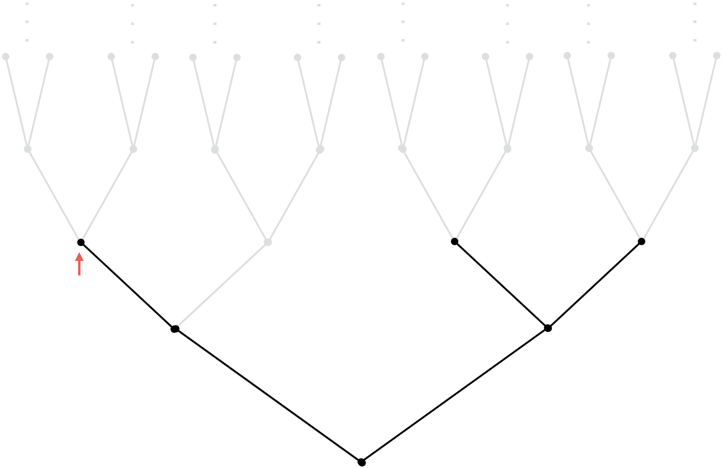
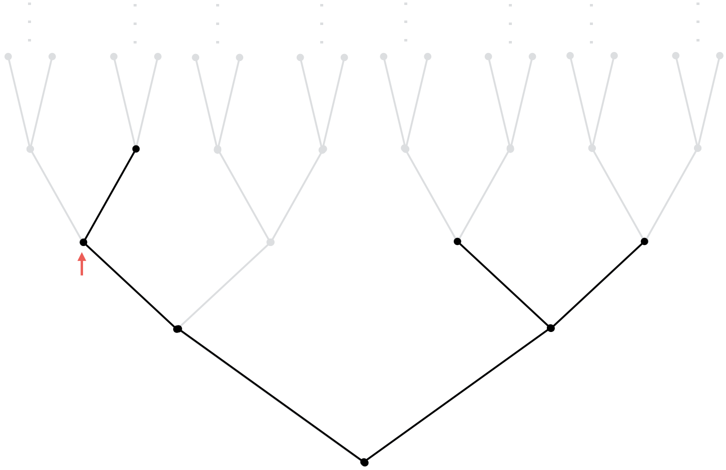


$$X = 2021122012\ldots$$



$$X = 2021122012\ldots$$



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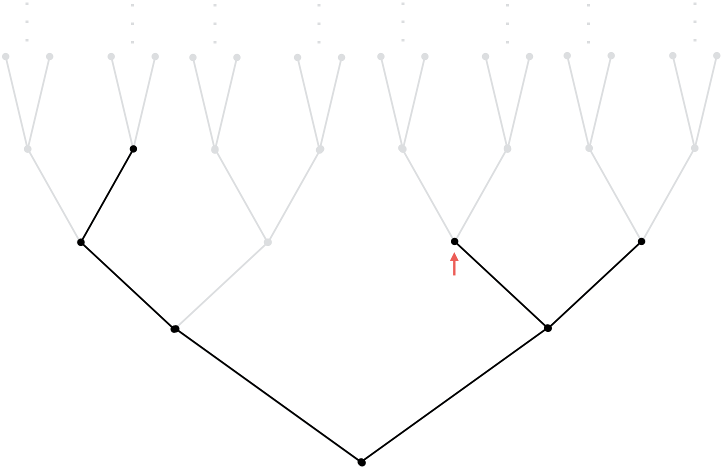

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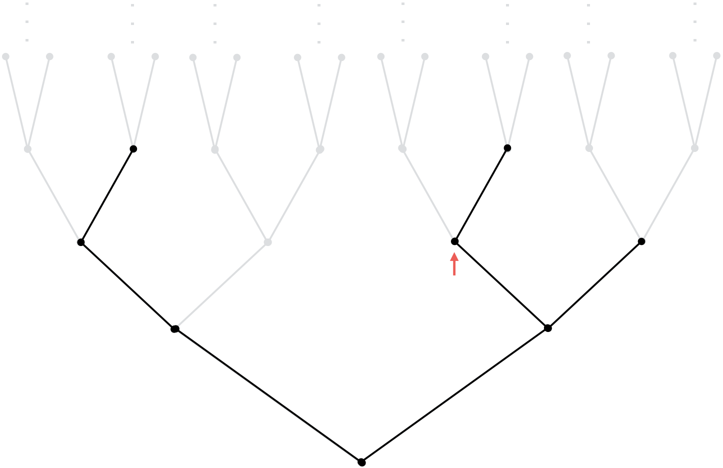

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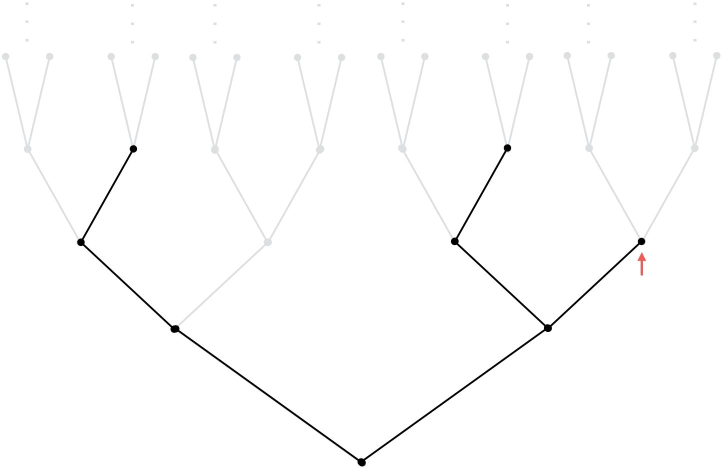

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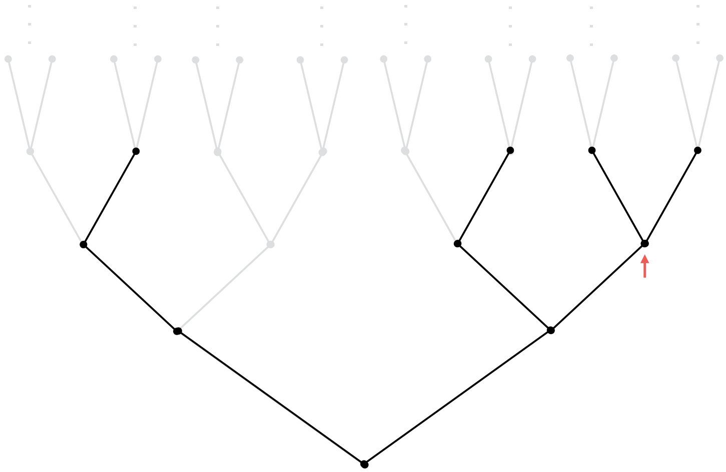

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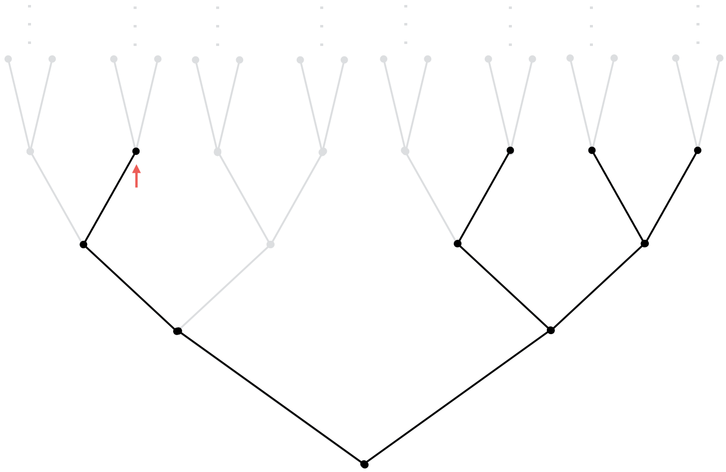

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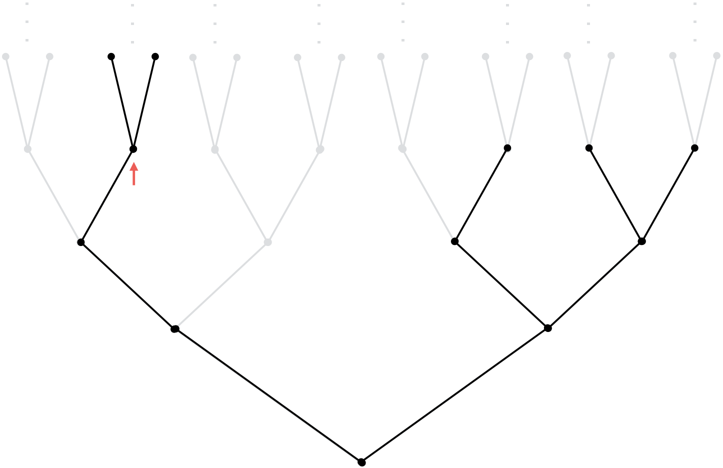
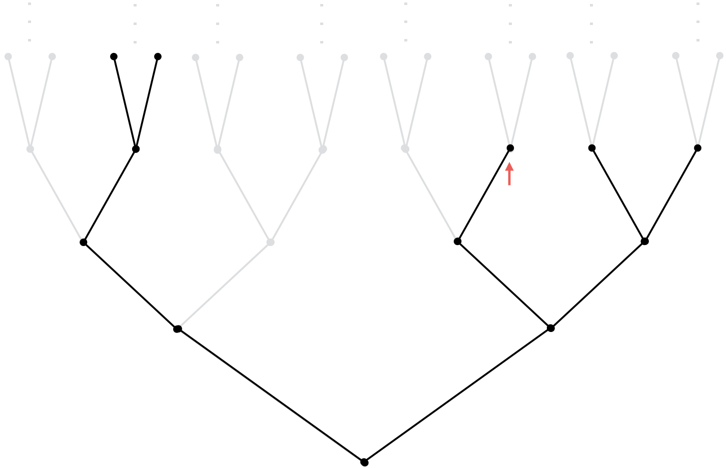
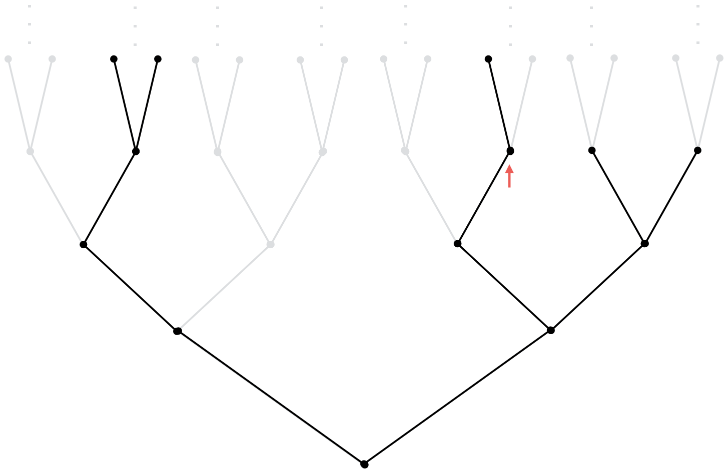


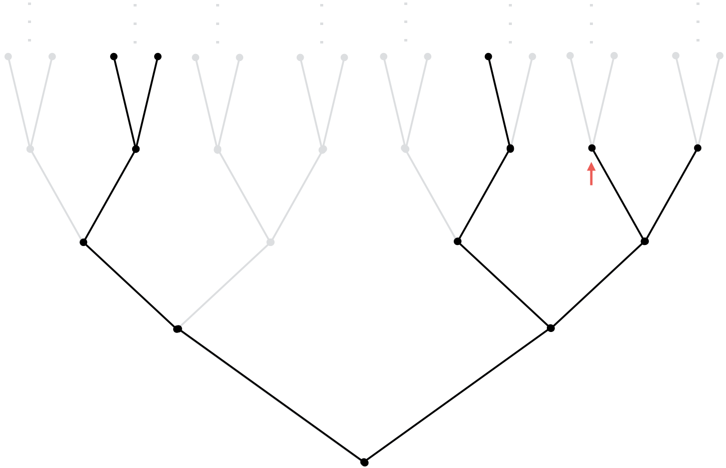
$X = 2021122012\ldots$



$X = 2021122\textcolor{red}{0}12\ldots$



$$X = 2021122012\ldots$$



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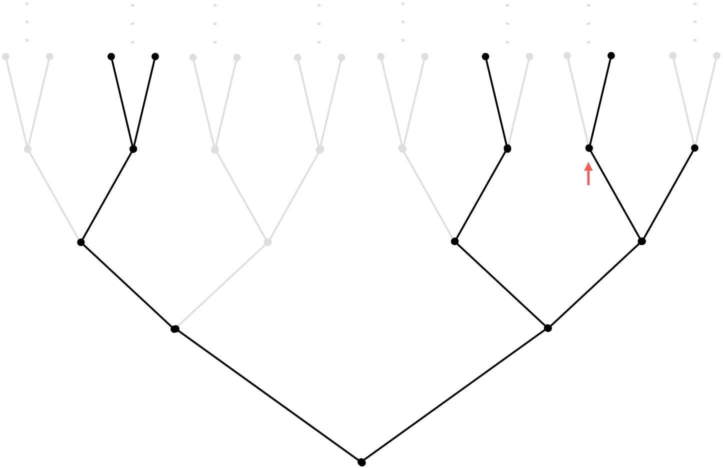

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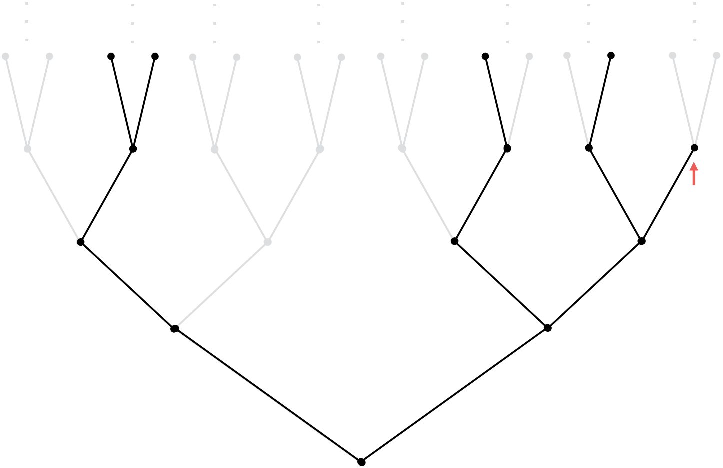

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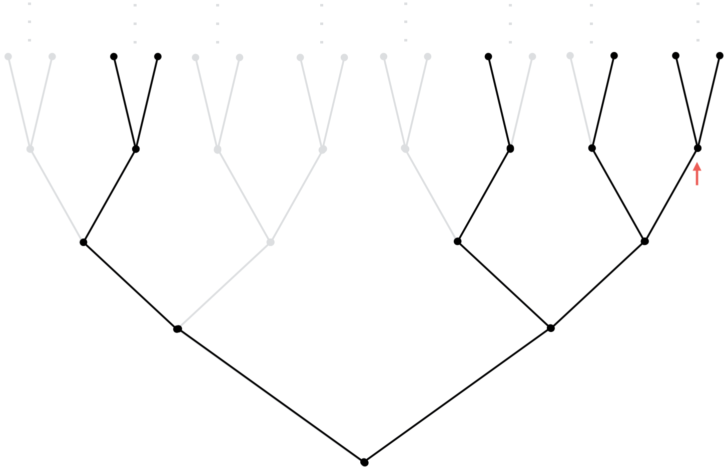

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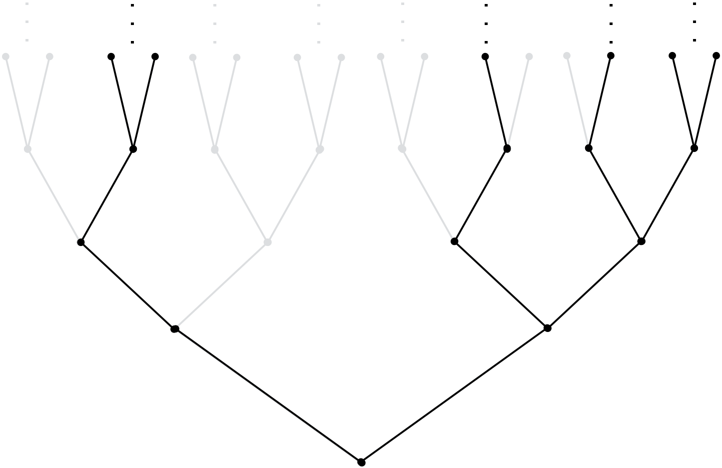
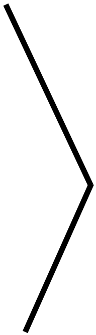


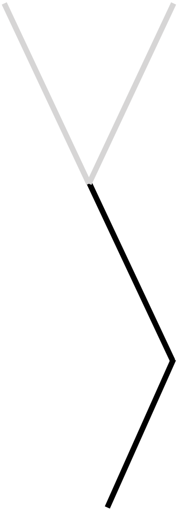
X = 2021122012...

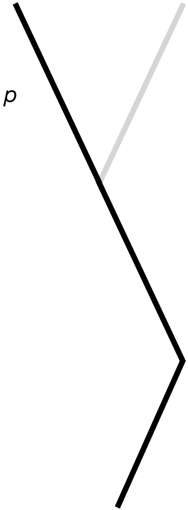
Uniformly random closed sets

This definition was originally given for the case $p = q = \frac{1}{3}$, i.e., with respect to the Lebesgue measure on 3^ω.

It was later extended to more general measures on 3^ω in a number of other Cenzer-led projects.



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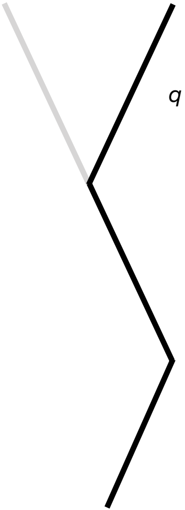
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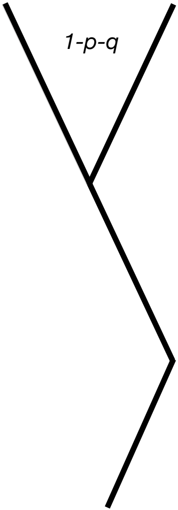

1-p-q

Convention about measures on $\mathcal{K}(2^\omega)$

If μ is a measure on 3^ω, then we write μ^* to stand for the corresponding measure on $\mathcal{K}(2^\omega)$.

That is, μ^*-random closed sets are those closed sets coded by a μ-random sequence in 3^ω.

Turing functionals and computable measures

Fix $n, m \in \omega$.

A computable measure μ on n^ω and a Turing functional $\Phi : n^\omega \to m^\omega$ satisfying $\mu(\mathrm{dom}(\Phi)) = 1$ together induce a computable probability measure on m^ω, denoted μ_Φ, defined by

$$\mu_\Phi(\mathcal{X}) = \mu(\Phi^{-1}(\mathcal{X}))$$

for every Borel $\mathcal{X} \subseteq m^\omega$.

Randomness preservation

Theorem (Levin)

Let μ be a computable measure on n^ω.

Suppose that $\Phi : n^\omega \to m^\omega$ is a Turing functional satisfying $\mu(\mathrm{dom}(\Phi)) = 1$ and $X \in n^\omega$ is μ-Martin-Löf random.

Then $\Phi(X) \in \mathrm{MLR}_{\mu_\Phi}$.

Heuristic: A random input yields a random output.

No randomness ex nihilo

Theorem (Shen)

Suppose that μ is a computable measure on n^ω and $\Phi : n^\omega \to m^\omega$ is a total Turing functional satisfying $\mu(\mathrm{dom}(\Phi)) = 1$.

Then for every μ_Φ-Martin-Löf random $Y \in m^\omega$, there is some μ-Martin-Löf random $X \in n^\omega$ such that $\Phi(X) = Y$.

Heuristic: Randomness in the codomain must come from some randomness in the domain.

Part 2: The intersection of random closed sets

Cenzer/Weber on the intersection of random closed sets

Theorem (Cenzer, Weber)

Suppose that $p, q, r, s \geq 0$ *are computable,* $0 \leq p + q \leq 1$ *and* $0 \leq r + s \leq 1$.

Suppose that $P \in \mathcal{K}(2^\omega)$ *is* $\mu^*_{\langle p,q \rangle}$*-random relative to* $Q \in \mathcal{K}(2^\omega)$ *and that* Q *is* $\mu^*_{\langle r,s \rangle}$*-random relative to* P.

Then one of three possibilities occurs:

The first possibility

- If $p+q+r+s \geq 2+pr+qs$, then $P \cap Q = \emptyset$.

This technical condition guarantees that neither P nor Q have a sufficient amount of branching to guarantee a non-empty intersection.

The second possibility

- If $p+q+r+s < 1+pr+qs$, then $P \cap Q = \emptyset$ with probability

$$\frac{ps+qr}{(1-p-q)(1-r-s)}.$$

In this case, there may be a sufficient amount of branching in P and Q, but we see that the intersection is empty due to some finite level of both P and Q.

The third possibility

- If $p + q + r + s < 1 + pr + qs$ and $P \cap Q \neq \emptyset$, then $P \cap Q$ is Martin-Löf random with respect to the measure $\mu^*_{\langle p+r-pr, q+s-qs \rangle}$.

Now we have a sufficient amount of branching in P and Q and some infinite path in their intersection.

The amount of branching in the resulting closed set is computable in the Bernoulli parameters of both P and Q.

What about the converse?

Question: Given computable $p, q, r, s \geq 0$ satisfying the conditions of the theorem, can every $\mu^*_{\langle p+r-pr, q+s-qs\rangle}$-random closed set be obtained as the intersection of relatively random closed sets, one $\mu^*_{\langle p,q\rangle}$-random and the other $\mu^*_{\langle r,s\rangle}$ in this way?

What about the converse?

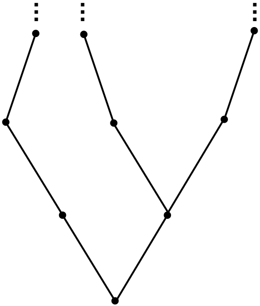
Question: Given computable $p, q, r, s \geq 0$ satisfying the conditions of the theorem, can every $\mu^*_{\langle p+r-pr, q+s-qs\rangle}$-random closed set be obtained as the intersection of relatively random closed sets, one $\mu^*_{\langle p,q\rangle}$-random and the other $\mu^*_{\langle r,s\rangle}$ in this way?

Yes!

A problem with the measure induced by intersections

Idea: Reprove the Cenzer/Weber result using randomness preservation, and then answer the question using no randomness ex nihilo.

There is a challenge with this approach, as illustrated by the next set of slides.



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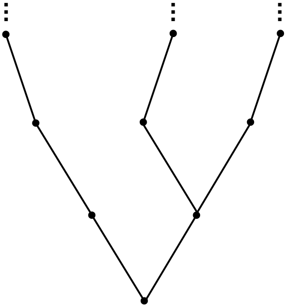

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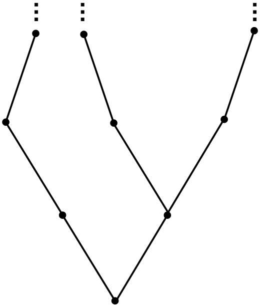

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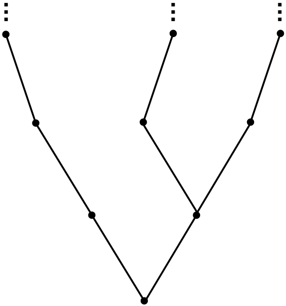

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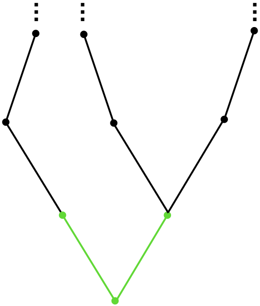
$$P \cap Q$$



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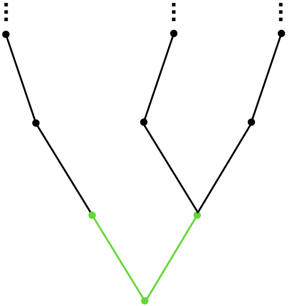

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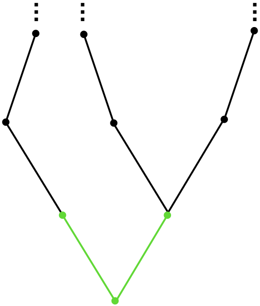

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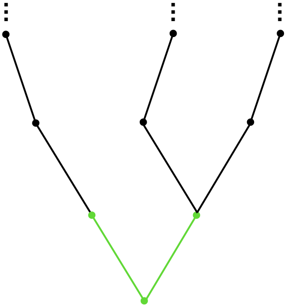

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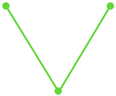

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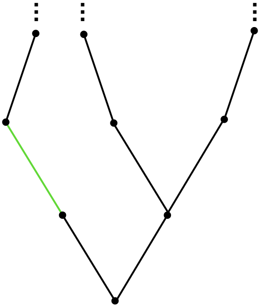

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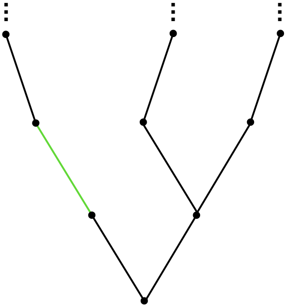

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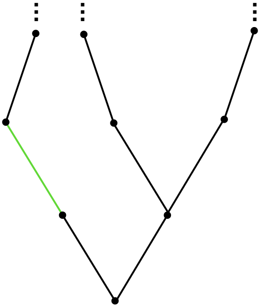

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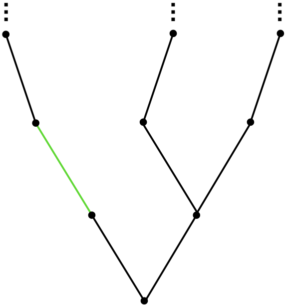

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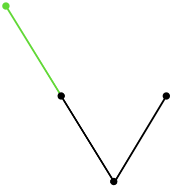

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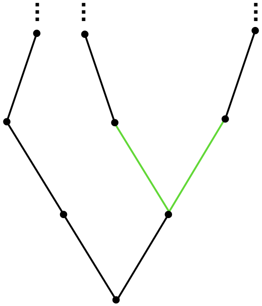

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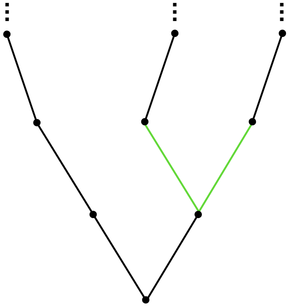

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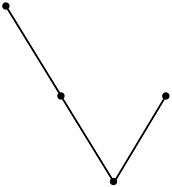

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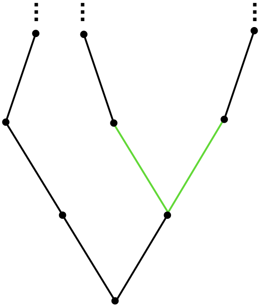

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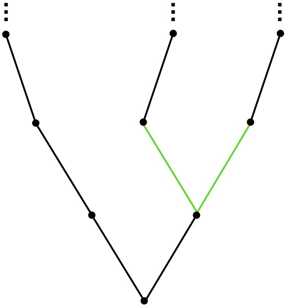

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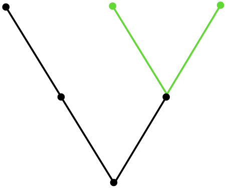

$P \cap Q$

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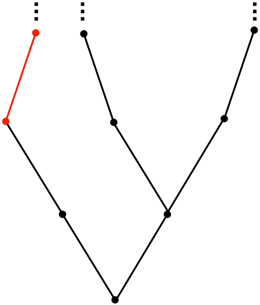

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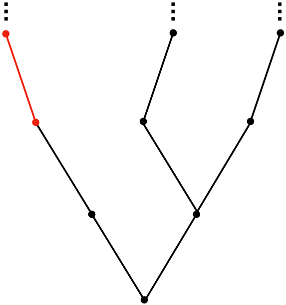

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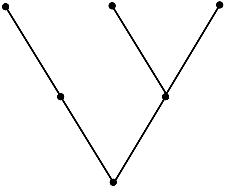

$P \cap Q$

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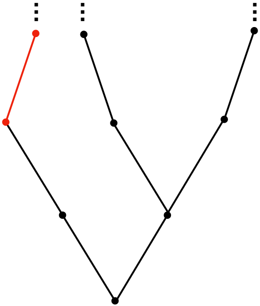

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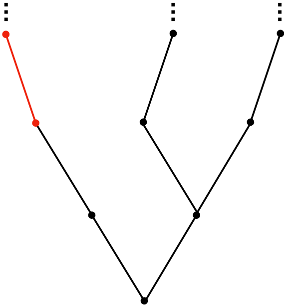
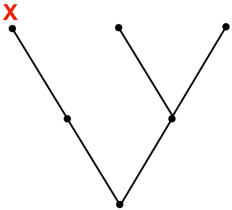


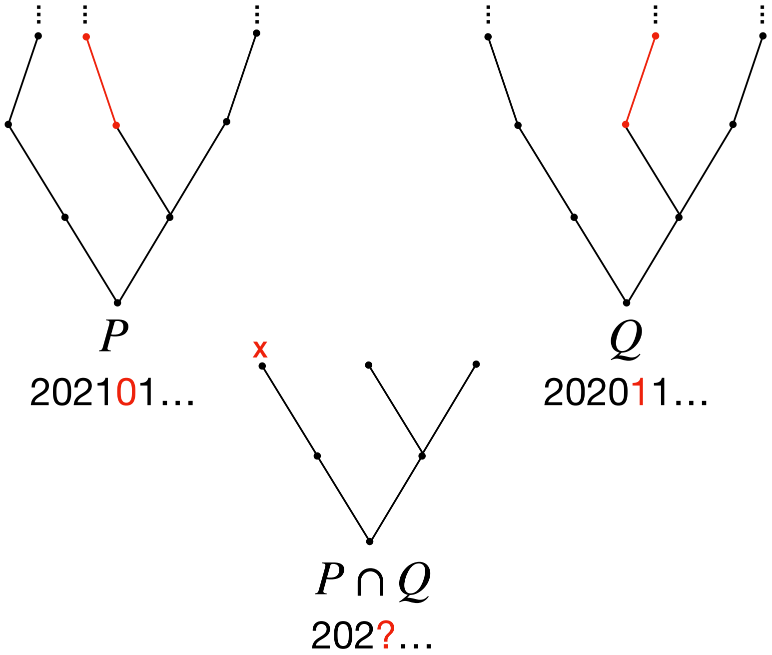
Q

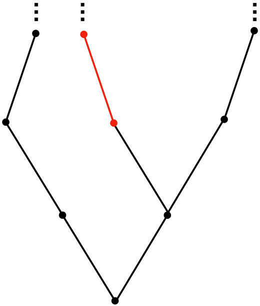
202**0**11…



$P \cap Q$

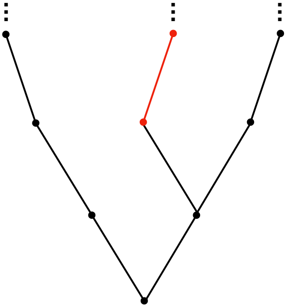
202**?**…


P
202101…
Q
202011…
x
$P \cap Q$
202?…



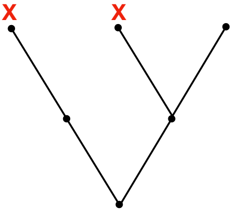
P

202101...



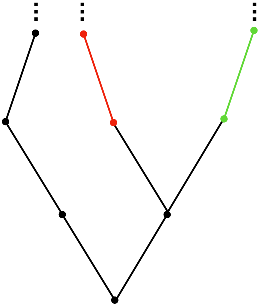
Q

202011...



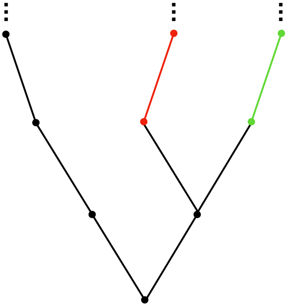
$P \cap Q$

202??...



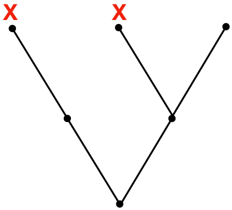
P

202101...



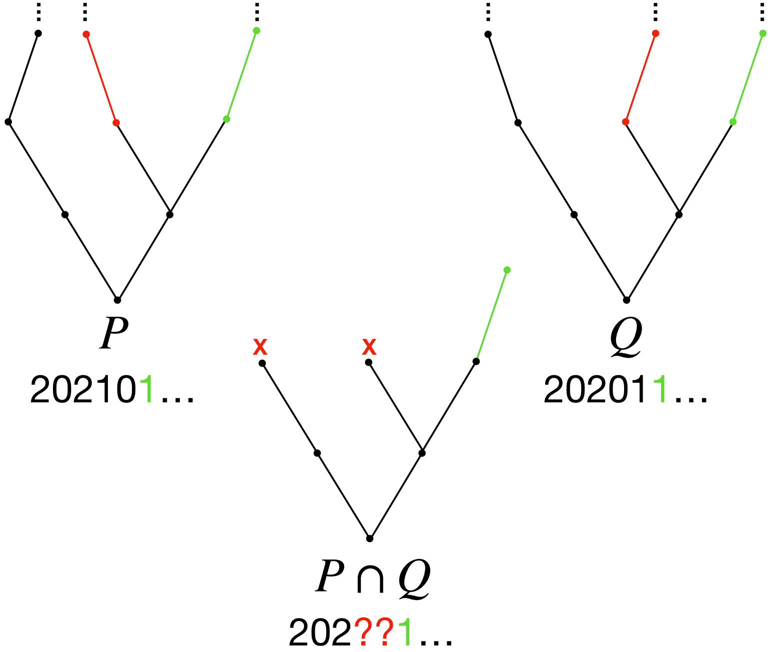
Q

202011...

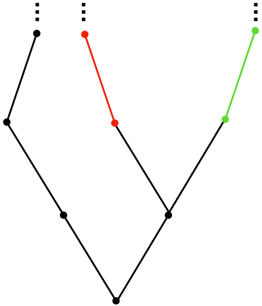


$P \cap Q$

202??...

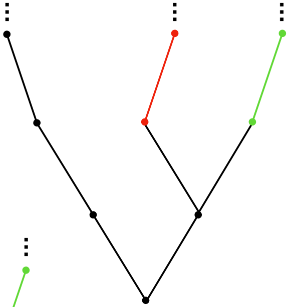


P
202101...
Q
202011...
x
x
$P \cap Q$
202??1...



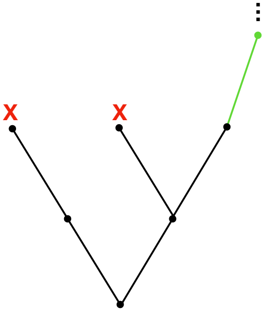
P

202101...



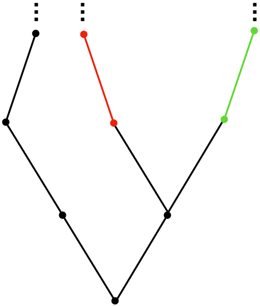
Q

202011...



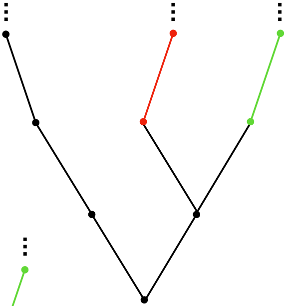
$P \cap Q$

202??1...



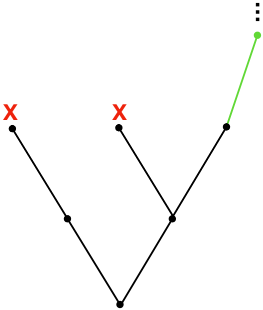
P

202101...



Q

202011...



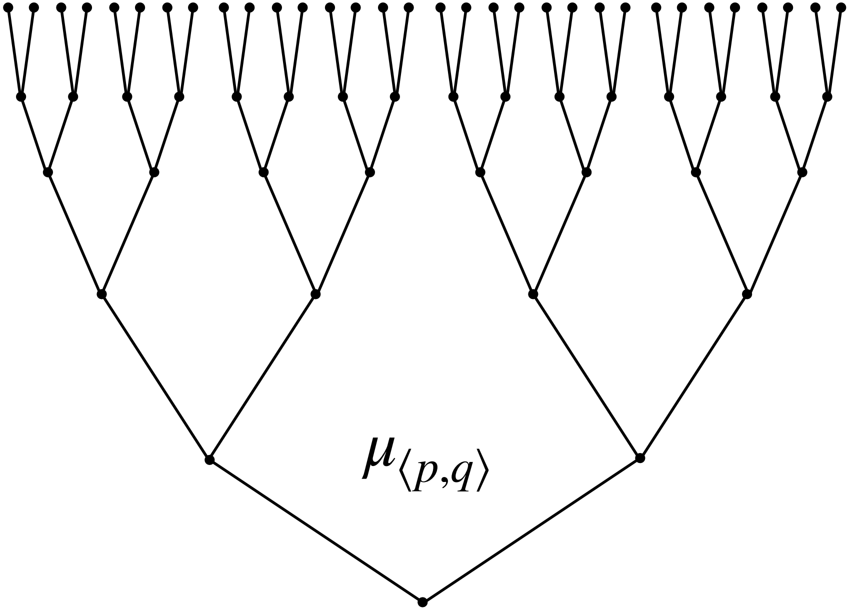
$P \cap Q$

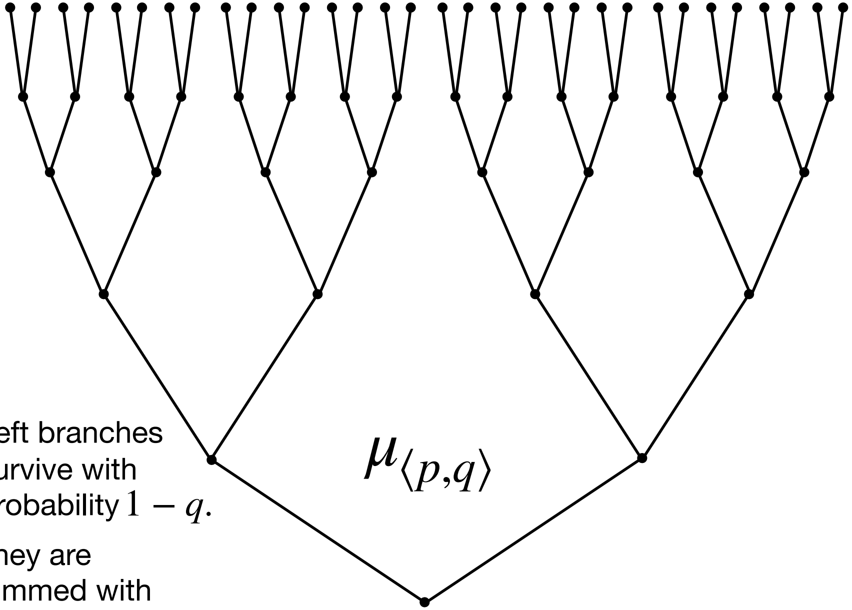
202331...

The solution

In order to answer the Cenzer/Weber question, we think of the intersection of random closed sets as given by a kind of Galton-Watson process, which may result in a tree with dead ends or even finite tree.

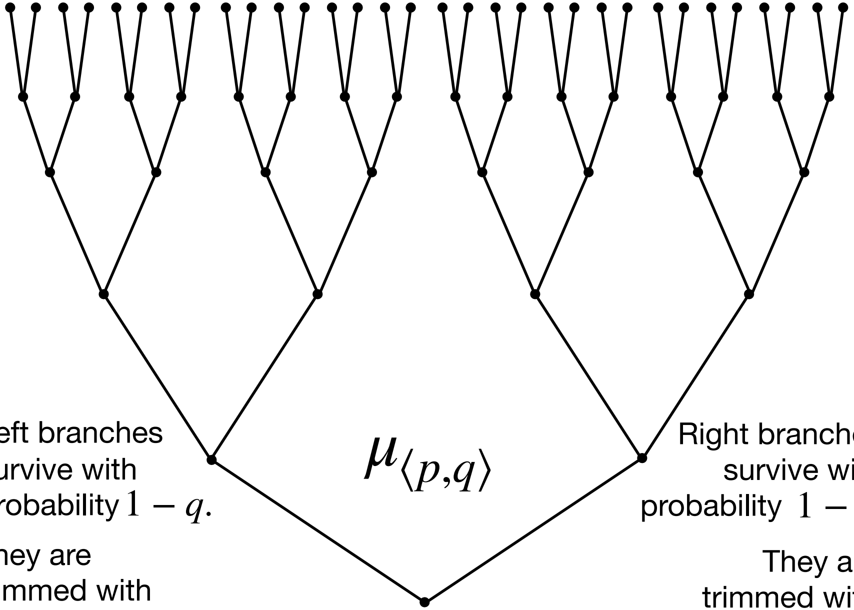
We then generalize work of Kjos-Hanssen and Diamondstone to show that infinite, random Galton-Watson correspond to random closed sets.


$\mu_{\langle p,q\rangle}$



Left branches survive with probability $1-q$.

They are trimmed with probability q.

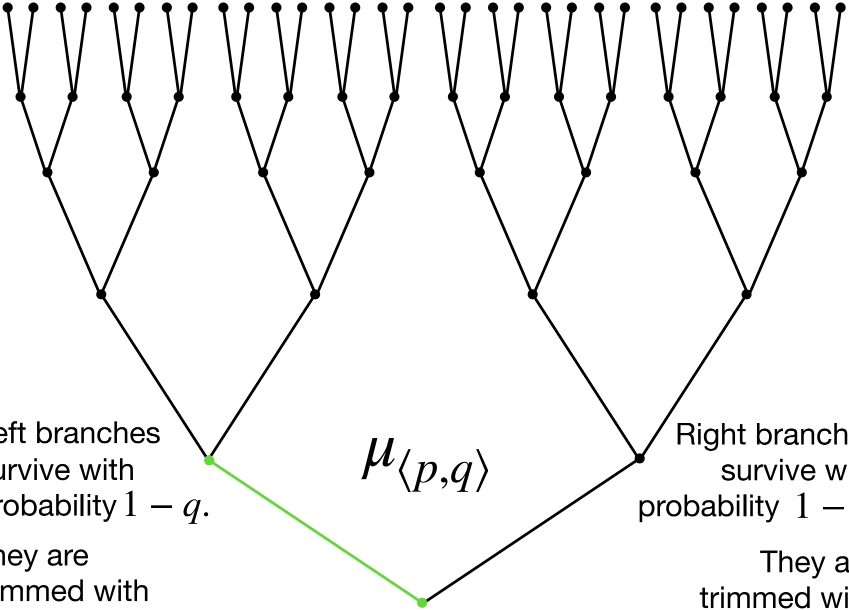


Left branches survive with probability $1 - q$.

They are trimmed with probability q.

Right branches survive with probability $1 - p$.

They are trimmed with probability p.



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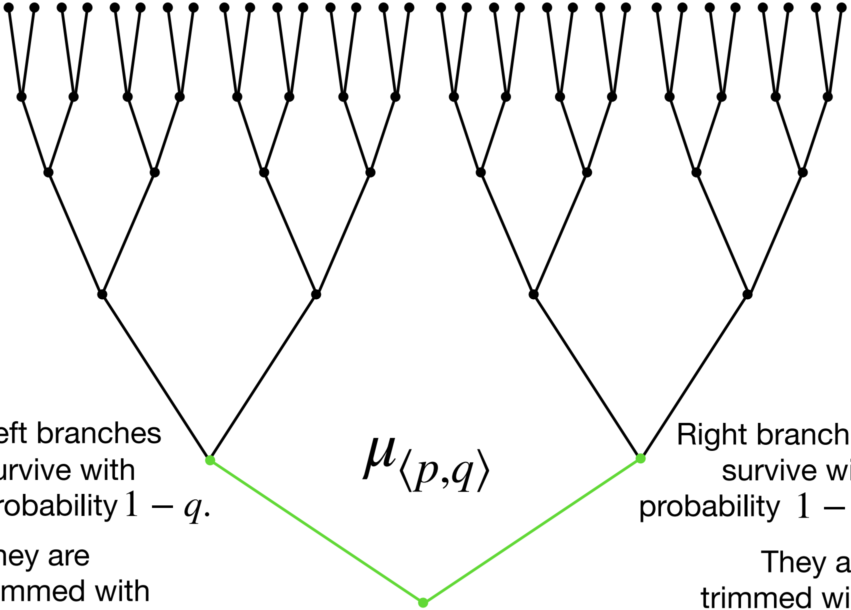

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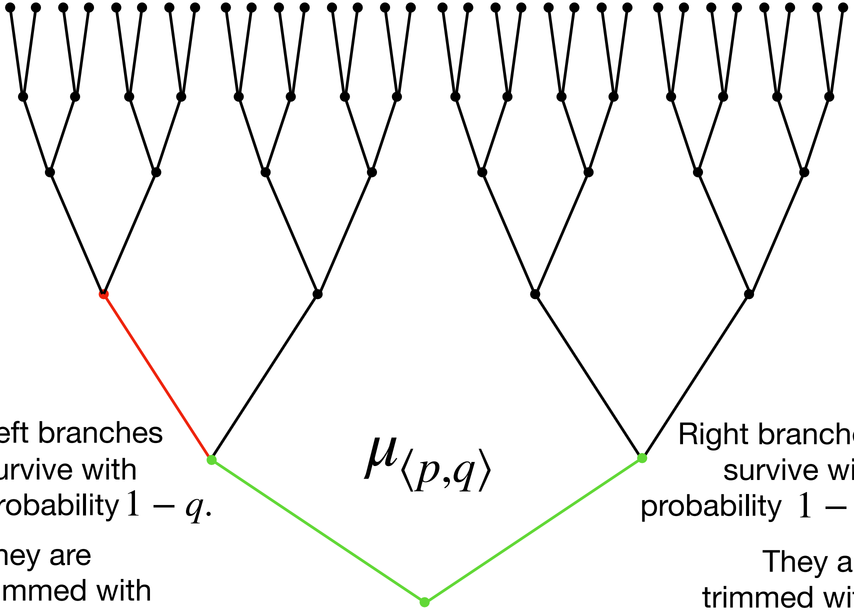

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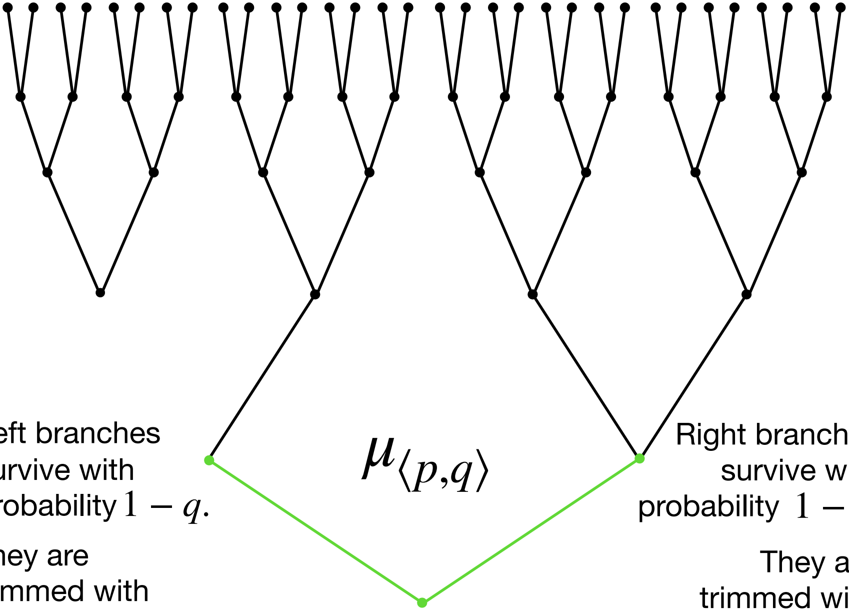

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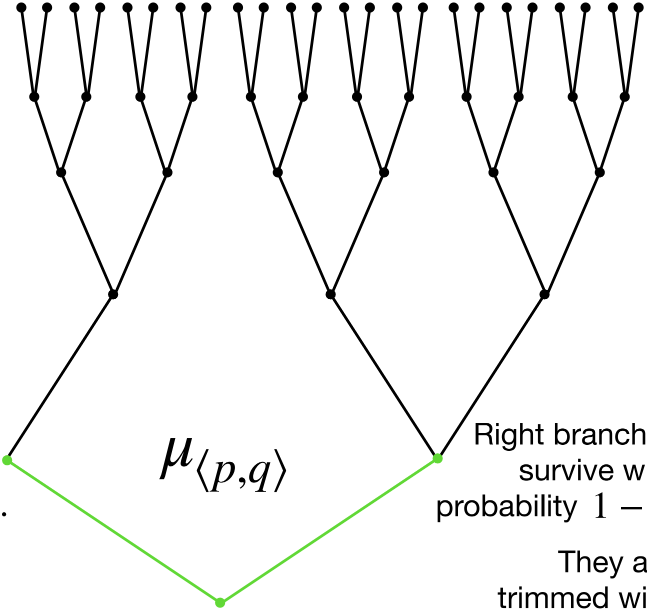
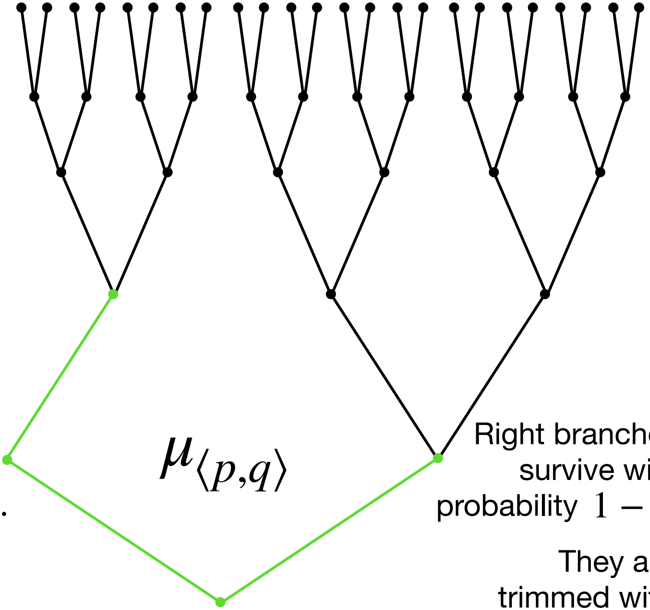


Right branches survive with probability $1 - p$.

They are trimmed with probability p.

Left branches survive with probability $1 - q$.

They are trimmed with probability q.

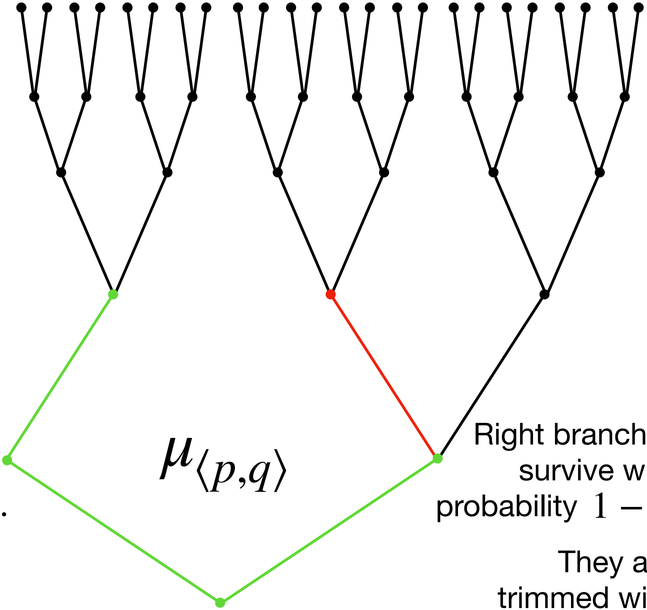


Right branches survive with probability $1 - p$.

They are trimmed with probability p.

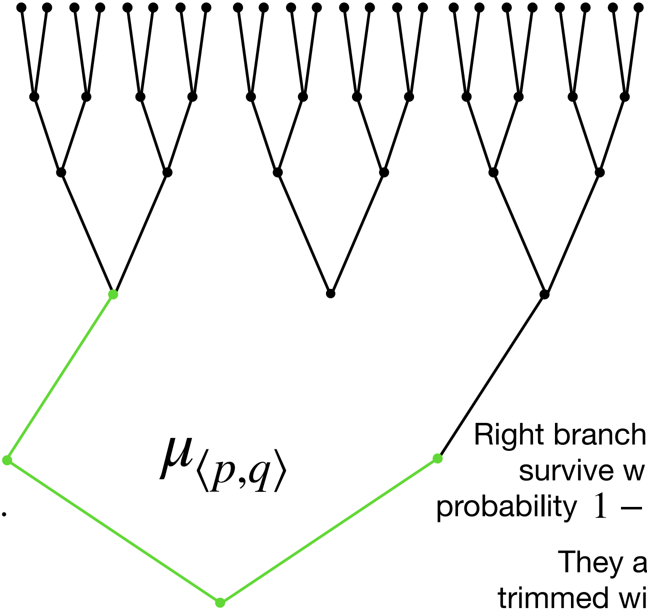
Left branches survive with probability $1 - q$.

They are trimmed with probability q.



Right branches survive with probability $1 - p$.

They are trimmed with probability p.



Left branches survive with probability $1-q$.

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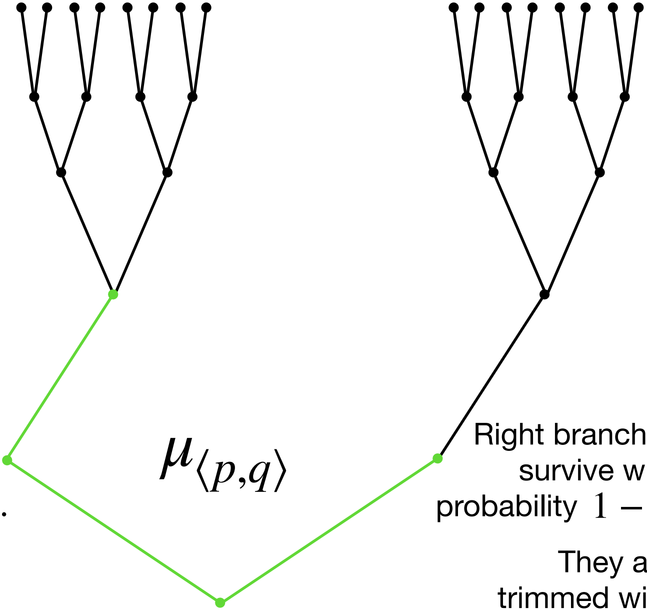

Left branches survive with probability $1 - q$.

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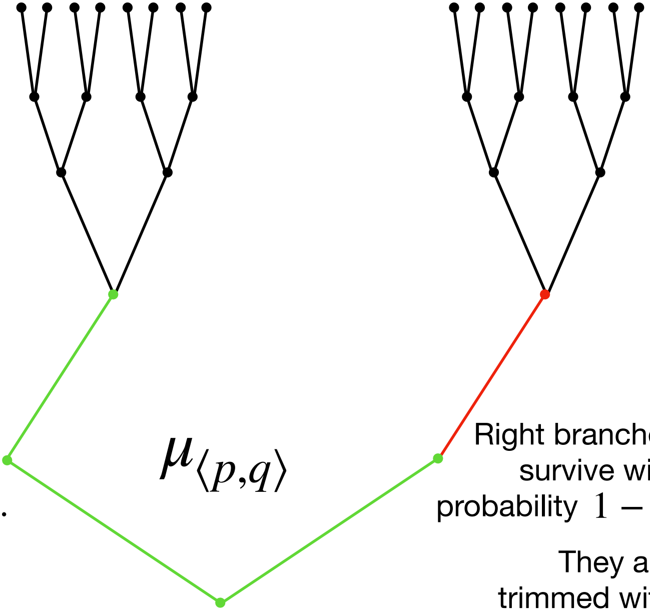

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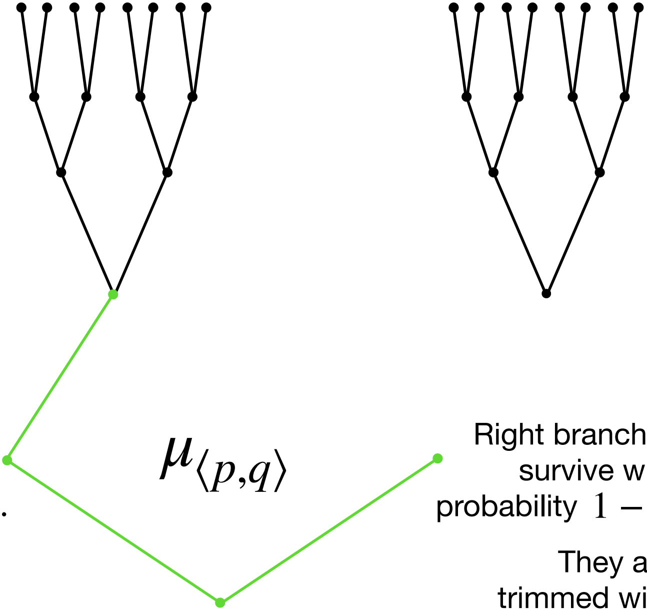

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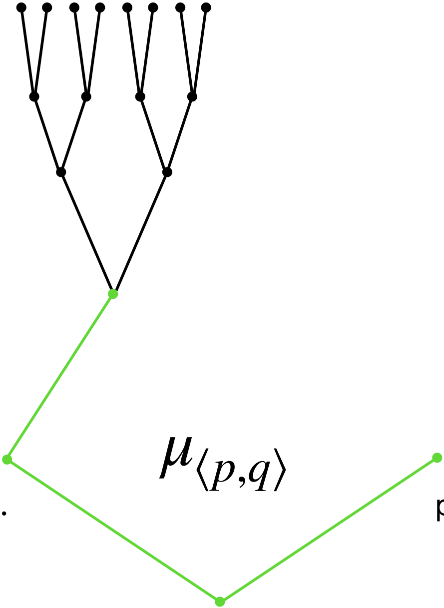

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Another problem

The map induced by the intersection of random closed sets can thus seen as a map from $3^\omega \times 3^\omega$ to 4^ω (where an element of 4^ω codes a tree $T \subseteq 2^{\leq\omega}$ possibly with dead ends).

However, this map will yield the empty set as output with positive probability, so we cannot immediately use the machinery of randomness preservation.

The solution

The solution is to use the machinery of layerwise computability, first developed by Hoyrup and Rojas.

We adopt a construction due to Bienvenu, Hoyrup, and Shen.

Layerwise computable transformations satisfy randomness preservation and no randomness ex nihilo, so we can apply these tools to derive the converse of the Cenzer/Weber theorem.

Part 3: Multiple intersections

A corollary of the Cenzer/Weber intersection theorem

Corollary (Cenzer, Weber)

For $p \in (0, 1/2)$, let $P, Q \in \mathcal{K}(2^\omega)$ be relatively μ_p^*-random.

1. *If $p \geq 1 - \frac{\sqrt{2}}{2}$, then $P \cap Q = \emptyset$.*
2. *If $p < 1 - \frac{\sqrt{2}}{2}$, then $P \cap Q = \emptyset$ with probability $\frac{2p^2}{(1-2p)^2}$.*
3. *If $p < 1 - \frac{\sqrt{2}}{2}$ and $P \cap Q \neq \emptyset$, then $P \cap Q$ is Martin-Löf random with respect to the measure $\mu_{2p-p^2}^*$.*

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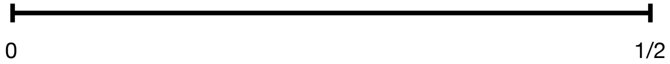
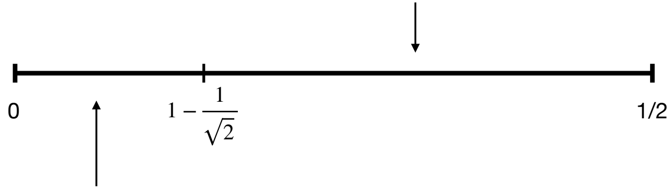
What's so special about $1 - \frac{\sqrt{2}}{2}$?

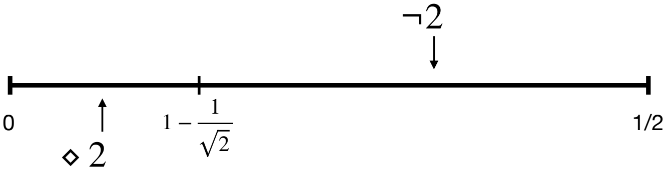

0
1/2


0
$1-\frac{1}{\sqrt{2}}$
1/2

For any parameter *p* in this interval,
the intersection of relatively μ_p^*-random closed sets is **always** empty.



For any parameter *p* in this interval,
the intersection of relatively μ_p^*-random closed may be non-empty.



$\neg n$: the intersection of n relatively random closed sets is empty

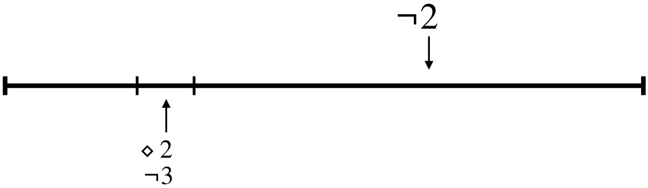
$\diamond n$: the intersection of n relatively random closed sets may be non-empty

$\neg n$: the intersection of n relatively random closed sets is empty

$\diamond\, n$: the intersection of n relatively random closed sets may be non-empty

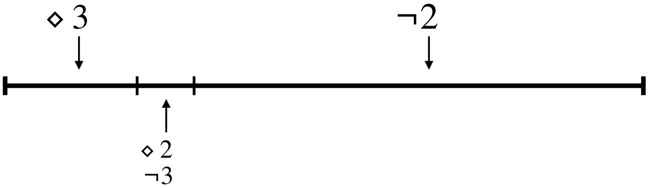
$\neg n$: the intersection of n relatively random closed sets is empty

$\diamond\, n$: the intersection of n relatively random closed sets may be non-empty



$\neg n$: the intersection of n relatively random closed sets is empty

$\Diamond\, n$: the intersection of n relatively random closed sets may be non-empty



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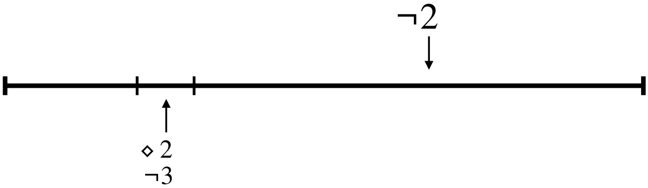

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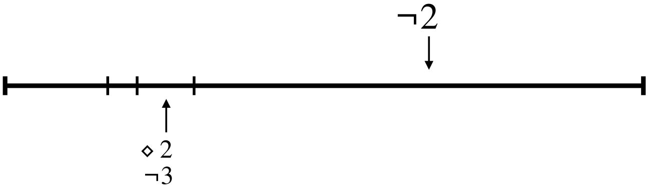

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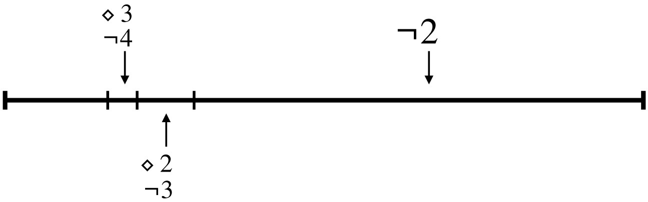

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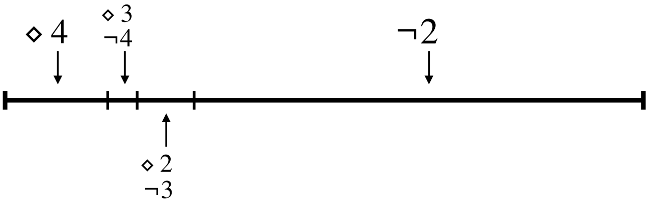

$\neg n$: the intersection of n relatively random closed sets is empty

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Generalizing the corollary

For $n \in \omega$, let $f_n(x) = 1 - (1-x)^n$.

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For $n \in \omega$, let $f_n(x) = 1 - (1-x)^n$.

Theorem

For $p \in (0, \frac{1}{2})$ and $n \geq 2$, given n mutually relatively μ_p^*-random closed sets $P_1, \ldots, P_n$, the following hold:

1. *If $p \geq 1 - \frac{1}{\sqrt[n]{2}}$, then $\bigcap_{i=1}^n P_i = \emptyset$.*
2. *If $p < 1 - \frac{1}{\sqrt[n]{2}}$, then $\bigcap_{i=1}^n P_i = \emptyset$ with probability $1 - \frac{1-2f_n(p)}{(1-2p)^n}$.*
3. *If $p < 1 - \frac{1}{\sqrt[n]{2}}$ and $\bigcap_{i=1}^n P_i \neq \emptyset$, then $\bigcap_{i=1}^n P_i$ is Martin-Löf random with respect to the measure $\mu_{f_n(p)}^*$.*

What about the converse?

The converse holds as well:

Theorem

For any $p \in [0, 1/2]$ and $n \geq 2$, a μ_p^*-random closed set can be written as the intersection of n mutually relatively random $\mu^*_{f_n^{-1}(p)}$-random closed sets.

Thank you!