Preserving P-points in definable forcing II*

Jindřich Zapletal †
University of Florida

September 13, 2008

Abstract

I will prove iteration and product preservation theorems related to P-point preservation based on the first part of the paper. I will also provide counterexamples to possible strengthenings of the presented results.

1 Introduction

In [8], I showed that in the context of the definable proper forcing, preservation of P-points is equivalent to the conjunction of two forcing properties: the weak Laver property and not adding a splitting real.

Definition 1.1. A forcing P adds a *splitting real* if in the extension there is a set $a \subset \omega$ which has nonempty intersection with every ground model infinite subset of ω.

Definition 1.2. A forcing P has the *weak Laver property* if for every function $f \in \omega^\omega$ in the extension, dominated by a function in the ground model, there are a ground model infinite set $a \subset \omega$ and a ground model function $h : \omega \to [\omega]^{<\aleph_0}$ such that for every number $n \in a$, $|h(n)| \leq 2^n$ and $f(n) \in h(n)$. The conjunction of the bounding property and the weak Laver property is referred to as the *weak Sacks property.*

This opens a possibility that various features of P-point preservation may be derivable from these two properties. In particular, P-point preservation is preserved under the countable support iteration [1, Theorem 6.2.6]; is it possible that the weak Laver property or not adding splitting real are so preserved on their own? In this paper, I provide both negative and positive results to this question.

Theorem 1.3. *Assume that suitable large cardinals exist. The weak Sacks property is preserved under the countable support iteration of definable proper forcing.*

*2000 AMS subject classification 03E17, 03E40.
†Partially supported by NSF grants DMS 0300201 and DMS 0801114.

I proved some time ago that the conjunction "bounding and adding no splitting real" is preserved by the countable support iteration of definable proper forcing. However, if the "bounding" conjunct is left out, the resulting statements fail badly. Neither the weak Laver property nor adding a splitting real is preserved even under two step iteration of definable proper forcing–Examples 2.2 and 2.1.

In the case of product, the situation is even more challenging.

Theorem 1.4. *Assume that suitable large cardinals exist. The weak Sacks property is preserved under the countable support product of definable proper forcing.*

The conjunction "bounding and not adding splitting reals" is not preserved even in a product of two definable proper forcings–Example 2.3. The main open question remains:

Question 1.5. Is the conjunction "bounding property and P-point preservation" preserved under the countable support product of definable proper forcing?

The proof of Theorem 1.3 develops an important technique introduced in [7]. It identifies a canonical c.c.c. forcing such that the regularity property in question can be restated in terms of the Fubini property of the c.c.c. forcing ideal. The ergodic iteration theorem [6, Theorem 6.3.3] then concludes the argument. It turns out that the weak Sacks property is in fact quite central to this method. It can be restated as the Fubini property with respect to all suitably definable σ-centered ideals.

The statements of the theorems include a large cardinal assumption and a definability assumption. To get the most general statement, interpret "definable forcing" as forcing with quotient algebra P_I of Borel I-positive sets ordered by inclusion, where I is a σ-ideal on a Polish space X such that for the universal analytic set $A \subset 2^\omega \times X$, the collection $\{x \in 2^\omega : A_x \in I\}$ is universally Baire. With this interpretation, the large cardinal assumption necessary is "a proper class of Woodin cardinals". However, there is a less general statement of essentially the same utility for which the necessary large cardinal strength reduces to zero. To get this statement, interpret "definable forcing" as forcing P consisting of an analytic family of binary trees, ordered by inclusion, closed under restriction below a node, and with the continuous reading of names: for every P-name $\dot{f}$ for a function in ω^ω and every condition $T \in P$ there is a condition $S \leq T$ and numbers $k_n : n \in \omega$ such that for every node $s \in S$ of length k_n the tree $S \restriction s$ decides the value of $\dot{f}(\check{n})$. These forcings can be represented as quotients of a σ-ideal I with a very simple definition: I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, meaning that for any analytic set $A \subset 2^\omega \times X$ the set $\{x \in 2^\omega : A_x \in I\}$ is coanalytic. With this interpretation, no extra large cardinal assumption is necessary. The parts of arguments that need the large cardinal axioms and/or definability do not appear in this paper; they are encapsulated in the references to various general iteration and product theorems from [6]. While many proper forcings adding a single real occurring in practice fall into the categories described above as explained in [6], there are some exceptions. For example, the generic filter of posets introduced in [4, Section 2.2] can be recovered from a single real, and the posets certainly have a nice definition, but they cannot be represented as quotients, and therefore cannot be treated using the methods of this paper.

The notation in this paper follows the set theoretic standard of [2]. For a subset A of a product of two Polish spaces, symbols A_x and A^x denote the horizontal or vertical section of the set A corresponidng to x.

2 Examples

In this section I introduce a couple of examples that show the limitations on possible iteration theorems regarding the preservation of P-points and related properties. In hindsight, it is interesting to note how exactly they manage to avoid the known iteration theorems.

Example 2.1. There is a two step iteration $P * \dot{Q}$ of definable proper forcings such that neither step adds a splitting real while the iteration does.

Proof. Let P be a forcing adding an unbounded real and not adding a splitting real, such as the Miller forcing. Let $0 = n_0 \in n_1 \in n_2 \in \dots$ be a fast increasing sequence of natural numbers in the P-extension such that for every infinite set $a \subset \omega$ in the ground model, there is a number $k \in \omega$ such that $|a \cap [n_k, n_{k+1})| \geq 2^k$. Now work in the P-extension and construct the forcing Q. For every number k consider the submeasure ϕ_k on the set $2^{[n_k, n_{k+1})}$ by setting $\phi_k(b) = \min\{|z| : z$ is a collection of subsets of $[n_k, n_{k+1})$ of size 2^k such that every function in b is constant at one of them$\}$. It is not difficult to see that $\phi_k(2^{[n_k,n_{k+1})}) \geq 2^{k-1}$. Now let Q be the forcing of all nonempty trees T such that $t \in T$ implies t is a finite sequence such that $\forall k \in |t|\ t(k) \in 2^{[n_k,n_{k+1})}$, and for every $l \in \omega$ there is $m \in \omega$ such that every sequence $t \in T$ of length $k > m$ splits into more than ϕ_k-mass l many immediate successors. The ordering on Q is that of inclusion.

Now, the forcing Q is a rather standard fat tree forcing of the type studied in [4] or [6, Section 4.4.3]. It is proper, bounding, and it does not add splitting reals by [6, Theorem 4.4.8]. This judgment is passed in the P-extension. However, the iteration $P * \dot{Q}$ does add splitting real, namely the set $\dot{x}_{gen}$ whose characteristic function is the union of all functions on the Q-generic branch. Suppose that $a \subset \omega$ is an infinite set and $\langle p, \dot{q}\rangle \in P * \dot{Q}$ is a condition. Strengthening the condition p if necessary we may find a number $m \in \omega$ such that sequences in the tree $\dot{q}$ are forced to branch into more than ϕ_k-mass 1 many immediate successors for every number $k > m$, and we may find number $k \in \omega$ greater than m such that p forces $|a \cap [n_k, n_{k+1})| \geq 2^k$. Then, there must be a node $t \in \dot{q}$ with $|t| > k$ and such that the function $t(k)$ is not constant on the set $a \cap [n_k, n_{k+1})$. Then the condition $\langle p, \dot{q} \restriction \dot{t}\rangle$ forces both $a \cap \dot{x}_{gen}$ and $a \setminus \dot{x}_{gen}$ to be nonempty, and therefore $\dot{x}_{gen}$ is forced to be a splitting real!

It is possible to adjust the construction in such a way that the forcing $\dot{Q}$ in the extension even preserves the Baire category.

□

Example 2.2. There is a two step iteration $P * \dot{Q}$ of definable proper forcings such that both steps have the weak Laver property while the iteration does not.

Proof. Let P be a forcing adding a dominating real with the weak Laver property, such as the Laver forcing. In the P-extension, find a sequence $0 = n_0 \in n_1 \in n_2 \in \dots$ of natural numbers such that for every infinite set $a \subset \omega$ there is a number $m \in \omega$ such that for every larger $k \in \omega$ the set $a \cap [n_k, n_{k+1})$ is nonempty. Now work in the P-extension and let Q be the forcing consisting of those nonempty trees $T \subset \omega^{<\omega}$ such that if $t \in T$ then $\forall k \in |t|\ t(k) \in 2^k$, and for every node $t \in T$ there is a number $k \in \omega$ such that for every number $i \in [n_k, n_{k+1})$ and every number $j \in 2^i$ there is an extension $s \supset t$ in T such that $s(i) = j$. The ordering on Q is that of inclusion.

Viewed from the P-extension, the forcing Q is a rather standard tree forcing of the type studied in [4] or [6, Section 4.1]. In particular, the associated ideal is σ-generated by a σ-compact family of compact sets, it is proper, bounding, and it preserves Baire category and P-points by [6, Theorem 4.1.8]. However, the iteration $P * \dot{Q}$ does not have the weak Laver property, as witnessed by the Q-generic function $\dot{x}_{gen} \in \omega^\omega$. If $\langle p, \dot{q} \rangle$ is a condition in the iteration, $a \subset \omega$ is an infinite set, and $h : \omega \to [\omega]^{<\aleph_0}$ is a function such that $\forall n\ |h(n)| < 2^n$, then strengthening the condition p if necessary it is possible to find a number m such that for all larger k, $a \cap [\dot{n}_k, \dot{n}_{k+1})$ is forced to be nonempty. It is also possible to find a number $k > m$ such that the tree $\dot{q}$ branches at k as in the definition of the forcing $\dot{Q}$. It is then possible to find an element $i \in a$ and a number $j \in 2^i$ such that $j \notin h(i)$, and a node $t \in \dot{q}$ such that $t(i) = j$. Clearly, the condition $\langle p, \dot{q} \restriction t \rangle$ forces the generic point to avoid the function h at the number $i \in a$. $\square$

Example 2.3. There is a product of two definable forcings, each of the proper, bounding, Baire category preserving and not adding a splitting real, while the product does add a splitting real.

Proof. The forcings are interesting in their own right.

- The forcing P_I adds a function $f \in \Pi_n 2^n$ such that for every ground model function $g : \omega \to [\omega^{<\omega}]^{<\aleph_0}$ with $\forall n\ g(n) \subset 2^n \wedge |g(n)| \leq n$ and for every ground model infinite set $a \subset \omega$ there is $n \in a$ such that $f(n) \notin g(n)$.
- The forcing P_J adds a function $h : \omega \to [\omega^{<\omega}]^{<\aleph_0}$ such that $\forall n\ h(n) \subset 2^n$ and for every ground model function $g : \omega \to [\omega^{<\omega}]^{<\aleph_0}$ such that $\forall n\ g(n) \subset 2^n \wedge |g(n)| \geq n$ and for every ground model infinite set $a \subset \omega$ there is $n \in a$ such that both $g(n) \setminus f(n)$ and $g(n) \cap f(n)$ are nonempty sets.

It is perhaps not clear immediately how to arrange such forcings P_I, P_J that are proper, bounding, preserving category, and not adding splitting reals. However, once they are obtained, it follows that the forcing $P_I \times P_J$ adds a splitting real $b \subset \omega$ given by $n \in b \leftrightarrow f(n) \in h(n)$. To see why neither b nor its complement contain an infinite set from the ground model, suppose $p \in P_I$ and

$q \in P_J$ are conditions and $c \subset \omega$ is infinite. Let $g(n) = \{i \in 2^n : \exists p' \leq p\ p' \Vdash \dot{f}(\check{n}) = i\}$ and $c_1 = \{n \in c : |g(n)| > n\}$. By the properties of the P_I-name $\dot{f}$ it follows that c_1 is an infinite set. By the properties of the P_J-name $\dot{h}$ it follows that there is a number $n \in c_1$ and a condition $q' \leq q$ which decides the value of $h(\check{n})$ to be some definite finite set such that both $g(n) \cap h(n)$ and $g(n) \setminus h(n)$ are nonempty sets. It follows that there are conditions $p', p'' \leq p$ such that $p' \Vdash f(n) \notin h(n)$ and $p'' \Vdash f(n) \in h(n)$. The condition $\langle p', q' \rangle \in P_I \times P_J$ forces $\check{c} \not\subset \dot{b}$ and the condition $\langle p'', q' \rangle \in P_I \times P_J$ forces $\check{c} \cap b \neq 0$ as required.

To construct the ideal I, consider the space $X = \Pi_n 2^n$, and for every number $n \in \omega$ and every set $s \subset 2^n$, write $O_s = \{x \in X : x(n) \in s\}$. Let I be σ-generated by the sets $C \subset X$ such that there is a number $k \in \omega$ such that for every number $n \in \omega$ there are sets $s_i \subset 2^{n_i} : i \in k$, where $n_i > n$ and $|s_i| = n_i$, such that $C \subset \bigcup_i O_{s_i}$. It is not difficult to see that $X \notin I$, as $\lim \frac{|2^n|}{n} = \infty$. It is also rather immediate that the ideal I is generated by a $G_{\sigma\delta}$ collection of compact sets, and therefore the forcing P_I is proper and preserves Baire category by [6, Section 4.1].

To prove that the forcing P_I is bounding and does not add splitting reals, I will observe that the ideal I in fact belongs to the class of Hausdorff submeasure ideals associated with a lower semicontinuous weight function [6, Section 4.4]. Indeed, set $U = \{O_s : s \text{ as above}\}$, let $\mathrm{diam}(O_s) = 2^{-n}$ if $s \subset 2^n$, and for every set $a \subset U$ let $w(a) = |a|$. A brief review of the definitions shows that [6, Theorem 4.4.8] can be applied to give the desired conclusion.

It is also not difficult to see that the P_I-generic function f has the desired properties. If g is a function and $a \subset \omega$ is an infinite set as in the first item above, the set $\{x \in X : \forall n \in a\ x(n) \in g(n)\}$ is clearly one of the generating sets in the ideal I and therefore the generic function f does not belong to it.

The construction of the ideal J is similar. Let Y be the space of all functions $y : \omega \to [\omega^{<\omega}]^{<\aleph_0}$ such that $\forall n\ y(n) \subset 2^n$. For every set $s \subset 2^n$ of size at least n let $O_s = \{y \in Y : y(n) \subset s \vee y(n) \cap s = 0\}$ and observe that for a fixed number n one needs at least $n/2$ many such sets to cover the whole space Y. Let J be the σ-ideal on the space Y generated by the sets C for which there is a number $k \in \omega$ such that for every number $n \in \omega$ there are sets $s_i \subset 2^{n_i} : i \in k$ such that $n_i > n$ and $|s_i| \geq n_i$, such that $C \subset \bigcup_i O_{s_i}$. The whole treatment then transfers from the previous paragraphs. $\square$

Note that the forcing P_I is explicitly constructed in such a way that it fails the weak Laver property. The main question left open in this paper is whether the conjunction of bounding and preservation of P-points (that is to say, the conjunction of bounding and no splitting reals and weak Laver property) is preserved under product of definable proper forcing.

Example 2.4. Assume CH. There is a (undefinable) proper bounding forcing with weak Laver property, adding no splitting real, which does not preserve P-points.

Proof. Let U be a Ramsey ultrafilter. Consider the forcing P consisting of all trees $T \subset 2^{<\omega}$ such that there is a set $a \in U$ such that a node in the tree T

branches if and only if its length is in the set a. The poset P is ordered by inclusion.

It is not difficult to see that the forcing P adds a set $b \subset \omega$ such that its characteristic function is a branch through all the trees in the generic filter. No set $c \in U$ can be a subset of either b or its complement: if for example a condition T forced $c \subset b$, then find a set $a \in U$ such that nodes in T branch if and only if their length is in a, find a node $t \in T$ of length in the set $a \cap c$, and note that the tree $T \restriction t^\frown 0$ forces $|t| \in \check{c} \setminus \dot{b}$.

The regularity properties of the forcing P are somewhat more difficult to verify, but in summary the poset P has all the Fubini properties of the symmetric Sacks forcing of [3]. In fact, the symmetric Sacks forcing decomposes into two step iteration $Q * R$ in which the first step is the $\mathcal{P}(\omega)/\mathrm{Fin}$ poset adding a Ramsey ultrafilter, and the second step is exactly the forcing P described here, derived from the Ramsey ultrafilter added in the first step.

$\square$

3 Main theorems

I will start with the proof of Theorem 1.3. The argument uses the ergodic preservation theorem [6, Theorem 6.3.3]. I will show that the weak Sacks property is equivalent to the Fubini property with all σ-centered ideals and then use the general preservation theorem. A couple of definitions are necessary for the argument.

Let P be the forcing notion consisting of pairs $p = \langle a_p, b_p \rangle$ where a_p is a finite partial function from ω to $[\omega]^{<\aleph_0}$ such that $\forall k \in \mathrm{dom}(a_p)\ |a_p(k)| \leq 2^k$, and $b_p \subset \omega^\omega$ is a finite set. The ordering is defined by $q \leq p$ iff $a_p \subset a_q$, $b_p \subset b_q$, and for every $k \in \mathrm{dom}(a_q \setminus a_p)$ and every $f \in b_p$ it is the case that $f(k) \in a_q(k)$. It is not difficult to see that P is a σ-centered notion of forcing (conditions with the same first coordinate are mutually compatible), and the generic filter is determined by the union of the first coordinates of conditions in it. I will call this union a_{gen}. It is a function with infinite domain, and for every ground model function $f \in \omega$, for all but finitely many numbers $n \in \mathrm{dom}(a_{gen})$, $f(n) \in a_{gen}(n)$. Thus the forcing P is designed to perform a job perpendicular to the violation of the weak Sacks property. I will reserve the letter J for the σ-ideal associated with the forcing P. The underlying Polish space Y is the collection of all functions $a : \omega \to [\omega]^{<\aleph_0}$ with infinite domain and such that for every $n \in \mathrm{dom}(a)$, $|a(n)| \leq 2^n$.

It is not difficult to see that the function $g \in \omega^\omega$ defined by $g(m) = \max(a_{gen}(n))$ where $n = \min(\mathrm{dom}(a_{gen}) \setminus m)$ modulo finite dominates all the ground model elements of ω^ω. Thus the poset P adds a dominating real, but it is not equivalent to the Hechler forcing. One rather roundabout way to see this is to note that the poset P_I from Example 2.3 is perpendicular to P, since it fails the weak Sacks property, while it is not perpendicular to the Hechler forcing, since it is bounding and preserves Baire category. In fact, the forcing P is the most complicated definably σ-centered forcing, as this section shows.

In order to simplify certain complexity calculations and identify interesting variations, a restricted version of the forcing P will be useful. Let $f \in \omega^\omega$ be a function. The forcing P_f is defined just as P is, except all the functions in b_p must be pointwise dominated by f, and $\forall p \forall n \in \mathrm{dom}(a_p)\ a_p(k) \subset f(k)$. The corresponding ideal on a Polish space Y_f will be denoted by J_f. The advantage of these restricted versions is that they do not add dominating reals, as the following claim and [6, Proposition 3.8.15] show.

Claim 3.1. *Let $f \in \omega^\omega$ be a function. The ideal J_f is ergodic and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. Recall that the ideal J_f is ergodic if there is a countable Borel equivalence relation E on the space Y_f such that every Borel E-invariant set is either in the ideal or or its complement is. Just let $h, k \in Y_f$ be E-equivalent if $\mathrm{dom}(k) = \mathrm{dom}(h)$ up to a finite set and $k = h$ at all but finitely many entries. This is certainly a Borel equivalence relation with countable classes, and it is not difficult to observe that if $h \in Y_f$ is a generic point, then all of its equivalents are generic as well. Now suppose that $B \subset Y_f$ is E-invariant. I will show that either $P_f \Vdash \dot{y}_{gen} \in \dot{B}$ or $P_f \Vdash \dot{y}_{gen}$
$notin\dot{B}$, and that will complete the proof of the ergodicity. Suppose for contradiction that there are conditions $p, q \in P_f$, $p \Vdash \dot{y}_{gen} \in \dot{B}$ and $q \Vdash \dot{y}_{gen} \notin \dot{B}$. Let M be a countable elementary submodel and let $h \in Y_f$ be an M-generic point meeting the condition p. It is not difficult to find a point $k \in Y_f$ such that kEh and k meets the condition q. Then k is an M-generic point as well, and the forcing theorem implies that $h \in B$ and $k \notin B$, contradicting the E-invariance of the set B.

The $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ part follows from [6, Proposition 3.8,11]. I just need to show that the poset P_f is very Suslin, that is, the collection $\{A \in P_f^\omega : A$ is a maximal antichain$\}$ is a Borel set. To see this, for every finite set a let $P_{a,n} = \{p \in P_f : a = a_p \wedge |b_p| = n\}$. For every condition $q \in P_f$, every a and every n the set $K_{q,a,n} = \{b \in [\omega^\omega]^n :$ every function in b is dominated by f and the condition $\langle a, b\rangle$ is incompatible with $q\}$ is compact. A countable collection A of conditions in P_f is a maximal antichain if and only if the conditions in A are pairwise incompatible and for every a and n, $\bigcap_{q \in A} K_{q,a,n} = 0$. The incompatibility is certainly a Borel condition, and checking a compact set for emptiness is Borel as well. Thus, the poset P_f is very Suslin, and the claim follows.

□

Finally, instead of the usual notion of σ-centeredness I will have to use its definable counterpart. I will call a forcing Q *definably σ-centered* if

- there is a Polish space Z such that Q consists of nonempty closed subsets of Z ordered by inclusion;
- Q is a Suslin forcing: the relations of compatibility and incompatibility are analytic;

- for every number $\varepsilon > 0$, the sets of radius $\leq \varepsilon$ form an open dense subset of Q;
- Q is separated: for any two conditions $p, q \in Q$ either $p \cap q \in Q$ or there is a condition $p' \leq p$ such that $p' \cap q = 0$;
- there are countably many analytic sets $Q_n \subset C(Z) : n \in \omega$ such that $Q = \bigcup_n Q_n$ and each Q_n is centered in Q.

The first four conditions are designed to ascertain that Q is a Suslin forcing adding a single point in the space Z such that a set is in the generic filter if and only if it contains this generic point. All Suslin forcings for adding a single real I know of are of this form, but I do not have a general theorem. The key condition is the last one. Clearly, the forcing P together with all its restricted versions is definably σ-centered. On the other hand, there are definable forcings which are σ-centered but not definably σ-centered; this is a rather unusual situation though.

With these definitions in hand, I can state the key lemma.

Lemma 3.2. *Suppose that I is a suitably definable σ-ideal on a Polish space X such that the quotient forcing P_I is proper. The following are equivalent:*

1. *P_I has the weak Sacks property;*
2. *$I \not\perp J$;*
3. *P_I is bounding and $I \not\perp J_f$ for every function $f \in \omega^\omega$;*
4. *for every definably σ-centered forcing and its associated σ-ideal K, $I \not\perp K$.*

Proof. The implications (4)→(2) and (4)→(3) are easy, noting that the posets P and P_f described above are definably σ-centered. To see why (2) implies (1), suppose that P_I fails the weak Sacks property, let $B \in P_I$ be a condition and $f : B \to \omega^\omega$ be a Borel function such that $B \Vdash \dot{f}(\dot{x}_{gen})$ cannot be predicted on a ground model infinite set, and let $D \subset B \times Y$ be the Borel set defined by $\langle x, a\rangle \in D$ iff $x \restriction n \notin a(n)$ for infinitely many numbers $n \in \mathrm{dom}(a)$. It is not difficult to verify that the Borel set D has J-small vertical sections, and its complement has I-small horizontal sections.

The key implication is (1)→(4). Suppose that P_I has the weak Sacks property, and K is a σ-ideal on a Polish space Y obtained from a definably σ-centered forcing Q, as witnessed by the centered families $Q_n : n \in \omega$. Let $\dot{y}_{gen}$ be the Q-name for the generic point in the space Y. Suppose that $B \subset X$ is an I-positive Borel set, $C \subset Y$ is a K-positive Borel set, and $D \subset B \times C$ is a Borel set with K-small vertical sections. I must find an I-positive horizontal section of the complement of the set D.

Let M be a countable elementary submodel of a large structure. A *wall* is a Borel function $f \in M$ such that $\mathrm{dom}(f) \subset B$ is a Borel I-positive set and (the coherence condition) $\mathrm{rng}(f) \subset Q_n$ for some number $n \in \omega$. Walls are ordered by $g \le f$ if $\mathrm{dom}(g) \subset \mathrm{dom}(f)$ and $\forall x \in \mathrm{dom}(g)\ g(x) \le f(x)$. I will find a decreasing sequence $f_0 \ge f_1 \ge f_2 \ge \dots$ of walls such that $\bigcap_n \mathrm{dom}(f_n)$ is an I-positive set, and for every point x in it the sequence $f_n(x) : x \in \omega$ is $M[x]$-generic for the poset Q. By the coherence condition, the generic point $y \in Y$ obtained from this generic sequence does not depend on the point x. Since the set $D \subset B \times C$ had K-small vertical sections, it must be the case that $\langle x, y \rangle \notin D$ for any point $x \in \bigcap_n \mathrm{dom}(f_n)$. Thus the horizontal section of the complement of the set D corresponding to the point y is I-positive as required.

Towards the construction of the decreasing sequence $f_n : n \in \omega$ of walls, first use the bounding property of the forcing P_I to find a compact I-positive set $B_0 \subset B$ consisting of M-generic reals only, such that all Borel subsets of the space X in the model M are relatively clopen in it. This is possible by [6, Theorem 3.3.2]. The following is the key claim. In order to simplify the notation, I assume $X = 2^\omega$ and for a finite set $a \subset 2^{<\omega}$ I write $O_a = \{x \in 2^\omega : \exists t \in a\ t \subset x\}$.

Claim 3.3. *Suppose that $f \in M$ is a wall, $\dot{O} \in M$ is a P_I-name for an open dense subset of Q and $n \in \omega$ is a number. Then there is a number $m \in \omega$ such that for every set $a \subset 2^m$ of size n either there is a wall $g \le f$ such that $\mathrm{dom}(g) \cap B_0 = \mathrm{dom}(f) \cap B_0 \cap O_a$ and $\mathrm{dom}(g) \Vdash g(\dot{x}_{gen}) \in \dot{O}$.*

Once this is showed, first find a wall f_0 such that $\mathrm{dom}(f_0) \Vdash_{P_I} f_0(\dot{x}_{gen}) \Vdash_Q \dot{y}_{gen} \in \dot{C}$, enumerate P_I-names for open dense subsets of the poset Q in the model M by $\dot{O}_k : k \in \omega$ and then by induction on $k \in \omega$ construct numbers $m_k : k \in \omega$ and walls $f_a : a \in [2^{m_k}]^{\le 2^k}$ such that $B_0 \cap O_a \subset \mathrm{dom}(f_a)$, $f_a \Vdash \dot{f}_a(\dot{x}_{gen}) \in \bigcap_{i \in k} \dot{O}_i$ where k is such that $a \subset 2^{m_k}$, and whenever $b < a$ then $f_b < f_a$. Once this is done, use the weak Sacks property to find a set $B_1 \subset B_0$ in the poset P_I and an infinite set $c \subset \omega$ such that for all $k \in c$, $|\{t \in 2^{m_k} : O_t \cap B_1 \ne 0\}| \le 2^k$. The walls $f_a : a = \{t \in 2^{m_k} : O_t \cap B_1 \ne 0\}, k \in c$ will have the required properties.

Thus only the claim remains to be shown. Suppose $f, \dot{O}$ and n are given. For every collection $x_i : i \in n$ of points in the compact set $\mathrm{dom}(f) \cap B_0$, repetitions allowed, consider the conditions $f(x_i) : i \in n$ in the poset Q. Since f was a wall, these conditions are all in the same centered set, and they have a lower bound, say p. The sets $\dot{O}/x_i$ are all predense in the poset Q, and therefore there is a condition $q \le p$ belonging to all of them; say $q \in Q_j$ for some number j. By the forcing theorem, for every number $i \in n$ there is a condition $\bar{B}_i \subset B$ in the model M and a Borel function $g_i : \bar{B}_i \to Q_j$ such that $x_i \in \bar{B}_i$ and $\bar{B}_i \Vdash \dot{g}_i(\dot{x}_{gen}) \in \dot{O}$. Note that the sets $\bar{B}_i : i \in n$ are all relatively clopen in B_0. It is clearly possible to choose the sets $\bar{B}_i$ in such a way that $x_i = x_j$ implies $\bar{B}_i = \bar{B}_j$ and $g_i = g_j$, and $x_i \ne x_j$ implies $\bar{B}_i \cap \bar{B}_j = 0$, and then it is possible to combine the functions $g_i : i \in n$ into a single wall. Thus the compact set $[B_0 \cap \mathrm{dom}(f)]^n$ is covered by relatively open sets for which there is a wall $g \le f$ such that $\mathrm{dom}(g) \Vdash \dot{g}(\dot{x}_{gen}) \in \dot{O}$. A compactness argument yields the required number m.

□

Theorem 1.3 now quickly follows from the ergodic iteration theorem [6, Theorem 6.3.3]. For every function $f \in \omega^\omega$, the ideal J_f is ergodic c.c.c. The Fubini property with respect to each J_f is preserved by the countable support iteration of definable proper forcings by the ergodic iteration theorem, and so is the bounding condition by [1, Theorem 6.3.5]. The weak Sacks property is just a conjunction of these properties by the lemma.

The proof of Theorem 1.4 appears on the surface to be simpler. For simplicity look at the case of a product of a single definable proper forcing P_I, where I is a σ-ideal on the space $X = 2^\omega$. The compact sets are dense in the quotient by [6, Theorem 3.3.2] and so I can represent the quotient forcing as a poset Q of binary trees closed under restriction.

Note that the weak Sacks property implies bounding and preservation of Baire category, the latter by [8, Proposition 3.2]. By [5] or [6, Theorem 5.2.6], this means that the countable support product is proper, bounding, preserves Baire category, and moreover, the rectangular Ramsey property holds: whenever $(2^\omega)^\omega = \bigcup_m B_m$ is a covering of the product space by countably many analytic sets, then there are trees $T_n \in Q : n \in \omega$ such that the product $\Pi_n[T_n]$ wholly contained in one of the pieces of the cover.

Now suppose that $\dot{f}$ is a product name for a function in ω^ω. The bounding and rectangular properties of the product imply that there is a condition $\langle T_n : n \in \omega\rangle$ in the product and numbers $m_k : k \in \omega$ such that for every $k \in \omega$ and every choice $t_n : n \in \omega$ of sequences of length k in the respective trees T_n, the condition $\langle T_n \restriction t_n : n \in \omega\rangle$ decides the value of $\dot{f}(n)$. Choose nondecreasing functions $g_n : n \in \omega$ in ω^ω diverging to infinity such that for every $k \in \omega$, $\Pi_n(g_n(k) + 1) \leq 2^k$. Use the weak Sacks property at each coordinate of the product to find trees $S_n \leq T_n$ and infinite sets $a_0 \supset a_1 \supset \dots$ such that $\forall n\ \forall k \in a_n\ |S_n| \cap 2^{m_k} \leq g_n(k) + 1$. Find sequences $s_n \in S_n$ of length m_k, where $k = \min(a_n)$, and consider the condition $\langle S_n \restriction s_n : n \in \omega\rangle$, and the infinite set $b = \{\min(a_n) : n \in \omega\}$. It is not difficult to see that for every number $k \in b$, the trees $S_n \restriction s_n$ contain at most $g_n(k) + 1$ many nodes at level m_k, and therefore there are at most $\Pi_n(g_n(k) + 1) \leq 2^k$ many possibilities left for the value of $\dot{f}(k)$. This verifies the weak Sacks property of the product.

While neither definability of the forcing nor the large cardinal assumptions are mentioned explicitly in the above argument, they are used in the proof of properness and rectangular property of the product.

References

[1] Tomek Bartoszynski and Haim Judah. *Set Theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[2] Thomas Jech. *Set Theory.* Academic Press, San Diego, 1978.

[3] Andrzej Rosłanowski. n-localization property. *Journal of Symbolic Logic*, 71:881–902, 2006.

[4] Andrzej Roslanowski and Saharon Shelah. Norms on possibilities I: forcing with trees and creatures. *Memoirs of the American Mathematical Society*, 141:xii + 167, 1999. math.LO/9807172.

[5] Jindřich Zapletal. Proper forcing and rectangular Ramsey theorems. *Israel Journal of Mathematics*, 152:29–47, 2006.

[6] Jindřich Zapletal. *Forcing Idealized*. Cambridge Tracts in Mathematics 174. Cambridge University Press, Cambridge, 2008.

[7] Jindřich Zapletal. n-localization property in iterations. submitted to Mathematical Logic Quarterly, 2008.

[8] Jindřich Zapletal. Preservation of p-points and definable forcing. submitted to Fundamenta Mathematicae, 2008.